Quantum mechanism angular momentum & graphical calculus

IV

Jinsong Yang (Guizhou university)

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Contents:

Applications of graphical calculus to LQG

1. The matrix element of the volume operator
2. The action of Euclidean Hamiltonian constraint operator on a SNF over trivalent vertices
3. The action of the inverse volume operator on a SNF over trivalent vertices

References:

[1] A.P. Yutsis, I.B. Levinson, V.V. Vanagas, *Mathematical Apparatus of the Theory of Angular Momentum* (Israel Program for Scientific Translations, Translated from Russian by A. Sen and R. N. Sen, 1962)

[2] D.M. Brink, G.R. Satchler, *Angular Momentum* (Clarendon Press, Oxford, 1968)

[3] D.A. Varshalovich, A.N. Moskalev, V.K. Khersonsky, *Quantum Theory of Angular Momentum: Irreducible Tensors, Spherical Harmonics, Vector Coupling Coefficients, 3nj Symbols* (World Scientific, Singapore, 1988)

[4] R. De Pietri, C. Rovelli, Geometry eigenvalues and scalar product from recoupling theory in loop quantum gravity. Phys. Rev. D **54**, 2664–2690 (1996). https://doi.org/10.1103/PhysRevD.54.2664. arXiv:gr-qc/9602023

[5] T. Thiemann, Closed formula for the matrix elements of the volume operator in canonical quantum gravity. J. Math. Phys. **39**, 3347–3371 (1998). https://doi.org/10.1063/1.532259. arXiv:gr-qc/9606091

[6] J. Brunnemann, T. Thiemann, Simplification of the spectral analysis of the volume operator in loop quantum gravity. Class. Quant. Grav. **23**, 1289–1346 (2006). https://doi.org/10.1088/0264-9381/23/4/014. arXiv:gr-qc/0405060

[7] R. Borissov, R. De Pietri, C. Rovelli, Matrix elements of Thiemann's Hamiltonian constraint in loop quantum gravity. Class. Quant. Grav. **14**, 2793–2823 (1997). https://doi.org/10.1088/0264-9381/14/10/008. arXiv:gr-qc/9703090

[8] E. Alesci, T. Thiemann, A. Zipfel, Linking covariant and canonical LQG: New solutions to the Euclidean scalar constraint. Phys. Rev. D **86**, 024017 (2012). https://doi.org/10.1103/PhysRevD.86.024017. arXiv:1109.1290

[9] J. Brunnemann, T. Thiemann, On (cosmological) singularity avoidance in loop quantum gravity. Class. Quant. Grav. **23**, 1395–1428 (2006). https://doi.org/10.1088/0264-9381/23/5/001. arXiv:gr-qc/0505032

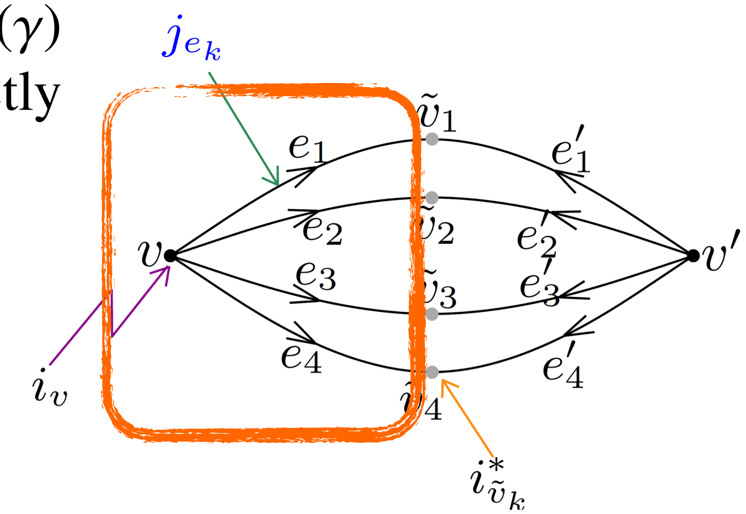
[10] J. Brunnemann, T. Thiemann, Unboundedness of triad-like operators in loop quantum gravity. Class. Quant. Grav. **23**, 1429–1484 (2006). https://doi.org/10.1088/0264-9381/23/5/002. arXiv:gr-qc/0505033

[11] J. Yang, Y. Ma, Graphical calculus of volume, inverse volume and Hamiltonian operators in loop quantum gravity. Eur. Phys. J. C **77**, 235 (2017). https://doi.org/10.1140/epjc/s10052-017-4713-0. arXiv:1505.00223

[12] E. Alesci, I. Mäkinen, J. Yang, Graphical calculus of spin networks, in *Handbook of Quantum Gravity*, edited by C. Bambi, L. Modesto, I. Shapiro (Springer, 2023). https://doi.org/10.1007/978-981-19-3079-9_101-1. arXiv:2403.18020

Review of Last Lesson's Key Points

standard graph



For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \ldots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v

$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

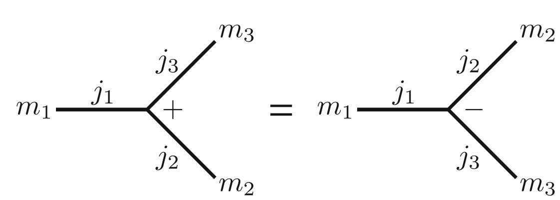
$$(i_v)_{m_1m_2\cdots m_n}{}^M \equiv \left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^M = (-1)^{j_1-\sum_{i=2}^n j_i - J}\sum_{k_2,\ldots,k_{n-1}}\langle a_2k_2|j_1m_1j_2m_2\rangle\cdots\langle JM|a_{n-1}k_{n-1}j_nm_n\rangle$$

$$\langle j_3m_3|j_1m_1j_2m_2\rangle = (-1)^{j_1-j_2-j_3}\sqrt{d_{j_3}}\sum_{m'_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m'_3\end{pmatrix} C^{m'_3m_3}_{(j_3)}$$

Wigner 3-j symbol "Metric"

Wigner 3j-symbol

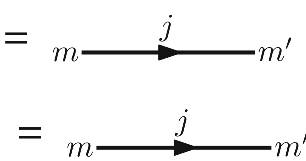
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} =$$



"Metric"

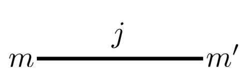
$$C^{(j)}_{m'm} = (-1)^{j-m}\delta_{m,-m'} = (-1)^{j+m'}\delta_{m,-m'} =$$

$$C^{mm'}_{(j)} = (-1)^{j-m}\delta_{m,-m'} = (-1)^{j+m'}\delta_{m,-m'} =$$



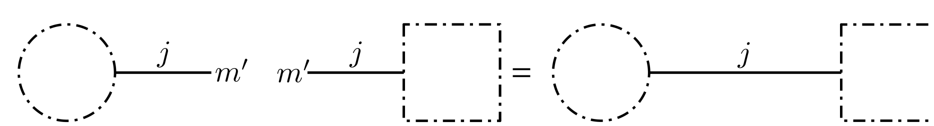
Kronecker delta symbol

$$\left(\delta^{(j)}\right)^m_{m'} = \left(\delta^{(j)}\right)^{m'}_m =$$

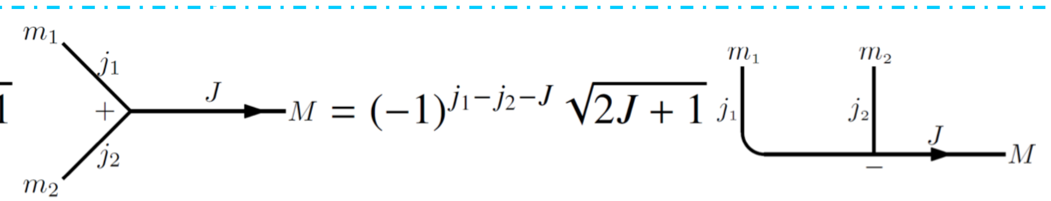


Summation over the same indices

$$\sum_{m'}$$



$$\langle JM|j_1m_1j_2m_2\rangle = (-1)^{j_1-j_2-J}\sqrt{2J+1}$$



$$\left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^M = \prod_{i=2}^{n-1}\sqrt{2a_i+1}\sqrt{2J+1}$$

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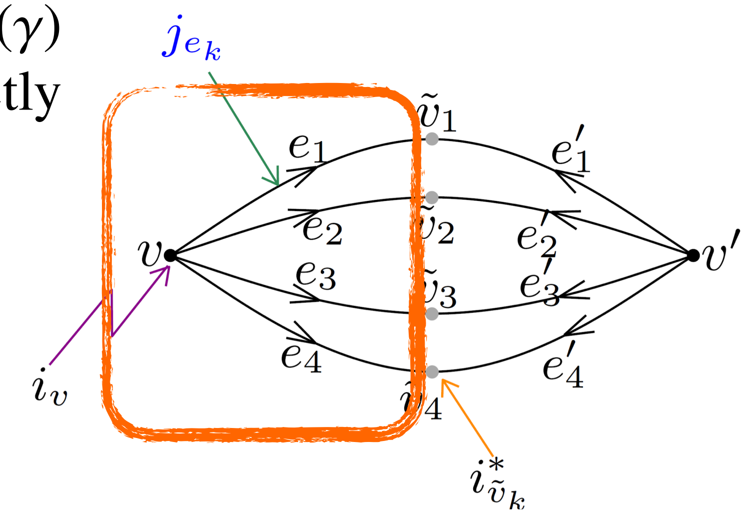
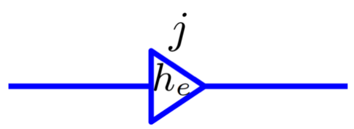
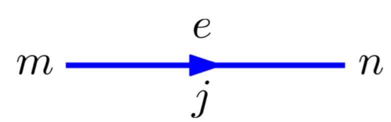
$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1 \cdots m_I \cdots m_n}{}^M \, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

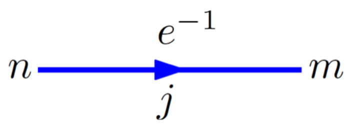
standard graph



Matrix element of holonomy with irrep.

$$\left[\pi_j(h_e(A))\right]^m{}_n = $$


$=:$


$$\left[\pi_j(h_e(A)^{-1})\right]^n{}_m = $$


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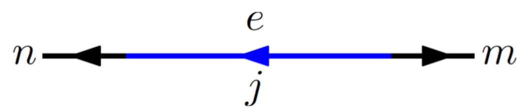
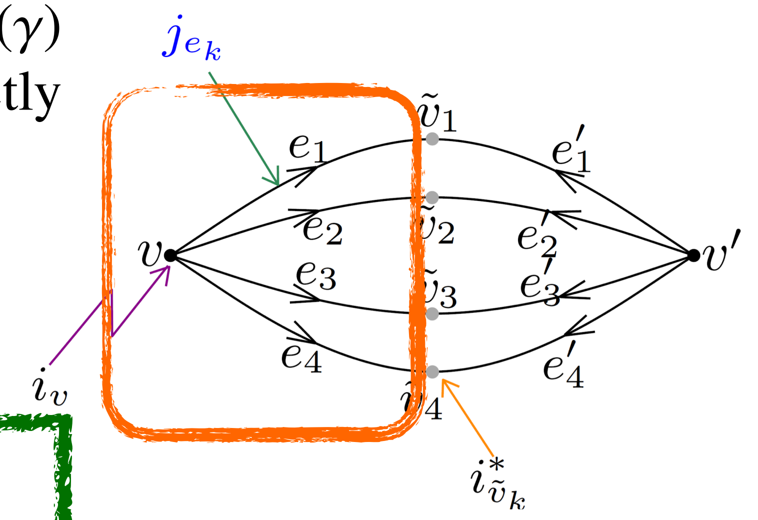


$$\left[\pi_j(h_e(A)^{-1})\right]^n{}_m = \left[\pi_j(h_{e^{-1}}(A))\right]^n{}_m = C^{(j)}_{mm'} \left[\pi_j(h_e(A))\right]^{m'}{}_{n'} C^{n'n}_{(j)}$$

Review of Last Lesson's Key Points

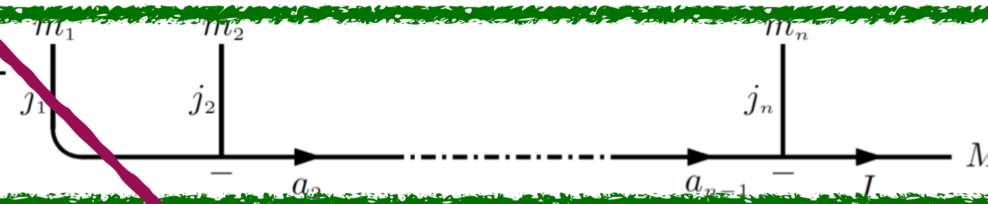
For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \dots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v

standard graph

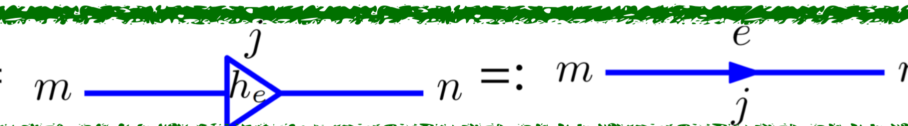


$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

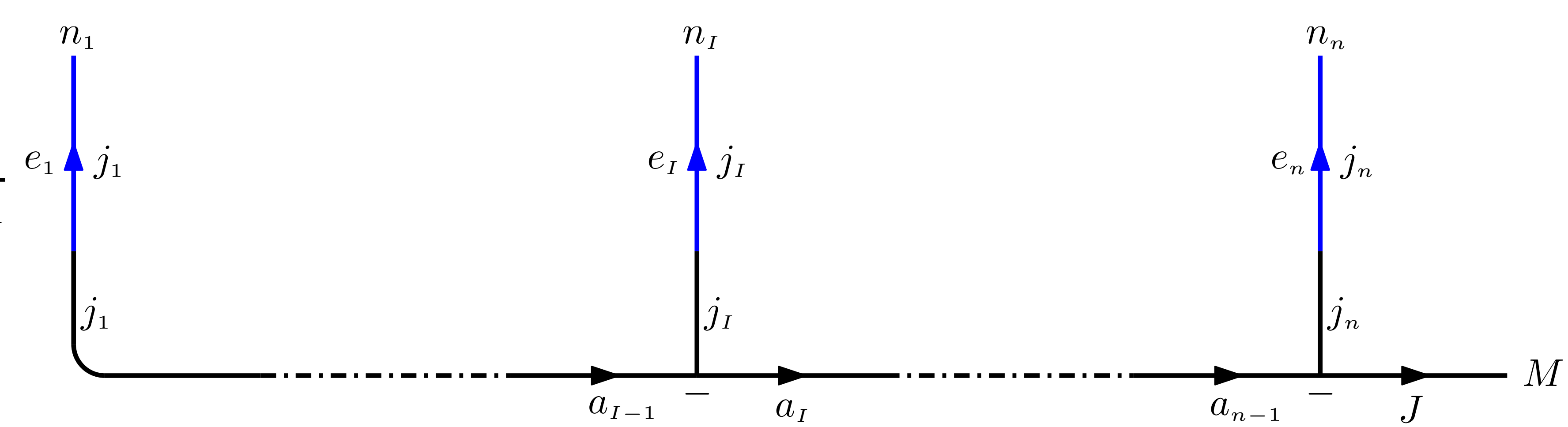
$$\left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^M = \prod_{i=2}^{n-1} \sqrt{2a_i+1}\,\sqrt{2J+1}$$



$$\left[\pi_j(h_e(A))\right]^m{}_n =$$

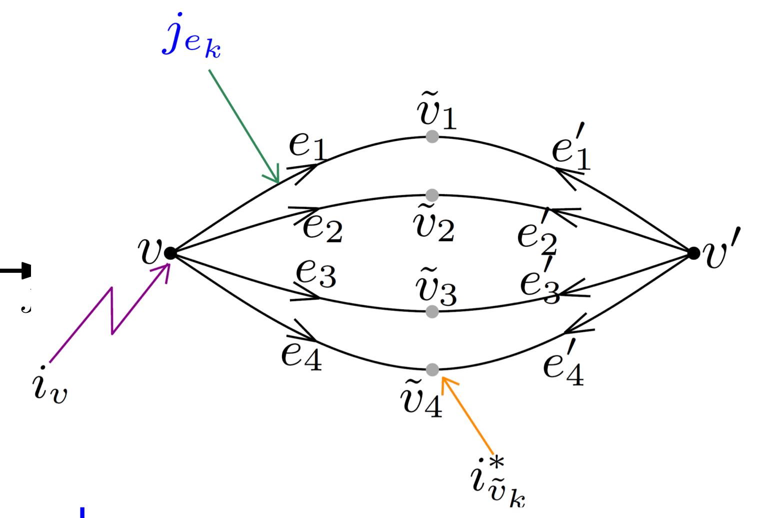


$$T^v_{\gamma,\vec{j},\vec{i}}(A) = \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\,\sqrt{d_J}$$

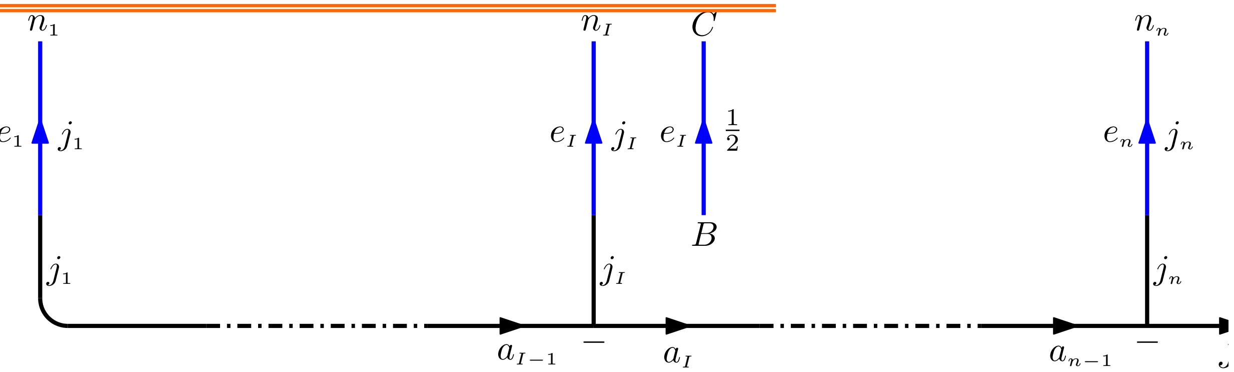


Review of Last Lesson's Key Points

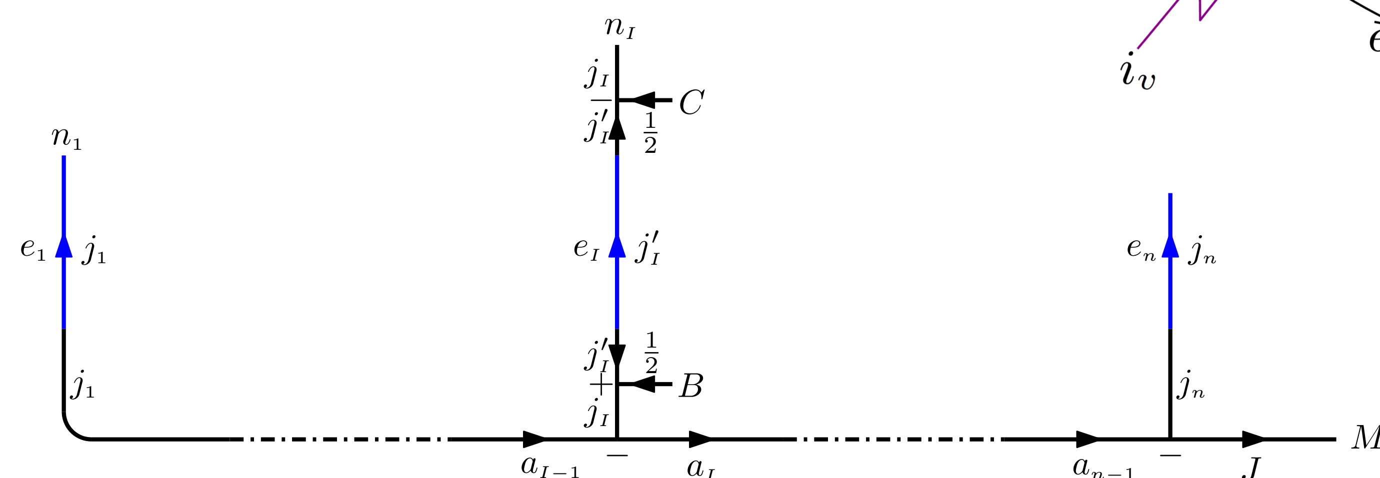
standard graph

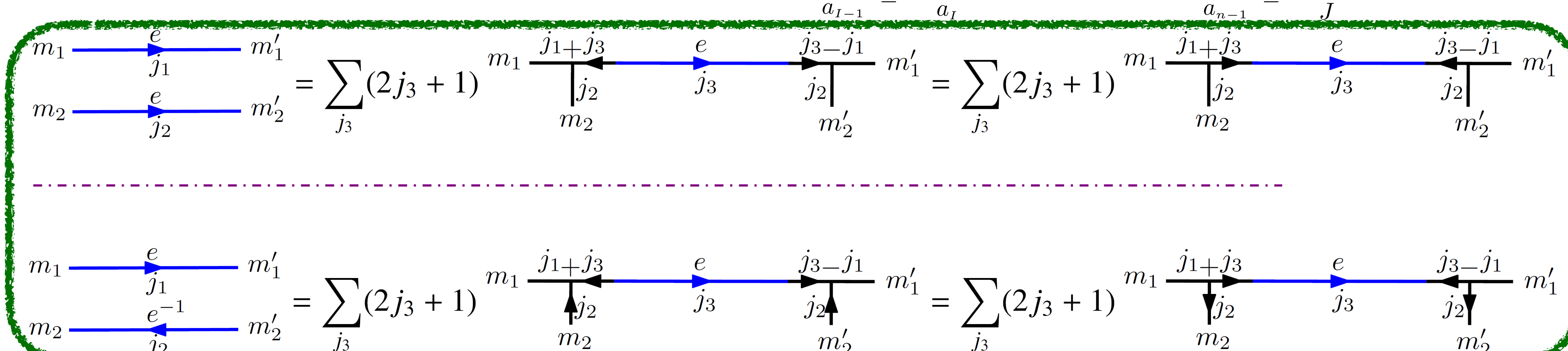


$$[\hat{h}_{e_I}]^B{}_C \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\,\sqrt{d_J}$$

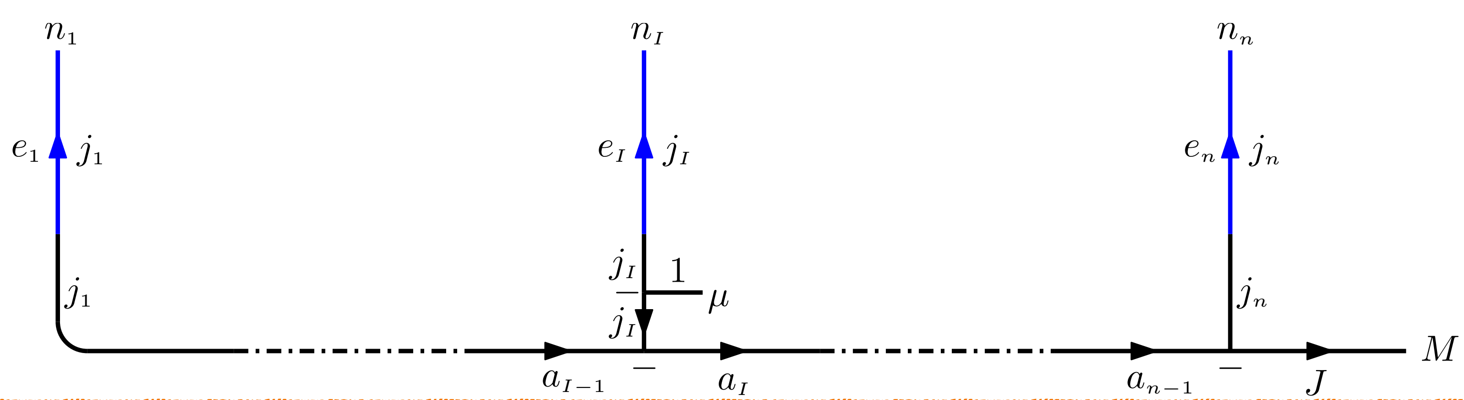


$$= \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\,\sqrt{d_J} \sum_{j_I'} d_{j_I'}$$





$$J^\mu_{e_I} \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = \chi(j_I) \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\,\sqrt{d_J}$$

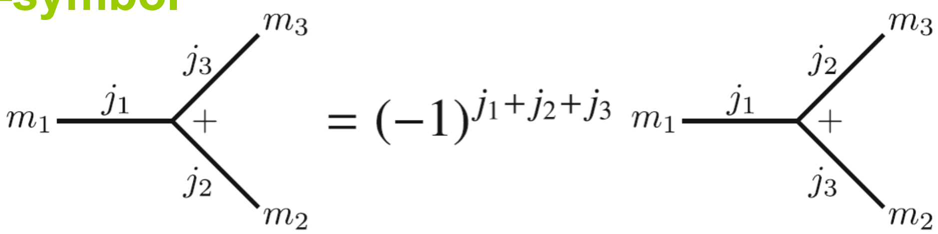


$\tau_i := -\mathrm{i}\sigma_i/2$, σ_i : Pauli matrices $i = 1,2,3$ ⟶ $\tau_\mu,\ \mu = 0, \pm 1$: $\tau_0 := \tau_3$, $\tau_{\pm 1} := \mp \frac{1}{\sqrt{2}}(\tau_1 \pm \mathrm{i}\tau_2)$

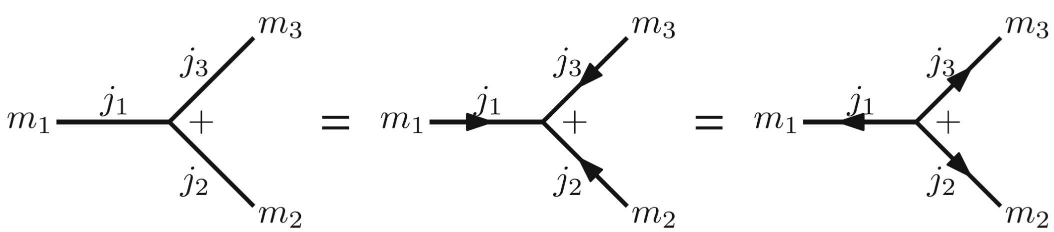
Graphical calculation: some useful rules of transforming graphs

Wigner 3j-symbol

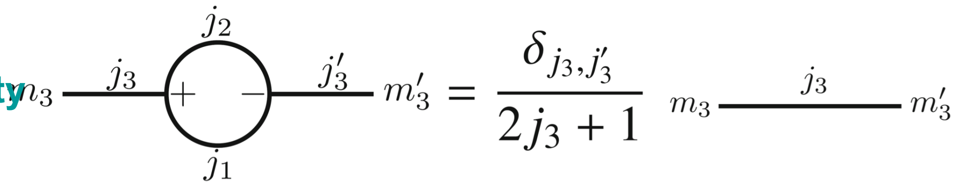
Symmetry



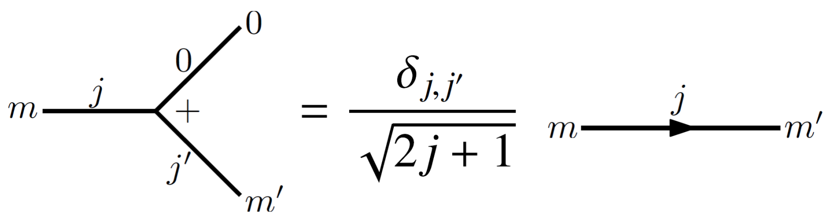
Reality (unitary)



Orthogonality



Special formula

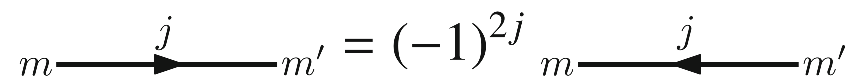


"Metric"

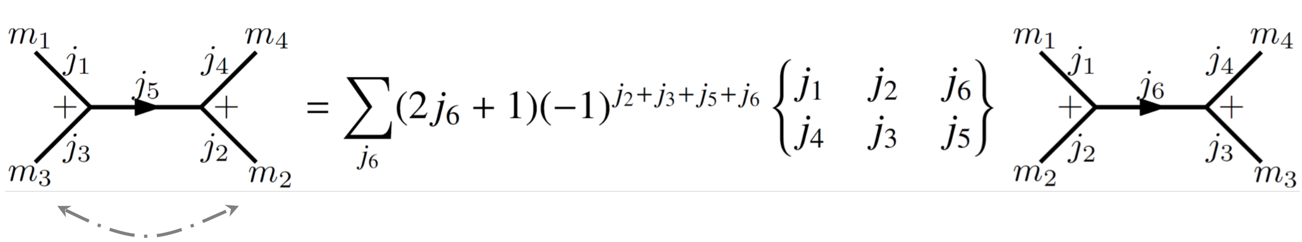
$$C^{(j)}_{mn}C^{nm'}_{(j)} = C^{m'n}_{(j)}C^{(j)}_{nm} = \delta^{m'}_{m}$$

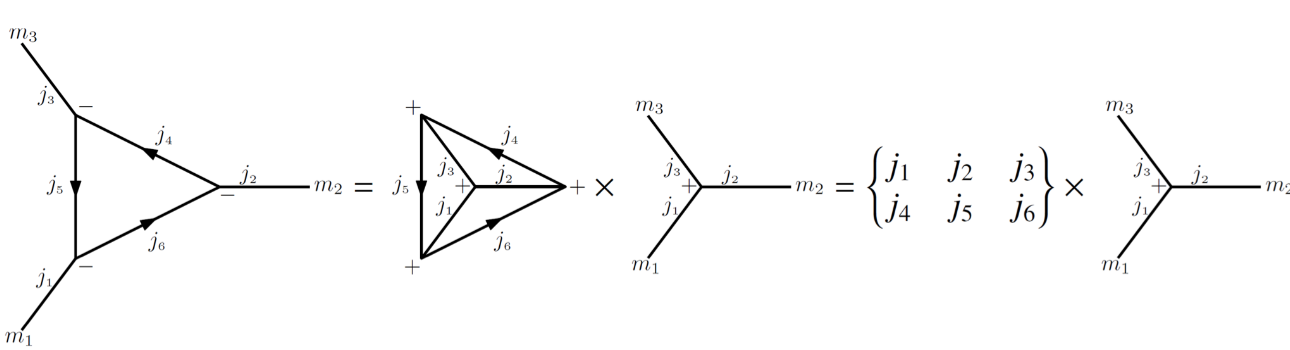
$$C^{(j)}_{nm}C^{nm'}_{(j)} = C^{m'n}_{(j)}C^{(j)}_{mn} = (-1)^{2j}\delta^{m'}_{m}$$

$$C^{(j)}_{m'm} = (-1)^{2j}C^{(j)}_{mm'}, \quad C^{mm'}_{(j)} = (-1)^{2j}C^{m'm}_{(j)}$$



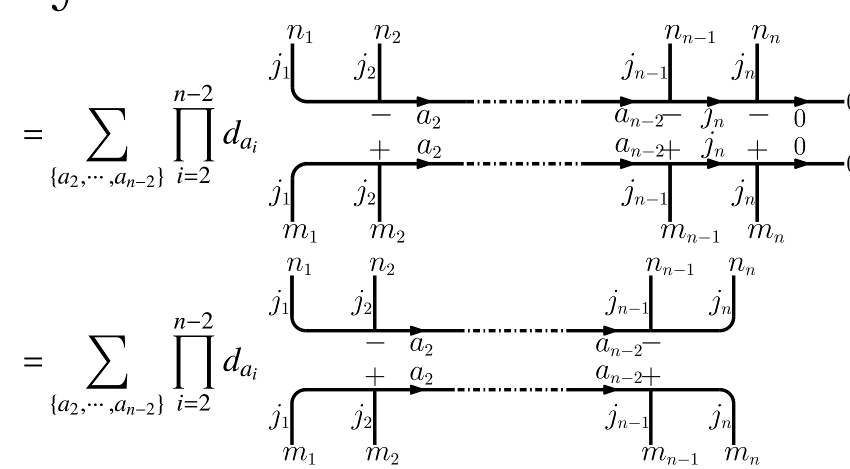
The rules involving the Wigner 6j-symbol

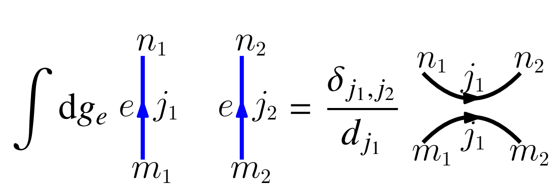


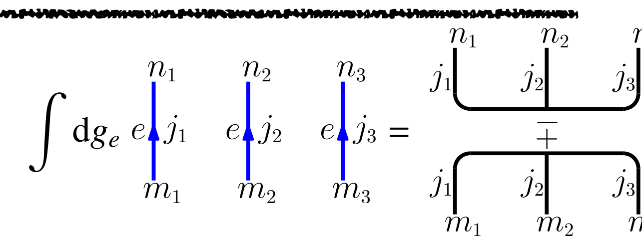


Integral over the product of irreducible representations

$$\int \mathrm{d}g_e [\pi_{j_1}(g_e)]^{m_1}{}_{n_1}[\pi_{j_2}(g_e)]^{m_2}{}_{n_2}\cdots[\pi_{j_n}(g_e)]^{m_n}{}_{n_n}$$

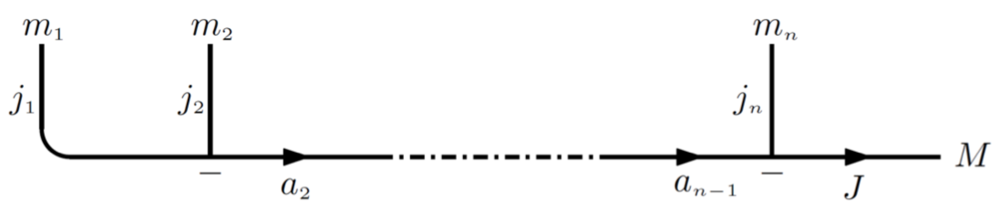




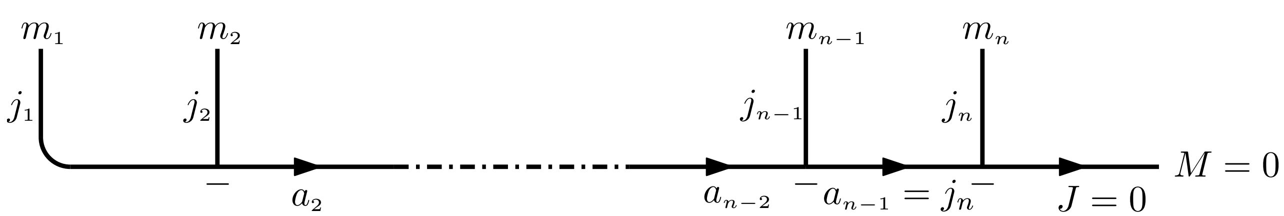


$$\boxed{\begin{array}{rcl} J=0, M=0 & \Leftrightarrow & \text{gauge-invariant} \\ J\neq 0 & \Leftrightarrow & \text{gauge-variant} \end{array}}$$

$$\left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^{M} = \prod_{i=2}^{n-1}\sqrt{2a_i+1}\,\sqrt{2J+1}$$

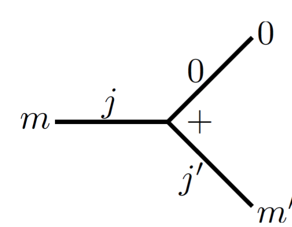


$$\left(i^{0;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^{0} = \prod_{i=2}^{n-1}\sqrt{2a_i+1}$$

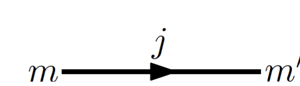


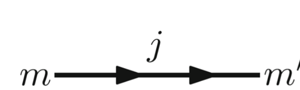
$$= (-1)^{2j_n}\prod_{i=2}^{n-2}\sqrt{2a_i+1}$$

Special formula



$$= \frac{\delta_{j,j'}}{\sqrt{2j+1}}$$





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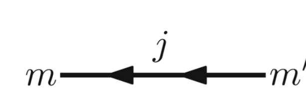
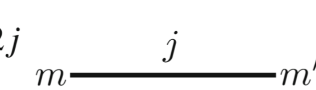


$= (-1)^{2j}$



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1. The matrix element of the volume operator

The volume operator (Ashtekar & Lewandowski, 1997)

$$V(R) := \int_R d^3x \sqrt{|\det(q)|} = \int_R d^3x \sqrt{\left|\frac{1}{3!}\epsilon^{ijk}\underset{\sim}{\epsilon}_{abc}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k\right|}$$

$$V^{\mathcal{P}_\epsilon}_{\mathrm{AL}}(R) := \sum_{C\in\mathscr{C}} \sqrt{\left|\frac{1}{3!}\epsilon^{ijk}\underset{\sim}{\epsilon}_{abc}\tilde{E}_i(S^a)\tilde{E}_j(S^b)\tilde{E}_k(S^c)\right|}$$

$$\hat{V}_\gamma(R)\cdot T_{\gamma,\vec{j},\vec{i}}(A) = \ell_{\mathrm{p}}^3\beta^{\frac{3}{2}} \sum_{v\in V(\gamma)\cap R} \sqrt{\left|\frac{1}{3!\times 2^3}\sum_{e_I\cap e_J\cap e_K=v} \varsigma(e_I,e_J,e_K)\epsilon_{ijk}J^i_{e_I}J^j_{e_J}J^k_{e_K}\right|}\cdot T_{\gamma,\vec{j},\vec{i}}(A)$$

$$= \ell_{\mathrm{p}}^3\beta^{\frac{3}{2}} \sum_{v\in V(\gamma)} \sqrt{\left|\frac{i}{8\times 4}\sum_{I<J<K,\, e_I\cap e_J\cap e_K=v} \varsigma(e_I,e_J,e_K)\,\hat{q}_{IJK}\right|}\cdot T_{\gamma,\vec{j},\vec{i}}(A)$$

$\varsigma(e_I,e_J,e_K) \equiv \mathrm{sgn}(\det(\dot{e}_I(0),\dot{e}_J(0),\dot{e}_K(0)))$

$$\hat{q}_{IJK} := -4i\epsilon_{ijk}J^i_{e_I}J^j_{e_J}J^k_{e_K} = 4\left[\delta_{ij}J^i_{e_I}J^j_{e_J},\delta_{lk}J^l_{e_J}J^k_{e_K}\right] = -\frac{1}{4}\left(16\,\delta_{lk}J^l_{e_J}J^k_{e_K}\delta_{ij}J^i_{e_I}J^j_{e_J} - 16\,\delta_{ij}J^i_{e_I}J^j_{e_J}\delta_{lk}J^l_{e_J}J^k_{e_K}\right)$$
$$=: -\frac{1}{4}\left(\hat{q}_{IJK}^{<JK;IJ>} - \hat{q}_{IJK}^{<IJ;JK>}\right).$$

(i) As a gauge-invariant operator, $\hat{q}_{IJK}$ does not alter J and M, but instead alters the intertwiner $(i_v^{j;\vec{a}})_{m_1\cdots m_n}{}^M$ by modifying the angular momenta $\vec{a}$ of the intermediate coupling.

(ii) Hence, the volume operator alters only the intertwiner $(i_v^{j;\vec{a}})_{m_1\cdots m_n}{}^M$ associated with v by modifying the angular momenta $\vec{a}$ of the intermediate coupling.

$$\hat{q}_{IJK} := -4i\epsilon_{ijk}J^i_{e_I}J^j_{e_J}J^k_{e_K} = 4\left[\delta_{ij}J^i_{e_I}J^j_{e_J},\, \delta_{lk}J^l_{e_J}J^k_{e_K}\right] = -\frac{1}{4}\left(16\,\delta_{lk}J^l_{e_J}J^k_{e_K}\delta_{ij}J^i_{e_I}J^j_{e_J} - 16\,\delta_{ij}J^i_{e_I}J^j_{e_J}\delta_{lk}J^l_{e_J}J^k_{e_K}\right)$$

$$=: -\frac{1}{4}\left(\hat{q}^{<JK;IJ>}_{IJK} - \hat{q}^{<IJ;JK>}_{IJK}\right).$$

Algebraic formula

$$\hat{q}^{<JK;IJ>}_{IJK}\cdot\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n} = 16\,\delta_{lk}J^{l}{}_{e_J}J^{k}{}_{e_K}\delta_{ij}J^{i}{}_{e_I}J^{j}{}_{e_J}$$

$$= 16\,C^{(1)}_{\rho\sigma}J^{\rho}{}_{e_J}J^{\sigma}{}_{e_K}C^{(1)}_{\mu\nu}J^{\mu}{}_{e_I}J^{\nu}{}_{e_J}\cdot\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}$$

$$= 16\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m'_I\cdots m'_J\cdots m'_K\cdots m_n}[\pi_{j_J}(\tau_\nu)]^{m''_J}{}_{m_J}C^{\nu'\nu}{}_{(1)}\,[\pi_{j_K}(\tau_{\nu'})]^{m'_K}{}_{m_K}\times[\pi_{j_I}(\tau_\mu)]^{m'_I}{}_{m_I}C^{\mu'\mu}{}_{(1)}\,[\pi_{j_J}(\tau_{\mu'})]^{m'_J}{}_{m''_J}$$

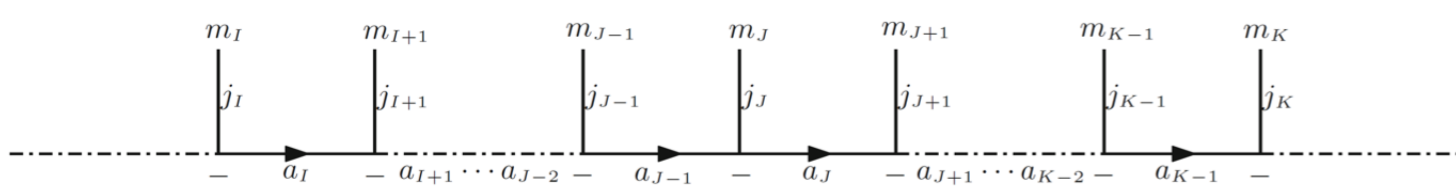
$$\hat{q}^{<IJ;JK>}_{IJK}\cdot\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n} = 16\,C^{(1)}_{\mu\nu}J^{\mu}_{e_I}J^{\nu}_{e_J}C^{(1)}_{\rho\sigma}J^{\rho}_{e_J}J^{\sigma}_{e_K}\cdot\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}$$

$$= 16\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m'_I\cdots m'_J\cdots m'_K\cdots m_n}[\pi_{j_I}(\tau_\mu)]^{m'_I}{}_{m_I}C^{\mu'\mu}_{(1)}\,[\pi_{j_J}(\tau_{\mu'})]^{m''_J}{}_{m_J}\times[\pi_{j_J}(\tau_\nu)]^{m'_J}{}_{m''_J}C^{\nu'\nu}_{(1)}\,[\pi_{j_K}(\tau_{\nu'})]^{m'_K}{}_{m_K}$$

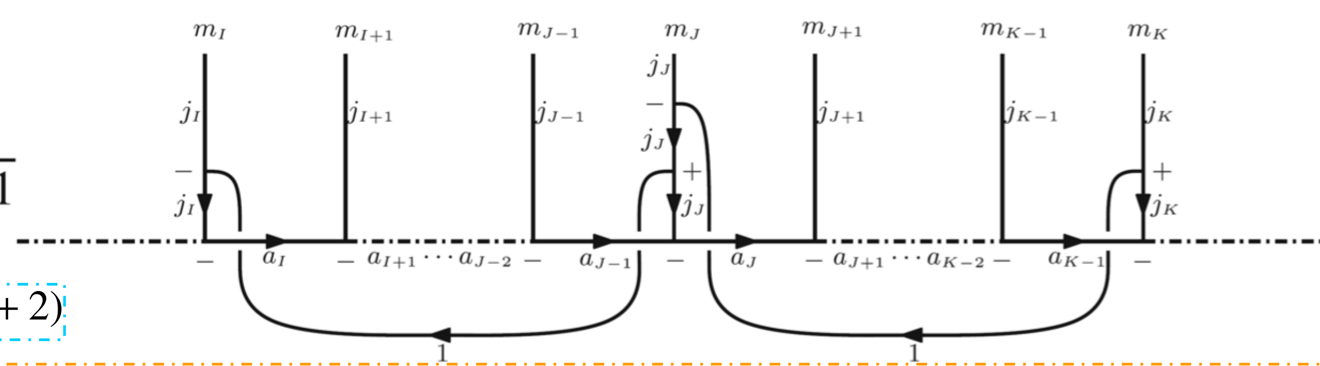
graphical formula

$$\hat{q}^{<JK;IJ>}_{IJK}\cdot\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}$$

$$= \hat{q}^{<JK;IJ>}_{IJK}\cdot\prod_{i=2}^{n-1}\sqrt{2a_i+1}\,\sqrt{2J+1}$$



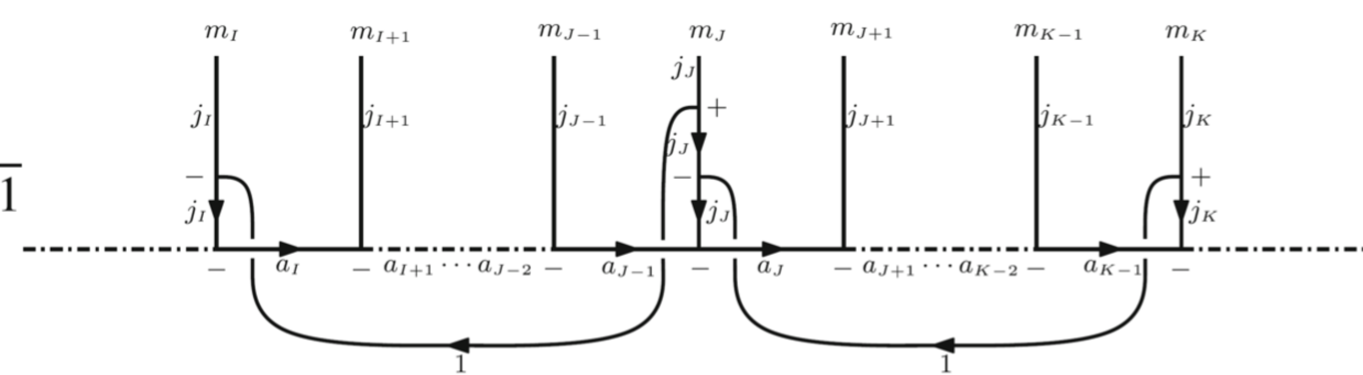
$$= X(j_I,j_J)^{\frac{1}{2}}X(j_J,j_K)^{\frac{1}{2}}\prod_{i=2}^{n-1}\sqrt{2a_i+1}\,\sqrt{2J+1}$$



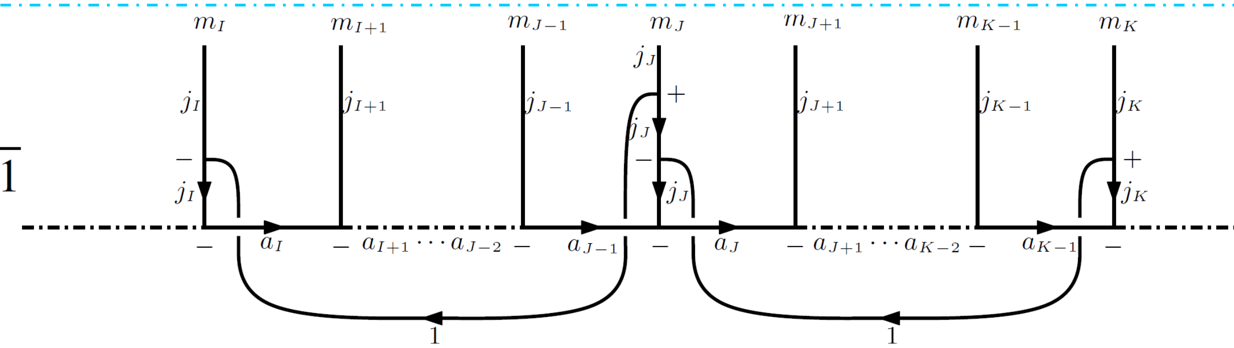
$$X(j_1,j_2) \equiv 2j_1(2j_1+1)(2j_1+2)2j_2(2j_2+1)(2j_2+2)$$

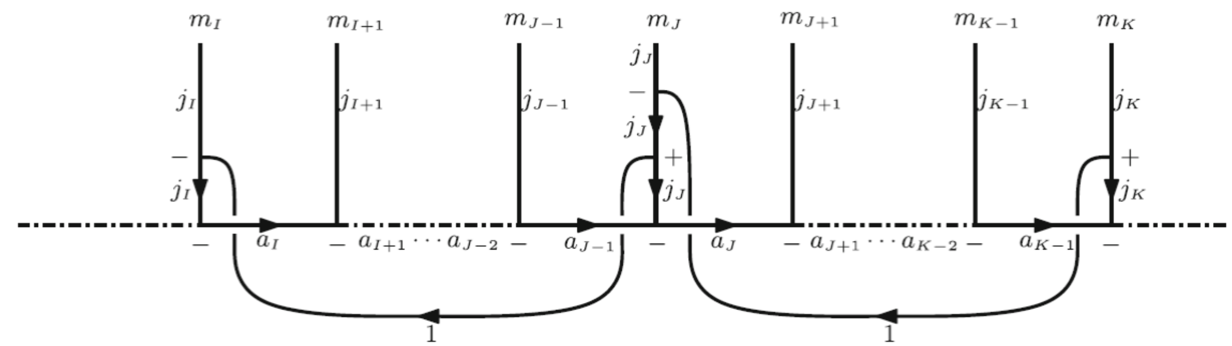
$$\hat{q}^{<IJ;JK>}_{IJK}\cdot\left(i^{J;\vec{a}}_{v}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}$$

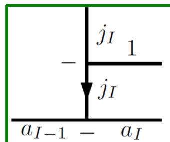
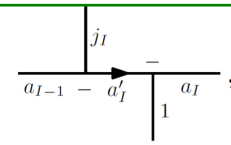
$$= X(j_I,j_J)^{\frac{1}{2}}X(j_J,j_K)^{\frac{1}{2}}\prod_{i=2}^{n-1}\sqrt{2a_i+1}\,\sqrt{2J+1}$$

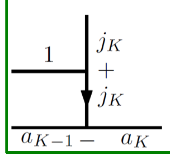
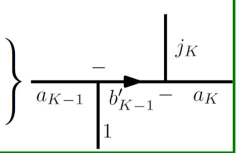


$$\hat{q}_{IJK}^{<IJ;JK>} \cdot \left(i_v^{J;\vec{a}}\right)_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}^{M} = X(j_I, j_J)^{\frac{1}{2}} X(j_J, j_K)^{\frac{1}{2}} \prod_{i=2}^{n-1} \sqrt{2a_i+1}\,\sqrt{2J+1}$$

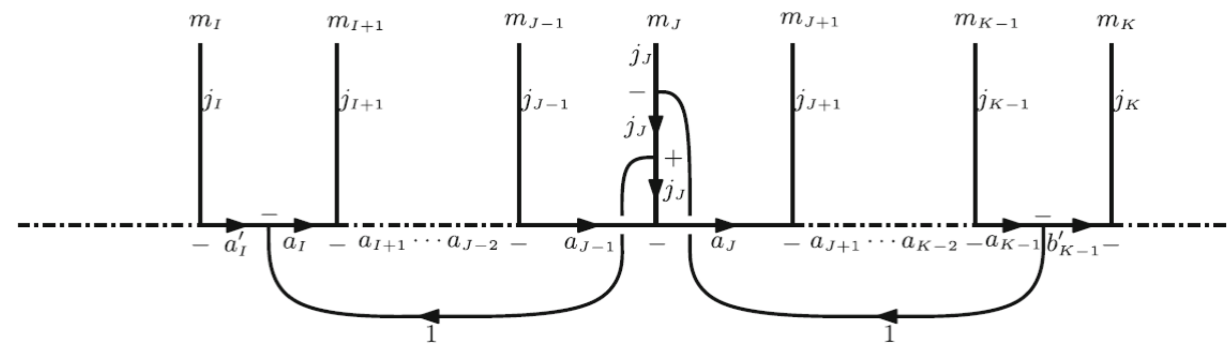


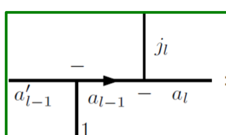
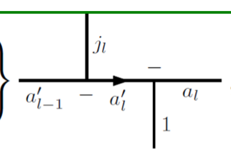


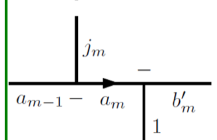
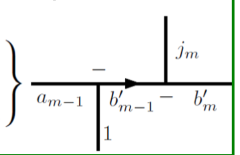
 $= \sum_{a'_I}(2a'_I+1)(-1)^{a_{I-1}-a'_I+j_I}\begin{Bmatrix} a_{I-1} & j_I & a_I \\ 1 & a'_I & j_I \end{Bmatrix}$  ,

 $= \sum_{b'_{K-1}}(2b'_{K-1}+1)(-1)^{a_K-b'_{K-1}+j_K+1}\begin{Bmatrix} a_K & j_K & a_{K-1} \\ 1 & b'_{K-1} & j_K \end{Bmatrix}$ 

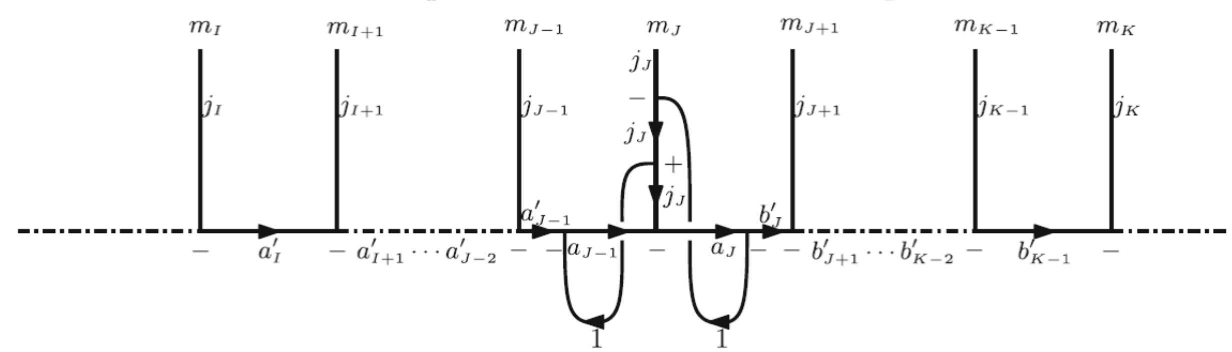
$\mapsto$



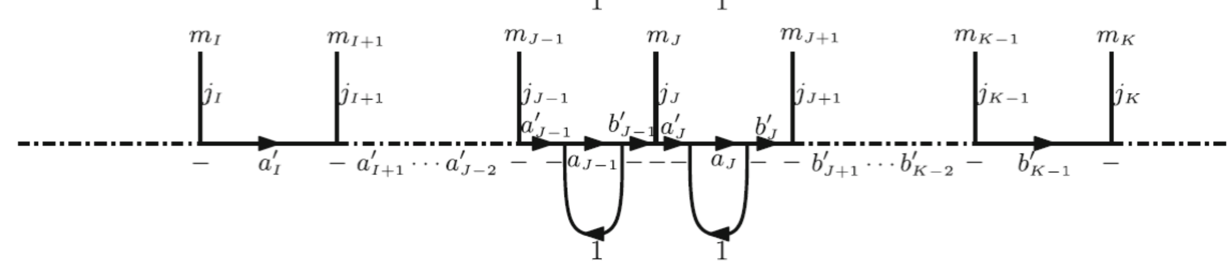
 $= \sum_{a'_l}(2a'_l+1)(-1)^{a'_{l-1}+a_{l-1}+1}(-1)^{a_{l-1}-a_l+j_l}\begin{Bmatrix} j_l & a'_{l-1} & a'_l \\ 1 & a_l & a_{l-1} \end{Bmatrix}$  ,

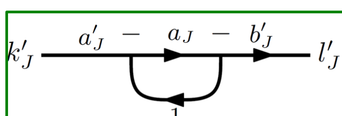
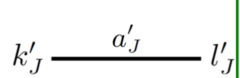
 $= \sum_{b'_{m-1}}(2b'_{m-1}+1)(-1)^{b'_{m-1}+a_{m-1}+1}(-1)^{b'_m-b'_{m-1}-j_m}\begin{Bmatrix} j_m & b'_{m-1} & b'_m \\ 1 & a_m & a_{m-1} \end{Bmatrix}$ 

$\mapsto$



$\mapsto$



 $= (-1)^{a'_J-a_J+1}\dfrac{\delta_{a'_J,b'_J}}{2a'_J+1}$ 

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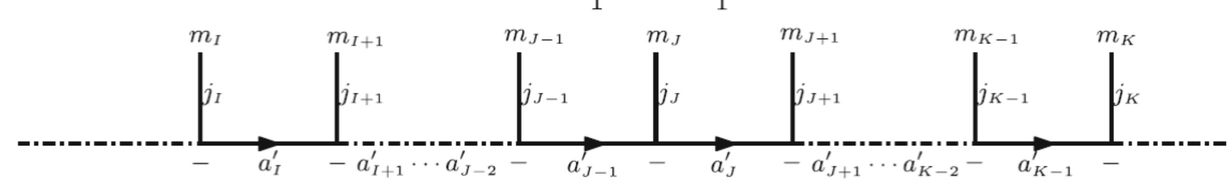
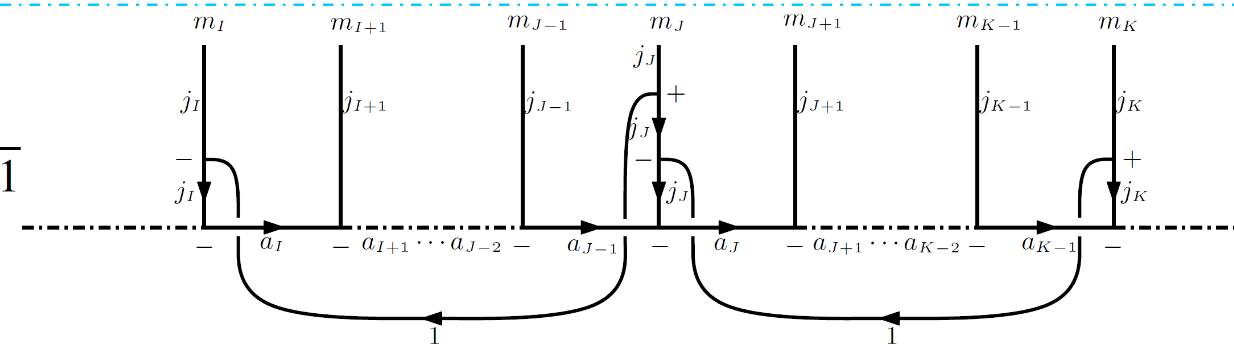
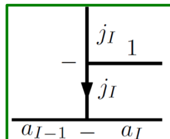
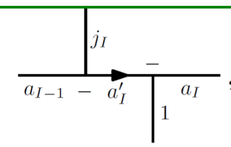
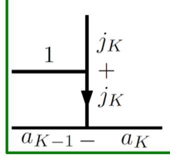


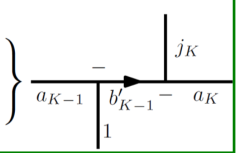
$$\hat{q}_{IJK}^{<IJ;JK>} \cdot \left(i_v^{J;\vec{a}}\right)_{m_1 \cdots m_I \cdots m_J \cdots m_K \cdots m_n}^{\quad M} = X(j_I, j_J)^{\frac{1}{2}} X(j_J, j_K)^{\frac{1}{2}} \prod_{i=2}^{n-1} \sqrt{2a_i+1}\sqrt{2J+1}$$





$= \sum_{a'_I} (2a'_I+1)(-1)^{a_{I-1}-a'_I+j_I} \begin{Bmatrix} a_{I-1} & j_I & a_I \\ 1 & a'_I & j_I \end{Bmatrix}$  ,



$= \sum_{b'_{K-1}} (2b'_{K-1}+1)(-1)^{a_K-b'_{K-1}+j_K+1} \begin{Bmatrix} a_K & j_K & a_{K-1} \\ 1 & b'_{K-1} & j_K \end{Bmatrix}$ 

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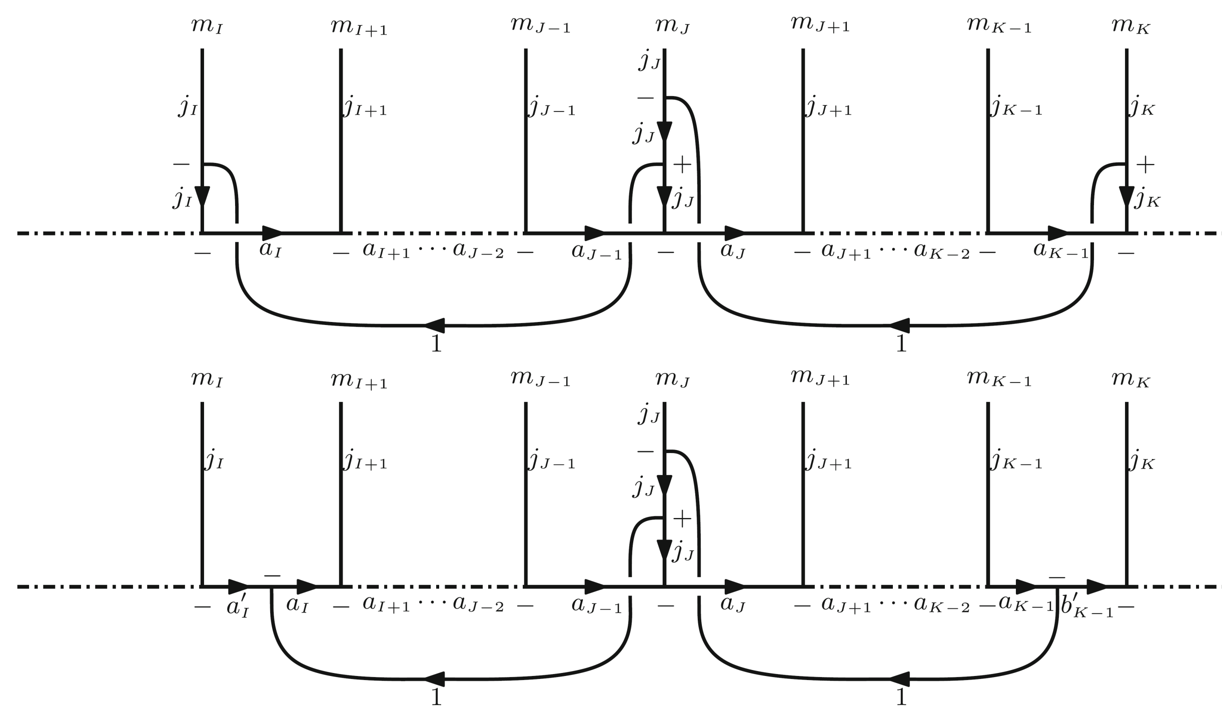
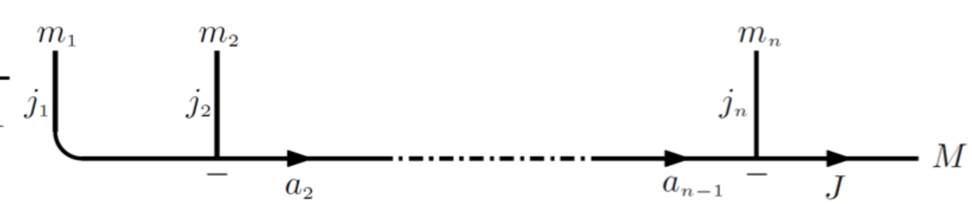
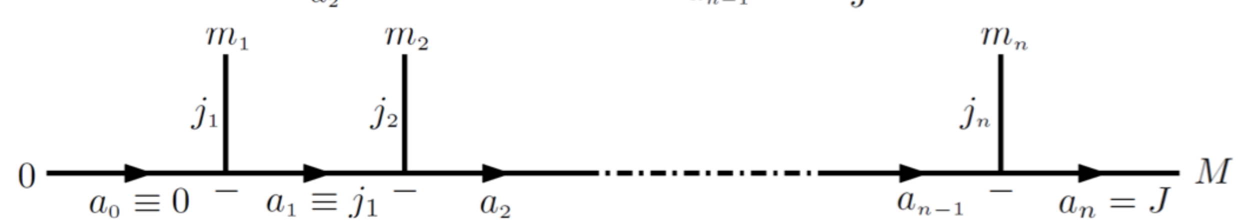
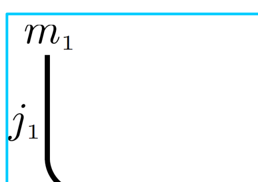


$$\left(i_v^{J;\vec{a}}\right)_{m_1 m_2 \cdots m_n}^{\quad M} \equiv \left(i_{j_1 \cdots j_n}^{J;\vec{a}}\right)_{m_1 \cdots m_n}^{\quad M} = \prod_{i=2}^{n-1} \sqrt{2a_i+1}\sqrt{2J+1}$$

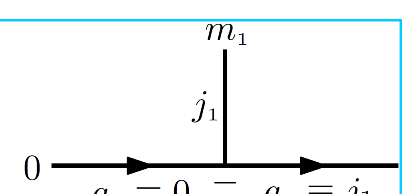


$$= (1)^{2j_1} \prod_{i=0}^{n} \sqrt{2a_i+1}$$





$= (1)^{2j_1} \sqrt{2a_0+1}\sqrt{2a_1+1}$



$$\hat{q}_{IJK}^{<IJ;JK>} \cdot \left(i_v^{J;\vec{a}}\right)_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}^{M} = \sum_{(a'_I,\ldots,a'_{K-1})} (-1)^{a_{I-1}+j_I+a_K+j_K}(-1)^{a_I-a'_I}(-1)^{\sum_{l=I+1}^{J-1} j_l}(-1)^{-\sum_{m=J+1}^{K-1} j_m} X(j_I,j_J)^{\frac{1}{2}} X(j_J,j_K)^{\frac{1}{2}}$$

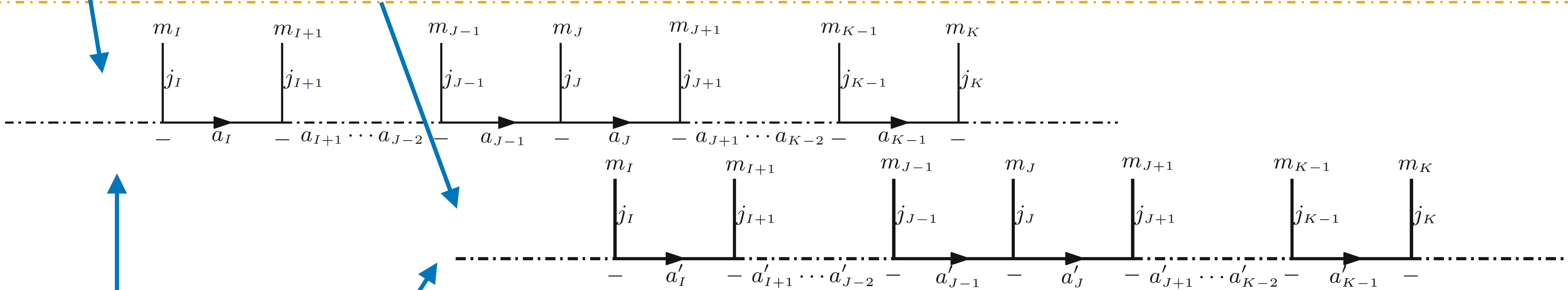
$$\times \sqrt{(2a'_I+1)(2a_I+1)}\sqrt{(2a'_J+1)(2a_J+1)} \begin{Bmatrix} a_{I-1} & j_I & a_I \\ 1 & a'_I & j_I \end{Bmatrix} \begin{Bmatrix} a_K & j_K & a_{K-1} \\ 1 & a'_{K-1} & j_K \end{Bmatrix}$$

$$\times \prod_{l=I+1}^{J-1} \sqrt{(2a'_l+1)(2a_l+1)}(-1)^{a'_{l-1}+a_{l-1}+1} \begin{Bmatrix} j_l & a'_{l-1} & a'_l \\ 1 & a_l & a_{l-1} \end{Bmatrix}$$

$$\times \prod_{m=J+1}^{K-1} \sqrt{(2a'_m+1)(2a_m+1)}(-1)^{a'_{m-1}+a_{m-1}+1} \begin{Bmatrix} j_m & a'_{m-1} & a'_m \\ 1 & a_m & a_{m-1} \end{Bmatrix}$$

$$\times (-1)^{a_{J-1}+a_J} \begin{Bmatrix} a'_J & j_J & a_{J-1} \\ 1 & a'_{J-1} & j_J \end{Bmatrix} \begin{Bmatrix} a_{J-1} & j_J & a_J \\ 1 & a'_J & j_J \end{Bmatrix}$$

$$\times \left(i_v^{J;\vec{a}'}\right)_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}^{M}$$



$$\hat{q}_{IJK}^{<JK;IJ>} \cdot \left(i_v^{J;\vec{a}}\right)_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}^{M}$$

$$= \sum_{(a'_I,\ldots,a'_{K-1})} (-1)^{a_{I-1}+j_I+a_K+j_K}(-1)^{a_I-a'_I}(-1)^{\sum_{l=I+1}^{J-1} j_l}(-1)^{-\sum_{m=J+1}^{K-1} j_m} X(j_I,j_J)^{\frac{1}{2}} X(j_J,j_K)^{\frac{1}{2}}$$

$$\times \sqrt{(2a'_I+1)(2a_I+1)}\sqrt{(2a'_J+1)(2a_J+1)} \begin{Bmatrix} a_{I-1} & j_I & a_I \\ 1 & a'_I & j_I \end{Bmatrix} \begin{Bmatrix} a_K & j_K & a_{K-1} \\ 1 & a'_{K-1} & j_K \end{Bmatrix}$$

$$\times \prod_{l=I+1}^{J-1} \sqrt{(2a'_l+1)(2a_l+1)}(-1)^{a'_{l-1}+a_{l-1}+1} \begin{Bmatrix} j_l & a'_{l-1} & a'_l \\ 1 & a_l & a_{l-1} \end{Bmatrix}$$

$$\times \prod_{m=J+1}^{K-1} \sqrt{(2a'_m+1)(2a_m+1)}(-1)^{a'_{m-1}+a_{m-1}+1} \begin{Bmatrix} j_m & a'_{m-1} & a'_m \\ 1 & a_m & a_{m-1} \end{Bmatrix}$$

$$\times (-1)^{a'_{J-1}+a'_J} \begin{Bmatrix} a_J & j_J & a_{J-1} \\ 1 & a'_{J-1} & j_J \end{Bmatrix} \begin{Bmatrix} a'_{J-1} & j_J & a_J \\ 1 & a'_J & j_J \end{Bmatrix}$$

$$\times \left(i_v^{J;\vec{a}'}\right)_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}^{M}$$

$$\hat{q}_{IJK}\cdot\left(i_v^{J;\vec{a}}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n} = -\frac{1}{4}\sum_{(a'_I\cdots a'_{K-1})}(-1)^{a_K+j_K+a_{I-1}+j_I}(-1)^{a_I-a'_I}(-1)^{\sum_{l=I+1}^{J-1}j_l}(-1)^{-\sum_{m=J+1}^{K-1}j_m}X(j_I,j_J)^{\frac{1}{2}}X(j_J,j_K)^{\frac{1}{2}}$$
$$\times\sqrt{(2a'_I+1)(2a_I+1)}\sqrt{(2a'_J+1)(2a_J+1)}\begin{Bmatrix}a_{I-1} & j_I & a_I\\ 1 & a'_I & j_I\end{Bmatrix}\begin{Bmatrix}a_K & j_K & a_{K-1}\\ 1 & a'_{K-1} & j_K\end{Bmatrix}$$
$$\times\prod_{l=I+1}^{J-1}\sqrt{(2a'_l+1)(2a_l+1)}(-1)^{a'_{l-1}+a_{l-1}+1}\begin{Bmatrix}j_l & a'_{l-1} & a'_l\\ 1 & a_l & a_{l-1}\end{Bmatrix}$$
$$\times\prod_{m=J+1}^{K-1}\sqrt{(2a'_m+1)(2a_m+1)}(-1)^{a'_{m-1}+a_{m-1}+1}\begin{Bmatrix}j_m & a'_{m-1} & a'_m\\ 1 & a_m & a_{m-1}\end{Bmatrix}$$
$$\times\left[(-1)^{a'_{J-1}+a'_J}\begin{Bmatrix}a_J & j_J & a_{J-1}\\ 1 & a'_{J-1} & j_J\end{Bmatrix}\begin{Bmatrix}a'_{J-1} & j_J & a_J\\ 1 & a'_J & j_J\end{Bmatrix}-(-1)^{a_{J-1}+a_J}\begin{Bmatrix}a'_J & j_J & a_{J-1}\\ 1 & a'_{J-1} & j_J\end{Bmatrix}\begin{Bmatrix}a_{J-1} & j_J & a_J\\ 1 & a'_J & j_J\end{Bmatrix}\right]$$
$$\times\left(i_v^{J;\vec{a}}\right)^{M}_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}$$

$$\left(T^{\text{norm}}_{\gamma',\vec{j}',\vec{i}'},\hat{q}_{IJK}\cdot T^{\text{norm}}_{\gamma,\vec{j},\vec{i}}\right)_{\mathcal{H}_{\text{kin}}} = \int_{SU(2)^{|E(\tilde{\gamma})|}}\prod_{e\in E(\tilde{\gamma})}\mathrm{d}\mu_H(h_e)\overline{T^{\text{norm}}_{\gamma',\vec{j}',\vec{i}'}(A)}\,\hat{q}_{IJK}\cdot T^{\text{norm}}_{\gamma,\vec{j},\vec{i}}(A)$$
$$=\delta_{\gamma,\gamma'}\int_{SU(2)^{|E(\gamma)|}}\prod_{e\in E(\gamma)}\mathrm{d}\mu_H(h_e)\overline{T^{\text{norm}}_{\gamma,\vec{j}',\vec{i}'}(A)}\,\hat{q}_{IJK}\cdot T^{\text{norm}}_{\gamma,\vec{j},\vec{i}}(A)$$
$$=\delta_{\gamma,\gamma'}\prod_{e\in E(\gamma)}\delta_{j_e,j'_e}\prod_{v\in V(\gamma)}\sum_{m_1,\ldots,m_n}\overline{\left(i_v^{J';\vec{a}'}\right)^{M'}_{m_1\cdots m_n}}\,\hat{q}_{IJK}\cdot\left(i_v^{J;\vec{a}}\right)^{M}_{m_1\cdots m_n}$$
$$=\delta_{\gamma,\gamma'}\prod_{e\in E(\gamma)}\delta_{j_e,j'_e}\prod_{v\in V(\gamma)}\langle\vec{a}';J',M'|\hat{q}_{IJK}|\vec{a};J,M\rangle_{\mathcal{H}^v_{j_1,\ldots,j_n}}$$
$$=:\delta_{\gamma,\gamma'}\prod_{e\in E(\gamma)}\delta_{j_e,j'_e}\prod_{v\in V(\gamma)}\langle\vec{a}'|\hat{q}_{IJK}|\vec{a}\rangle\delta_{J,J'}\delta_{M,M'}.$$

$$\hat{q}_v\equiv\frac{i\ell_p^6\beta^3}{8\times4}\sum_{I<J<K,\,e_I\cap e_J\cap e_K=v}\varsigma(e_I,e_J,e_K)\,\hat{q}_{IJK}$$

$$\hat{q}_v|\lambda_{\hat{q}_v}\rangle=\lambda_{\hat{q}_v}|\lambda_{\hat{q}_v}\rangle$$

The intertwiner associated to v in Dirac's notation

$$\Rightarrow\hat{V}_v|i_v\rangle=\sqrt{|\hat{q}_v|}\,|i_v\rangle=\sum_{\lambda_{\hat{q}_v}}\sqrt{|\hat{q}_v|}\,|\lambda_{\hat{q}_v}\rangle\langle\lambda_{\hat{q}_v}|i_v\rangle=\sum_{\lambda_{\hat{q}_v}}\left[\sqrt{|\lambda_{\hat{q}_v}|}\,\langle\lambda_{\hat{q}_v}|i_v\rangle\right]|\lambda_{\hat{q}_v}\rangle$$

The eigenstate is a linear combination of intertwiners associated to v

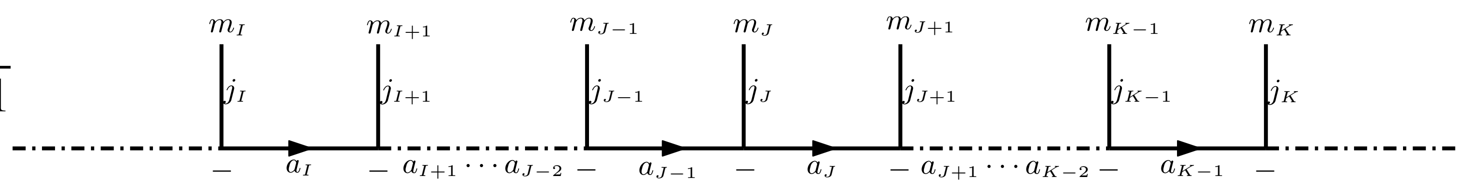
$$\hat{q}_{IJK}\cdot\left(i_v^{J;\vec{a}}\right)_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}^{\quad M}=-4i\epsilon_{ijk}J^i_{e_I}J^j_{e_J}J^k_{e_K}\cdot\left(i_v^{J;\vec{a}}\right)_{m_1\cdots m_I\cdots m_J\cdots m_K\cdots m_n}^{\quad M}$$

$$=\left(i_v^{J;\vec{a}}\right)_{m_1\cdots m'_I\cdots m'_J\cdots m'_K\cdots m_n}^{\quad M}4\epsilon_{ijk}[\pi_{j_I}(\tau_i)]^{m'_I}{}_{m_I}[\pi_{j_J}(\tau_j)]^{m'_J}{}_{m_J}[\pi_{j_K}(\tau_k)]^{m'_K}{}_{m_K}$$

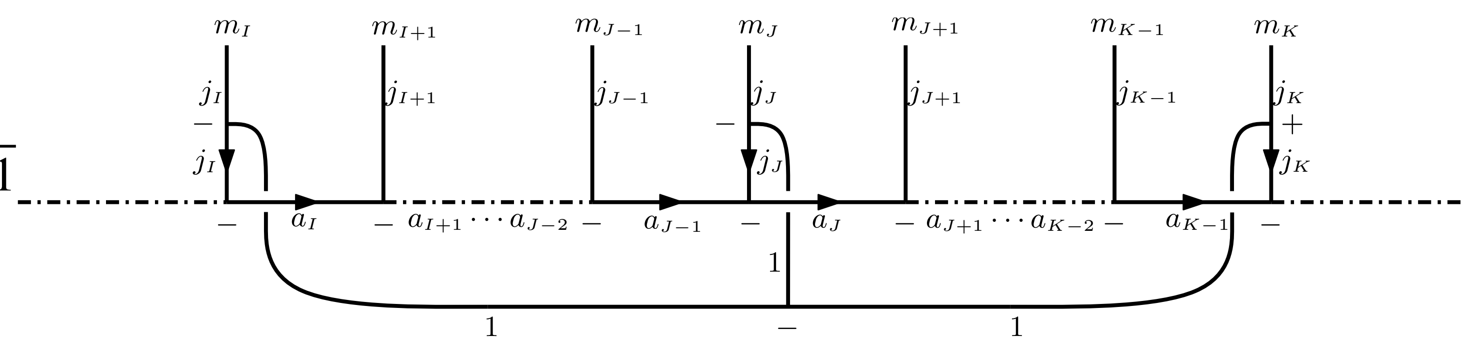
$$=\left(i_v^{J;\vec{a}}\right)_{m_1\cdots m'_I\cdots m'_J\cdots m'_K\cdots m_n}^{\quad M}(-4i)\epsilon_{\mu\nu\rho}[\pi_{j_I}(\tau_\mu)]^{m'_I}{}_{m_I}[\pi_{j_J}(\tau_\nu)]^{m'_J}{}_{m_J}[\pi_{j_K}(\tau_\rho)]^{m'_K}{}_{m_K}$$

$$\epsilon_{\mu\nu\rho}=\sqrt{6}\begin{pmatrix}1 & 1 & 1\\ \mu & \nu & \rho\end{pmatrix}$$

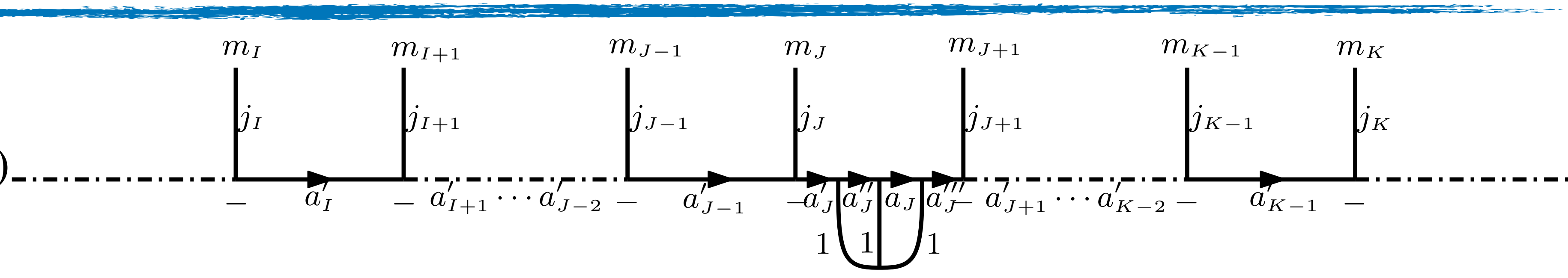
$$\hat{q}_{IJK}\cdot\prod_{i=2}^{n-1}\sqrt{2a_i+1}\sqrt{2J+1}$$

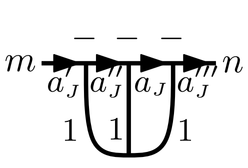


$$=-\frac{\sqrt{6}}{2}X(j_I,j_J,j_K)^{\frac{1}{2}}\prod_{i=2}^{n-1}\sqrt{2a_i+1}\sqrt{2J+1}$$

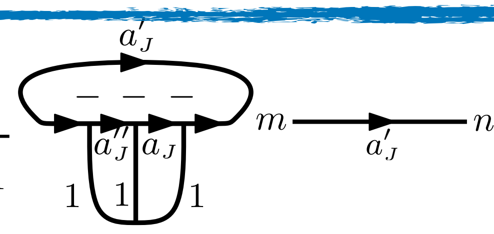


$$\sum_{a',a'',a'''}F(a,a',a'',a''')$$

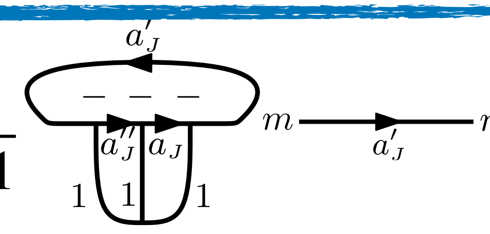




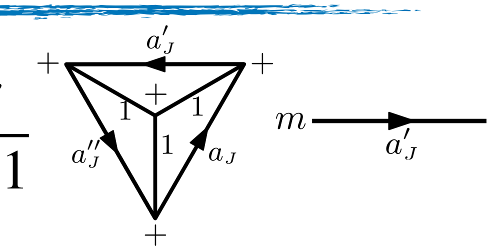
$$=\frac{\delta_{a'_J,a'''_J}}{2a'_J+1}$$



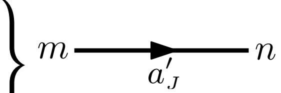
$$=\frac{\delta_{a'_J,a'''_J}}{2a'_J+1}$$



$$=\frac{\delta_{a'_J,a'''_J}}{2a'_J+1}$$



$$=\frac{\delta_{a'_J,a'''_J}}{2a'_J+1}\begin{Bmatrix}1 & 1 & 1\\ a_J & a'_J & a''_J\end{Bmatrix}$$



2. The action of Euclidean Hamiltonian constraint operator

The Hamiltonian constraint operator (Thiemann, 1998)

$$H_{\mathrm{gr}}(N) = \frac{1}{2\kappa}\int_\Sigma \mathrm{d}^3x\, N \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{\det(q)}}\left[\epsilon_{ijk}F^k_{ab} - 2(1+\beta^2)K^i_{[a}K^j_{b]}\right] =: H^E(N) - 2(1+\beta^2)T(N)$$

$$H^E(N) = \frac{1}{2\kappa}\int_\Sigma \mathrm{d}^3x\, N\epsilon_{ijk}\frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{\det(q)}}F^k_{ab} = -\frac{2}{\kappa^2\beta}\sum_{\Delta\in T(\epsilon)}\int_\Delta \mathrm{d}^3x\,\bar{N}\,\tilde{\epsilon}^{abc}\mathrm{tr}(F_{ab}\{A_c, V\})$$

$$= \lim_{\epsilon\to 0}\frac{2}{3\kappa^2\beta}\sum_{\Delta\in T(\epsilon)}\bar{N}(v(\Delta))\epsilon^{IJK}\mathrm{tr}(h_{\alpha_{IJ}(\Delta)}h_{s_K(\Delta)}\{h^{-1}_{s_K(\Delta)}, V\})$$

$$=: \lim_{\epsilon\to 0} H^E_{T(\epsilon)}(N)$$

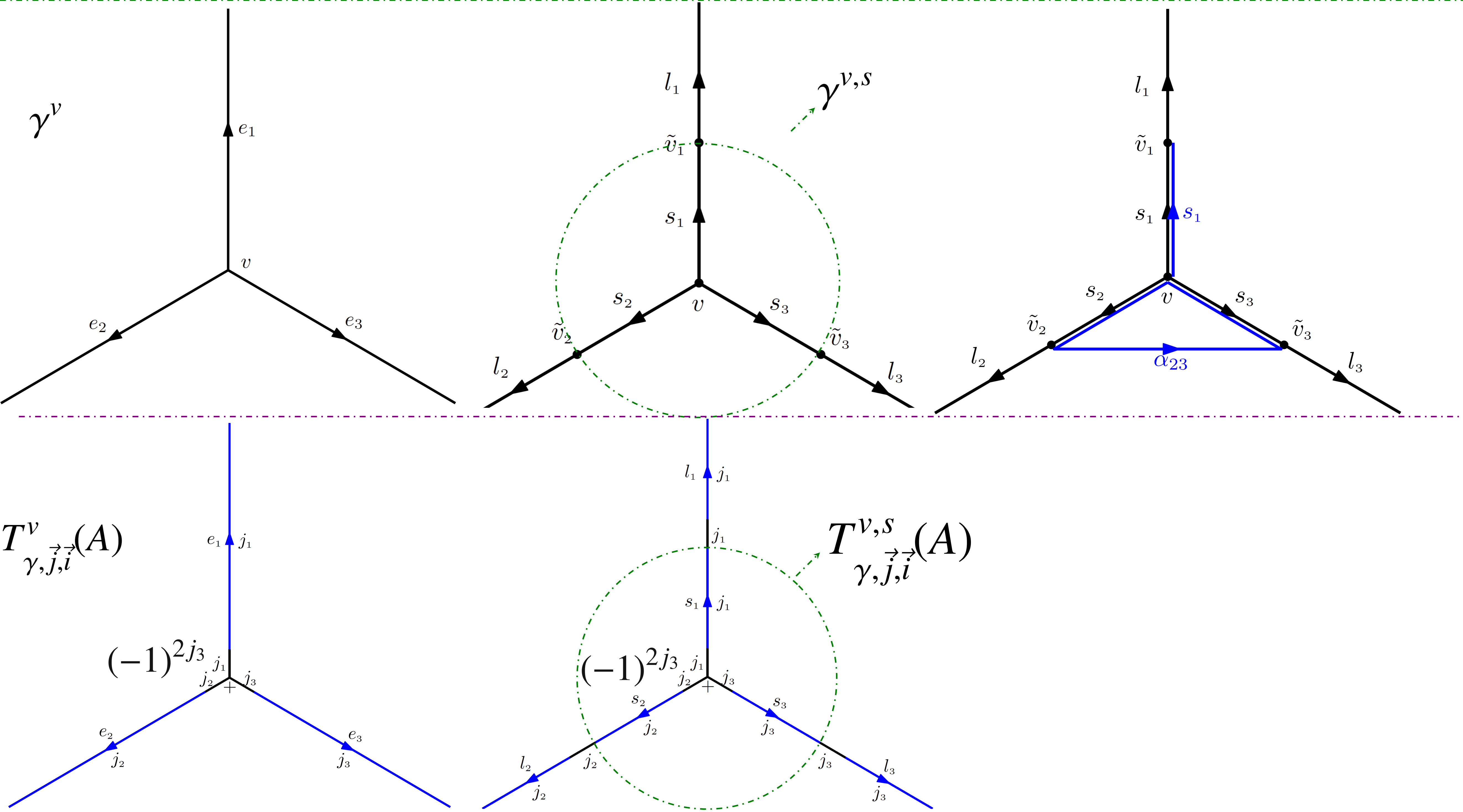
$$\hat{H}^E_\gamma(N)\cdot f_\gamma := \hat{H}^E_{T(\gamma)}(N)\cdot f_\gamma = -\frac{2}{3i\hbar\kappa^2\beta}\sum_{v\in V(\gamma)}N(v)\frac{8}{E(v)}\sum_{v(\Delta)=v}\hat{H}^E_\Delta\cdot f_\gamma =: \sum_{v\in V(\gamma)}N'_v\hat{H}^E_v\cdot f_\gamma$$

$$\hat{H}^E_v := \sum_{v(\Delta)=v}\epsilon^{IJK}\mathrm{tr}(h_{\alpha_{IJ}(\Delta)}h_{s_K(\Delta)}\hat{V}h^{-1}_{s_K(\Delta)})$$

$$\hat{H}^E_\gamma(N)\cdot f_\gamma := \hat{H}^E_{T(\gamma)}(N)\cdot f_\gamma = -\frac{2}{3i\hbar\kappa^2\beta}\sum_{v\in V(\gamma)} N(v)\frac{8}{E(v)}\sum_{v(\Delta)=v}\hat{H}^E_\Delta\cdot f_\gamma =: \sum_{v\in V(\gamma)} N'_v\hat{H}^E_v\cdot f_\gamma$$

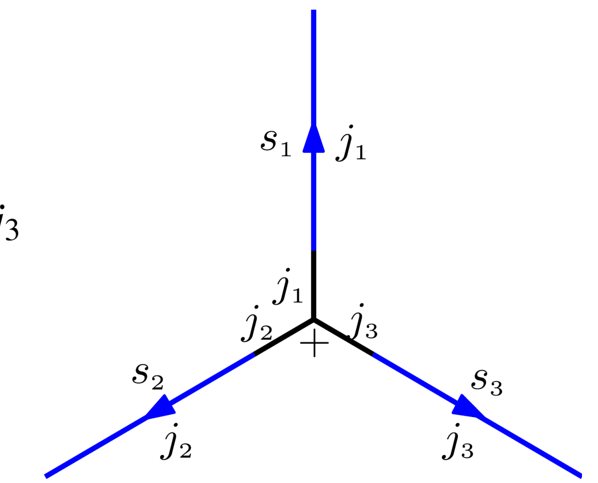
$$\hat{H}^E_v := \sum_{v(\Delta)=v}\epsilon^{IJK}\mathrm{tr}(h_{\alpha_{IJ}(\Delta)}h_{s_K(\Delta)}\hat{V}h^{-1}_{s_K(\Delta)})$$

Consider a trivalent non-planar vertex v at which three edges e_1, e_2, e_3 incident.

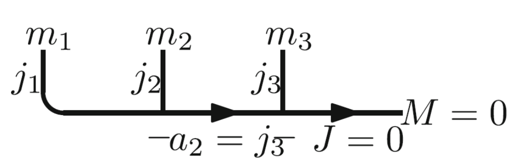


$$
\begin{aligned}
\hat{H}^E_v \cdot T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) &= \epsilon^{IJK}\mathrm{tr}\left[h_{\alpha_{IJ}}(A) h_{s_K}(A)\hat{V} h_{s_K}(A)^{-1}\right]\cdot T^{v,s}_{\gamma,\vec{j},\vec{i}}(A)\\
&= [h_{\alpha_{23}}(A)-h_{\alpha_{32}}(A)]^A{}_B [h_{s_1}(A)]^B{}_C \hat{V}[h_{s_1}(A)^{-1}]^C{}_A \cdot T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) + [h_{\alpha_{31}}(A)-h_{\alpha_{13}}(A)]^A{}_B [h_{s_2}(A)]^B{}_C \hat{V}[h_{s_2}(A)^{-1}]^C{}_A \cdot T^{v,s}_{\gamma,\vec{j},\vec{i}}(A)\\
&\quad + [h_{\alpha_{12}}(A)-h_{\alpha_{21}}(A)]^A{}_B [h_{s_3}(A)]^B{}_C \hat{V}[h_{s_3}(A)^{-1}]^C{}_A \cdot T^{v,s}_{\gamma,\vec{j},\vec{i}}(A)\\
&\equiv \left(\underline{\hat{H}^E_{v,s_2 s_3 s_1}} + \hat{H}^E_{v,s_3 s_1 s_2} + \hat{H}^E_{v,s_1 s_2 s_3}\right)\cdot \underline{T^{v,s}_{\gamma,\vec{j},\vec{i}}(A)}
\end{aligned}
$$

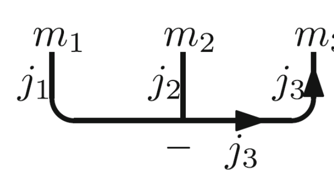
$$
T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1 m_2 m_3}\,[\pi_{j_1}(h_{s_1})]^{m_1}{}_{l_1}[\pi_{j_2}(h_{s_2})]^{m_2}{}_{l_2}[\pi_{j_3}(h_{s_3})]^{m_3}{}_{l_3} = (-1)^{2j_3}
$$



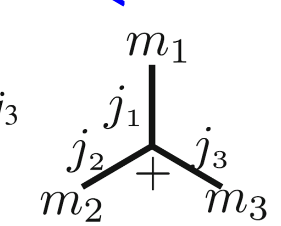
$$
(i_v)_{m_1 m_2 m_3} \equiv \left(i^{J=0;\,\vec{a}\equiv\{a_2=j_3\}}_{j_1,j_2,j_3}\right)^{M=0}_{m_1 m_2 m_3} = \sqrt{d_{j_3}}
$$



$=$



$= (-1)^{2j_3}$



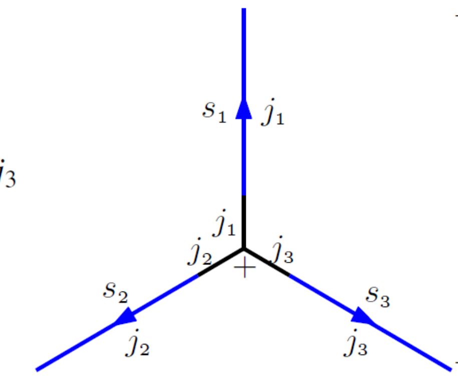
(i) The action of operator $\hat{H}^E_v$ on SNF can be expressed as three operators acting on SNF, with the three operators being mutually symmetric.

(ii) SNF is symmetric with respect to its three edges.

(iii) The action of one term, e.g., $\hat{H}^E_{v,\,s_2 s_3 s_1}$, on SNF suffices to determine the actions of the remaining two terms, due to symmetry.

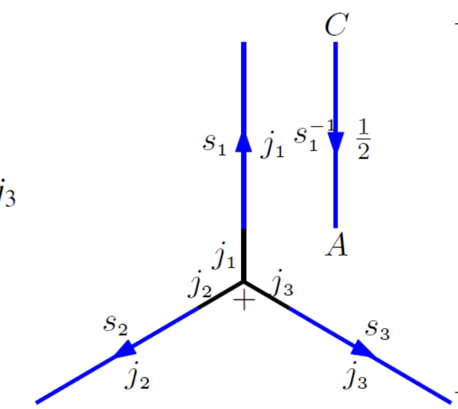
Step I

$$\hat{H}^E_{v,s_2s_3s_1} T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) = \underbrace{[h_{\alpha_{23}}(A) - h_{\alpha_{32}}(A)]^A{}_B}_{\text{Step III}} \underbrace{[h_{s_1}(A)]^B{}_C}_{\text{Step II}} \underbrace{\hat{V}[h_{s_1}(A)^{-1}]^C{}_A}_{\text{Step I}} \left[(-1)^{2j_3} \right.$$

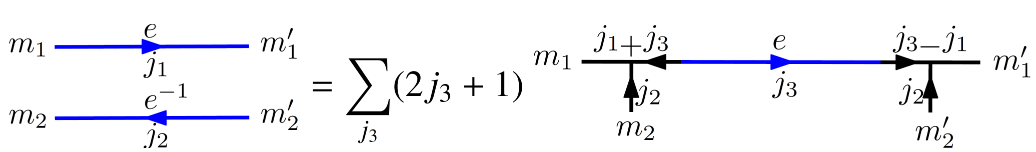


$$\left. \right]$$

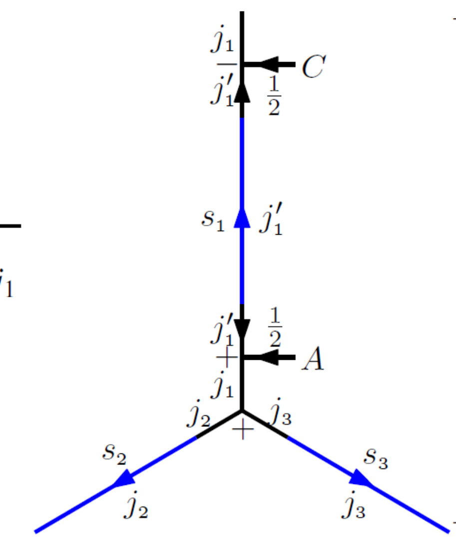
$$= [h_{\alpha_{23}}(A) - h_{\alpha_{32}}(A)]^A{}_B [h_{s_1}(A)]^B{}_C \hat{V} \left[(-1)^{2j_3} \right.$$



$$\left. \right]$$



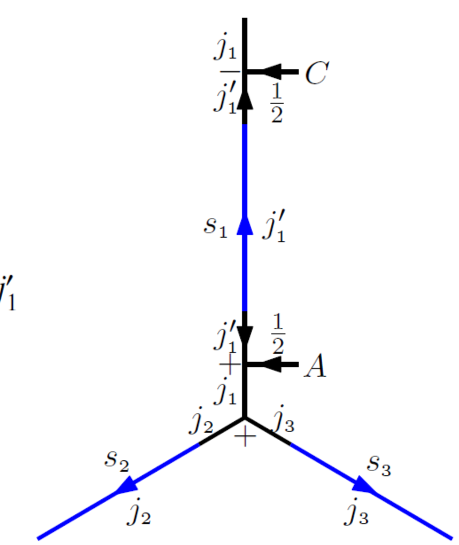
$$= [h_{\alpha_{23}}(A) - h_{\alpha_{32}}(A)]^A{}_B [h_{s_1}(A)]^B{}_C \sum_{j_1'} \frac{d_{j_1'}}{\sqrt{d_{j_1}}} \hat{V} \left[(-1)^{2j_3} \sqrt{d_{j_1}} \right.$$



$$\left. \right]$$

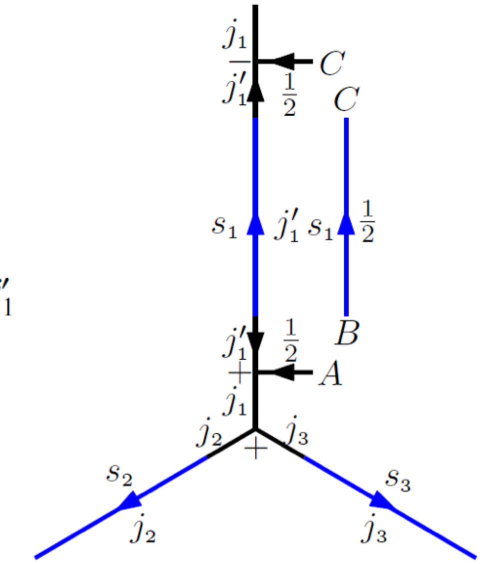
The volume operator is automatically diagonal on the 2-dimensional intertwiner space.

$$= [h_{\alpha_{23}}(A) - h_{\alpha_{32}}(A)]^A{}_B [h_{s_1}(A)]^B{}_C \sum_{j_1'} V(j_1', j_2, j_3)(-1)^{2j_3} d_{j_1'}$$

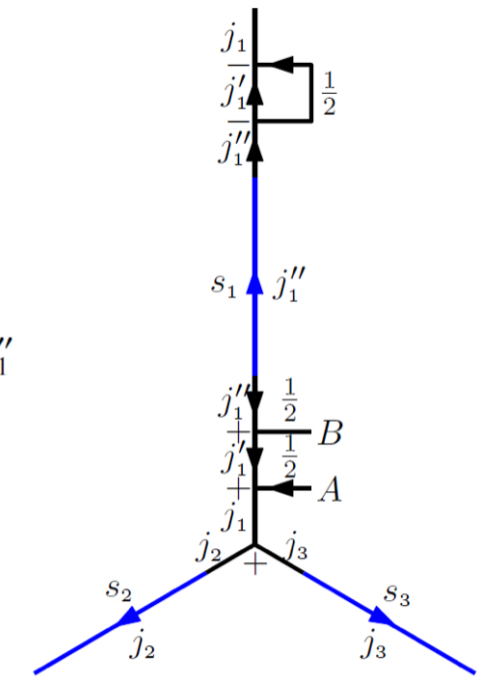


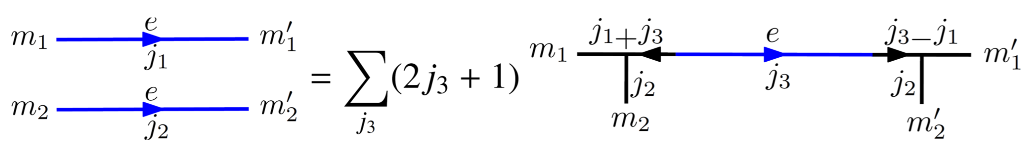
Step II

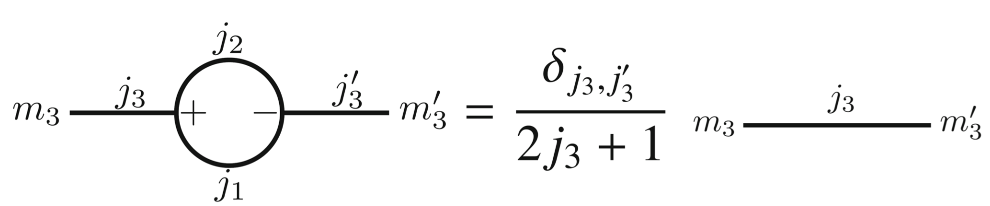
$$\hat{H}^E_{v,s_2 s_3 s_1} T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) = [h_{\alpha_{23}}(A) - h_{\alpha_{32}}(A)]^A{}_B \sum_{j_1'} V(j_1', j_2, j_3)(-1)^{2j_3} d_{j_1'}$$



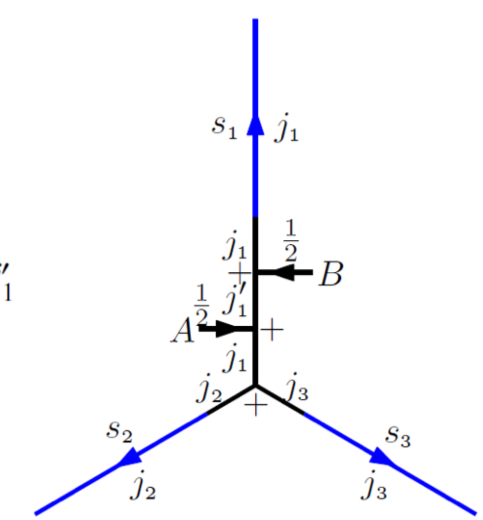
$$= [h_{\alpha_{23}}(A) - h_{\alpha_{32}}(A)]^A{}_B \sum_{j_1'} V(j_1', j_2, j_3)(-1)^{2j_3} d_{j_1'} \sum_{j_1''} d_{j_1''}$$





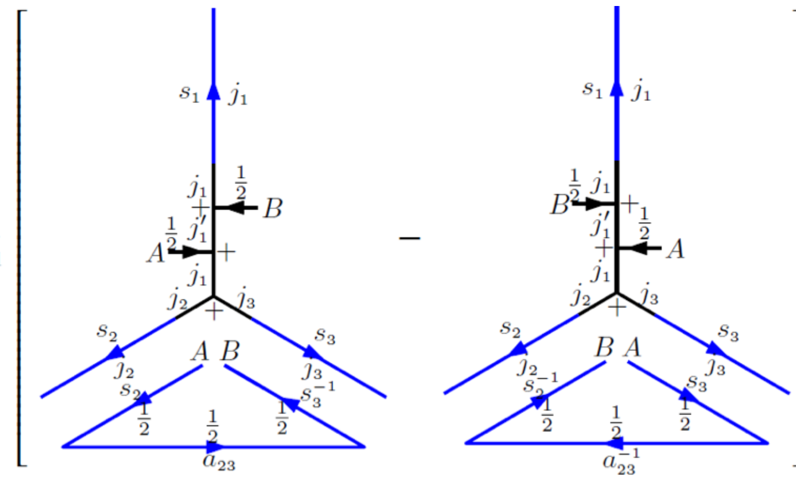


$$= [h_{\alpha_{23}}(A) - h_{\alpha_{32}}(A)]^A{}_B \sum_{j_1'} V(j_1', j_2, j_3)(-1)^{2j_3} d_{j_1'}$$

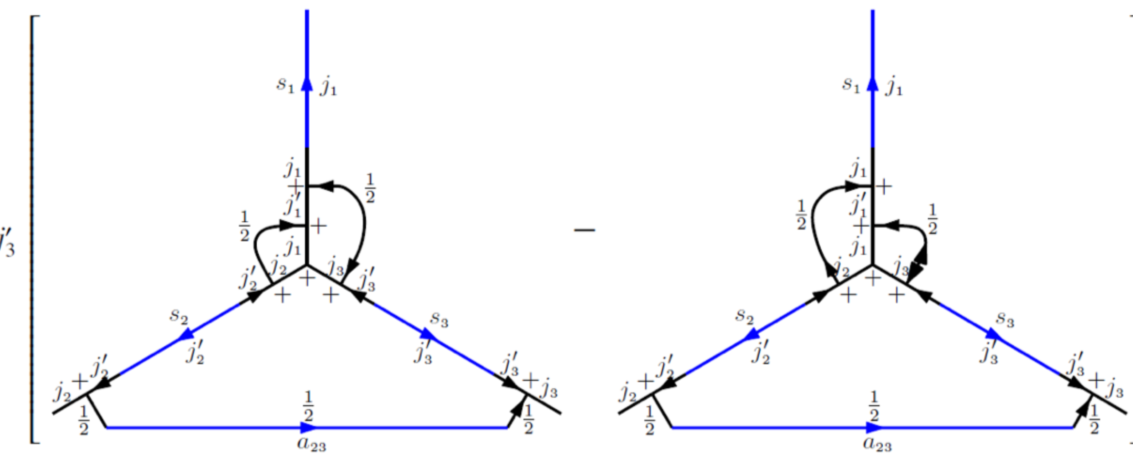


Step III

$$\hat{H}^E_{v,s_2s_3s_1} T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) = \sum_{j'_1} V(j'_1, j_2, j_3)(-1)^{2j_3} d_{j'_1}$$



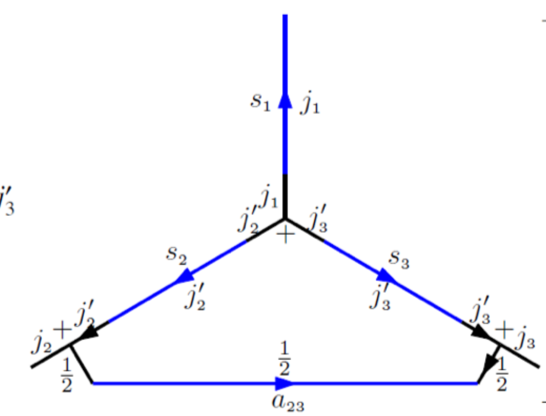
$$= \sum_{j'_1} V(j'_1, j_2, j_3)(-1)^{2j_3} d_{j'_1} \sum_{j'_2, j'_3} d_{j'_2} d_{j'_3}$$

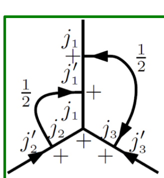


$$= \sum_{j'_1, j'_2, j'_3} V(j'_1, j_2, j_3) d_{j'_1} d_{j'_2} d_{j'_3}$$

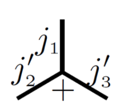
$$\times \left[-(-1)^{j_1+j'_1+\frac{1}{2}}(-1)^{j_3+j'_3+\frac{1}{2}} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j'_2 & j_3 & j_2 \end{Bmatrix} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j_3 & j'_2 & j'_3 \end{Bmatrix} - (-1)^{j_1-j'_1+\frac{1}{2}}(-1)^{j_2+j'_2+\frac{1}{2}} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j_2 & j'_3 & j'_2 \end{Bmatrix} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j'_3 & j_2 & j_3 \end{Bmatrix} \right]$$

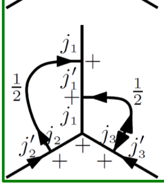
$$\times \left[(-1)^{2j'_3} \right.$$



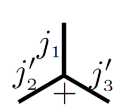


$$= -(-1)^{j_1+j'_1+\frac{1}{2}}(-1)^{j_3+j'_3+\frac{1}{2}} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j'_2 & j_3 & j_2 \end{Bmatrix} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j_3 & j'_2 & j'_3 \end{Bmatrix}$$

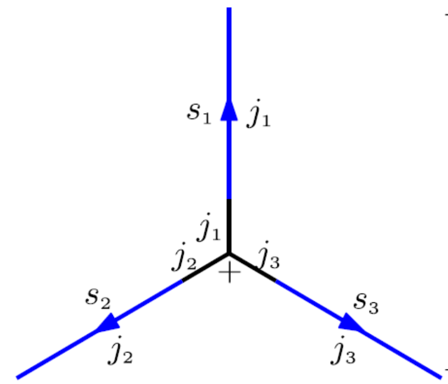
,



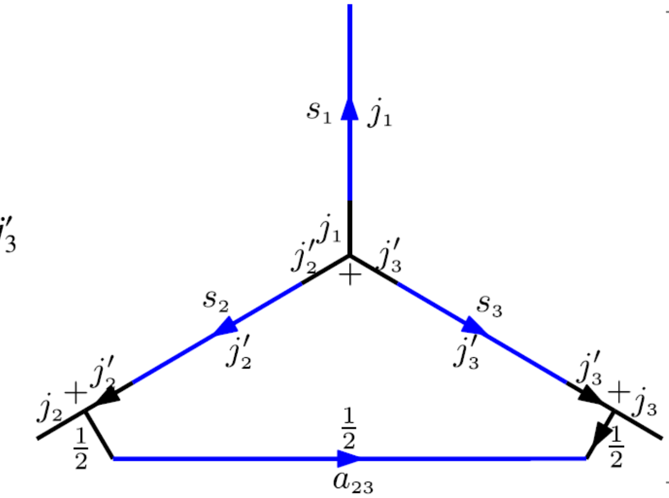
$$= (-1)^{j_1-j'_1+\frac{1}{2}}(-1)^{j_2+j'_2+\frac{1}{2}} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j_2 & j'_3 & j'_2 \end{Bmatrix} \begin{Bmatrix} j_1 & \frac{1}{2} & j'_1 \\ j'_3 & j_2 & j_3 \end{Bmatrix}$$

.

$$\hat{H}_v^E \Bigg[(-1)^{2j_3}$$

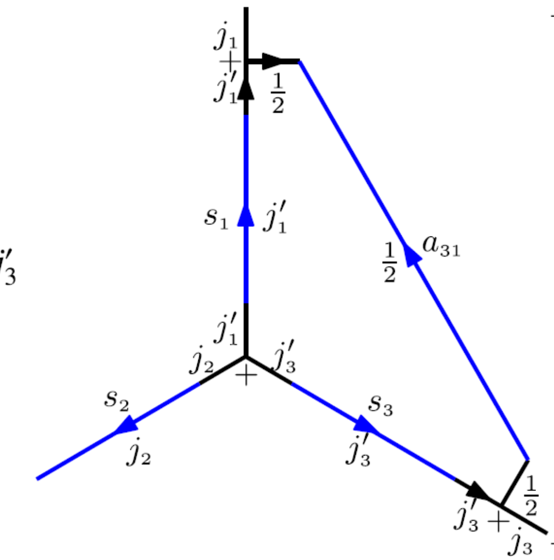


$$\Bigg] = -\sum_{j'_2, j'_3} H(j'_2, j'_3, j_1) \Bigg[(-1)^{2j'_3}$$



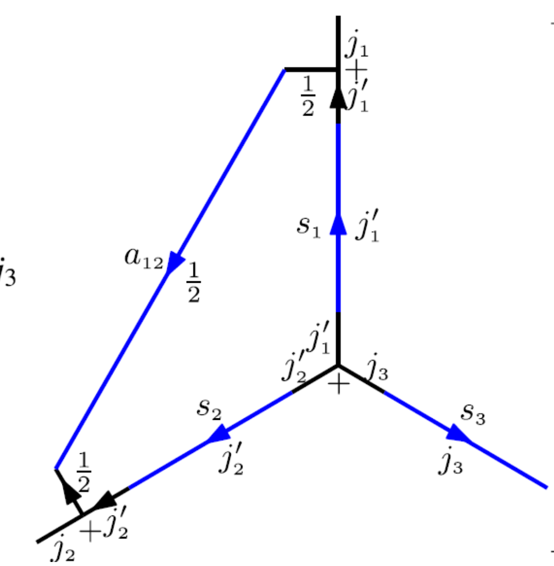
$$\Bigg]$$

$$-\sum_{j'_3, j'_1} H(j'_3, j'_1, j_2) \Bigg[(-1)^{2j'_3}$$



$$\Bigg]$$

$$+\sum_{j'_1, j'_2} H(j'_1, j'_2, j_3) \Bigg[(-1)^{2j_3}$$



$$\Bigg] .$$

$$H(j'_I, j'_J, j_K) = \sum_{j'_K} V(j'_K, j_I, j_J) d_{j'_K} d_{j'_I} d_{j'_J} \Bigg[(-1)^{j_K + j'_K + \frac{1}{2}} (-1)^{j_J + j'_J + \frac{1}{2}} \begin{Bmatrix} j_K & \frac{1}{2} & j'_K \\ j'_I & j_J & j_I \end{Bmatrix} \begin{Bmatrix} j_K & \frac{1}{2} & j'_K \\ j_J & j'_I & j'_J \end{Bmatrix}$$
$$+ (-1)^{j_K - j'_K + \frac{1}{2}} (-1)^{j_I + j'_I + \frac{1}{2}} \begin{Bmatrix} j_K & \frac{1}{2} & j'_K \\ j_I & j'_J & j'_I \end{Bmatrix} \begin{Bmatrix} j_K & \frac{1}{2} & j'_K \\ j'_J & j_I & j_J \end{Bmatrix} \Bigg] .$$

3. The action of the inverse volume operator

The matter Hamiltonian constraint operator (Thiemann, 1998)

$$H(N) = H_{\text{gr}}(N) + H_\phi$$

$$H_\phi(N) = \frac{1}{2}\int_\Sigma d^3x\, N(x)\left[\frac{\pi^2}{\sqrt{\det(q)}} + \sqrt{\det(q)}\, q^{ab}(\partial_a\phi)\partial_b\phi\right](x) \equiv \frac{1}{2}\left[H_{\text{kin},\phi}(N) + H_{\text{der},\phi}(N)\right]$$

$$\hat{H}_{\text{kin},\phi}(N)_\gamma = \frac{2^{23}}{3\hbar^4\kappa^4\beta^6}\sum_{v\in V(\gamma)}\frac{N(v)}{E(v)^2}X(v)X(v)\widehat{V^{-1}}_{\text{alt},v}$$

The inverse volume operator :
(gauge-invariant)

$$\widehat{V^{-1}}_{\text{alt},v}\cdot f_\gamma = \sum_{\substack{s_I\cap s_J\cap s_K=v \\ s_L\cap s_M\cap s_N=v}} \epsilon^{IJK}\epsilon^{LMN}\hat{q}_{IL}(v)\hat{q}_{JM}(v)\hat{q}_{KN}(v)\cdot f_\gamma$$

Gauge-invariant :

$$\begin{aligned}\hat{q}_{IJ}(v) &:= \delta^{ij}\operatorname{tr}\left(\tau_i h_I \hat{V}^{\frac{1}{2}} h_I^{-1}\right)\operatorname{tr}\left(\tau_j h_J \hat{V}^{\frac{1}{2}} h_J^{-1}\right) \\ &= -\operatorname{tr}\left(\tau_{\mu'} h_I \hat{V}^{\frac{1}{2}} h_I^{-1}\right) C^{\mu'\mu}{}_{(1)}\operatorname{tr}\left(\tau_\mu h_J \hat{V}^{\frac{1}{2}} h_J^{-1}\right) \\ &= -\left[\operatorname{tr}\left(\tau_\mu h_I \hat{V}^{\frac{1}{2}} h_I^{-1}\right)\right]^\dagger \operatorname{tr}\left(\tau_\mu h_J \hat{V}^{\frac{1}{2}} h_J^{-1}\right) \\ &=: -\left[{}^{(\frac{1}{2})}\hat{e}^\mu_I(v)\right]^\dagger\, {}^{(\frac{1}{2})}\hat{e}^\mu_J(v)\end{aligned}$$

A gauge-invariant SNF at a trivalent vertex
(non-trivial eigenstate with zero-eigenvalue)

$$T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}}(A) := \sqrt{d_{j_1}d_{j_2}d_{j_3}}\; T^{v,s}_{\gamma,\vec{j},\vec{i}}(A)$$

The intertwiner space associated to v is 1-dim

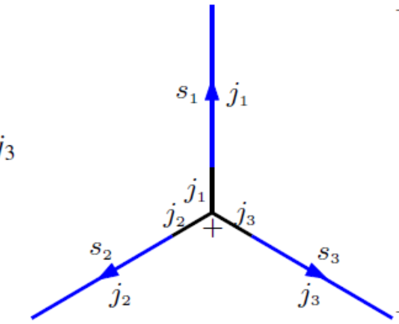
$$\hat{q}_{IJ}(v)\cdot T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}}(A) = Q_{IJ}T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}}(A)\,,$$

$$\widehat{V^{-1}}_{\text{alt},v}\cdot T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}}(A) = \epsilon^{IJK}\epsilon^{LMN}Q_{IL}Q_{JM}Q_{KN}T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}}(A)\,,$$

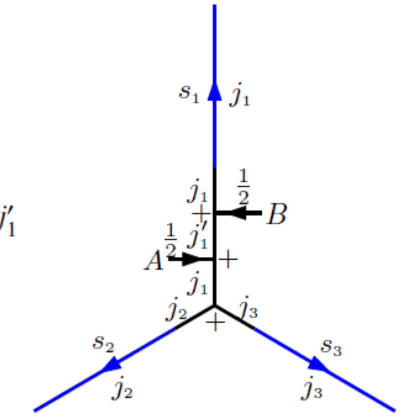
$$Q_{IJ} = \left(T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}},\, \hat{q}_{IJ}(v)\cdot T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}}\right)_{\mathcal{H}_{\text{kin}}} = -\left({}^{(\frac{1}{2})}\hat{e}^\mu_I(v)\cdot T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}},\, {}^{(\frac{1}{2})}\hat{e}^\mu_J(v)\cdot T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}}\right)_{\mathcal{H}_{\text{kin}}}$$

$$Q_{11} = -\left({}^{(\frac{1}{2})}\hat{e}^{\mu}_{1}(v) T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}},\ {}^{(\frac{1}{2})}\hat{e}^{\mu}_{1}(v) T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}} \right)_{\mathcal{H}_{kin}}$$

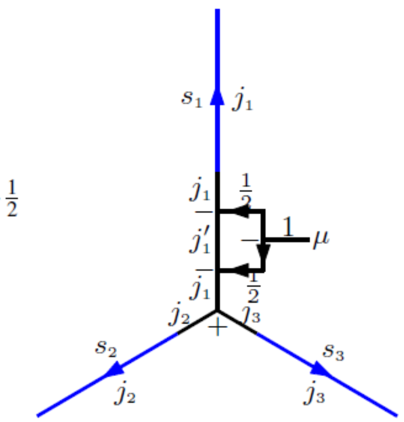
$${}^{(\frac{1}{2})}\hat{e}^{\mu}_{1}(v) \cdot T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) = (\tau_\mu)^{A}{}_{B} [h_{s_1}]^{B}{}_{C} \hat{V}^{\frac{1}{2}} [h^{-1}_{s_1}]^{C}{}_{A} \left[(-1)^{2j_3} \right.$$



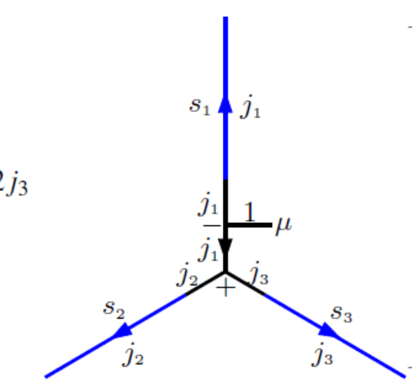
$$= (\tau_\mu)^{A}{}_{B} \sum_{j_1'} [V(j_1', j_2, j_3)]^{\frac{1}{2}} (-1)^{2j_3} d_{j_1'}$$



$$= i\frac{\sqrt{6}}{2} \sum_{j_1'} [V(j_1', j_2, j_3)]^{\frac{1}{2}} (-1)^{2j_3} d_{j_1'} (-1)^{j_1 + j_1' + \frac{1}{2}}$$



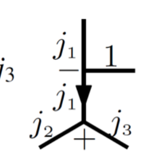
$$= i\frac{\sqrt{j_1(j_1+1)}}{2j_1+1} \left(V^{\frac{1}{2}}_{1A} - V^{\frac{1}{2}}_{1B} \right) \left[\sqrt{d_{j_1}} (-1)^{2j_3} \right.$$



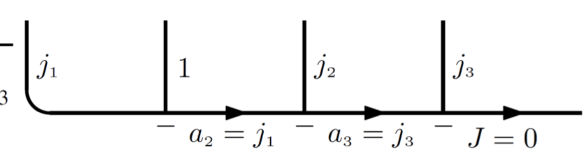
$$V^{\frac{1}{2}}_{1A} := [V(j_1' = j_1 - 1/2, j_2, j_3)]^{\frac{1}{2}}$$

$$V^{\frac{1}{2}}_{1B} := [V(j_1' = j_1 + 1/2, j_2, j_3)]^{\frac{1}{2}}$$

$$\sqrt{d_{j_1}} (-1)^{2j_3}$$



$$= \sqrt{d_{j_1} d_{j_3}}$$

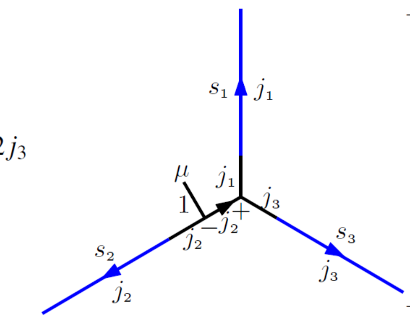


$$= i^{J=0;\ \vec{a}' \equiv \{a_2 = j_1, a_3 = j_3\}}_{j_1, 1, j_2, j_3}$$

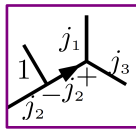
$$
\begin{aligned}
Q_{11} &= -\left({}^{(\frac{1}{2})}\hat{e}^{\mu}_{1}(v) T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}}, {}^{(\frac{1}{2})}\hat{e}^{\mu}_{1}(v) T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}} \right)_{\mathcal{H}_{kin}} \\
&= -\left(i \frac{\sqrt{j_1(j_1+1)}}{2j_1+1} \left(V^{\frac{1}{2}}_{1A} - V^{\frac{1}{2}}_{1B} \right) T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}'}, i \frac{\sqrt{j_1(j_1+1)}}{2j_1+1} \left(V^{\frac{1}{2}}_{1A} - V^{\frac{1}{2}}_{1B} \right) T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}'} \right)_{\mathcal{H}_{kin}} \\
&= -\frac{j_1(j_1+1)}{(2j_1+1)^2} \left(V^{\frac{1}{2}}_{1A} - V^{\frac{1}{2}}_{1B} \right)^2 \int_{SU(2)^3} \prod_{I=1,2,3} \mathrm{d}\mu_H(h_{s_I}) \overline{T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}'}(A)}\, T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}'}(A) \\
&= -\frac{j_1(j_1+1)}{(2j_1+1)^2} \left(V^{\frac{1}{2}}_{1A} - V^{\frac{1}{2}}_{1B} \right)^2 \mathrm{tr}\left(\overline{i^{J=0;\,\vec{a}'\equiv\{a_2=j_1,a_3=j_3\}}_{j_1,1,j_2,j_3}} \cdot i^{J=0;\,\vec{a}'\equiv\{a_2=j_1,a_3=j_3\}}_{j_1,1,j_2,j_3} \right) \\
&= -\frac{j_1(j_1+1)}{(2j_1+1)^2} \left(V^{\frac{1}{2}}_{1A} - V^{\frac{1}{2}}_{1B} \right)^2 \mathrm{tr}\left(i^{J=0;\,\vec{a}'\equiv\{a_2=j_1,a_3=j_3\}}_{j_1,1,j_2,j_3} \cdot i^{J=0;\,\vec{a}'\equiv\{a_2=j_1,a_3=j_3\}}_{j_1,1,j_2,j_3} \right) \\
&= -\frac{j_1(j_1+1)}{(2j_1+1)^2} \left(V^{\frac{1}{2}}_{1A} - V^{\frac{1}{2}}_{1B} \right)^2
\end{aligned}
$$

$$Q_{12} = -\left({}^{(\frac{1}{2})}\hat{e}^{\mu}_{1}(v) \cdot T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}},\ {}^{(\frac{1}{2})}\hat{e}^{\mu}_{2}(v) \cdot T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}} \right)_{\mathcal{H}_{kin}}$$

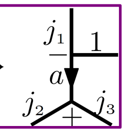
$$ {}^{(\frac{1}{2})}\hat{e}^{\mu}_{2}(v) \cdot T^{v,s}_{\gamma,\vec{j},\vec{i}}(A) = i\frac{\sqrt{j_2(j_2+1)}}{2j_2+1}\left(V^{\frac{1}{2}}_{2A} - V^{\frac{1}{2}}_{2B}\right)\left[\sqrt{2j_2+1}(-1)^{2j_3} \right.$$



$$\left.\right]$$

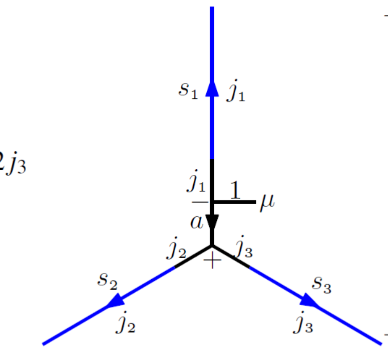


$$= (-1)^{j_1+j_2+j_3} \sum_a (2a+1) \begin{Bmatrix} j_3 & j_2 & a \\ 1 & j_1 & j_2 \end{Bmatrix}$$



$$= i\frac{\sqrt{j_2(j_2+1)}}{2j_2+1}\left(V^{\frac{1}{2}}_{2A} - V^{\frac{1}{2}}_{2B}\right)(-1)^{j_1+j_2+j_3} \sum_a \sqrt{(2a+1)(2j_2+1)} \begin{Bmatrix} j_3 & j_2 & a \\ 1 & j_1 & j_2 \end{Bmatrix}$$

$$\times \left[\sqrt{2a+1}(-1)^{2j_3} \right.$$

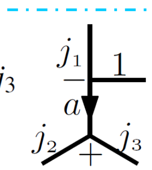


$$\left.\right]$$

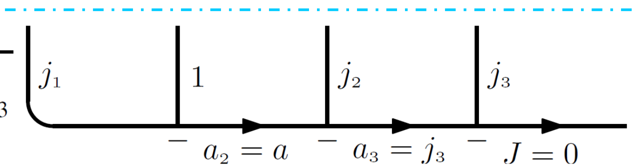
$$V^{\frac{1}{2}}_{2A} := \left[V(j_2' = j_2 - 1/2, j_3, j_1)\right]^{\frac{1}{2}}$$

$$V^{\frac{1}{2}}_{2B} := \left[V(j_2' = j_2 + 1/2, j_3, j_1)\right]^{\frac{1}{2}}$$

$$\sqrt{d_a}\,(-1)^{2j_3}$$



$$= \sqrt{d_a d_{j_3}}$$



$$= i^{J=0;\,\vec{a}''\equiv\{a_2=a,a_3=j_3\}}_{j_1,1,j_2,j_3}$$

$$
\begin{aligned}
Q_{12} &= -\left({}^{(\frac{1}{2})}\hat{e}_1^{\mu}(v) \cdot T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}} , {}^{(\frac{1}{2})}\hat{e}_2^{\mu}(v) \cdot T^{v,s,\text{norm}}_{\gamma,\vec{j},\vec{i}} \right)_{\mathcal{H}_{kin}} \\
&= -\frac{\sqrt{j_1(j_1+1)}}{2j_1+1}\left(V_{1A}^{\frac{1}{2}} - V_{1B}^{\frac{1}{2}}\right)\frac{\sqrt{j_2(j_2+1)}}{2j_2+1}\left(V_{2A}^{\frac{1}{2}} - V_{2B}^{\frac{1}{2}}\right)(-1)^{j_1+j_2+j_3} \\
&\qquad \times \sum_a \sqrt{(2a+1)(2j_2+1)} \begin{Bmatrix} j_3 & j_2 & a \\ 1 & j_1 & j_2 \end{Bmatrix} \operatorname{tr}\left(i_{j_1,1,j_2,j_3}^{J=0;\vec{a}'\equiv\{a_2=j_1,a_3=j_3\}} \cdot i_{j_1,1,j_2,j_3}^{J=0;\vec{a}''\equiv\{a_2=a,a_3=j_3\}} \right) \\
&= -\frac{\sqrt{j_1(j_1+1)j_2(j_2+1)}}{(2j_1+1)(2j_2+1)}\left(V_{1A}^{\frac{1}{2}} - V_{1B}^{\frac{1}{2}}\right)\left(V_{2A}^{\frac{1}{2}} - V_{2B}^{\frac{1}{2}}\right)(-1)^{j_1+j_2+j_3} \sum_a \sqrt{(2a+1)(2j_2+1)} \begin{Bmatrix} j_3 & j_2 & a \\ 1 & j_1 & j_2 \end{Bmatrix} \delta_{a,j_1} \\
&= -\frac{\sqrt{j_1(j_1+1)j_2(j_2+1)}}{(2j_1+1)(2j_2+1)}\left(V_{1A}^{\frac{1}{2}} - V_{1B}^{\frac{1}{2}}\right)\left(V_{2A}^{\frac{1}{2}} - V_{2B}^{\frac{1}{2}}\right)(-1)^{j_1+j_2+j_3} \sqrt{(2j_1+1)(2j_2+1)} \begin{Bmatrix} j_3 & j_2 & j_1 \\ 1 & j_1 & j_2 \end{Bmatrix} \\
&= -\frac{\sqrt{j_1(j_1+1)j_2(j_2+1)}}{(2j_1+1)(2j_2+1)}\left(V_{1A}^{\frac{1}{2}} - V_{1B}^{\frac{1}{2}}\right)\left(V_{2A}^{\frac{1}{2}} - V_{2B}^{\frac{1}{2}}\right)(-1)^{j_1+j_2+j_3} \sqrt{(2j_1+1)(2j_2+1)} \\
&\qquad \times (-1)^{j_1+j_2+j_3+1}\frac{2[j_1(j_1+1)+j_2(j_2+1)-j_3(j_3+1)]}{\sqrt{2j_1(2j_1+1)(2j_1+2)2j_2(2j_2+1)(2j_2+2)}} \\
&= \frac{j_1(j_1+1)+j_2(j_2+1)-j_3(j_3+1)}{2(2j_1+1)(2j_2+1)}\left(V_{1A}^{\frac{1}{2}} - V_{1B}^{\frac{1}{2}}\right)\left(V_{2A}^{\frac{1}{2}} - V_{2B}^{\frac{1}{2}}\right)
\end{aligned}
$$

$$Q_{IJ} = \left(T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}}, \hat{q}_{IJ}(v) \cdot T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}}\right)_{\mathcal{H}_{\mathrm{kin}}} = -\left({}^{(\frac{1}{2})}\hat{e}^{\mu}_{I}(v) \cdot T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}}, {}^{(\frac{1}{2})}\hat{e}^{\mu}_{J}(v) \cdot T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}}\right)_{\mathcal{H}_{\mathrm{kin}}} \quad \surd$$

$$\widehat{V^{-1}}_{\mathrm{alt},v} \cdot T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}}(A) = \epsilon^{IJK}\epsilon^{LMN} Q_{IL} Q_{JM} Q_{KN} T^{v,s,\mathrm{norm}}_{\gamma,\vec{j},\vec{i}}(A) = 0 \quad \text{!!!}$$

Thanks for your attention!