Quantum mechanism angular momentum & graphical calculus

III

Jinsong Yang (Guizhou university)

Loops'26 summer school
Yangzhou University, May 11-12, 2026

Contents:

References:

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1. Elementary operators and their actions

The holonomy operator acts as multiplication:

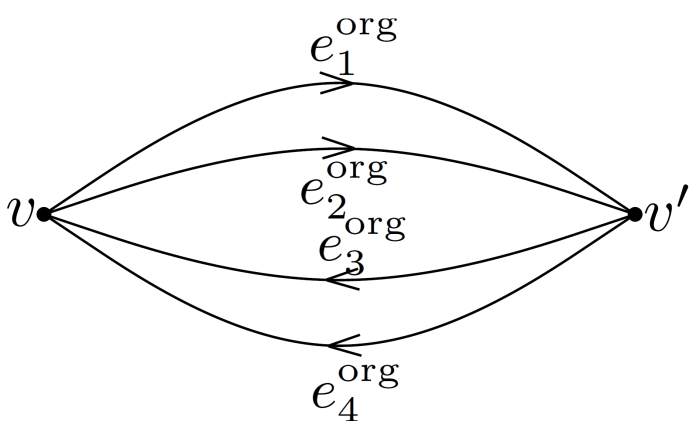
$$[\hat{h}_{e_I}(A)]^B{}_C \cdot f_\gamma(A) := \left[\pi_{\frac{1}{2}}(h_{e_I}(A))\right]^B{}_C f_\gamma(A)$$

The flux operator acts on a cylindrical function f_γ with respect to the graph γ adapted to S as a derivative operator,

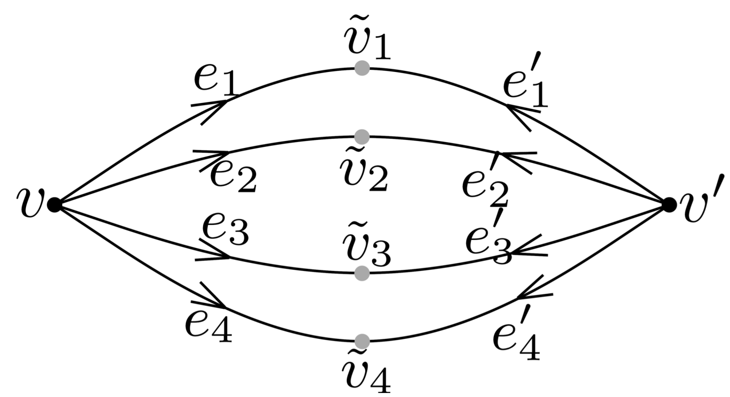
$$\begin{aligned}\hat{\tilde{E}}_i(S) \cdot f_\gamma(A) &:= -\mathrm{i}\hbar \left\{f_\gamma(A), \tilde{E}_i(S)\right\} \\ &= \frac{\hbar\kappa\beta}{2} \sum_{e_I \in E(\gamma)} \kappa(e, S)\, \underline{J^i_{e_I} \cdot f_\gamma(A)} \\ &= \frac{\hbar\kappa\beta}{2} \sum_{v \in V(\gamma)} \left(\sum_{b(e)=v} \kappa(e, S) J^i_{e,R} + \sum_{f(e)=v} \kappa(e, S) J^i_{e,L}\right) f_\gamma(A)\end{aligned}$$

2. Spin-network states (based on standard graphs)

The original graph ⟹ standard graph



(original graph)



(standard graph)

1. Adding a (pseudo-)vertex $\tilde{v}_i$ to each edge e_i^{org} of the original graph.
2. Splitting each edge of the original graph into two edges (two segments) starting from the original graph's vertices.

$$e_i^{\text{org}} = e_i \circ e_i'^{-1}$$

A spin-network state (based on a standard graph)

$$T_{\gamma,\vec{j},\vec{i}}(A) := \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}},$$

$$(i_v)_{m_1 m_2\cdots m_n}{}^{M} \equiv \left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^{M} = (-1)^{j_1-\sum_{i=2}^{n} j_i - J} \sum_{k_2,\ldots,k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2\rangle \cdots \langle JM | a_{n-1} k_{n-1} j_n m_n\rangle$$

$$(i^*_{\tilde{v}})_N{}^{n_1 n_2} \equiv \left(i^{J}_{j_1 j_2}\right)_N{}^{n_1 n_2} := (-1)^{j_1-j_2-J} \langle j_1 n_1 j_2 n_2 | JN\rangle = (i_{\tilde{v}})_{n_1 n_2}{}^{N}$$

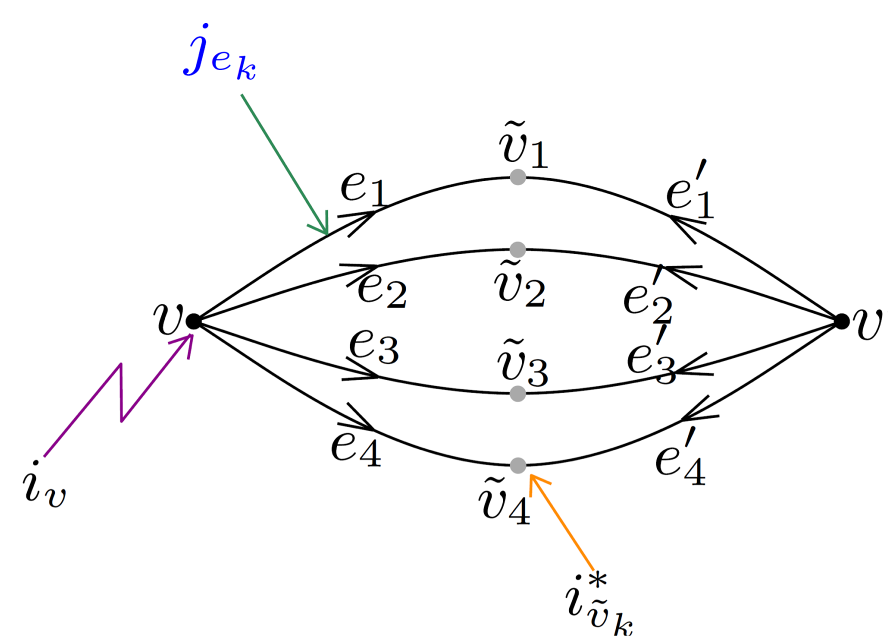
$J = 0, M = 0$	$\Leftrightarrow$	gauge-invariant
$J \neq 0$	$\Leftrightarrow$	gauge-variant

The normalized spin-network state

$$T^{\text{norm}}_{\gamma,\vec{j},\vec{i}}(A) := \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}}\, \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}}$$

A spin-network

$$(\gamma, \vec{j}, \vec{i})$$



2. Spin-network states (based on standard graphs)

A spin-network state (based on a standard graph)

$$T_{\gamma,\vec{j},\vec{i}}(A) := \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}},$$

$$(i_v)_{m_1 m_2 \cdots m_n}{}^{M} \equiv \left(i^{J;\,\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^{M} = (-1)^{j_1-\sum_{i=2}^{n} j_i - J} \sum_{k_2,\ldots,k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2\rangle \cdots \langle JM | a_{n-1} k_{n-1} j_n m_n\rangle$$

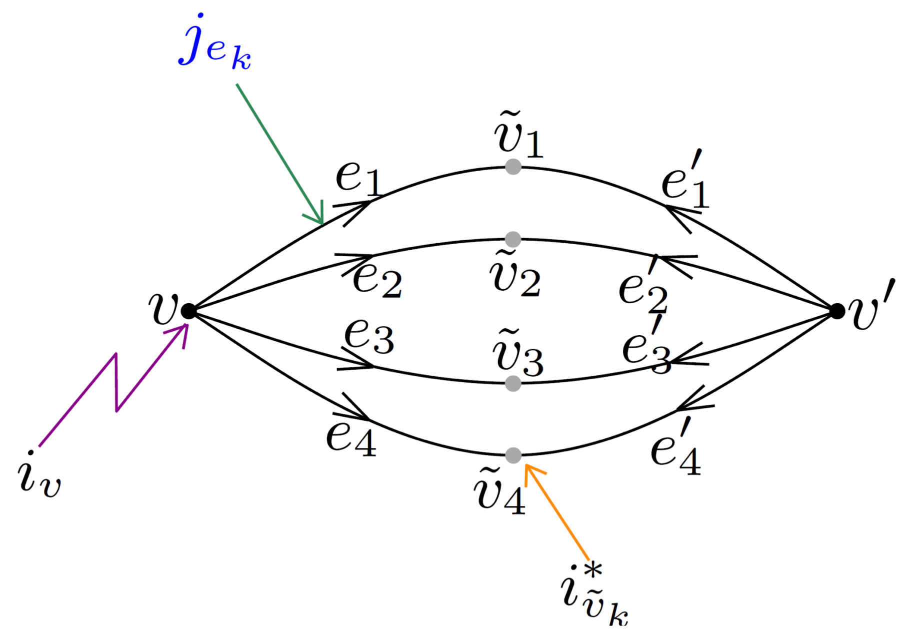
$$(i^*_{\tilde{v}})_{N}{}^{n_1 n_2} \equiv \left(i^{J}_{j_1 j_2}\right)_{N}{}^{n_1 n_2} := (-1)^{j_1-j_2-J}\langle j_1 n_1 j_2 n_2 | JN\rangle = (i_{\tilde{v}})_{n_1 n_2}{}^{N}$$

$J=0, M=0$	$\Leftrightarrow$	gauge-invariant
$J\neq 0$	$\Leftrightarrow$	gauge-variant

The normalized spin-network state

$$T^{\text{norm}}_{\gamma,\vec{j},\vec{i}}(A) := \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}}\, \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}}$$

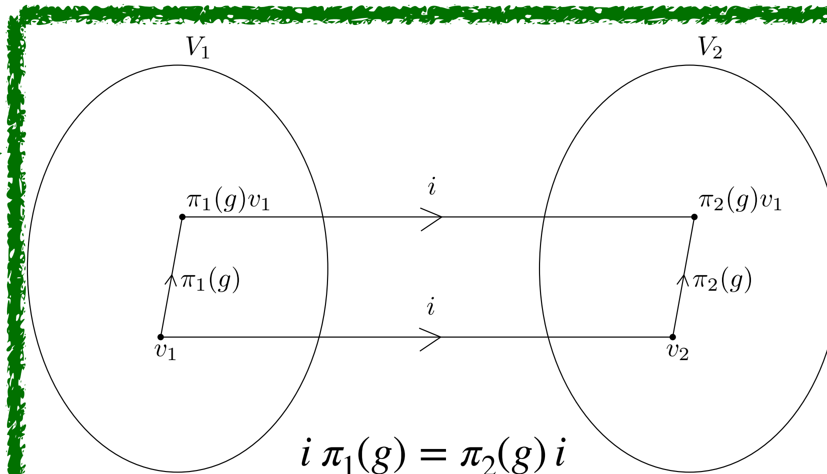
A spin-network



$$\begin{aligned}
T^{\text{norm}}_{\gamma,\vec{j},\vec{i}}(A^g) &= \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}}\, \pi_{j_e}(h_e(A^g)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}} \\
&= \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}}\, \pi_{j_e}(g(b(e)))\cdot \pi_{j_e}(h_e(A))\cdot \pi_{j_e}(g(f(e))^{-1}) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}} \\
&= \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{b(e)=v} \pi_{j_e}(g(v)) \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}} \cdot \pi_{j_e}(h_e(A)) \cdot \bigotimes_{f(e)=\tilde{v}} \pi_{j_e}(g(\tilde{v})^{-1}) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}} \\
&= \bigotimes_{v\in V(\gamma)} \left[i_v \cdot \bigotimes_{b(e)=v} \pi_{j_e}(g(v)) \right] \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}} \cdot \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} \left[i^*_{\tilde{v}} \cdot \bigotimes_{f(e)=\tilde{v}} \pi_J(g(\tilde{v})^{-1}) \right] \\
&= \bigotimes_{v\in V(\gamma)} [\pi_J(g(v))\cdot i_v] \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}} \cdot \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} [\pi^*_J(g(\tilde{v}))\cdot i^*_{\tilde{v}}] \\
&\overset{J=0}{=} \bigotimes_{v\in V(\gamma)} i_v \cdot \bigotimes_{e\in E(\gamma)} \sqrt{d_{j_e}}\, \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v}\in V(\gamma)} i^*_{\tilde{v}}
\end{aligned}$$

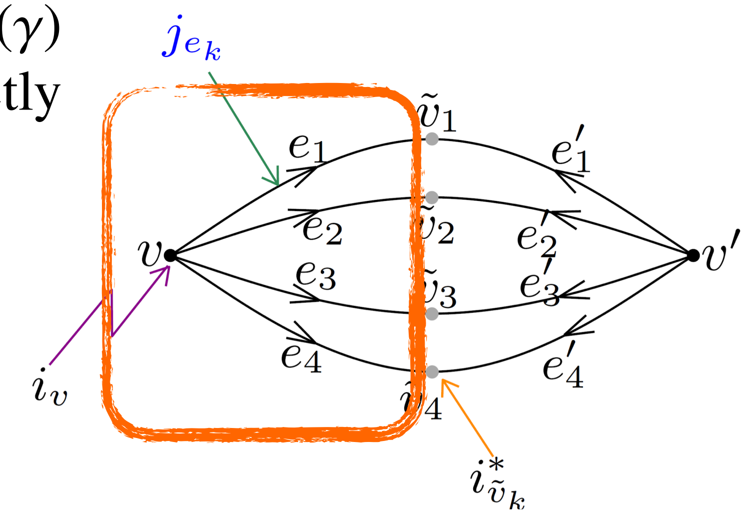
$$A \mapsto A^g = -(\mathrm{d}g)g^{-1} + gAg^{-1}$$

$$h_e(A^g) = g(b(e))h_e(A)g(f(e))^{-1}$$



3. The actions on spin-network states (based on standard graphs)

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \dots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v



$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I \cdots m_n}{}^M [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$[\pi_{j_1}(h_{e_1}(A)]^{m_1}{}_{n_1}$$

$$[\hat{h}_{e_I}]^B{}_C \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I \cdots m_n}{}^M [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \left[\pi_{\frac{1}{2}}(h_{e_I})\right]^B{}_C \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= (i_v)_{m_1\cdots m_I \cdots m_n}{}^M [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots \sum_{j_I', m_I', n_I'} \left(i^{j_I'}_{j_I \frac{1}{2}}\right)_{m_I'}{}^{m_I B} [\pi_{j_I'}(h_{e_I})]^{m_I'}{}_{n_I'} \left(i^{j_I'}_{j_I \frac{1}{2}}\right)^{n_I'}{}_{n_I C} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

acting at an edge — coupling with holonomy of an edge

$$J^i_{e_I} \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I \cdots m_n}{}^M [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots \left(\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} [\pi_{j_I}(e^{-t\tau_i} h_{e_I})]^{m_I}{}_{n_I} \right) \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= \left[(i_v)_{m_1\cdots m_I' \cdots m_n}{}^M \left(-\mathrm{i}\, [\pi_{j_I}(\tau_i)]^{m_I'}{}_{m_I} \right) \right] [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$\tau_i := -\mathrm{i}\sigma_i/2$, σ_i : Pauli matrices $i = 1,2,3$

$$J^i_{e_I} \cdot (i_v)_{m_1\cdots m_I \cdots m_n}{}^M = (i_v)_{m_1\cdots m_I' \cdots m_n}{}^M \left(-\mathrm{i}\, [\pi_{j_I}(\tau_i)]^{m_I'}{}_{m_I} \right)$$

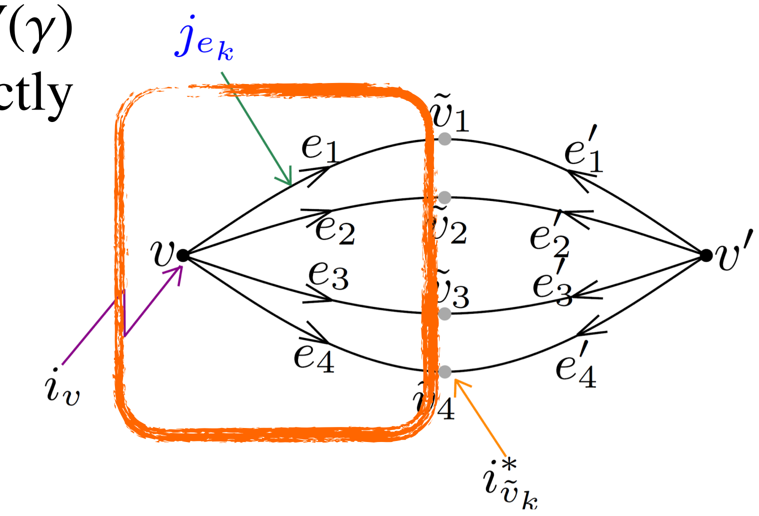
$$J^\mu_{e_I} \cdot (i_v)_{m_1\cdots m_I \cdots m_n}{}^M = (i_v)_{m_1\cdots m_I' \cdots m_n}{}^M \left(-\mathrm{i}\, [\pi_{j_I}(\tau_\mu)]^{m_I'}{}_{m_I} \right)$$

$$\tau_0 := \tau_3, \quad \tau_{\pm 1} := \mp \frac{1}{\sqrt{2}} (\tau_1 \pm \mathrm{i}\tau_2)$$

acting at a vertex — contracting with intertwiner of a vertex

3. The actions on spin-network states (based on standard graphs)

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \ldots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v



$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M \,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$(i_v)_{m_1 m_2\cdots m_n}{}^M \equiv \begin{pmatrix} i^{J;\,\vec{a}}_{j_1\cdots j_n}\end{pmatrix}_{m_1\cdots m_n}{}^M = (-1)^{j_1-\sum_{i=2}^n j_i - J} \sum_{k_2,\ldots,k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2\rangle \cdots \langle JM | a_{n-1} k_{n-1} j_n m_n\rangle$$

$$\langle j_3 m_3 | j_1 m_1 j_2 m_2\rangle = (-1)^{j_1-j_2-j_3}\sqrt{d_{j_3}} \sum_{m'_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m'_3\end{pmatrix} C^{m'_3 m_3}_{(j_3)}$$

Wigner 3-j symbol "Metric"

Properties of 3j-symbol

1. Symmetry:

odd-numbered $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2\end{pmatrix} = \ldots$

even-numbered $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1\end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2\end{pmatrix}$ **(cyclic symmetry)**

2. Reality (unitary):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = \sum_{m'_1,m'_2,m'_3} C^{m_1 m'_1}_{(j_1)} C^{m_2 m'_2}_{(j_2)} C^{m_3 m'_3}_{(j_3)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3\end{pmatrix} = \sum_{m'_1,m'_2,m'_3} C^{(j_1)}_{m_1 m'_1} C^{(j_2)}_{m_2 m'_2} C^{(j_3)}_{m_3 m'_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3\end{pmatrix}$$

3. Orthogonality:

$$\sum_{m_1,m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j'_3 \\ m_1 & m_2 & m'_3\end{pmatrix} = \frac{\delta_{j_3,j'_3}}{2j_3+1}\delta^{(j_3)}_{m_3,m'_3}$$

"Metric" $C^{(j)}_{mn}$ **on** V_j

$$\overleftarrow{C}^{(j)}_{mn} := (-1)^{j-n}\delta_{n,-m} = (-1)^{j+m}\delta_{m,-n},$$

$$\overrightarrow{C}^{mn}_{(j)} \equiv (C^{(j)\,-1})^{mn} := (-1)^{j-m}\delta_{m,-n} = (-1)^{j+n}\delta_{n,-m},$$

satisfying

$$C^{(j)}_{m'm} = (-1)^{2j} C^{(j)}_{mm'}$$

$$C^{mm'}_{(j)} = (-1)^{2j} C^{m'm}_{(j)}$$

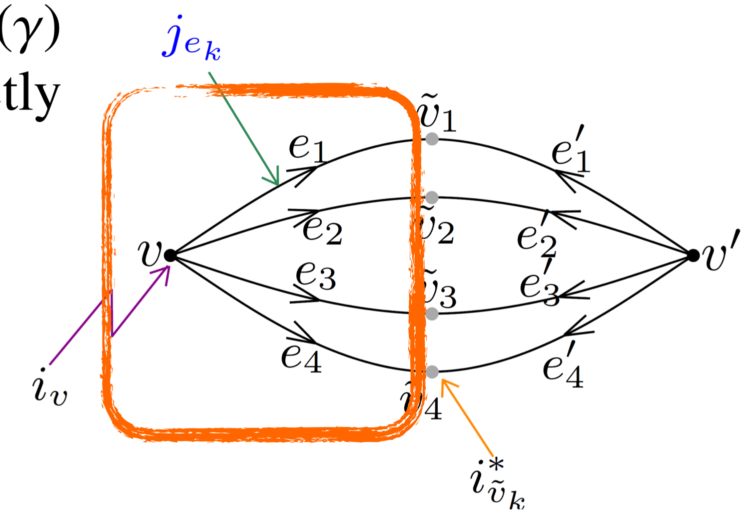
$$C^{(j)}_{mn} C^{nm'}_{(j)} = C^{m'n}_{(j)} C^{(j)}_{nm} = \delta^{m'}_m$$

$$C^{(j)}_{nm} C^{nm'}_{(j)} = C^{m'n}_{(j)} C^{(j)}_{mn} = (-1)^{2j}\delta^{m'}_m$$

$$\begin{pmatrix} j & j' & 0 \\ m & m' & 0\end{pmatrix} = \frac{\delta_{j,j'}}{\sqrt{d_j}} C^{(j)}_{m'm}$$

4. Graphical representation: intertwiner

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \dots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v



$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

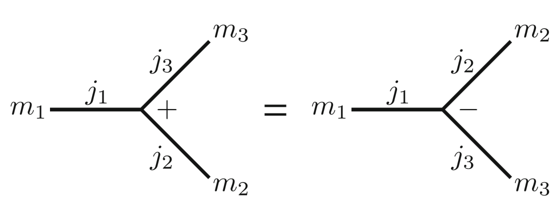
$$(i_v)_{m_1m_2\cdots m_n}{}^M \equiv \left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^M = (-1)^{j_1-\sum_{i=2}^n j_i - J}\sum_{k_2,\dots,k_{n-1}}\langle a_2k_2|j_1m_1j_2m_2\rangle\cdots\langle JM|a_{n-1}k_{n-1}j_nm_n\rangle$$

$$\langle j_3m_3|j_1m_1j_2m_2\rangle = (-1)^{j_1-j_2-j_3}\sqrt{d_{j_3}}\sum_{m'_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m'_3\end{pmatrix} C^{m'_3m_3}_{(j_3)}$$

Wigner 3-j symbol "Metric"

Wigner 3j-symbol

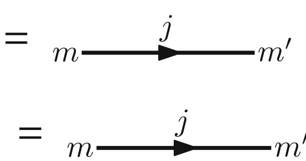
$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} =$$



"Metric"

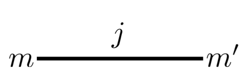
$$C^{(j)}_{m'm} = (-1)^{j-m}\delta_{m,-m'} = (-1)^{j+m'}\delta_{m,-m'} =$$

$$C^{mm'}_{(j)} = (-1)^{j-m}\delta_{m,-m'} = (-1)^{j+m'}\delta_{m,-m'} =$$

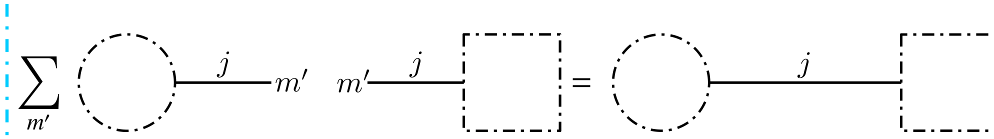


Kronecker delta symbol

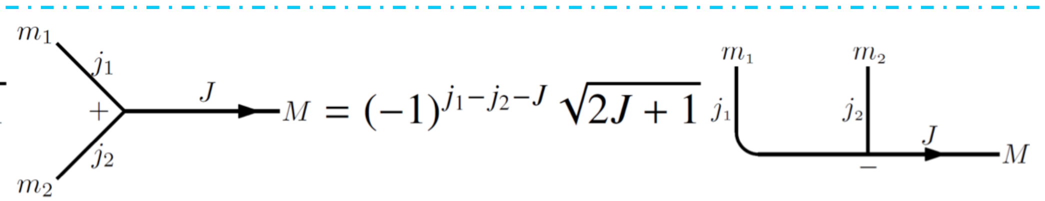
$$\left(\delta^{(j)}\right)^m{}_{m'} = \left(\delta^{(j)}\right)_m{}^{m'} =$$



Summation over the same indices



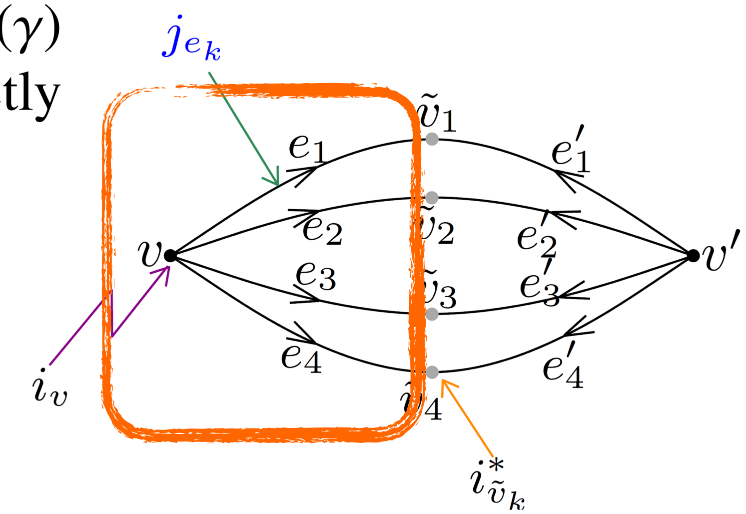
$$\langle JM|j_1m_1j_2m_2\rangle = (-1)^{j_1-j_2-J}\sqrt{2J+1}$$



$$\left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^M = \prod_{i=2}^{n-1}\sqrt{2a_i+1}\,\sqrt{2J+1}$$

4. Graphical representation: holonomy

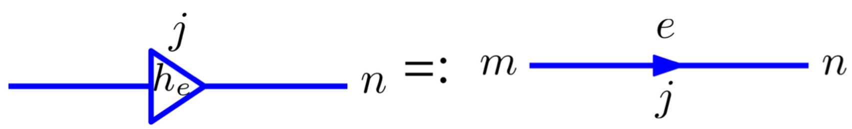
For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \dots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v



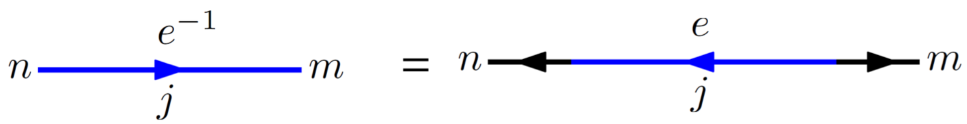
$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M \, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

Matrix element of holonomy with irrep.

$\left[\pi_j(h_e(A))\right]^m{}_n =$



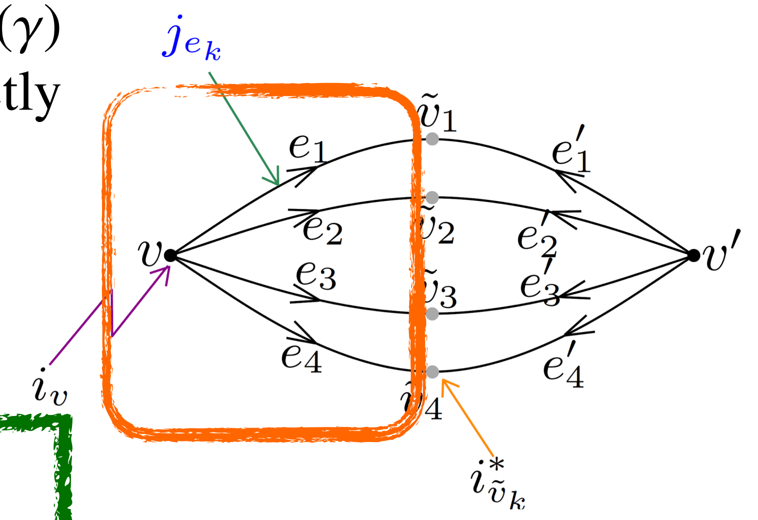
$\left[\pi_j(h_e(A)^{-1})\right]^n{}_m =$



$$\left[\pi_j(h_e(A)^{-1})\right]^n{}_m = \left[\pi_j(h_{e^{-1}}(A))\right]^n{}_m = C^{(j)}_{mm'} \left[\pi_j(h_e(A))\right]^{m'}{}_{n'} C^{n'n}_{(j)}$$

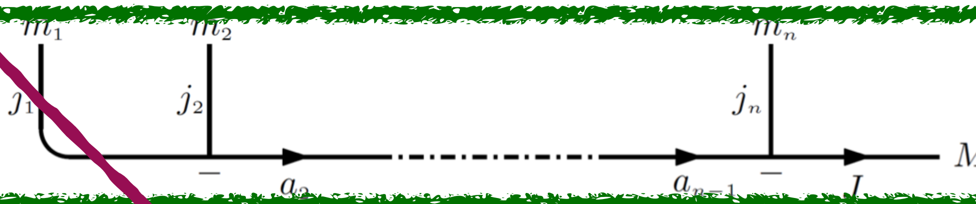
4. Graphical representation: spin network states

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \ldots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v

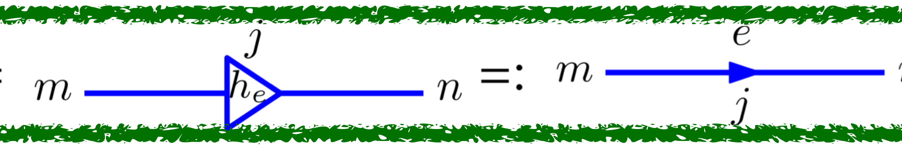


$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

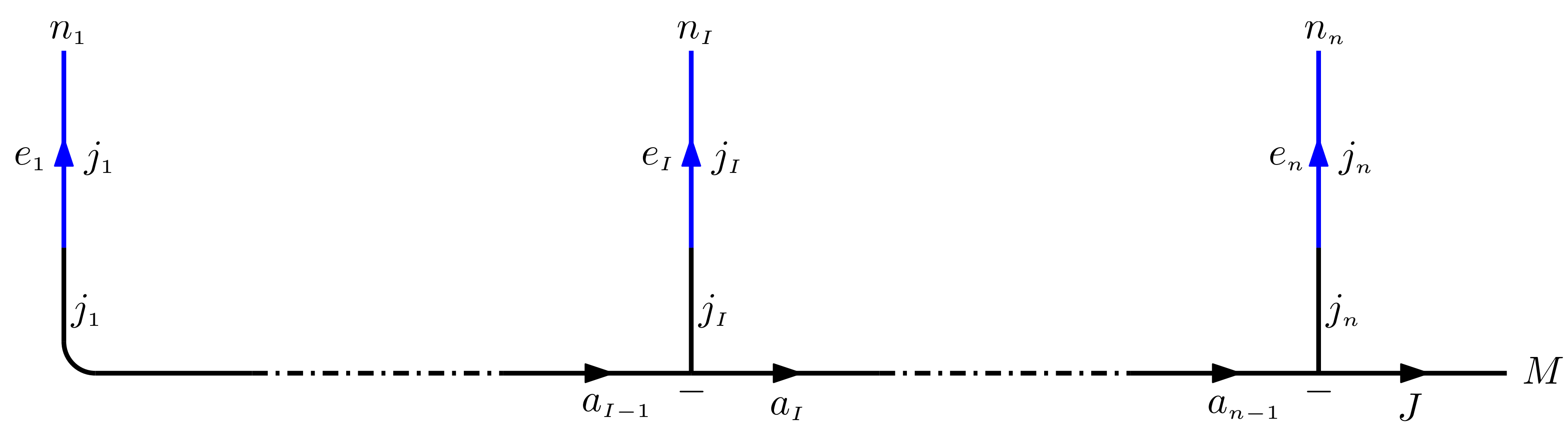
$$\left(i^{J;\vec{a}}_{j_1\cdots j_n}\right)_{m_1\cdots m_n}{}^M = \prod_{i=2}^{n-1}\sqrt{2a_i+1}\,\sqrt{2J+1}$$



$$[\pi_j(h_e(A))]^m{}_n =$$

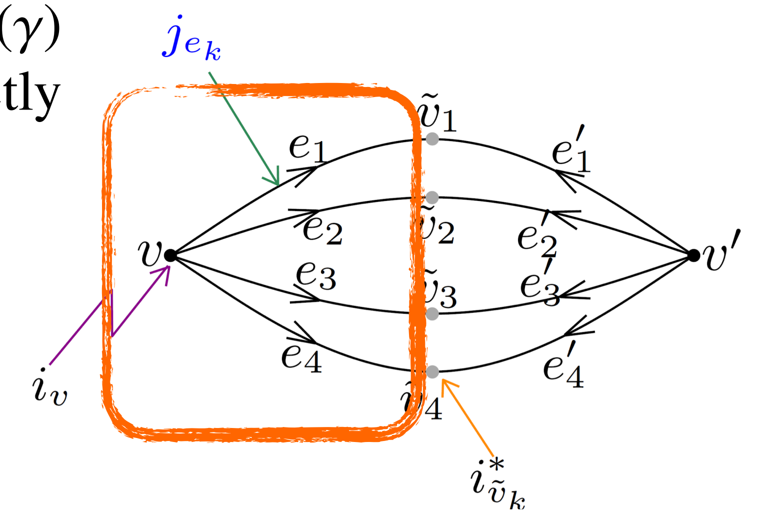


$$T^v_{\gamma,\vec{j},\vec{i}}(A) = \prod_{i=2}^{n-1}\sqrt{d_{a_i}}\,\sqrt{d_J}$$



5. Graphical calculus: the actions of elementary operators

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \ldots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v



$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$[\hat{h}_{e_I}]^B{}_C \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \left[\pi_{\frac{1}{2}}(h_{e_I})\right]^B{}_C \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots \sum_{j'_I, m'_I, n'_I} \left(i^{j'_I}_{j_I \frac{1}{2}}\right)_{m'_I}{}^{m_I B} [\pi_{j'_I}(h_{e_I})]^{m'_I}{}_{n'_I} \left(i^{j'_I}_{j_I \frac{1}{2}}\right)^{n'_I}{}_{n_I C} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

acting at an edge — coupling with holonomy of an edge

$$J^i_{e_I} \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots \left(\mathrm{i}\, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} [\pi_{j_I}(e^{-t\tau_i} h_{e_I})]^{m_I}{}_{n_I} \right) \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= \left[(i_v)_{m_1\cdots m'_I\cdots m_n}{}^M \left(-\mathrm{i}\, [\pi_{j_I}(\tau_i)]^{m'_I}{}_{m_I} \right) \right] [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$\tau_i := -\mathrm{i}\sigma_i/2$, σ_i : Pauli matrices $\quad i = 1,2,3$

$$J^i_{e_I} \cdot (i_v)_{m_1\cdots m_I\cdots m_n}{}^M = (i_v)_{m_1\cdots m'_I\cdots m_n}{}^M \left(-\mathrm{i}\, [\pi_{j_I}(\tau_i)]^{m'_I}{}_{m_I} \right)$$

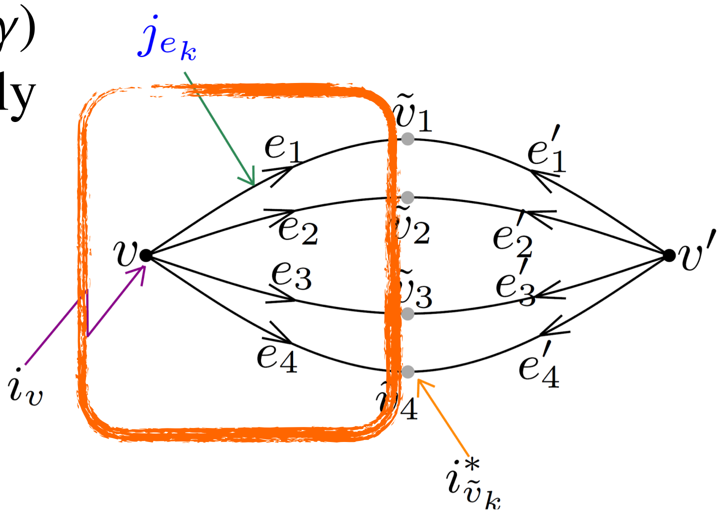
$$J^\mu_{e_I} \cdot (i_v)_{m_1\cdots m_I\cdots m_n}{}^M = (i_v)_{m_1\cdots m'_I\cdots m_n}{}^M \left(-\mathrm{i}\, [\pi_{j_I}(\tau_\mu)]^{m'_I}{}_{m_I} \right)$$

$$\tau_0 := \tau_3, \quad \tau_{\pm 1} := \mp \frac{1}{\sqrt{2}} (\tau_1 \pm \mathrm{i}\tau_2)$$

acting at a vertex — contracting with intertwiner of a vertex

5. Graphical calculus: the actions of elementary operators

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1,\dots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v



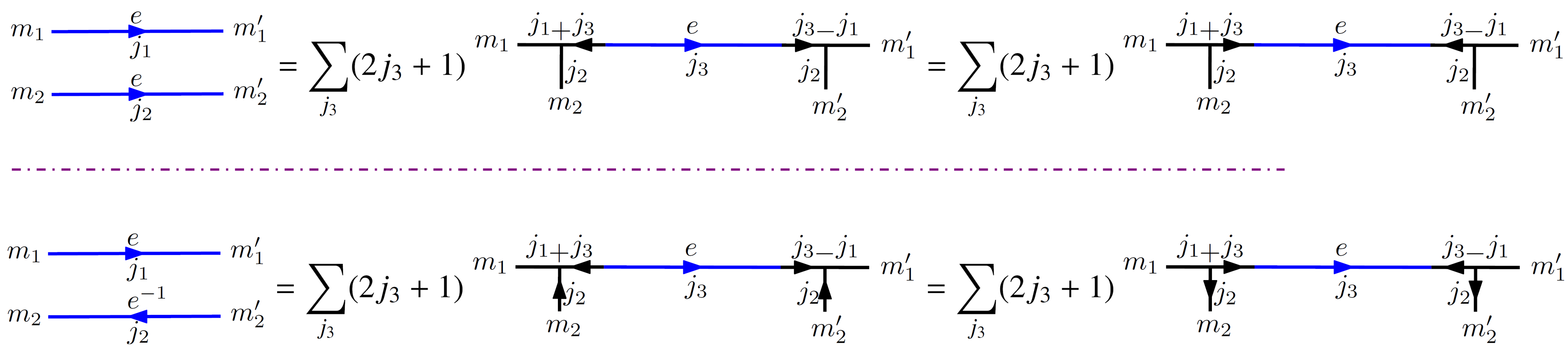
$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$[\hat{h}_{e_I}]^B{}_C\cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\left[\pi_{\frac{1}{2}}(h_{e_I})\right]^B{}_C\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots\sum_{j'_I,m'_I,n'_I}\left(i^{j'_I}_{j_I\frac{1}{2}}\right)_{m'_I}{}^{m_I B}[\pi_{j'_I}(h_{e_I})]^{m'_I}{}_{n'_I}\left(i^{j'_I}_{j_I\frac{1}{2}}\right)_{n_I C}{}^{n'_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

Clebsch-Gordan series

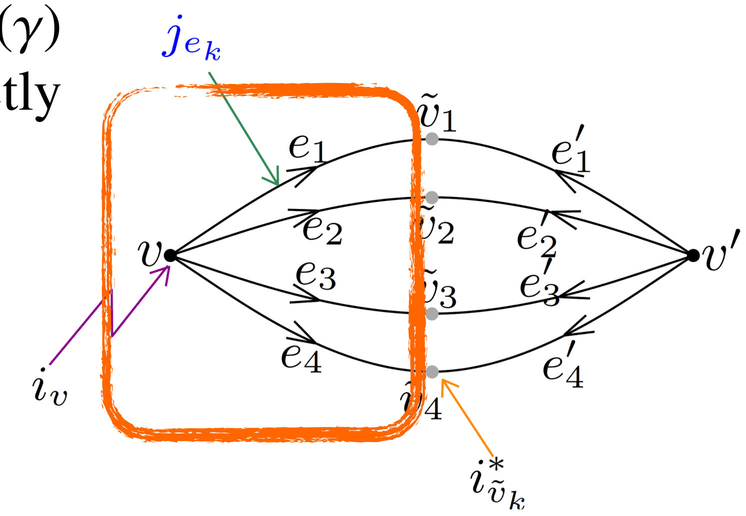
$$[\pi_{j_1}(g)]^{m_1}{}_{n_1}[\pi_{j_2}(g)]^{m_2}{}_{n_2} = \sum_{J,M,N}\left((i^J_{j_1j_2})^{-1}\right)_M{}^{m_1m_2}[\pi_J(g)]^M{}_N\left(i^J_{j_1j_2}\right)_{n_1n_2}{}^N$$



5. Graphical calculation: the actions of elementary operators

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \dots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v

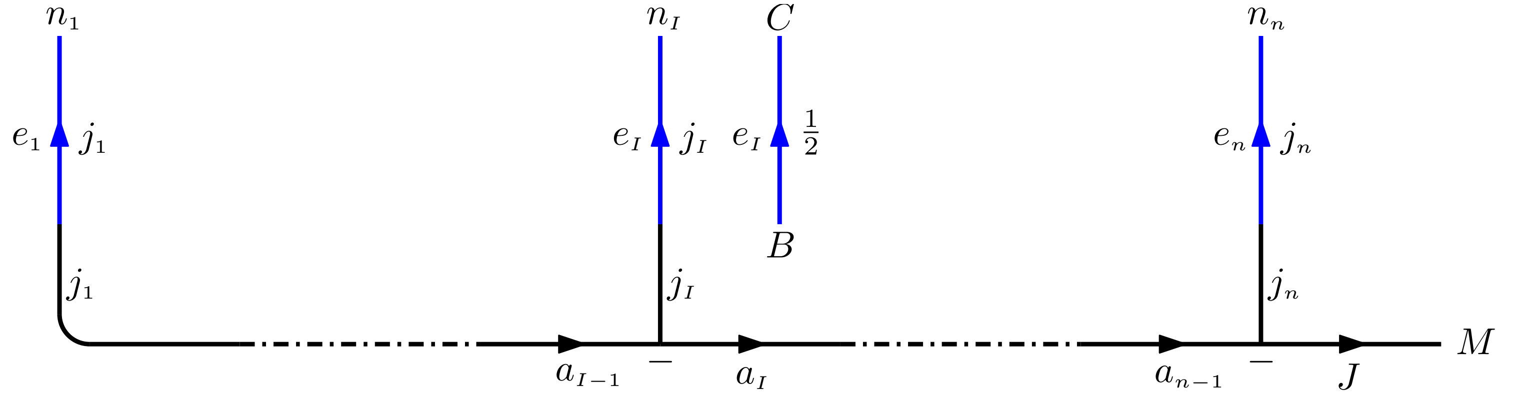
$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$



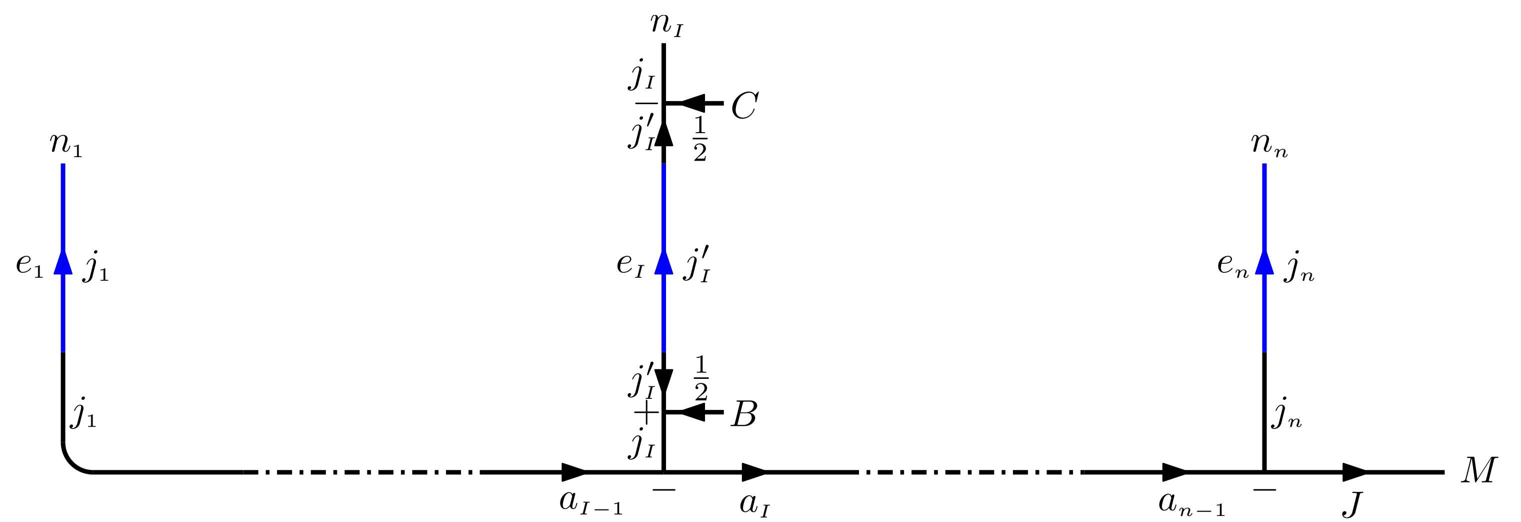
$$[\hat{h}_{e_I}]^B{}_C \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I\cdots m_n}{}^M [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \left[\pi_{\frac{1}{2}}(h_{e_I})\right]^B{}_C \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= (i_v)_{m_1\cdots m_I\cdots m_n}{}^M [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots \sum_{j_I',m_I',n_I'} \left(i^{j_I'}_{j_I \frac{1}{2}}\right)_{m_I'}{}^{m_I B} [\pi_{j_I'}(h_{e_I})]^{m_I'}{}_{n_I'} \left(i^{j_I'}_{j_I \frac{1}{2}}\right)^{n_I'}{}_{n_I C} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$[\hat{h}_{e_I}]^B{}_C \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\sqrt{d_J}$$

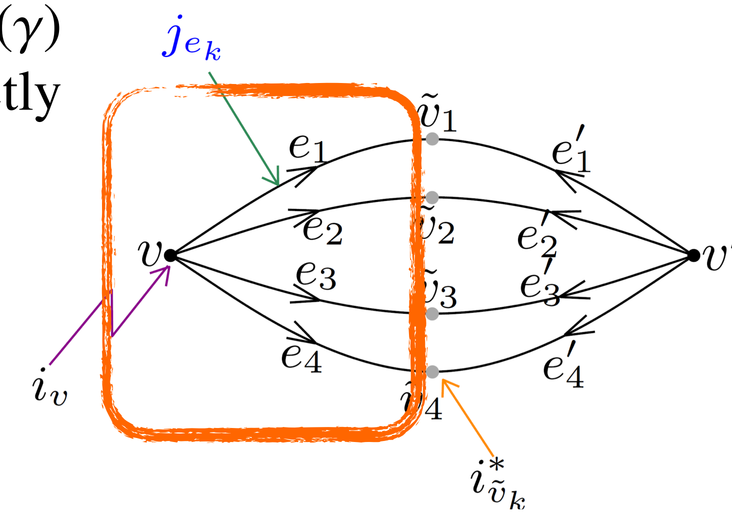


$$= \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\sqrt{d_J}\sum_{j_I'} d_{j_I'}$$



5. Graphical calculation: the actions of elementary operators

For a spin-network state $T_{\gamma,\vec{j},\vec{i}}(A)$ on a graph γ, we consider a true vertex $v \in V(\gamma)$ at which n edges $e_1, \ldots e_n$ incident and denote $T^v_{\gamma,\vec{j},\vec{i}}(A)$ the terms, in $T_{\gamma,\vec{j},\vec{i}}(A)$, directly associated to v



$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$J^i_{e_I}\cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots\left(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}[\pi_{j_I}(e^{-t\tau_i}h_{e_I})]^{m_I}{}_{n_I}\right)\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= \left[(i_v)_{m_1\cdots m_I'\cdots m_n}{}^M\left(-\mathrm{i}\,[\pi_{j_I}(\tau_i)]^{m_I'}{}_{m_I}\right)\right][\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$\tau_i := -\,\mathrm{i}\sigma_i/2,\;\; \sigma_i$: Pauli matrices $\quad i = 1,2,3$

$$J^i_{e_I}\cdot\;(i_v)_{m_1\cdots m_I\cdots m_n}{}^M = (i_v)_{m_1\cdots m_I'\cdots m_n}{}^M\left(-\mathrm{i}\,[\pi_{j_I}(\tau_i)]^{m_I'}{}_{m_I}\right)$$

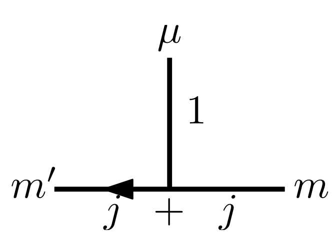
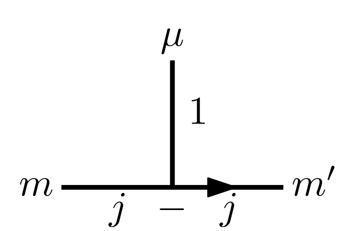
$$J^\mu_{e_I}\cdot\;(i_v)_{m_1\cdots m_I\cdots m_n}{}^M = (i_v)_{m_1\cdots m_I'\cdots m_n}{}^M\left(-\mathrm{i}\,[\pi_{j_I}(\tau_\mu)]^{m_I'}{}_{m_I}\right)$$

$$\tau_0 := \tau_3,\qquad \tau_{\pm1} := \mp\frac{1}{\sqrt{2}}\left(\tau_1 \pm \mathrm{i}\tau_2\right)$$

acting at a vertex — contracting with intertwiner of a vertex

$$[\pi_j(\tau_\mu)]^{m'}{}_m = \mathrm{i}\,\chi(j)\begin{pmatrix} j & 1 & j \\ m & \mu & m'' \end{pmatrix} C^{m''m'}_{(j)} = \mathrm{i}\,\chi(j)\begin{pmatrix} 1 & j & j \\ \mu & m'' & m \end{pmatrix} C^{m''m'}_{(j)}$$

$\chi(j) \equiv \sqrt{j(j+1)(2j+1)}$

$= \mathrm{i}\,\chi(j)$  $= \mathrm{i}\,\chi(j)$ 

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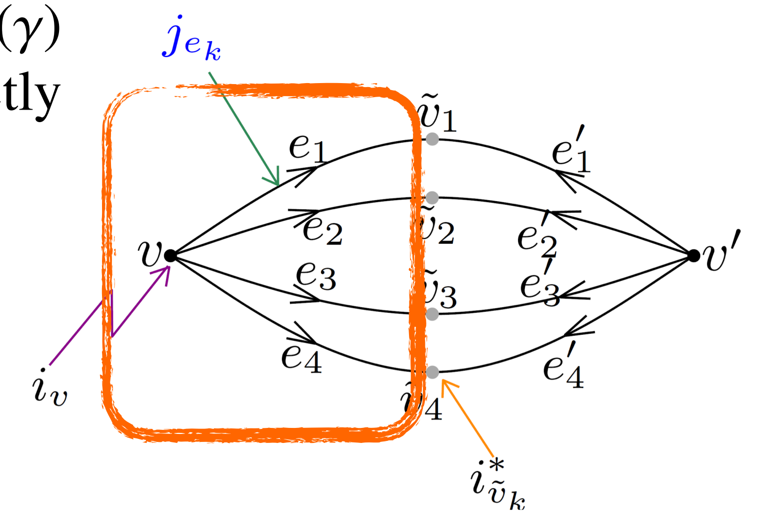


$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$J^i_{e_I}\cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = (i_v)_{m_1\cdots m_I\cdots m_n}{}^M\,[\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots\left(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}[\pi_{j_I}(e^{-t\tau_i}h_{e_I})]^{m_I}{}_{n_I}\right)\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$= \left[(i_v)_{m_1\cdots m'_I\cdots m_n}{}^M\left(-\mathrm{i}\,[\pi_{j_I}(\tau_i)]^{m'_I}{}_{m_I}\right)\right][\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1}\cdots[\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I}\cdots[\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

$$\tau_i := -\mathrm{i}\sigma_i/2,\ \ \sigma_i : \text{Pauli matrices}\quad i=1,2,3$$

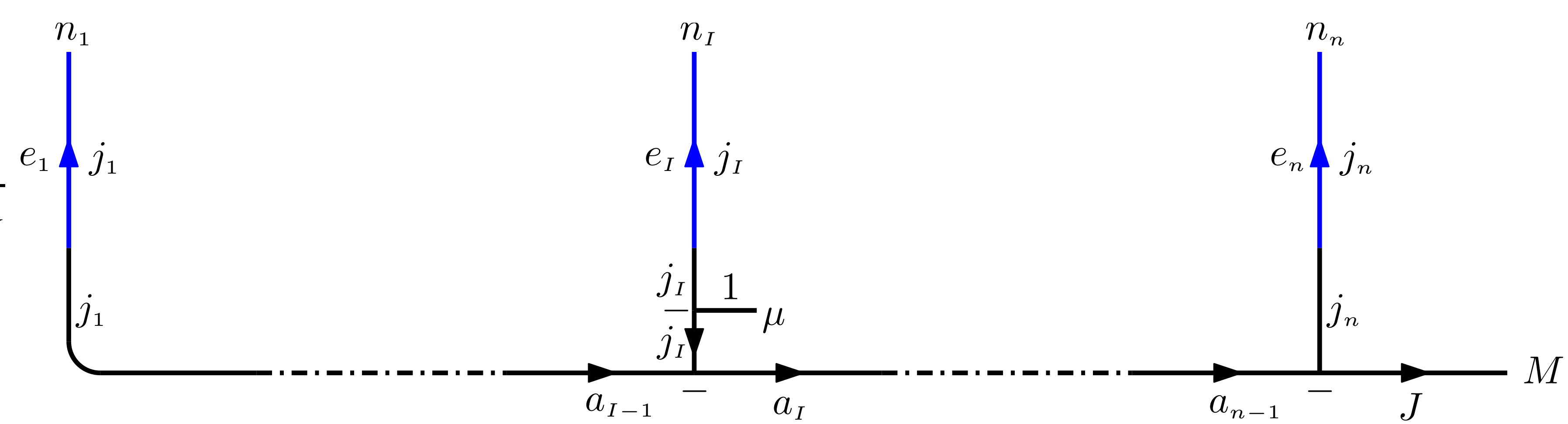
$$J^i_{e_I}\cdot\ (i_v)_{m_1\cdots m_I\cdots m_n}{}^M = (i_v)_{m_1\cdots m'_I\cdots m_n}{}^M\left(-\mathrm{i}\,[\pi_{j_I}(\tau_i)]^{m'_I}{}_{m_I}\right)$$

$$J^\mu_{e_I}\cdot\ (i_v)_{m_1\cdots m_I\cdots m_n}{}^M = (i_v)_{m_1\cdots m'_I\cdots m_n}{}^M\left(-\mathrm{i}\,[\pi_{j_I}(\tau_\mu)]^{m'_I}{}_{m_I}\right)$$

$$\tau_0 := \tau_3,\quad \tau_{\pm1} := \mp\frac{1}{\sqrt{2}}\left(\tau_1 \pm \mathrm{i}\tau_2\right)$$

acting at a vertex — contracting with intertwiner of a vertex

$$J^\mu_{e_I}\cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = \chi(j_I)\prod_{i=2}^{n-1}\sqrt{d_{a_i}}\,\sqrt{d_J}$$



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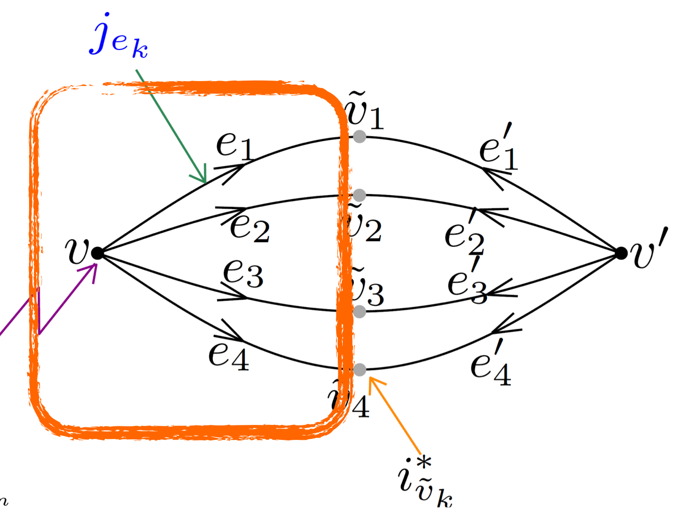
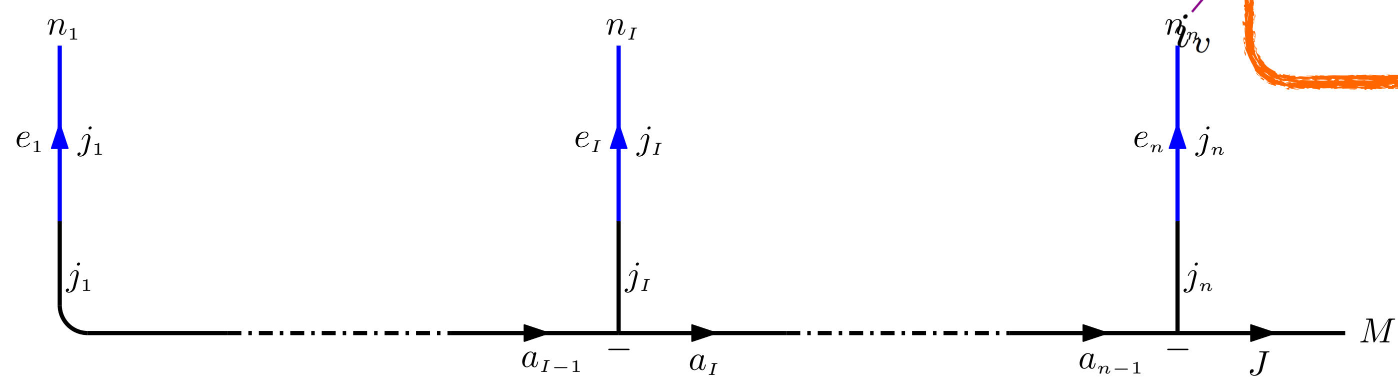
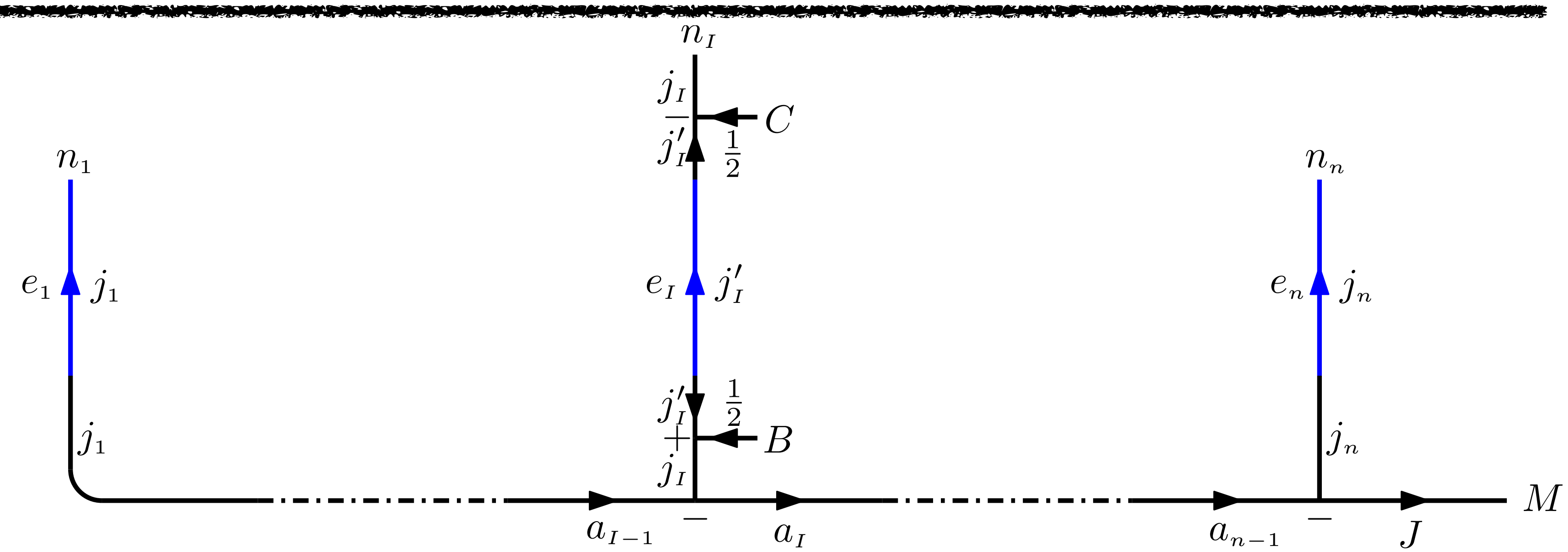


$$T^v_{\gamma,\vec{j},\vec{i}}(A) := (i_v)_{m_1\cdots m_I\cdots m_n}{}^M \, [\pi_{j_1}(h_{e_1})]^{m_1}{}_{n_1} \cdots [\pi_{j_I}(h_{e_I})]^{m_I}{}_{n_I} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}{}_{n_n}$$

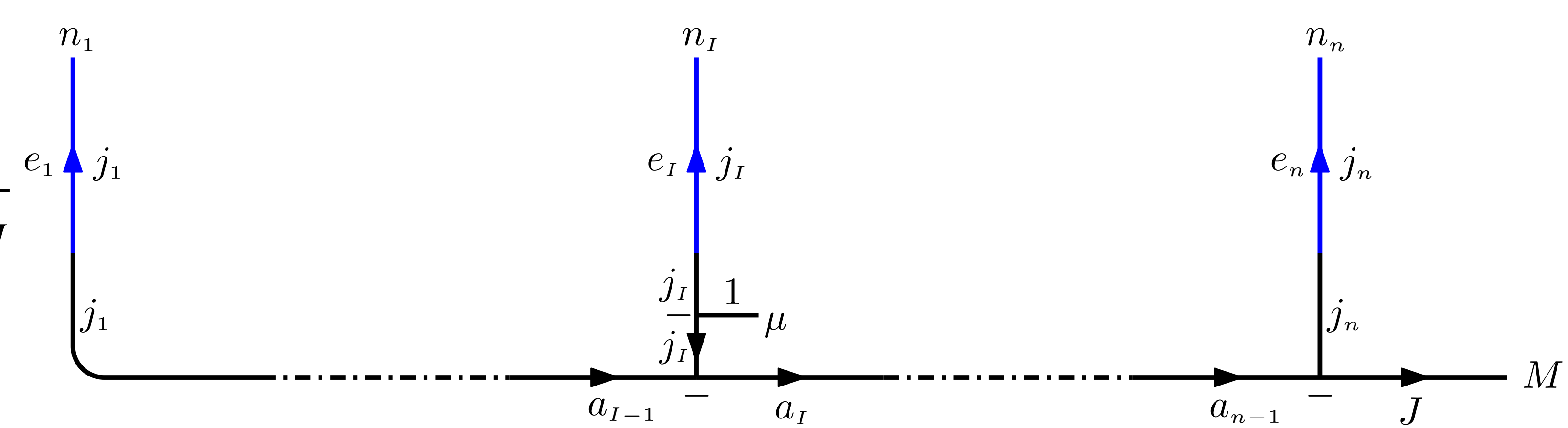
$$= \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\sqrt{d_J}$$



$$[\hat{h}_{e_I}]^B{}_C \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\sqrt{d_J} \sum_{j_I'} d_{j_I'}$$



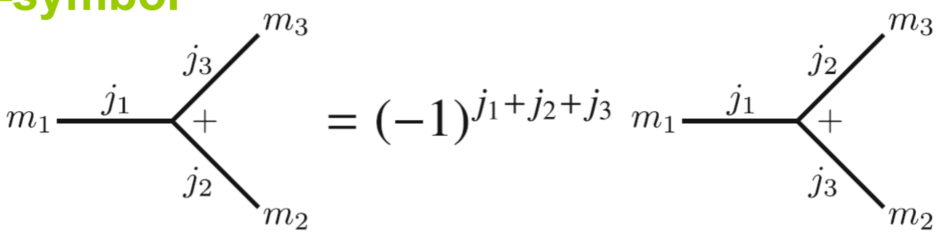
$$J^\mu_{e_I} \cdot T^v_{\gamma,\vec{j},\vec{i}}(A) = \chi(j_I) \prod_{i=2}^{n-1} \sqrt{d_{a_i}}\sqrt{d_J}$$



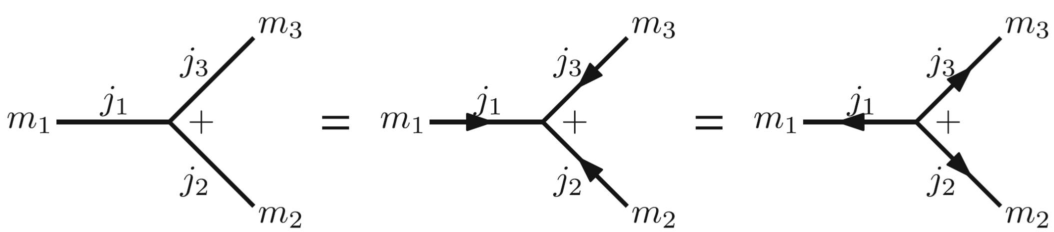
6. Graphical calculation: some useful rules of transforming graphs

Wigner 3j-symbol

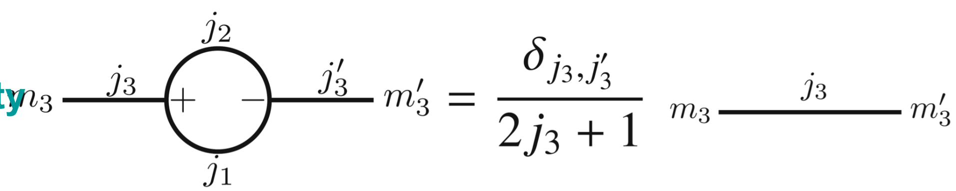
Symmetry



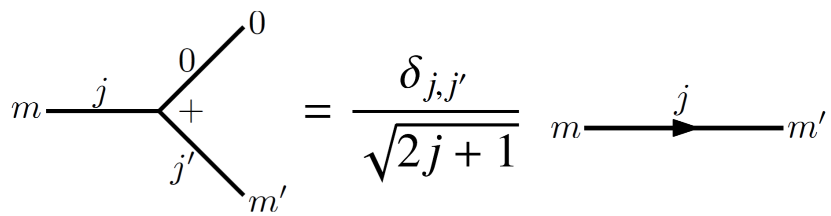
Reality (unitary)



Orthogonality



Special formula

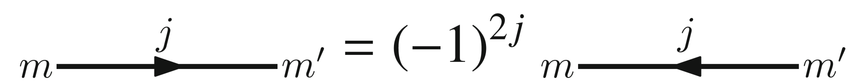


"Metric"

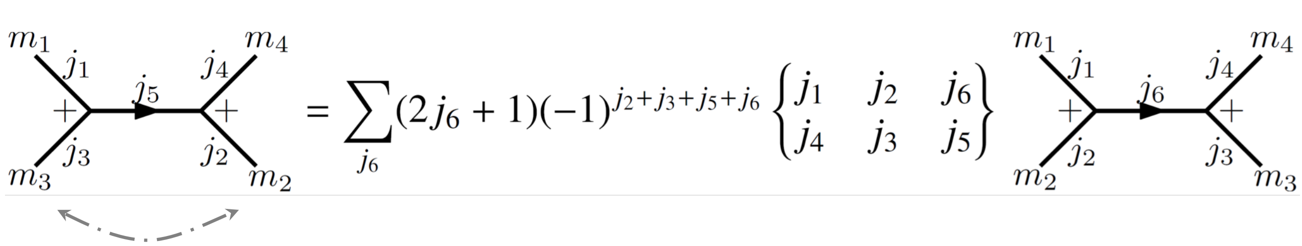
$$C^{(j)}_{mn}C^{nm'}_{(j)} = C^{m'n}_{(j)}C^{(j)}_{nm} = \delta^{m'}_{m}$$

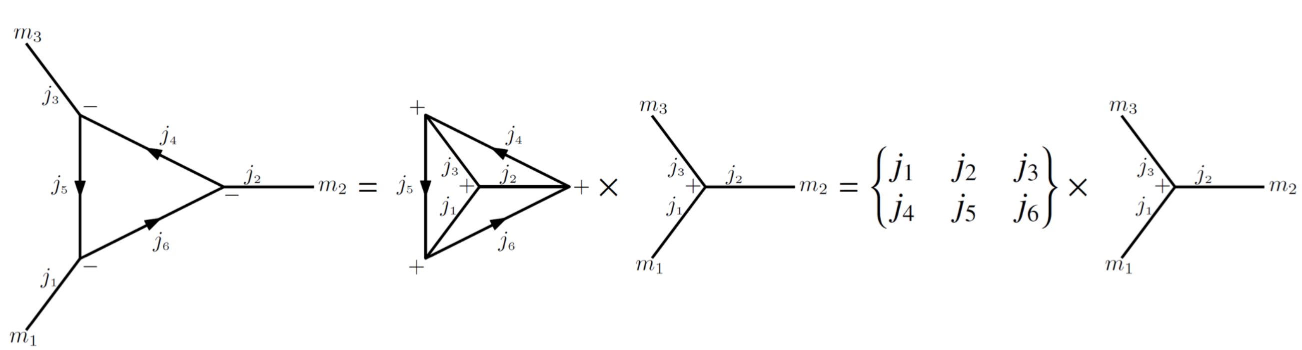
$$C^{(j)}_{nm}C^{nm'}_{(j)} = C^{m'n}_{(j)}C^{(j)}_{mn} = (-1)^{2j}\delta^{m'}_{m}$$

$$C^{(j)}_{m'm} = (-1)^{2j}C^{(j)}_{mm'}, \quad C^{mm'}_{(j)} = (-1)^{2j}C^{m'm}_{(j)}$$



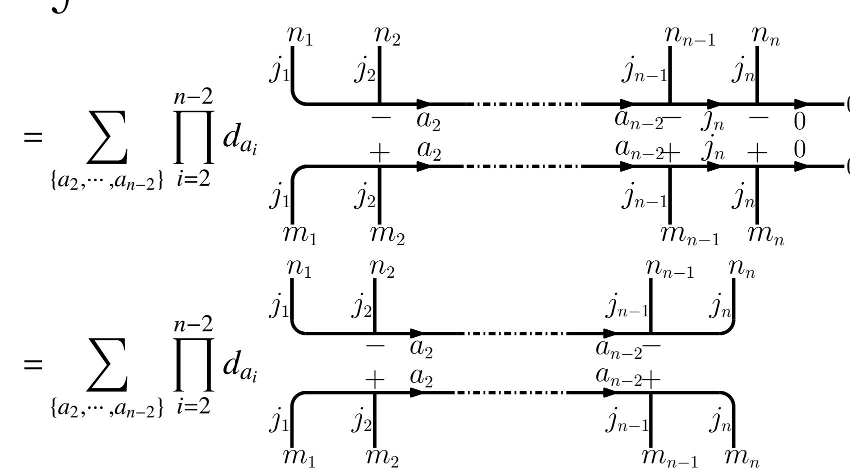
The rules involving the Wigner 6j-symbol

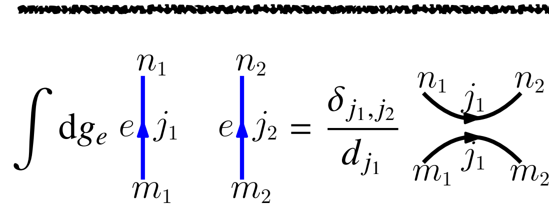


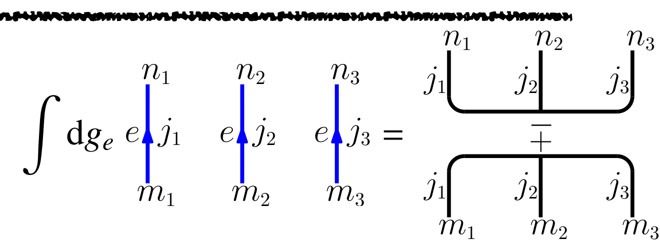


Integral over the product of irreducible representations

$$\int \mathrm{d}g_e [\pi_{j_1}(g_e)]^{m_1}{}_{n_1}[\pi_{j_2}(g_e)]^{m_2}{}_{n_2}\cdots[\pi_{j_n}(g_e)]^{m_n}{}_{n_n}$$







Thanks for your attention!