Quantum mechanism angular momentum & graphical calculus

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Loops'26 summer school
Yangzhou University, May 11-12, 2026

Contents:

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1. Group theory

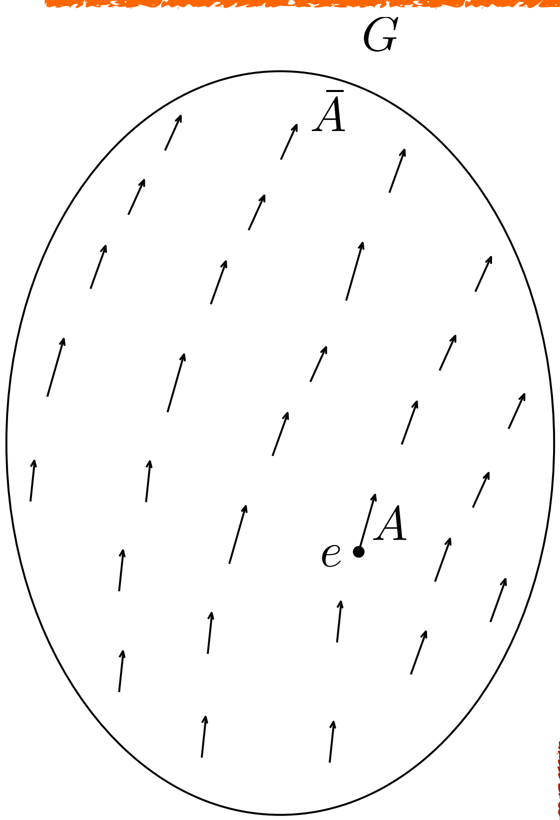
Def. (Group): a set G equipped with a mapping (group multiplication) $G \times G \to G$ satisfying the following conditions is called a group:

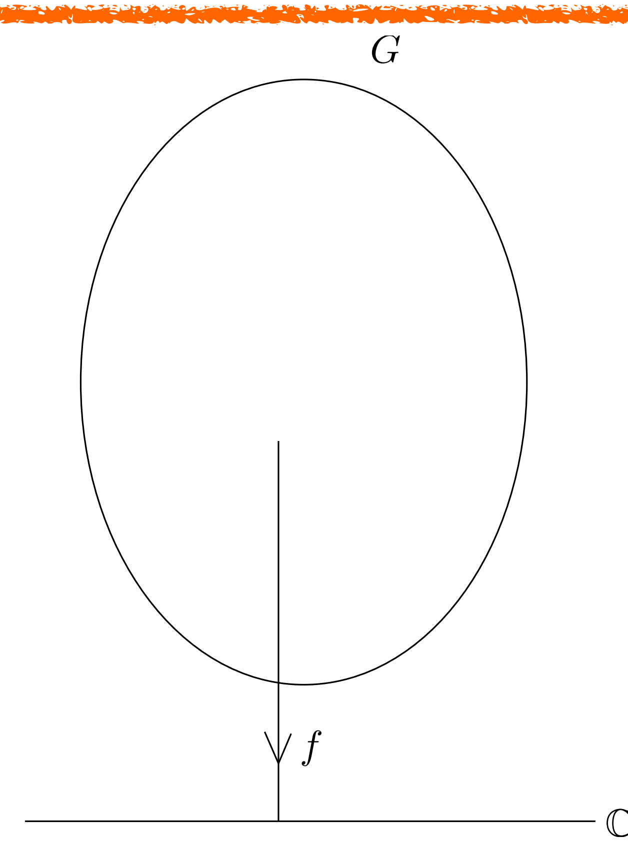
(i) $(g_1 g_2) g_3 = g_1 (g_2 g_3), \quad \forall g_1, g_2, g_3 \in G$

(ii) $\exists$ identity element e such that $ge = eg = g, \quad \forall g \in G$

(iii) $\forall g \in G$ $\exists$ its inverse g^{-1} such that $gg^{-1} = g^{-1}g = e$

Def. (Lie group): if G both an n-dimensional manifold and a group, and the group multiplication mapping $G \times G \to G$ and the inverse mapping $G \to G$ are both C^∞, then G is called an n-dimensional Lie group.





vector fields → left-invariant vector field → (↑) angular moment operator

scalar fields → Hilbert space → (↑) quantum state

1. Group theory

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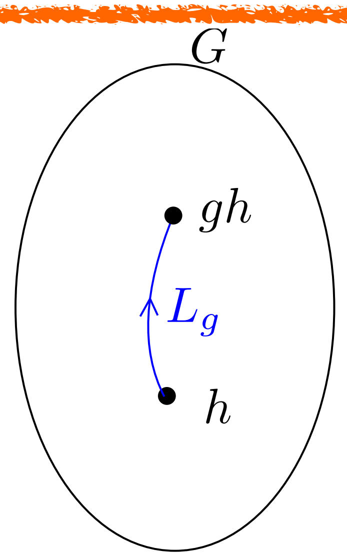
Def. (Lie group): if G both an n-dimensional manifold and a group , and the group multiplication mapping $G \times G \to G$ and the inverse mapping $G \to G$ are both C^∞, then G is called an n-dimensional Lie group.

Def. (Left translation): Let G be a Lie group, $\forall g \in G$, a map

$$L_g : h \mapsto gh \quad \forall h \in G$$

is called the left translation generated by g .



1. Group theory

Def. (Left-invariant vector field): A vector field $\bar{A}$ on a Lie group G is called left-invariant if

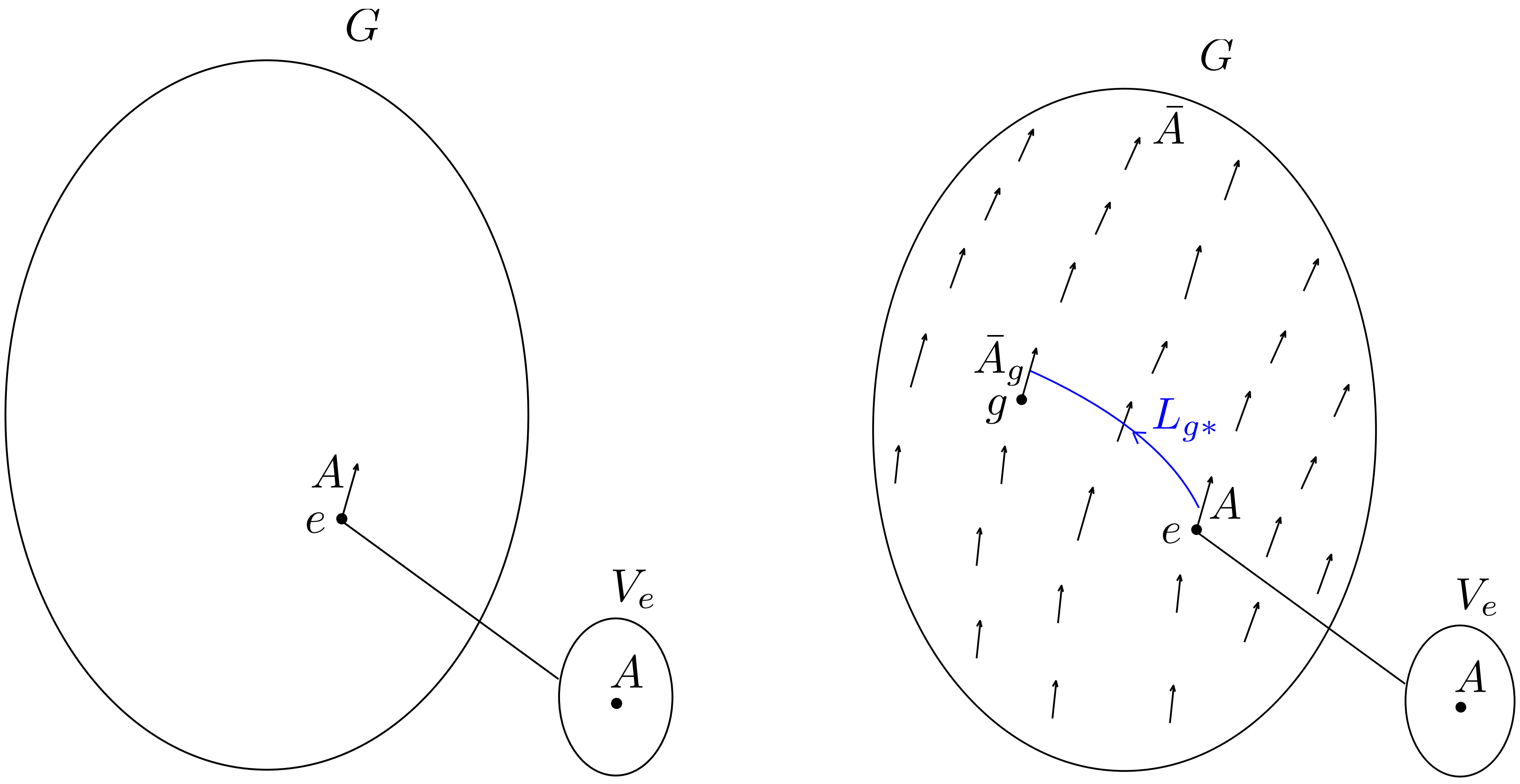
$$L_{g*}\bar{A} = \bar{A}, \quad \forall g \in G,$$

where L_{g*} is the pushforward mapping induced by the left translation $L_g : G \to G$.

Thm. Any vector $A \in V_e$ on a Lie group G determines a vector field by:

$$\bar{A}_g := L_{g*}A, \quad \forall g \in G,$$

which is left-invariant, and thus is called the left-invariant vector field generated by $A \in V_e$.



1. Group theory

Def. (One-parameter subgroup): A C^∞ curve $\gamma : \mathbb{R} \to G$ is called a one-parameter subgroup of a Lie group G if

$$\gamma(s+t) = \gamma(s)\gamma(t), \quad \forall s, t \in \mathbb{R}$$

where $\gamma(s)\gamma(t)$ denotes the group product of the group elements $\gamma(s)$ and $\gamma(t)$.

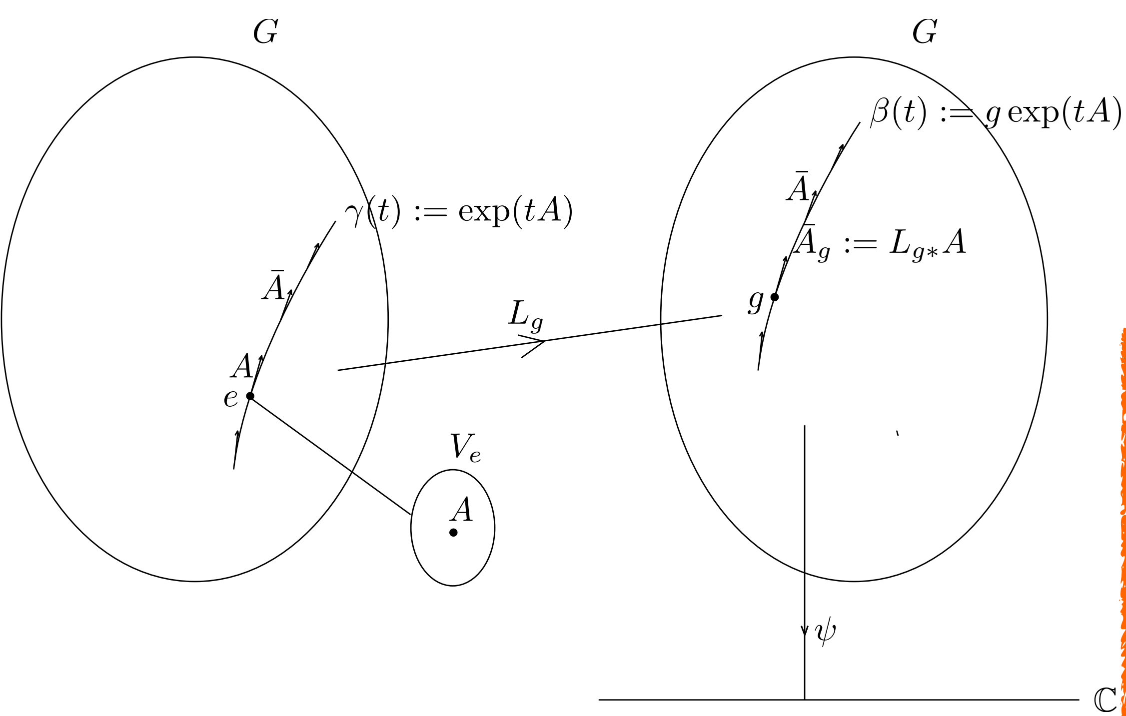
Thm. Any vector $A \in V_e$ on a Lie group G determines a one-parameter subgroup by exponential map as:

$$\gamma(t) := \exp(tA), \quad \forall t \in \mathbb{R}, A \in V_e,$$

which is an integral curve of the left-invariant vector field $\bar{A}$ generated by $A \in V_e$.
Let

$$\beta(t) \equiv g\gamma(t), \quad \forall g \in G,$$

then $\beta : \mathbb{R} \to G$ is also an integral curve of $\bar{A}$ passing through g .



$$\bar{A}_g\psi(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi \circ \beta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(\beta(t))$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(g\exp(tA))$$

Hence

$$\bar{A}_g\psi(g) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(g\exp(tA))$$

is often used to give the definition of the left-invariant vector field $\bar{A}$ generated by $A \in V_e$.

1. Group theory

Quantum mechanics on $\mathrm{SU}(2)$

$G = \mathrm{SU}(2)$: a Lie group

$\dim(\mathrm{SU}(2)) = \dim(su(2)) = 2^2 - 1 = 3$

$\forall g \in SU(2)$ can be uniquely written as

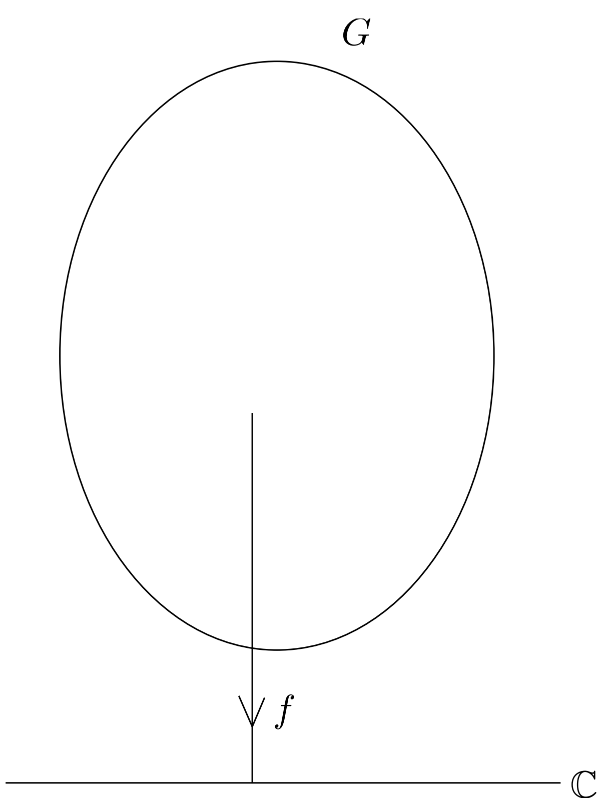
$$g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad \text{with } a, b \in \mathbb{C}, \quad a\bar{a} + b\bar{b} = 1$$

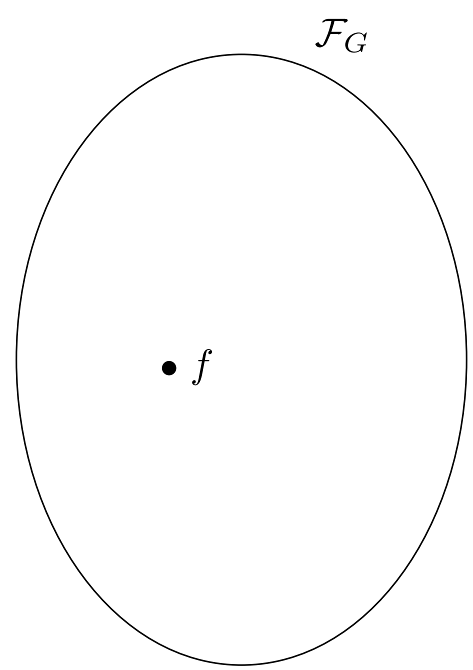
Thm. There exists a unique Harr measure μ_H over $\mathrm{SU}(2)$ which satisfies

(i) Invariant: $\mu_H(h) = \mu_H(hg) = \mu_H(gh) = \mu_H(h^{-1})$

(ii) Normalized: $\mu_H(\mathrm{SU}(2)) = 1$

The Hilbert space $\mathscr{H}_{\mathrm{SU}(2)}$ composed of complex-valued functions f over $\mathrm{SU}(2)$





$$(f_1, f_2) := \int_{\mathrm{SU}(2)} \overline{f_1(g)} f_2(g) \mathrm{d}\mu_H(g)$$

$$\mathscr{H}_{\mathrm{SU}(2)} := \overline{(\mathscr{F}_{\mathrm{SU}(2)}, (\,\cdot\,,\,\cdot\,))} = L^2(\mathrm{SU}(2), \mathrm{d}\mu_H)$$

Completion of $\mathscr{F}_{\mathrm{SU}(2)}$ w.r.t. the inner product

1. Group theory

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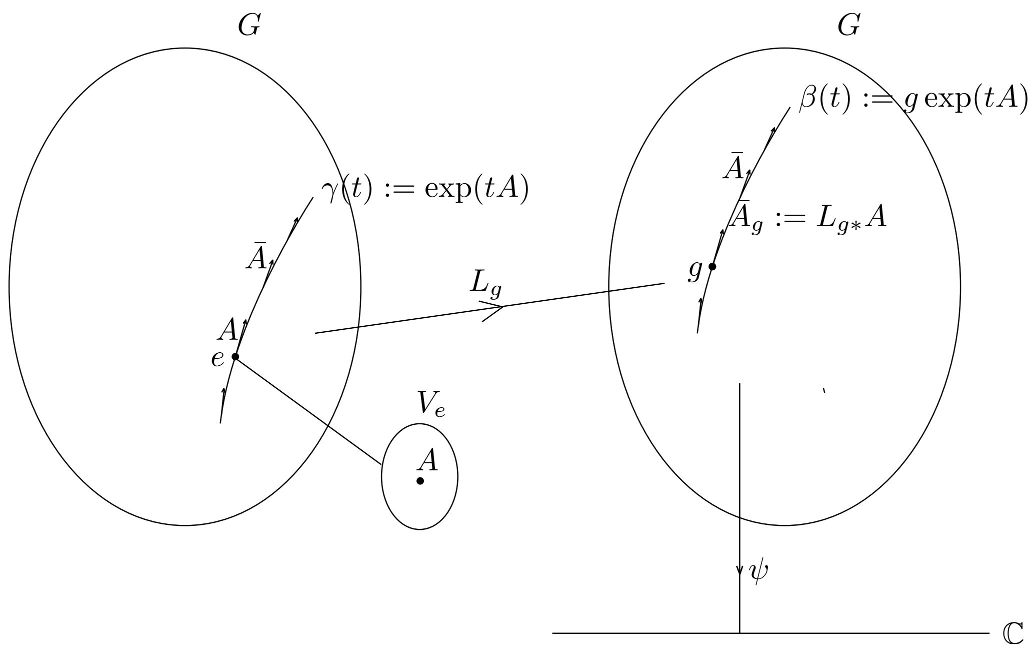


Hence

$$\bar{A}_g \psi(g) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(g\exp(tA))$$

is often used to give the definition of the left-invariant vector field $\bar{A}$ generated by $A \in V_e$.

The **left-invariant vector field** $L^{(\tau_i)}$ and the **right-invariant vector field** $R^{(\tau_i)}$ on SU(2), determined by the generator, $\tau_i = -\frac{i}{2}\sigma_i \in V_e$ with σ_i being the Pauli matrices, are defined by

$$L^{(\tau_i)}\psi(g) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi\left(g e^{t\tau_i}\right) \qquad L^{(\tau_i)} g^A{}_B = (g\tau_i)^A{}_B \quad \Leftrightarrow \quad L^{(\tau_i)} g = g\tau_i$$

$$R^{(\tau_i)}\psi(g) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi\left(e^{-t\tau_i} g\right) \qquad R^{(\tau_i)} g^A{}_B = -(\tau_i g)^A{}_B \quad \Leftrightarrow \quad R^{(\tau_i)} g = -\tau_i g$$

Then one can define the so-called angular momentum operators on $\mathcal{H}_{\mathrm{SU}(2)}$ by

$$\hat{J}_i^{(L)} := \mathrm{i}L^{(\tau_i)} \quad \text{and} \quad \hat{J}_i^{(R)} := \mathrm{i}R^{(\tau_i)},$$

satisfying $\quad [\hat{J}_i^{(L)}, \hat{J}_j^{(L)}] = \mathrm{i}\epsilon^k{}_{ij}\hat{J}_k^{(L)}, \quad [\hat{J}_i^{(R)}, \hat{J}_j^{(R)}] = \mathrm{i}\epsilon^k{}_{ij}\hat{J}_k^{(R)}, \quad [\hat{J}_i^{(L)}, \hat{J}_j^{(R)}] = 0.$

The Casimir operator on $\mathcal{H}_{\mathrm{SU}(2)}$ can be expressed as $\quad \hat{J}^2 := \delta^{ij}\hat{J}_i^{(L)}\hat{J}_j^{(L)} = \delta^{ij}\hat{J}_i^{(R)}\hat{J}_j^{(R)}.$

2. Representation theory of a group

Def. (A representation of a group). *A representation of a group G is a map*

$$\begin{aligned} \pi : G &\to \mathcal{B}(V) \\ g &\mapsto \pi(g) \end{aligned}$$

where $\mathcal{B}(V)$ denotes the bounded linear operators on some Hilbert space V, called the representation space, satisfying

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \quad \forall g_1, g_2 \in G$$

Def. (Faithful and trivial representation). *A representation π is called faithful if it is injective, equivalently,*

$$\pi(g) = \mathbb{I}_V \quad \Rightarrow \quad g = e$$

and it called trival if

$$\pi(g) = \mathbb{I}_V, \quad \forall g \in G$$

Def. (Dual representation). *Let V^* be the space dual to V, that is, the space of continuous linear functionals on V (since V is a Hilbert space, $V^* = V$ by the Riesz lemma). Then the representation π^* dual (or contragredient) to π is defined by*

$$[\pi^*(g)f](v) := f(\pi(g^{-1})v)$$

In a Hilbert space V we have $f(\cdot) = \langle f, \cdot \rangle$ so that $\pi^*(g) = [\pi(g^{-1})]\dagger$ where $\dagger$ denotes the adjoint with respective to $\langle \cdot, \cdot \rangle$.

Def. (Unitary representation). *A representation $\pi : G \to \mathcal{B}(V)$ is called unitary if $\pi(g)$ is a unitary operator on V for all $g \in G$, that is, $[\pi(g)]^\dagger = [\pi(g)]^{-1}$.*

Def. (Tensor product representation). *Let $\pi_i : G \to \mathcal{B}(V_i)$, $i = 1, 2$, be representations. The tensor product $\pi_1 \otimes \pi_2 : G \to \mathcal{B}(V_1 \otimes V_2)$ is defined by*

$$[\pi_1 \otimes \pi_2](g) \cdot v_1 \otimes v_2 := (\pi_1(g)v_1) \otimes (\pi_2(g)v_2)$$

2. Representation theory of a group

Def. (Invariant subspace & irreducible/reducible representation)**.** *A closed subspace* $V_1 \subset V$ *is called invariant for a representation* $\pi : G \to \mathcal{B}(V)$ *iff*

$$\pi(g)V_1 \subset V_1, \quad \forall g \in G.$$

A representation is called irreducible if it has no invariant subspace except for the trivial invariant subspaces V *and* $\{0\}$*, otherwise reducible.*

Def. (Completely reducible representation)**.** *A representation* $\pi : G \to \mathcal{B}(V)$ *is said to be completely reducible if it decomposes into a direct sum of irreducible representations* π_i *on the spaces* V_i*, that is,*

$$\pi = \oplus_i \pi_i$$

where $V = \oplus_i V_i$ *and the set of indices i is countable.*

Thm. *Every finite-dimensional unitary representation is completely reducible.*

2. Representation theory of a group

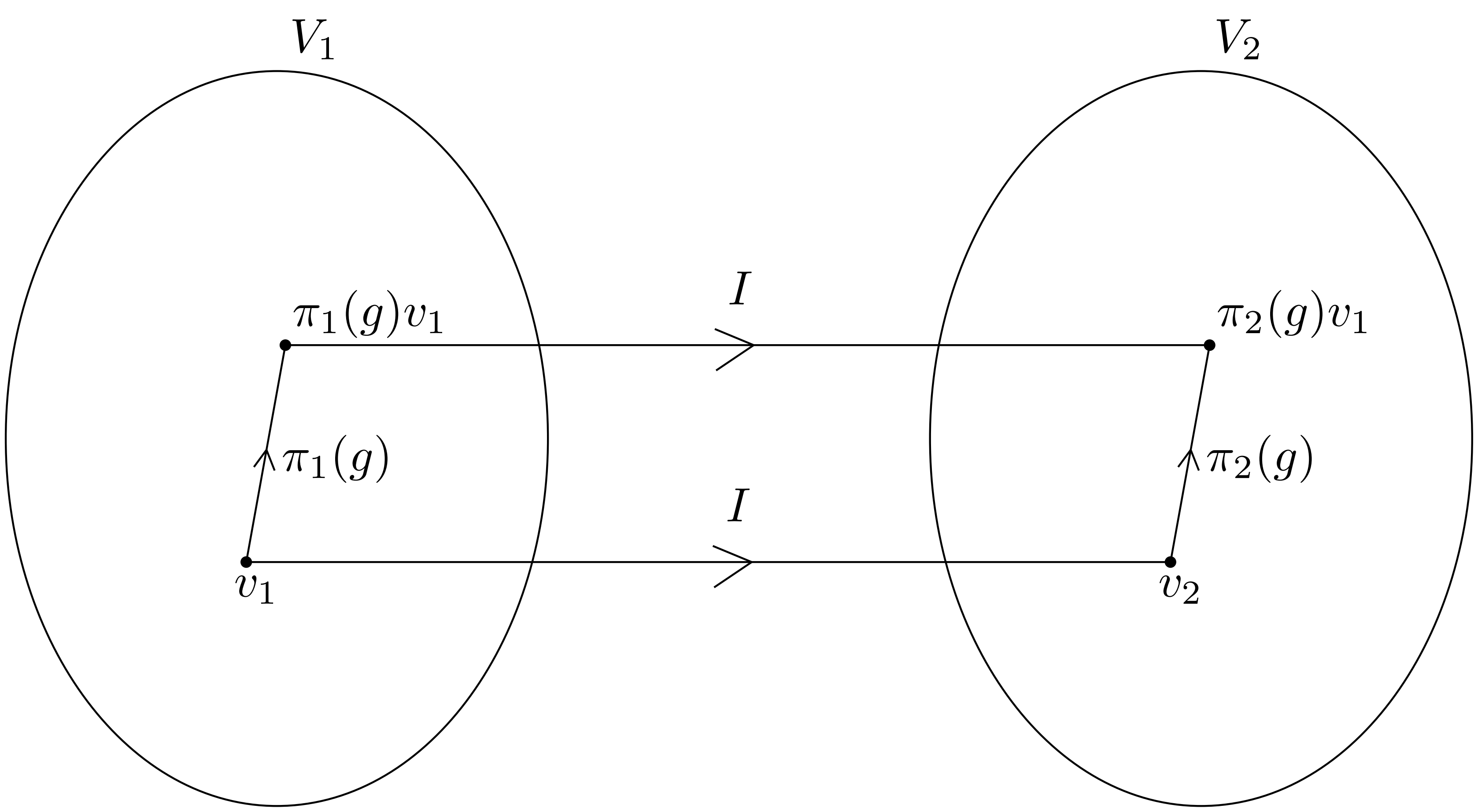
Def. (Equivalent representations & intertwiner)**.** *Two representations $\pi_i : G \to \mathcal{B}(V_i)$, $i = 1, 2$, are called equivalent iff there exists an invertible (unitary) linear map*

$$I : V_1 \to V_2$$

satisfying [it "commutes" with the corresponding operators of the representations]

$$I\,\pi_1(g) = \pi_2(g)\,I \quad \Leftrightarrow \quad \pi_1(g) = I^{-1}\,\pi_2(g)\,I \quad \Leftrightarrow \quad \pi_2(g)^{-1} I\,\pi_1(g) = I, \quad \forall g \in G$$

Such a map I is called an intertwiner (intertwining operator, invariant tersor) for these two representations.



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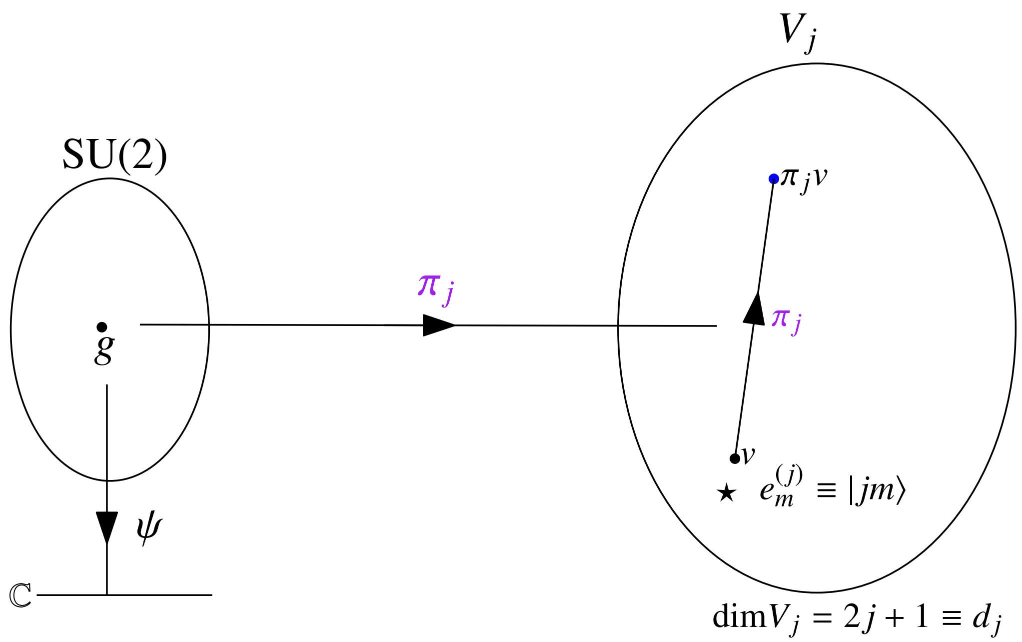
The irreducible unitary representation of SU(2) group (characterized by spin quantum number $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$)



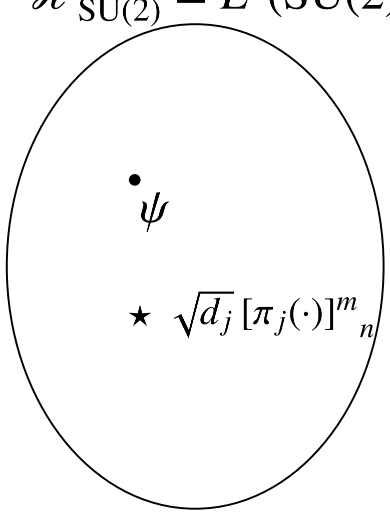
$$[\pi_j(g)]^m{}_n \equiv \langle jm | \pi_j(g) | jn \rangle$$

$$= \sum_{\ell} \frac{\sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}}{(j-m-\ell)!(j+n-\ell)!(m-n+\ell)!\ell!} a^{j+n-\ell} (\bar{a})^{j-m-\ell} b^{\ell} (-\bar{b})^{m-n+\ell}$$

Thm (Peter and Weyl). *Let G be a compact Lie group (such as* SU(2)*). The system of (spin network) functions on G, consisting of matrix elements* $\sqrt{d_j}\,[\pi_j(\cdot)]^m{}_n$ *in finite dimensional irreducible representations* π_j *labeled by half-integers* j *with dimension* d_j*, is a complete orthonormal basis for* $\mathcal{H}_G \equiv L^2(G, \mathrm{d}\mu_H)$,

$$\int_G d_j \overline{[\pi_{j'}(g)]^{m'}{}_{n'}} [\pi_j(g)]^m{}_n \mathrm{d}\mu_H(g) = \delta_{j'j} \delta^{m'm} \delta_{n'n}$$

$\mathscr{H}_{\mathrm{SU}(2)} = L^2(\mathrm{SU}(2), \mathrm{d}\mu_H)$



$$\psi(g) = \sum_{j,m,n} \psi^{jn}_m \sqrt{d_j} [\pi_j(g)]^m{}_n, \quad \forall \psi \in \mathscr{H}_{SU(2)}$$

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Angular momentum operators on $\mathscr{H}_{\mathrm{SU}(2)}$

$$\hat{J}_i^{(L)} := \mathrm{i}L^{(\tau_i)} \quad \text{and} \quad \hat{J}_i^{(R)} := \mathrm{i}R^{(\tau_i)},$$

$$L^{(\tau_i)}\psi(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi\left(ge^{t\tau_i}\right)$$

$$R^{(\tau_i)}\psi(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi\left(e^{-t\tau_i}g\right)$$

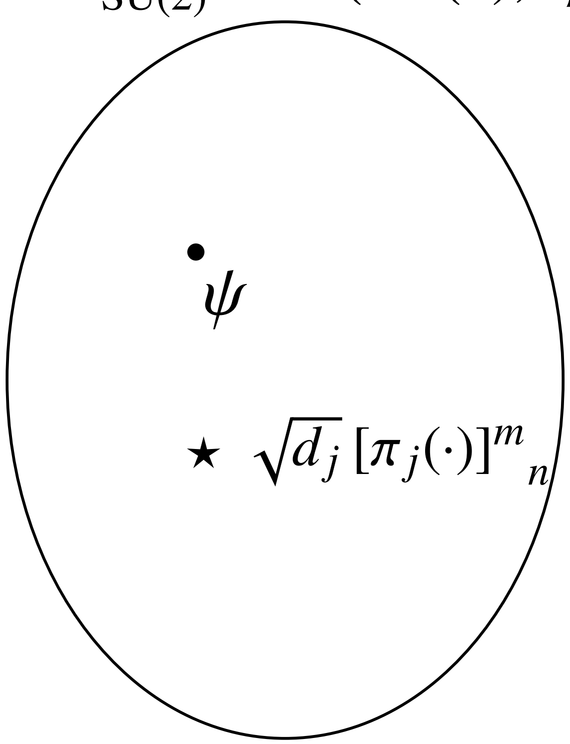
$$[\pi_j(\tau_k)]^m{}_n := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left[\pi_j(e^{t\tau_k})\right]^m{}_n, \quad e^{t\tau_i} = \cos(t)\mathbb{I}_2 + 2\sin(t)\tau_i \quad \longrightarrow$$

$$[\pi_j(\tau_1)]]^m{}_n = -\frac{\mathrm{i}}{2}\sqrt{j(j+1) - m(m-1)}\,\delta_{m-n,1} - \frac{\mathrm{i}}{2}\sqrt{j(j+1) - m(m+1)}\,\delta_{m-n,-1}$$

$$[\pi_j(\tau_2)]]^m{}_n = -\frac{1}{2}\sqrt{j(j+1) - m(m-1)}\,\delta_{m-n,1} + \frac{1}{2}\sqrt{j(j+1) - m(m+1)}\,\delta_{m-n,-1}$$

$$[\pi_j(\tau_3)]^m{}_n = -\mathrm{i}m\,\delta_{m-n,0}$$

$\mathscr{H}_{\mathrm{SU}(2)} = L^2(\mathrm{SU}(2), \mathrm{d}\mu_H)$

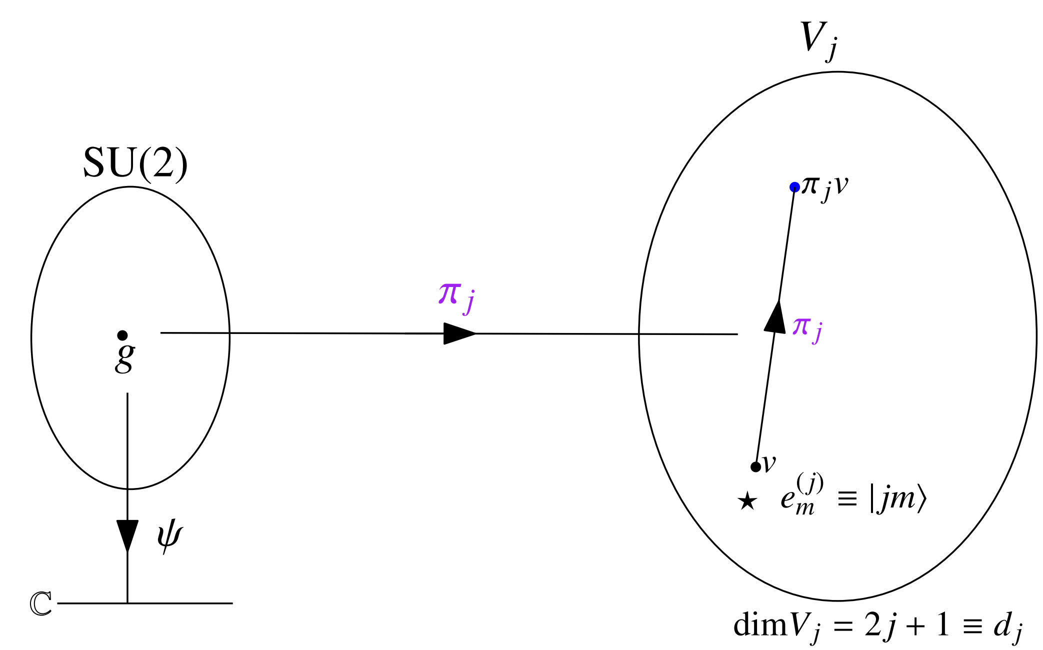


$$\hat{J}^2[\pi_j(g)]^m{}_n = j(j+1)[\pi_j(g)]^m{}_n$$

$$\hat{J}_3^{(R)}[\pi_j(g)]^m{}_n = -m[\pi_j(g)]^m{}_n$$

$$\hat{J}_3^{(L)}[\pi_j(g)]^m{}_n = n[\pi_j(g)]^m{}_n$$

The irreducible unitary representation of SU(2) group



$$[\pi_j(g)]^m{}_n = \langle jm \,|\, \pi_j(g) \,|\, jn \rangle$$

$$\hat{J}_3^{(R)}\langle jm| = -m\langle jm|, \qquad \hat{J}_3^{(L)}|jn\rangle = n|jn\rangle$$

2. Representation theory of a group

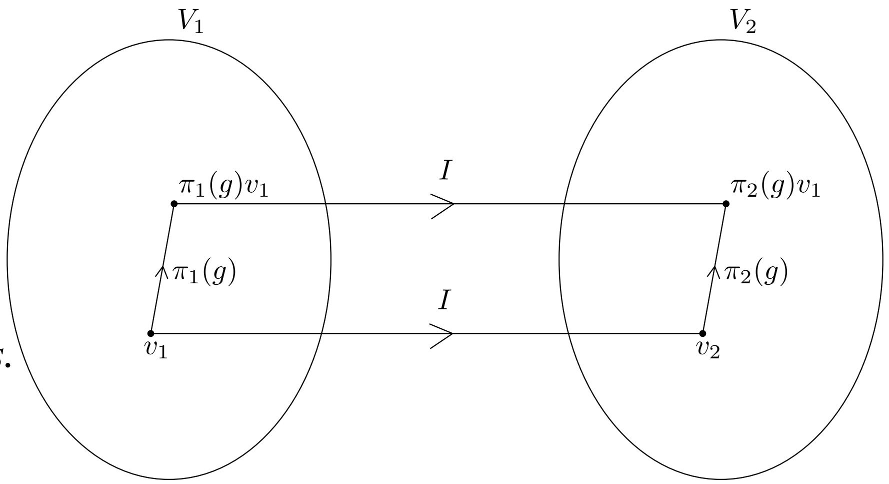
Def. (Equivalent representations & intertwiner). *Two representations $\pi_i : G \to \mathcal{B}(V_i)$, $i = 1, 2$, are called equivalent iff there exists an invertible (unitary) linear map*

$$I : V_1 \to V_2$$

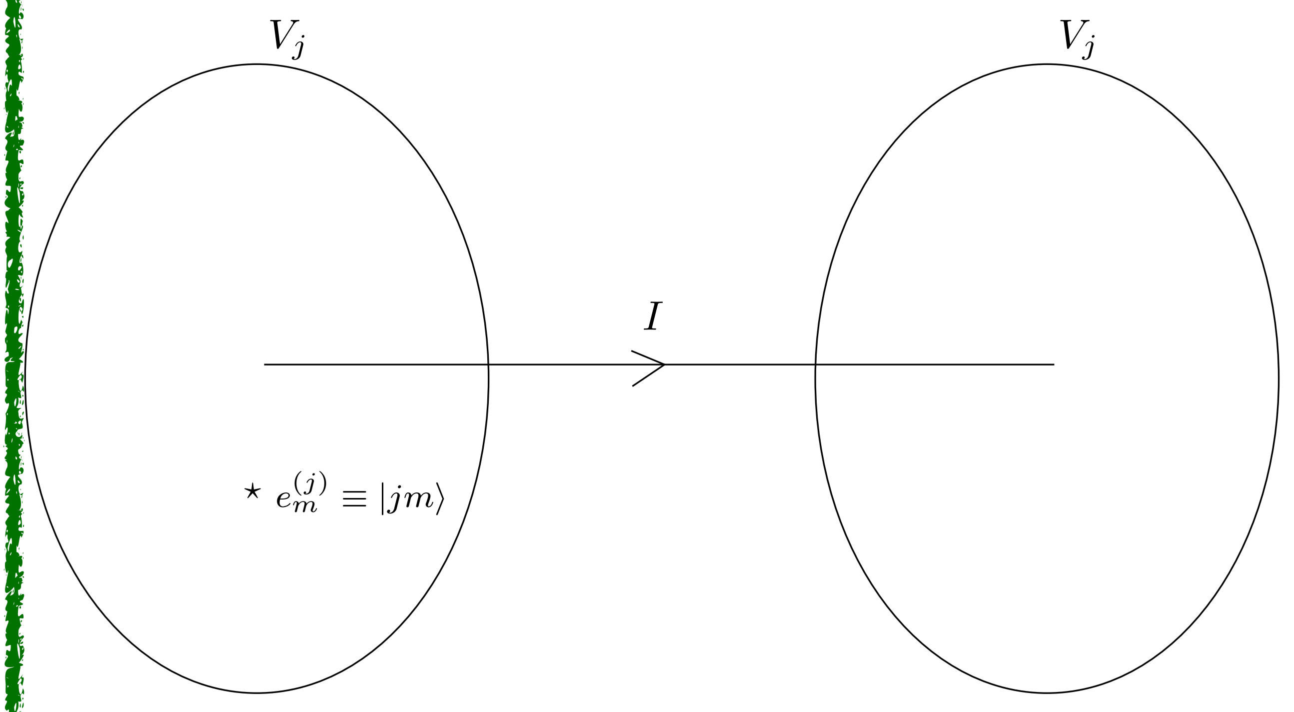
satisfying [it "commutes" with the corresponding operators of the representations]

$$I\,\pi_1(g) = \pi_2(g)\,I \quad \Leftrightarrow \quad \pi_1(g) = I^{-1}\,\pi_2(g)\,I \quad \Leftrightarrow \quad \pi_2(g)^{-1} I\,\pi_1(g) = I, \quad \forall g \in G$$

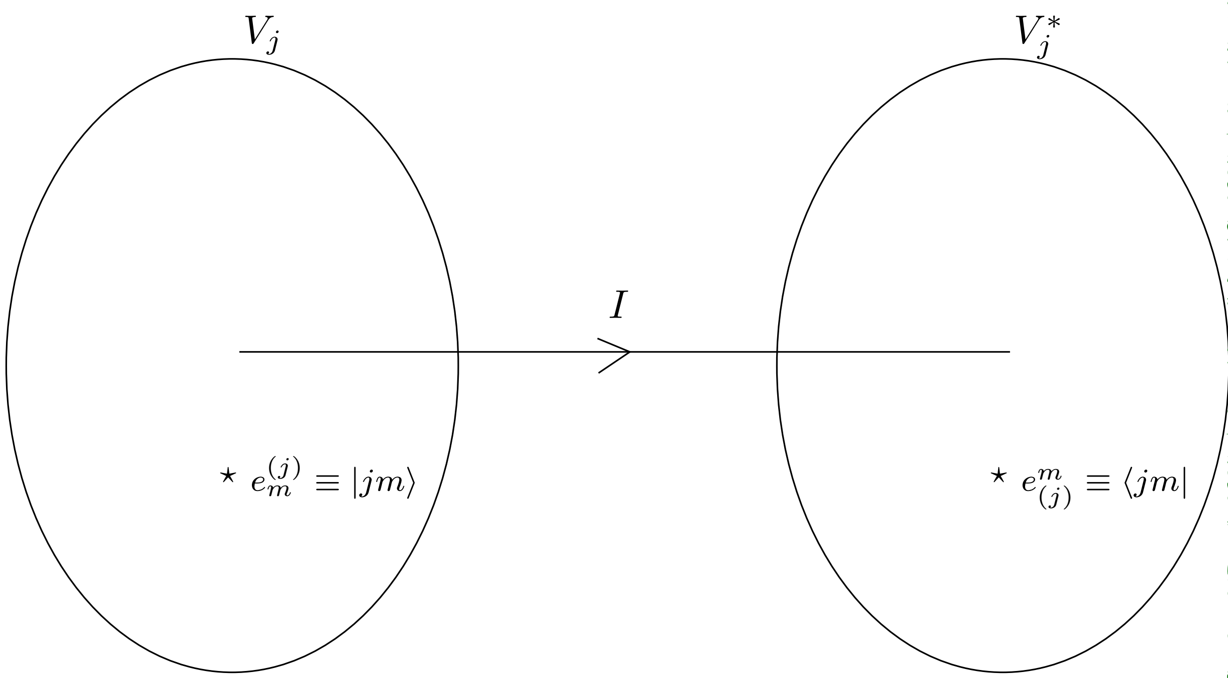
Such a map I is called an intertwiner (intertwining operator, invariant tersor) for these two representations.



Two frequently used **intertwiners** in irrep. of SU(2)



$$I^m{}_n = \delta^m{}_n$$



Notation in Wigner's book

$$I_{m'm} \equiv C^{(j)}_{m'm} := (-1)^{j-m}\delta_{m,-m'}$$

$$I^{mm'} \equiv C^{mm'}_{(j)} := (-1)^{j-m}\delta_{m,-m'}$$

2. Representation theory of a group

Let $\pi_{j_i} : G \to \mathcal{B}(V_{j_i})$, $i = 1, 2$, be two irreducible unitary representations of SU(2). Consider the direct product representation:

$$\pi_1 \otimes \pi_2 : G \to \mathcal{B}(V_1 \otimes V_2)$$

Thm.. *Every finite-dimensional unitary representation is completely reducible.*

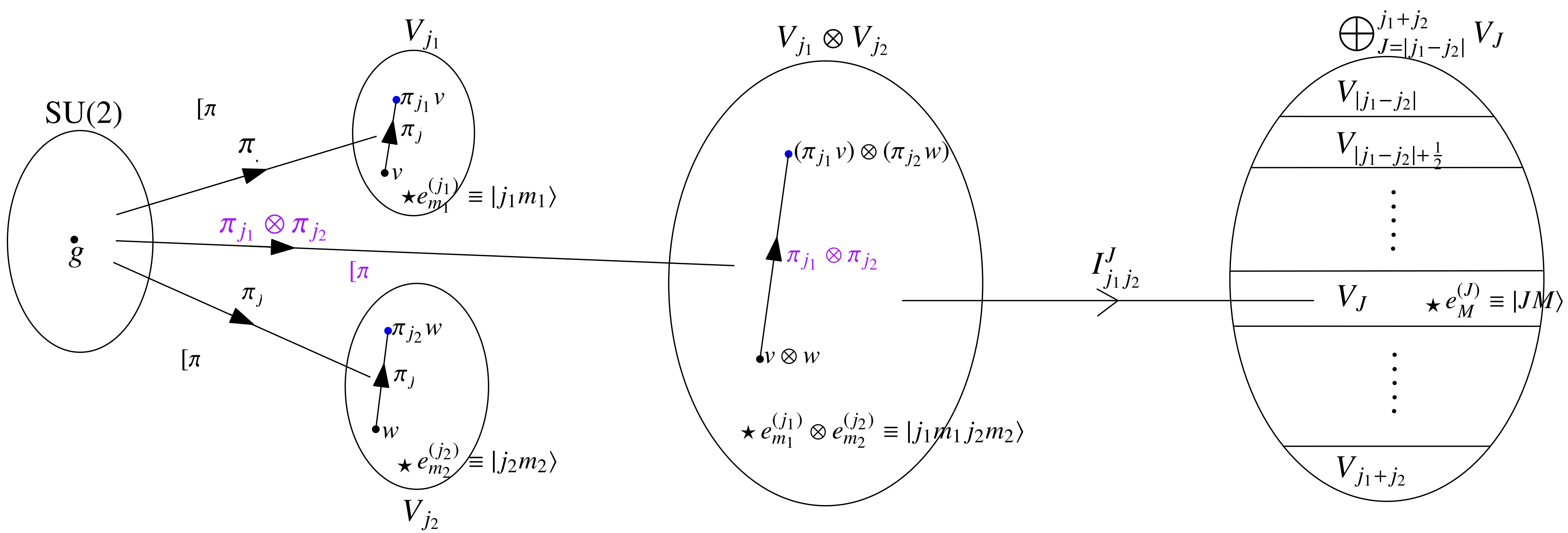
Means: The direct product rep. can be written as a direct sum of irreps.

i.e. there exists a bijective intertwiner that maps the tensor product to a direct sum of irrps.

Physics: angular momentum coupling **Mathematics: recoupling theory**

$$V_{j_1} \otimes V_{j_2} \cong \oplus_J V_J$$

equivlance of reps.



$\dim(V_{j_1} \otimes V_{j_2}) = \dim V_{j_1} \times \dim V_{j_2}$ $\dim(V_{j_1} \otimes V_{j_2}) = \sum_J \dim V_J$

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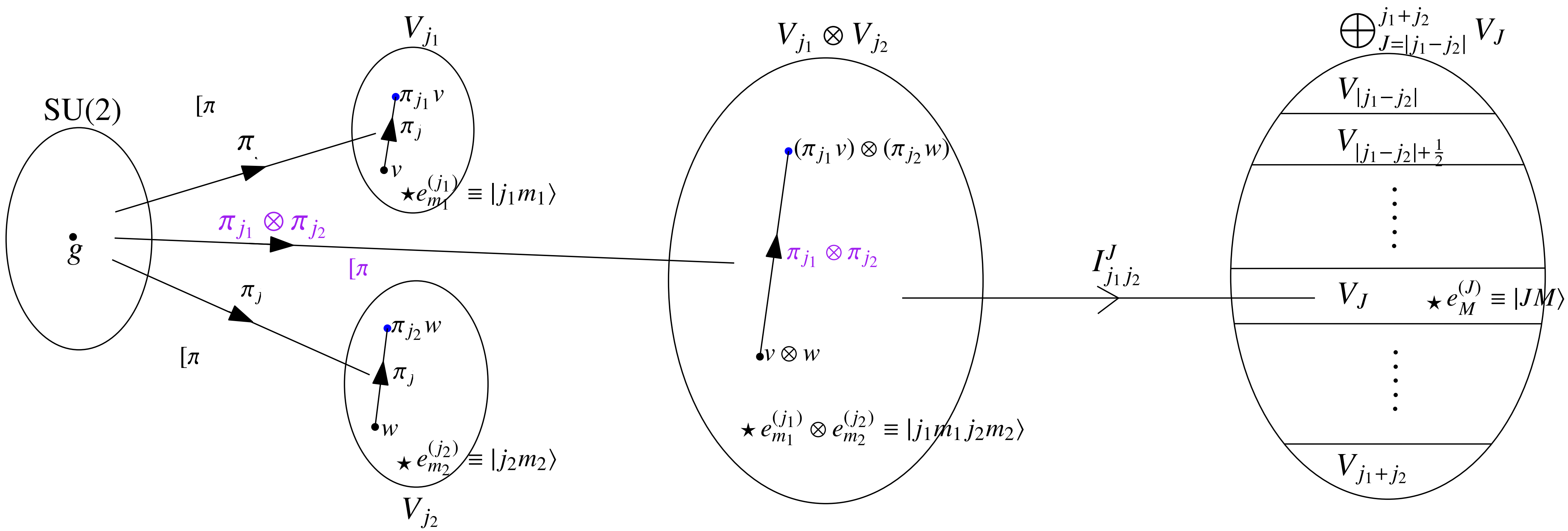


$\dim(V_{j_1} \otimes V_{j_2}) = \dim V_{j_1} \times \dim V_{j_2}$ $\dim(V_{j_1} \otimes V_{j_2}) = \sum_J \dim V_J$

The intertwiner: $I^J_{j_1 J_2} : V_1 \otimes V_2 \to \otimes_J V_J$

$$e^{(j_1)}_{m_1} \otimes e^{j_2}_{m_2} \mapsto I^J_{j_1 j_2}(e^{(j_1)}_{m_1} \otimes e^{j_2}_{m_2}) = \sum_M \langle JM | I^J_{j_1 j_2} | j_1 m_1 j_2 m_2 \rangle e^{(J)}_M$$
$$= \sum_M \langle JM | j_1 m_1 j_2 m_2 \rangle e^{(J)}_M$$

The Clebsch-Gordan coefficients (CGCs) are chosen as real numbers in the Condon-Shortley gauge

$$\langle j_1 m_1 j_2 m_2 | JM \rangle = \langle JM | j_1 m_1 j_2 m_2 \rangle$$

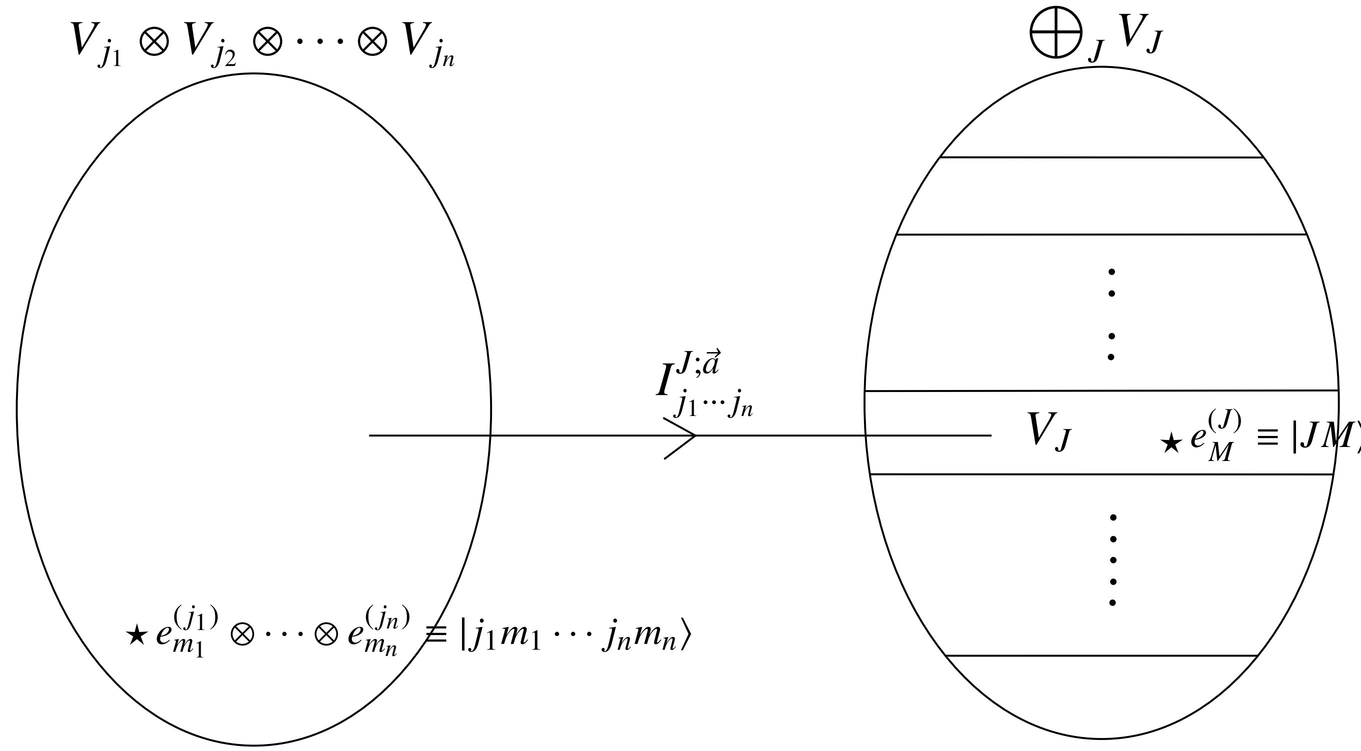
2. Representation theory of a group

Physics: angular momentum coupling **Mathematics: recoupling theory**

Let $\pi_{j_i} : G \to \mathcal{B}(V_{j_i}),\ i = 1, 2, \ldots, n$, be n irreducible unitary representations of SU(2)

$$V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes \cdots \otimes V_{j_{n-1}} \otimes V_{j_n} \cong \oplus_J V_J$$

$$\underbrace{V_{j_1} \otimes V_{j_2}}_{V_{a_2}}, \quad \underbrace{V_{a_2} \otimes V_{j_3}}_{V_{a_3}}, \quad \cdots, \quad \underbrace{V_{a_{n-1}} \otimes V_{j_n}}_{V_J}$$



The intertwiner:

$$\left(I^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}^{\quad M} = \sum_{k_2,\cdots,k_{n-1}} \left(I^{a_2}_{j_1 j_2}\right)_{m_1 m_2}^{\quad k_2} \cdots \left(I^{a_{i+1}}_{a_i j_{i+1}}\right)_{k_i m_{i+1}}^{\quad k_{i+1}} \cdots \left(I^{J}_{a_{n-1} j_n}\right)_{m_n}^{\quad M}$$

$$\langle JM; \vec{a}\,|\, j_1 m_1 j_2 m_2 \cdots j_n m_n\rangle = \sum_{k_2,\cdots,k_{n-1}} \langle a_2 k_2 \,|\, j_1 m_1 j_2 m_2\rangle \langle a_3 k_3 \,|\, a_2 k_2 j_3 m_3\rangle \cdots \langle JM \,|\, a_{n-1} k_{n-1} j_n m_n\rangle$$

where $\vec{a} \equiv \{a_2, \cdots, a_{n-1}\}$ denotes the set of the angular momenta appeared in the intermediate coupling.

The Clebsch-Gordan series: $\pi_{j_1}(g) \otimes \pi_{j_2}(g) \otimes \cdots \otimes \pi_{j_n}(g) = (I^J_{j_1 j_2})^{-1}\, \pi_J(g)\, I^J_{j_1 j_2}$

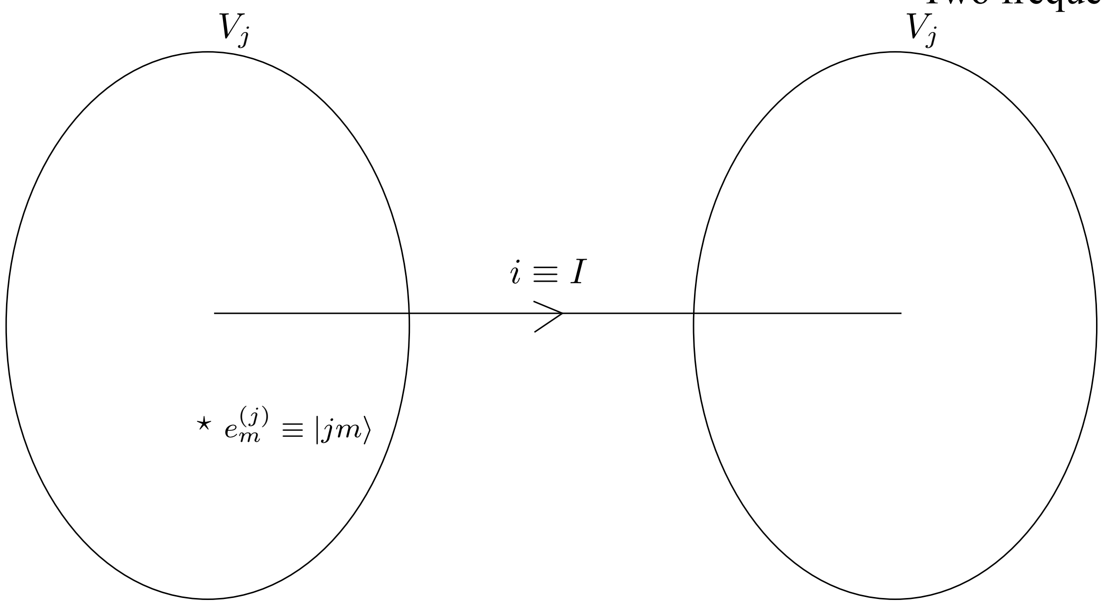
$$[\pi_{j_1}(g)]^{m_1}_{\quad n_1} \cdots [\pi_{j_n}(g)]^{m_n}_{\quad n_n} = \sum_{J,M,N} \left((I^{J;\vec{a}}_{j_1\cdots j_n})^{-1}\right)_M^{\quad m_1\cdots m_n} [\pi_J(g)]^M_{\quad N} \left(I^{J;\vec{a}}_{j_1\cdots j_n}\right)_{n_1\cdots n_n}^{\quad N}$$

$$\left((I^{J;\vec{a}}_{j_1\cdots j_n})^{-1}\right)_M^{\quad m_1\cdots m_n} = \left((I^{J;\vec{a}}_{j_1\cdots j_n})^{\dagger}\right)_M^{\quad m_1\cdots m_n} = \left(I^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}^{\quad M}$$

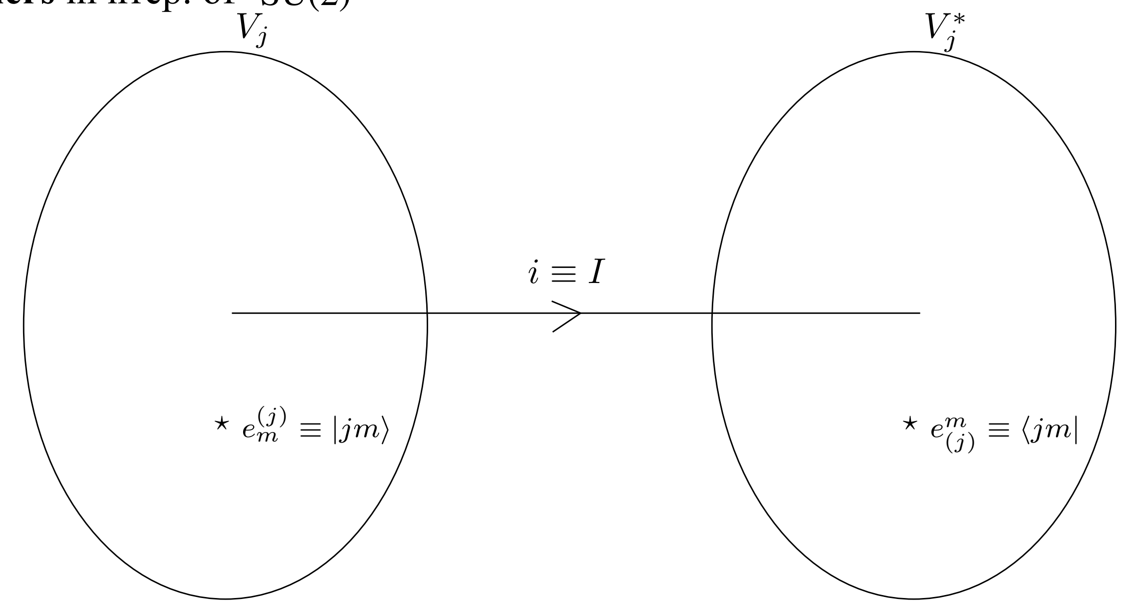
2. Representation theory of a group

To be convenient to be suitable for graphical formula, we induce i to represent intertwiner, instead of I .

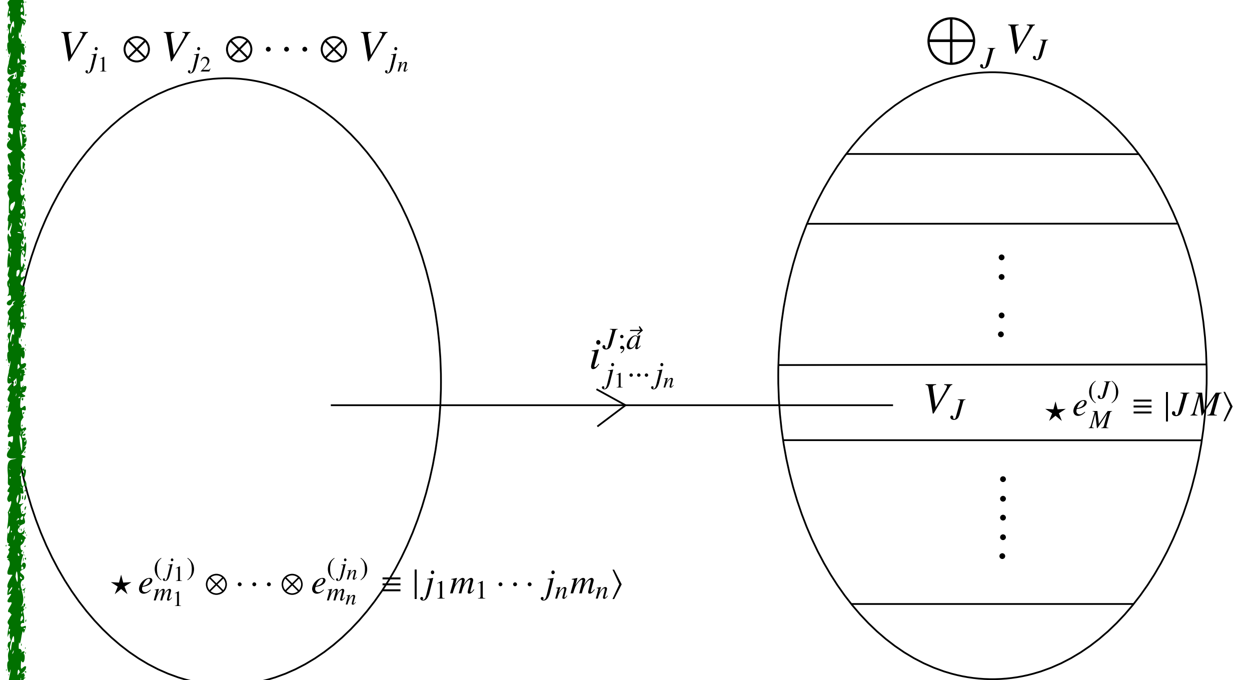
Two frequently used **intertwiners** in irrep. of SU(2)



$$i^m{}_n := I^m{}_n = \delta^m{}_n$$



$$i_{m'm} := I_{m'm} \equiv C^{(j)}_{m'm} := (-1)^{j-m}\delta_{m,-m'}$$
$$i^{mm'} := I^{mm'} \equiv C^{mm'}_{(j)} := (-1)^{j-m}\delta_{m,-m'}$$



The intertwiner:

$$\left(i^{J;\,\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^{M} := (-1)^{j_1-\sum_{i=2}^n j_i - J}\left(I^{J;\,\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^{M}$$
$$= (-1)^{j_1-\sum_{i=2}^n j_i - J}\langle JM;\,\vec{a}\,|\,j_1 m_1 j_2 m_2 \cdots j_n m_n\rangle$$

The Clebsch-Gordan series:

$$[\pi_{j_1}(g)]^{m_1}{}_{n_1}\cdots[\pi_{j_n}(g)]^{m_n}{}_{n_n} = \sum_{J,M,N}\left((i^{J;\,\vec{a}}_{j_1\cdots j_n})^{-1}\right)_M{}^{m_1\cdots m_n}[\pi_J(g)]^M{}_N\left(i^{J;\,\vec{a}}_{j_1\cdots j_n}\right)_{n_1\cdots n_n}{}^{N}$$

Thanks for your attention!