

Quantum mechanism angular momentum & graphical calculus

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Jinsong Yang (Guizhou university)

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- [4] J. Yang, Y. Ma, Graphical calculus of volume, inverse volume and Hamiltonian operators in loop quantum gravity. *Eur. Phys. J. C* **77**, 235 (2017). <https://doi.org/10.1140/epjc/s10052-017-4713-0>. [arXiv:1505.00223](https://arxiv.org/abs/1505.00223)
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1. Elementary operators and their actions

The holonomy operator acts as multiplication:

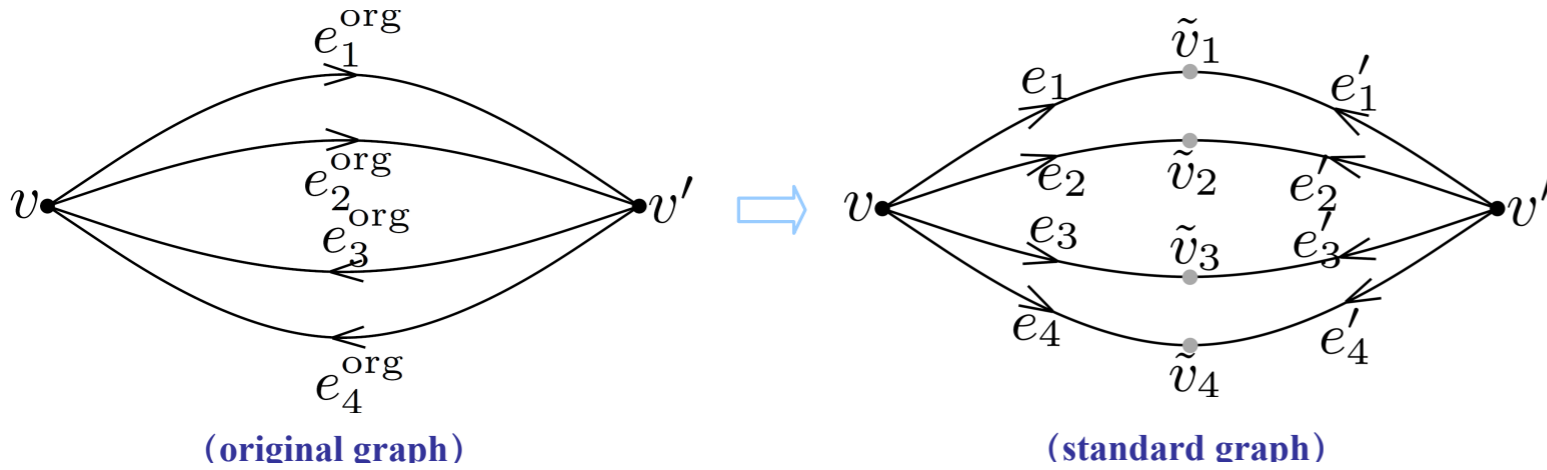
$$[\hat{h}_{e_I}(A)]^B_C \cdot f_\gamma(A) := \left[\pi_{\frac{1}{2}}(h_{e_I}(A)) \right]^B_C f_\gamma(A)$$

The flux operator acts on a cylindrical function f_γ with respect to the graph γ adapted to S as a derivative operator,

$$\begin{aligned} \hat{E}_i(S) \cdot f_\gamma(A) &:= -i\hbar \{f_\gamma(A), \tilde{E}_i(S)\} \\ &= \frac{\hbar\kappa\beta}{2} \sum_{e_I \in E(\gamma)} \kappa(e, S) \underline{J_{e_I}^i} \cdot f_\gamma(A) \\ &= \frac{\hbar\kappa\beta}{2} \sum_{v \in V(\gamma)} \left(\sum_{b(e)=v} \kappa(e, S) J_{e,R}^i + \sum_{f(e)=v} \kappa(e, S) J_{e,L}^i \right) f_\gamma(A) \end{aligned}$$

2. Spin-network states (based on standard graphs)

The original graph \Rightarrow standard graph



1. Adding a (pseudo-)vertex \tilde{v}_i to each edge e_i^{org} of the original graph.
2. Splitting each edge of the original graph into two edges (two segments) starting from the original graph's vertices.

$$e_i^{\text{org}} = e_i \circ e_i'^{-1}$$

A spin-network state (based on a standard graph)

$$T_{\gamma, \vec{j}, \vec{i}}(A) := \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{e \in E(\gamma)} \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^*$$

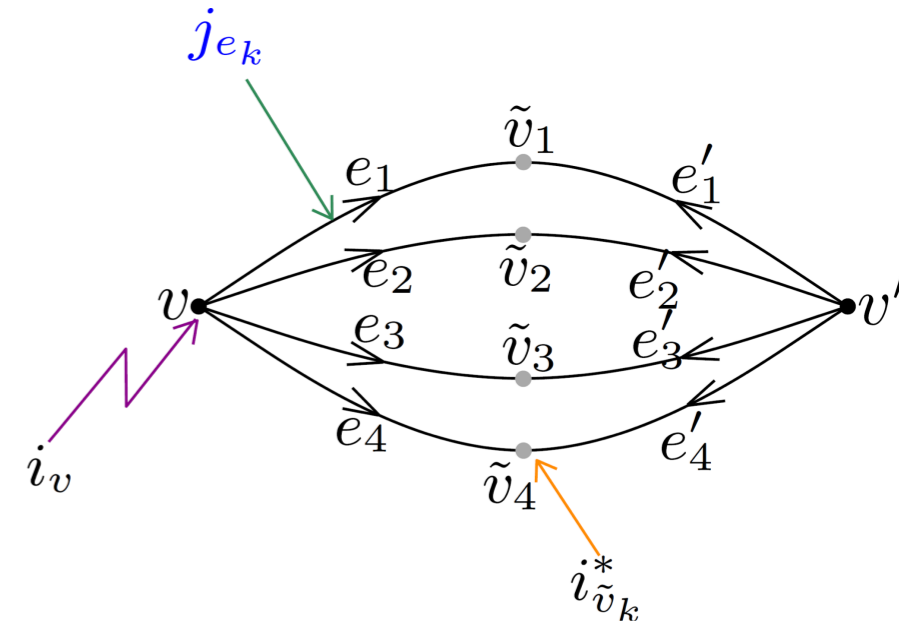
$$(i_v)_{m_1 m_2 \dots m_n}^M \equiv \left(i_{j_1 \dots j_n}^J; \vec{a} \right)_{m_1 \dots m_n}^M = (-1)^{j_1 - \sum_{i=2}^n j_i - J} \sum_{k_2, \dots, k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2 \rangle \dots \langle JM | a_{n-1} k_{n-1} j_n m_n \rangle$$

$$(i_{\tilde{v}}^*)_{n_1 n_2} \equiv \left(i_{j_1 j_2}^J \right)_{n_1 n_2} := (-1)^{j_1 - j_2 - J} \langle j_1 n_1 j_2 n_2 | JN \rangle = (i_{\tilde{v}})_{n_1 n_2}^N$$

$$\begin{aligned} J = 0, M = 0 &\Leftrightarrow \text{gauge-invariant} \\ J \neq 0 &\Leftrightarrow \text{gauge-variant} \end{aligned}$$

A spin-network

$$(\gamma, \vec{j}, \vec{i})$$



The normalized spin-network state

$$T_{\gamma, \vec{j}, \vec{i}}^{\text{norm}}(A) := \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^*$$

2. Spin-network states (based on standard graphs)

A spin-network state (based on a standard graph)

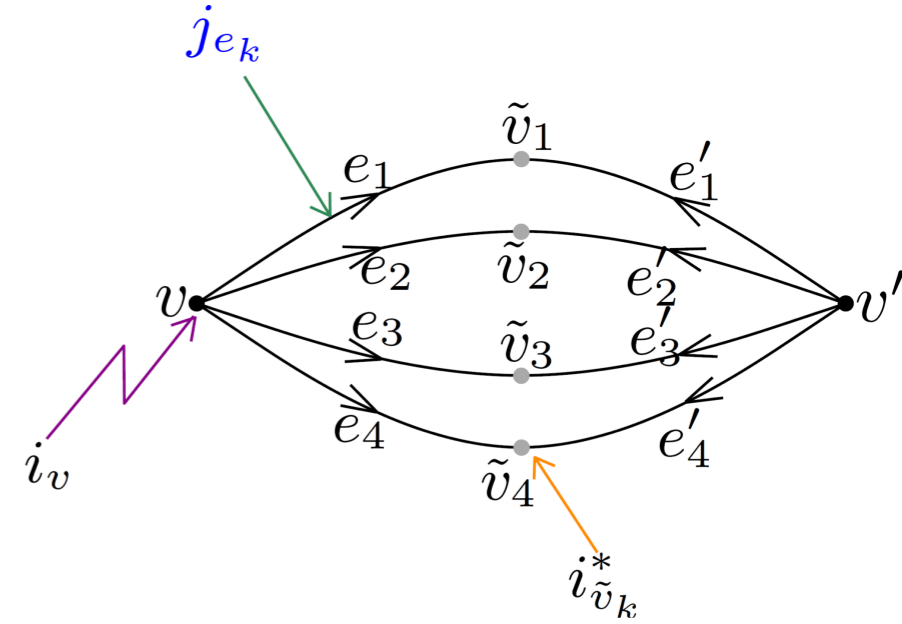
$$T_{\gamma, \vec{j}, \vec{i}}(A) := \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{e \in E(\gamma)} \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^*$$

$$(i_v)_{m_1 m_2 \dots m_n}^M \equiv \left(i_{j_1 \dots j_n}^J; \vec{a} \right)_{m_1 \dots m_n}^M = (-1)^{j_1 - \sum_{i=2}^n j_i - J} \sum_{k_2, \dots, k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2 \rangle \dots \langle JM | a_{n-1} k_{n-1} j_n m_n \rangle$$

$$(i_{\tilde{v}}^*)_{n_1 n_2} \equiv \left(i_{j_1 j_2}^J \right)_{n_1 n_2} := (-1)^{j_1 - j_2 - J} \langle j_1 n_1 j_2 n_2 | JN \rangle = (i_{\tilde{v}})_{n_1 n_2}^N$$

$J = 0, M = 0$	\Leftrightarrow	gauge-invariant
$J \neq 0$	\Leftrightarrow	gauge-variant

A spin-network

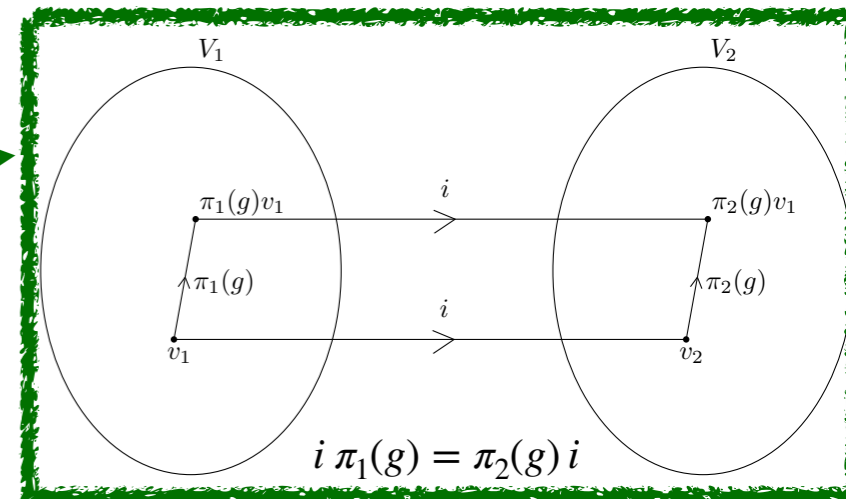


The normalized spin-network state

$$T_{\gamma, \vec{j}, \vec{i}}^{\text{norm}}(A) := \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^*$$

$$\begin{aligned} T_{\gamma, \vec{j}, \vec{i}}^{\text{norm}}(A^g) &= \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \pi_{j_e}(h_e(A^g)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^* \\ &= \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \pi_{j_e}(g(b(e))) \cdot \pi_{j_e}(h_e(A)) \cdot \pi_{j_e}(g(f(e))^{-1}) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^* \\ &= \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{b(e)=v} \pi_{j_e}(g(v)) \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \cdot \pi_{j_e}(h_e(A)) \cdot \bigotimes_{f(e)=\tilde{v}} \pi_{j_e}(g(\tilde{v})^{-1}) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^* \\ &= \bigotimes_{v \in V(\gamma)} \left[i_v \cdot \bigotimes_{b(e)=v} \pi_{j_e}(g(v)) \right] \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \cdot \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} \left[i_{\tilde{v}}^* \cdot \bigotimes_{f(e)=\tilde{v}} \pi_{j_e}(g(\tilde{v})^{-1}) \right] \\ &= \bigotimes_{v \in V(\gamma)} [\pi_J(g(v)) \cdot i_v] \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \cdot \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} [\pi_J^*(g(\tilde{v})) \cdot i_{\tilde{v}}^*] \\ &\stackrel{J=0}{=} \bigotimes_{v \in V(\gamma)} i_v \cdot \bigotimes_{e \in E(\gamma)} \sqrt{d_{j_e}} \pi_{j_e}(h_e(A)) \cdot \bigotimes_{\tilde{v} \in V(\gamma)} i_{\tilde{v}}^* \end{aligned}$$

$$\begin{aligned} A &\mapsto A^g = -(dg)g^{-1} + gAg^{-1} \\ h_e(A^g) &= g(b(e))h_e(A)g(f(e))^{-1} \end{aligned}$$

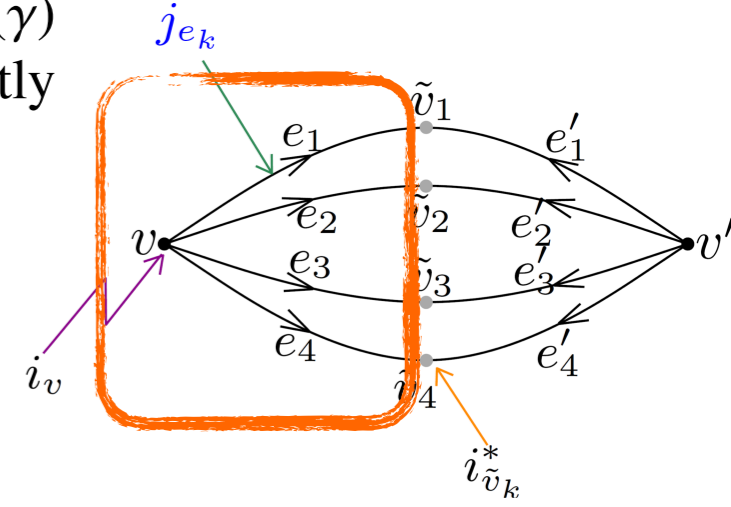


3. The actions on spin-network states (based on standard graphs)

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}^v(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$

$$[\pi_{j_1}(h_{e_1(A)})]_{n_1}^{m_1}$$



$$[\hat{h}_{e_I}]_C^B \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) = (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} [\pi_{\frac{1}{2}}(h_{e_I})]_C^B \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$

$$= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots \sum_{j'_I, m'_I, n'_I} \left(i_{j'_I}^{j_I} \right)_{m'_I}^{n'_I} [\pi_{j'_I}(h_{e_I})]_{n'_I}^{m'_I} \left(i_{j_I}^{j'_I} \right)_{n_I}^{m_I} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$

acting at an edge — coupling with holonomy of an edge

$$J_{e_I}^i \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) = (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots \left(i \frac{d}{dt} \Big|_{t=0} [\pi_{j_I}(e^{-t\tau_i} h_{e_I})]_{n_I}^{m_I} \right) \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$

$$= \left[(i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_i)]_{m_I}^{m'_I} \right) \right] [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$

$$\tau_i := -i\sigma_i/2, \quad \sigma_i : \text{Pauli matrices} \quad i = 1, 2, 3$$

$$J_{e_I}^i \cdot (i_v)_{m_1 \dots m_I \dots m_n}^M = (i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_i)]_{m_I}^{m'_I} \right)$$

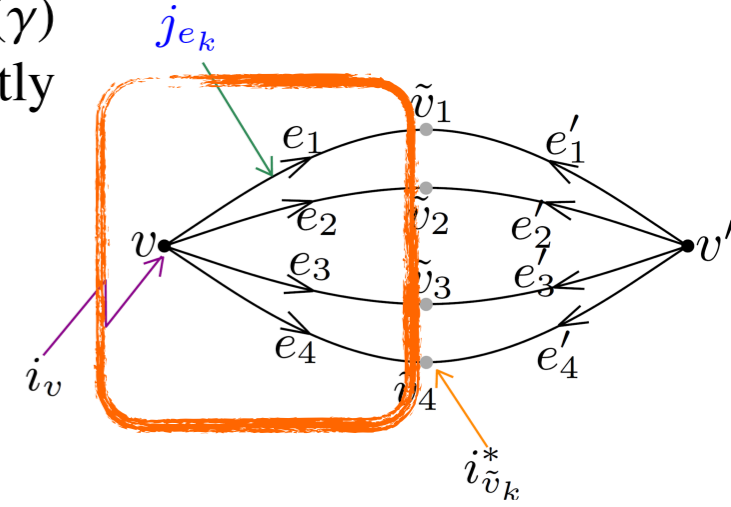
$$J_{e_I}^\mu \cdot (i_v)_{m_1 \dots m_I \dots m_n}^M = (i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_\mu)]_{m_I}^{m'_I} \right)$$

$$\tau_0 := \tau_3, \quad \tau_{\pm 1} := \mp \frac{1}{\sqrt{2}} (\tau_1 \pm i\tau_2)$$

acting at a vertex — contracting with intertwiner of a vertex

3. The actions on spin-network states (based on standard graphs)

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v



$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_n}^M [\pi_{j_1}(h_{e_1})]^{m_1}_{n_1} \cdots [\pi_{j_n}(h_{e_n})]^{m_n}_{n_n}$$

$$(i_v)_{m_1 m_2 \dots m_n}^M \equiv \left(i_{j_1 \dots j_n}^J; \vec{a} \right)_{m_1 \dots m_n}^M = (-1)^{j_1 - \sum_{i=2}^n j_i - J} \sum_{k_2, \dots, k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2 \rangle \cdots \langle JM | a_{n-1} k_{n-1} j_n m_n \rangle$$

$$\langle j_3 m_3 | j_1 m_1 j_2 m_2 \rangle = (-1)^{j_1 - j_2 - j_3} \sqrt{d_{j_3}} \sum_{m'_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} C_{(j_3)}^{m'_3 m_3}$$

Wigner 3-j symbol "Metric"

Properties of 3j-symbol

1. Symmetry:

odd-numbered $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = \dots$

even-numbered $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$ (cyclic symmetry)

2. Reality (unitary):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sum_{m'_1, m'_2, m'_3} C_{(j_1)}^{m_1 m'_1} C_{(j_2)}^{m_2 m'_2} C_{(j_3)}^{m_3 m'_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} = \sum_{m'_1, m'_2, m'_3} C_{(j_1)}^{(j_1)} C_{(j_2)}^{(j_2)} C_{(j_3)}^{(j_3)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}$$

3. Orthogonality:

$$\sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j'_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} = \frac{\delta_{j_3, j'_3}}{2j_3 + 1} \delta_{m_3, m'_3}$$

"Metric" $C_{mn}^{(j)}$ on V_j

$$C_{mn}^{(j)} := (-1)^{j-n} \delta_{n, -m} = (-1)^{j+m} \delta_{m, -n}$$

$$C_{(j)}^{mn} \equiv (C^{(j)})^{-1}{}^{mn} := (-1)^{j-m} \delta_{m, -n} = (-1)^{j+n} \delta_{n, -m}$$

satisfying

$$C_{m'm}^{(j)} = (-1)^{2j} C_{mm'}^{(j)}$$

$$C_{(j)}^{mm'} = (-1)^{2j} C_{(j)}^{m'm}$$

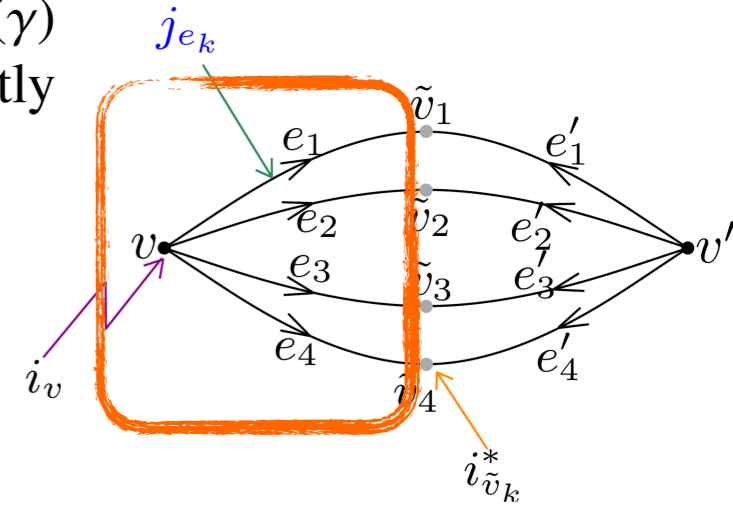
$$C_{mn}^{(j)} C_{(j)}^{nm'} = C_{(j)}^{m'n} C_{nm}^{(j)} = \delta_m^{m'}$$

$$C_{nm}^{(j)} C_{(j)}^{nm'} = C_{(j)}^{m'n} C_{mn}^{(j)} = (-1)^{2j} \delta_m^{m'}$$

$$\begin{pmatrix} j & j' & 0 \\ m & m' & 0 \end{pmatrix} = \frac{\delta_{j, j'}}{\sqrt{d_j}} C_{m'm}^{(j)}$$

4. Graphical representation: intertwiner

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v



$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_n}^M [\pi_{j_1}(h_{e_1})]^{m_1}_{n_1} \dots [\pi_{j_l}(h_{e_l})]^{m_l}_{n_l} \dots [\pi_{j_n}(h_{e_n})]^{m_n}_{n_n}$$

$$(i_v)_{m_1 m_2 \dots m_n}^M \equiv \left(i_{j_1 \dots j_n}^J; \vec{a} \right)_{m_1 \dots m_n}^M = (-1)^{j_1 - \sum_{i=2}^n j_i - J} \sum_{k_2, \dots, k_{n-1}} \langle a_2 k_2 | j_1 m_1 j_2 m_2 \rangle \dots \langle JM | a_{n-1} k_{n-1} j_n m_n \rangle$$

$$\langle j_3 m_3 | j_1 m_1 j_2 m_2 \rangle = (-1)^{j_1 - j_2 - j_3} \sqrt{d_{j_3}} \sum_{m'_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} C_{(j_3)}^{m'_3 m_3}$$

Wigner 3-j symbol "Metric"

Wigner 3j-symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{array}{c} j_3 \nearrow \\ m_1 \text{---} j_1 \text{---} + \\ j_2 \searrow \\ m_2 \end{array} = \begin{array}{c} j_2 \nearrow \\ m_1 \text{---} j_1 \text{---} - \\ j_3 \searrow \\ m_3 \end{array}$$

"Metric"

$$C_{m'm}^{(j)} = (-1)^{j-m} \delta_{m,-m'} = (-1)^{j+m'} \delta_{m,-m'} = \begin{array}{c} j \\ m \text{---} \rightarrow \text{---} m' \end{array}$$

$$C_{(j)}^{mm'} = (-1)^{j-m} \delta_{m,-m'} = (-1)^{j+m'} \delta_{m,-m'} = \begin{array}{c} j \\ m \text{---} \rightarrow \text{---} m' \end{array}$$

Kronecker delta symbol

$$(\delta^{(j)})_{m'}^m = (\delta^{(j)})_m^{m'} = \begin{array}{c} j \\ m \text{---} \text{---} m' \end{array}$$

Summation over the same indices

$$\sum_{m'} \begin{array}{c} \text{---} j \text{---} m' \\ \text{---} m' \text{---} j \text{---} \end{array} = \begin{array}{c} \text{---} j \text{---} \\ \text{---} j \text{---} \end{array}$$

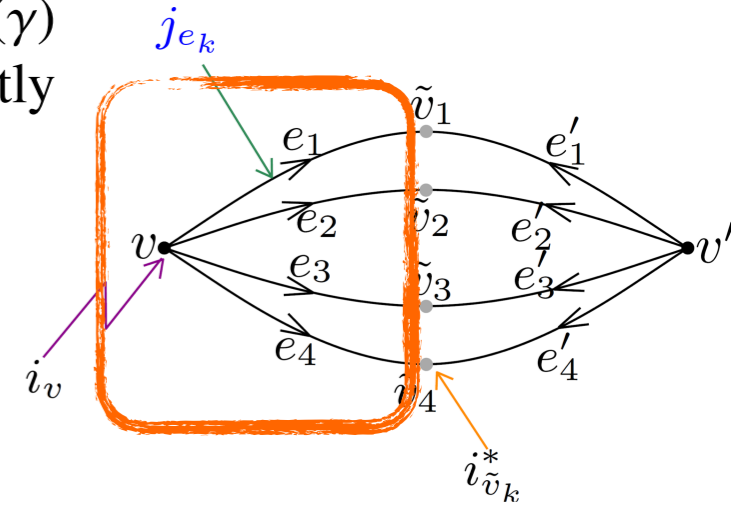
$$\langle JM | j_1 m_1 j_2 m_2 \rangle = (-1)^{j_1 - j_2 - J} \sqrt{2J+1} \begin{array}{c} j_1 \nearrow \\ m_1 \text{---} + \\ j_2 \searrow \\ m_2 \end{array} \xrightarrow{J} M = (-1)^{j_1 - j_2 - J} \sqrt{2J+1} \begin{array}{c} m_1 \text{---} j_1 \text{---} \\ j_2 \text{---} \\ m_2 \end{array} \xrightarrow{J} M$$

$$\left(i_{j_1 \dots j_n}^J; \vec{a} \right)_{m_1 \dots m_n}^M \equiv \prod_{i=2}^{n-1} \sqrt{2a_i + 1} \sqrt{2J + 1} \begin{array}{c} m_1 \text{---} j_1 \text{---} \\ j_2 \text{---} \\ m_2 \end{array} \text{---} a_2 \text{---} \dots \text{---} a_{n-1} \text{---} \begin{array}{c} m_n \text{---} j_n \text{---} \\ j_n \text{---} \\ m_n \end{array} \xrightarrow{J} M$$

4. Graphical representation: holonomy

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_l \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_l}(h_{e_l})]_{n_l}^{m_l} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



Matrix element of holonomy with irrep.

$$[\pi_j(h_e(A))]_{n}^m = m \text{ --- } \begin{array}{c} j \\ \triangleleft h_e \triangleright \\ j \end{array} \text{ --- } n =: m \text{ --- } \begin{array}{c} e \\ \triangleright \\ j \end{array} \text{ --- } n$$

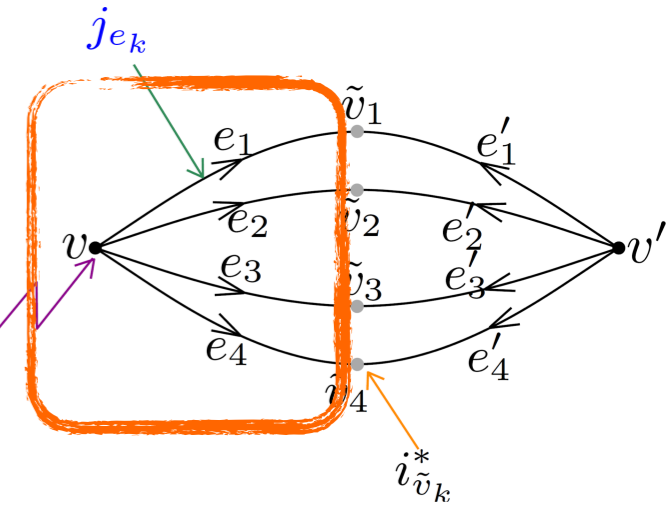
$$[\pi_j(h_e(A)^{-1})]_{m}^n = n \text{ --- } \begin{array}{c} e^{-1} \\ \triangleright \\ j \end{array} \text{ --- } m = n \text{ --- } \begin{array}{c} e \\ \triangleleft \\ j \end{array} \text{ --- } m$$

$$[\pi_j(h_e(A)^{-1})]_{m}^n = [\pi_j(h_{e^{-1}}(A))]_{m}^n = C_{mm'}^{(j)} [\pi_j(h_e(A))]_{n'}^{m'} C_{(j)}^{n'n}$$

4. Graphical representation: spin network states

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



$$(i_{j_1 \dots j_n}^{J; \vec{a}})_{m_1 \dots m_n}^M = \prod_{i=2}^{n-1} \sqrt{2a_i + 1} \sqrt{2J + 1}$$

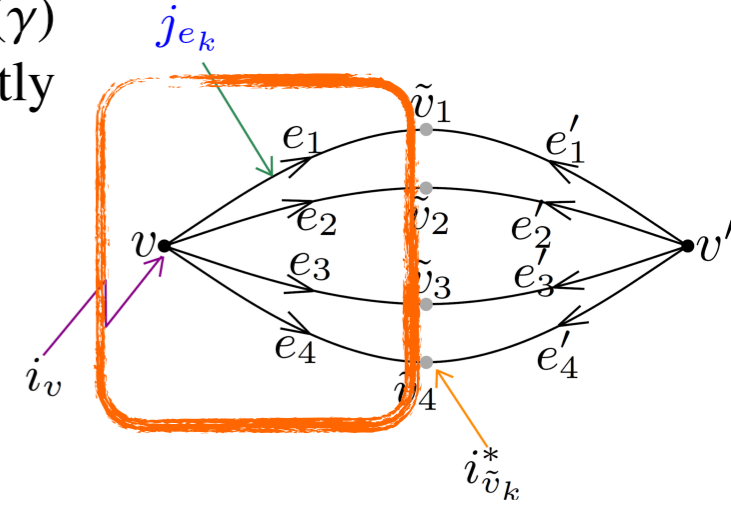
$$[\pi_j(h_e(A))]_n^m = m \text{ --- } \begin{matrix} j \\ \triangle \\ h_e \end{matrix} \text{ --- } n =: m \text{ --- } \begin{matrix} e \\ \text{---} \\ j \end{matrix} \text{ --- } n$$

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) = \prod_{i=2}^{n-1} \sqrt{d_{a_i}} \sqrt{d_J}$$

5. Graphical calculus: the actions of elementary operators

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



$$\begin{aligned} [\hat{h}_{e_I}]_C^B \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \underbrace{[\pi_{\frac{1}{2}}(h_{e_I})]_C^B}_{\text{green box}} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \\ &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots \underbrace{\sum_{j'_I, m'_I, n'_I} \binom{j'_I}{j_I \frac{1}{2}}_{m'_I}^{m_I B}}_{\text{green box}} [\pi_{j'_I}(h_{e_I})]_{n'_I}^{m'_I} \binom{j'_I}{j_I \frac{1}{2}}_{n_I C}^{n'_I} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \end{aligned}$$

acting at an edge — coupling with holonomy of an edge

$$\begin{aligned} J_{e_I}^i \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots \left(i \frac{d}{dt} \Big|_{t=0} [\pi_{j_I}(e^{-t\tau_i} h_{e_I})]_{n_I}^{m_I} \right) \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \\ &= \left[(i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_i)]_{m_I}^{m'_I} \right) \right] [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \end{aligned}$$

$$\tau_i := -i\sigma_i/2, \quad \sigma_i : \text{Pauli matrices} \quad i = 1, 2, 3$$

$$J_{e_I}^i \cdot (i_v)_{m_1 \dots m_I \dots m_n}^M = (i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_i)]_{m_I}^{m'_I} \right)$$

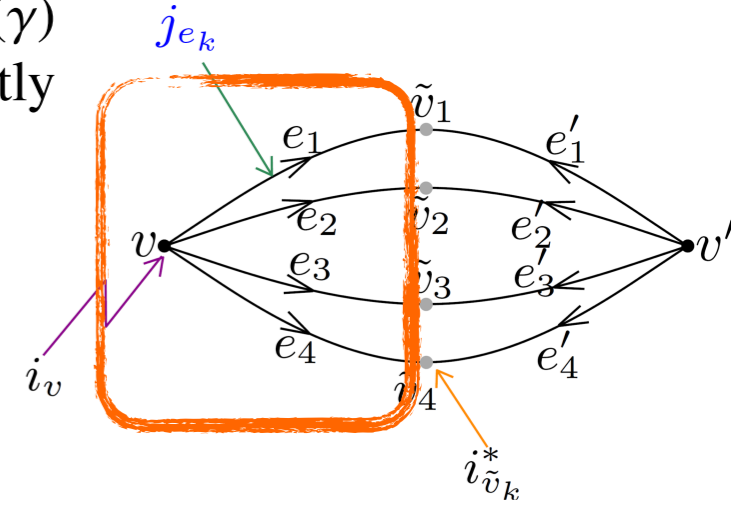
$$\begin{aligned} J_{e_I}^\mu \cdot (i_v)_{m_1 \dots m_I \dots m_n}^M &= (i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_\mu)]_{m_I}^{m'_I} \right) \\ \tau_0 &:= \tau_3, \quad \tau_{\pm 1} := \mp \frac{1}{\sqrt{2}} (\tau_1 \pm i\tau_2) \end{aligned}$$

acting at a vertex — contracting with intertwiner of a vertex

5. Graphical calculus: the actions of elementary operators

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



$$\begin{aligned} [\hat{h}_{e_I}]_C^B \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \underbrace{[\pi_{\frac{1}{2}}(h_{e_I})]_C^B}_{\text{green underline}} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \\ &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots \underbrace{\sum_{j'_I, m'_I, n'_I} \binom{j'_I}{j_I \frac{1}{2}}_{m'_I} [\pi_{j'_I}(h_{e_I})]_{n'_I}^{m'_I} \binom{j'_I}{j_I \frac{1}{2}}_{n'_I C}}_{\text{green underline}} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \end{aligned}$$

Clebsch-Gordan series

$$[\pi_{j_1}(g)]_{n_1}^{m_1} [\pi_{j_2}(g)]_{n_2}^{m_2} = \sum_{J, M, N} \left((i_{j_1 j_2}^J)^{-1} \right)_M^{m_1 m_2} [\pi_J(g)]_N^M (i_{j_1 j_2}^J)_{n_1 n_2}^N$$

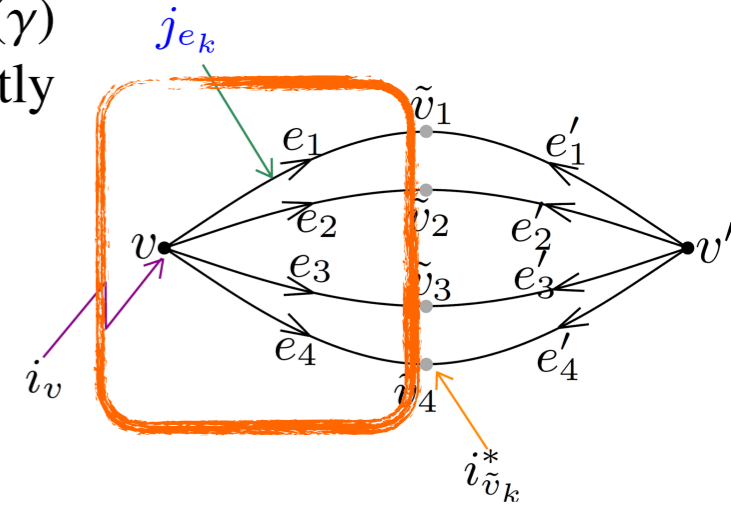
$$m_1 \begin{array}{c} \xrightarrow{e} \\ j_1 \end{array} m'_1 = \sum_{j_3} (2j_3 + 1) \begin{array}{c} j_1 + j_3 \\ \leftarrow \quad \xrightarrow{e} \quad \rightarrow \\ j_2 \quad j_3 \quad j_2 \end{array} m'_1 = \sum_{j_3} (2j_3 + 1) \begin{array}{c} j_1 + j_3 \\ \leftarrow \quad \xrightarrow{e} \quad \rightarrow \\ j_2 \quad j_3 \quad j_2 \end{array} m'_1$$

$$m_1 \begin{array}{c} \xrightarrow{e} \\ j_1 \end{array} m'_1 = \sum_{j_3} (2j_3 + 1) \begin{array}{c} j_1 + j_3 \\ \uparrow \quad \xrightarrow{e} \quad \downarrow \\ j_2 \quad j_3 \quad j_2 \end{array} m'_1 = \sum_{j_3} (2j_3 + 1) \begin{array}{c} j_1 + j_3 \\ \downarrow \quad \xrightarrow{e} \quad \uparrow \\ j_2 \quad j_3 \quad j_2 \end{array} m'_1$$

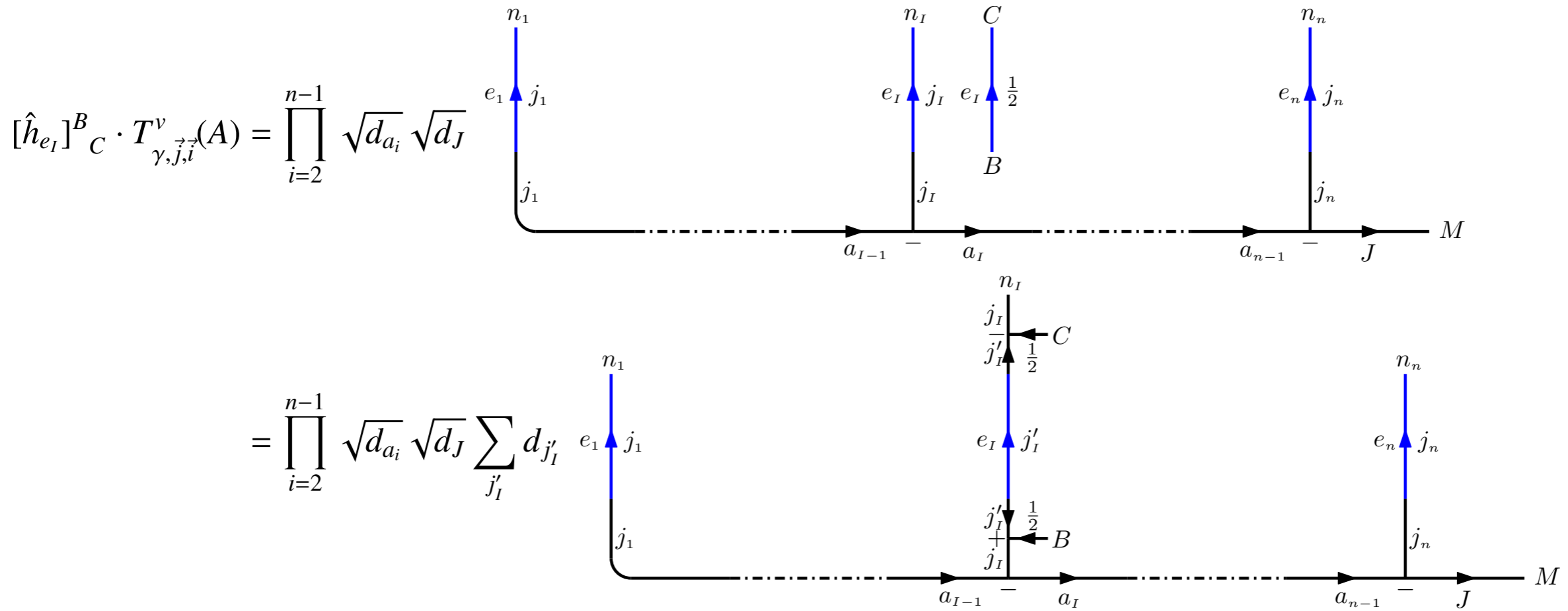
5. Graphical calculation: the actions of elementary operators

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



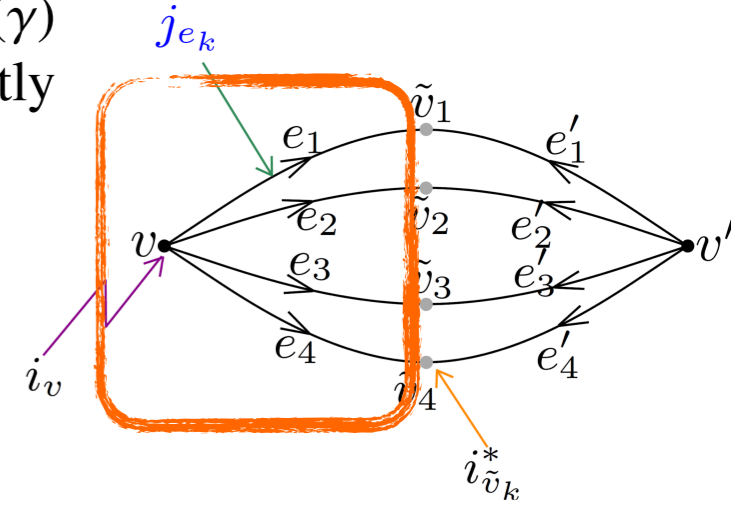
$$\begin{aligned} [\hat{h}_{e_I}]^B_C \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \underbrace{[\pi_{\frac{1}{2}}(h_{e_I})]_C^B}_{\text{green underline}} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \\ &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots \sum_{j'_I, m'_I, n'_I} \underbrace{\left(i_{j'_I}^{j_I} \right)_{m'_I}^{n'_I}}_{\text{green underline}} [\pi_{j'_I}(h_{e_I})]_{n'_I}^{m'_I} \left(i_{j_I}^{j'_I} \right)_{n_I}^{m_I} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \end{aligned}$$



5. Graphical calculation: the actions of elementary operators

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_l \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_l}(h_{e_l})]_{n_l}^{m_l} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



$$\begin{aligned} J_{e_l}^i \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) &= (i_v)_{m_1 \dots m_l \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots \left(i \frac{d}{dt} \Big|_{t=0} [\pi_{j_l}(e^{-t\tau_i} h_{e_l})]_{n_l}^{m_l} \right) \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \\ &= \left[(i_v)_{m_1 \dots m'_l \dots m_n}^M \left(-i [\pi_{j_l}(\tau_i)]_{m_l}^{m'_l} \right) \right] [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \cdots [\pi_{j_l}(h_{e_l})]_{n_l}^{m_l} \cdots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \end{aligned}$$

$$\tau_i := -i\sigma_i/2, \quad \sigma_i : \text{Pauli matrices } i = 1, 2, 3$$

$$\begin{aligned} J_{e_l}^\mu \cdot (i_v)_{m_1 \dots m_l \dots m_n}^M &= (i_v)_{m_1 \dots m'_l \dots m_n}^M \left(-i [\pi_{j_l}(\tau_\mu)]_{m_l}^{m'_l} \right) \\ \tau_0 &:= \tau_3, \quad \tau_{\pm 1} := \mp \frac{1}{\sqrt{2}} (\tau_1 \pm i\tau_2) \end{aligned}$$

$$J_{e_l}^i \cdot (i_v)_{m_1 \dots m_l \dots m_n}^M = (i_v)_{m_1 \dots m'_l \dots m_n}^M \left(-i [\pi_{j_l}(\tau_i)]_{m_l}^{m'_l} \right)$$

acting at a vertex — contracting with intertwiner of a vertex

$$[\pi_j(\tau_\mu)]_m^{m'} = i\chi(j) \begin{pmatrix} j & 1 & j \\ m & \mu & m' \end{pmatrix} C_{(j)}^{m''m'} = i\chi(j) \begin{pmatrix} 1 & j & j \\ \mu & m'' & m \end{pmatrix} C_{(j)}^{m''m'}$$

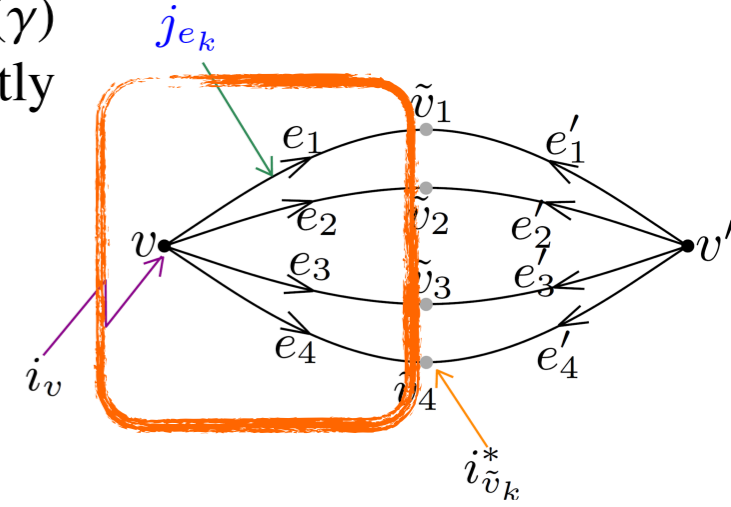
$$\chi(j) \equiv \sqrt{j(j+1)(2j+1)}$$

$$= i\chi(j) \begin{array}{c} \mu \\ | \\ 1 \\ | \\ m' \xleftarrow{j} + \xrightarrow{j} m \end{array} = i\chi(j) \begin{array}{c} \mu \\ | \\ 1 \\ | \\ m \xleftarrow{j} - \xrightarrow{j} m' \end{array}$$

5. Graphical calculation: the actions of elementary operators

For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v

$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



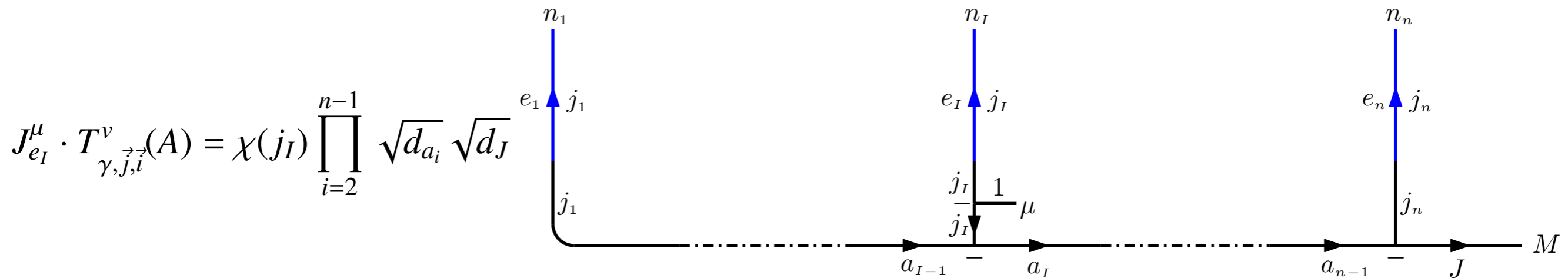
$$\begin{aligned} J_{e_1}^i \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) &= (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots \left(i \frac{d}{dt} \Big|_{t=0} [\pi_{j_I}(e^{-t\tau_i} h_{e_I})]_{n_I}^{m_I} \right) \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \\ &= \left[(i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_i)]_{m_I}^{m'_I} \right) \right] [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n} \end{aligned}$$

$$\tau_i := -i\sigma_i/2, \quad \sigma_i : \text{Pauli matrices } i = 1, 2, 3$$

$$\begin{aligned} J_{e_1}^\mu \cdot (i_v)_{m_1 \dots m_I \dots m_n}^M &= (i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_\mu)]_{m_I}^{m'_I} \right) \\ \tau_0 &:= \tau_3, \quad \tau_{\pm 1} := \mp \frac{1}{\sqrt{2}} (\tau_1 \pm i\tau_2) \end{aligned}$$

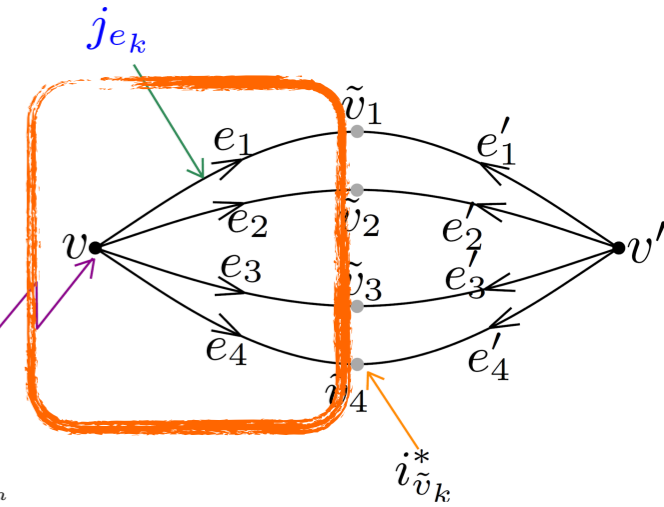
$$J_{e_1}^i \cdot (i_v)_{m_1 \dots m_I \dots m_n}^M = (i_v)_{m_1 \dots m'_I \dots m_n}^M \left(-i [\pi_{j_I}(\tau_i)]_{m_I}^{m'_I} \right)$$

acting at a vertex — contracting with intertwiner of a vertex

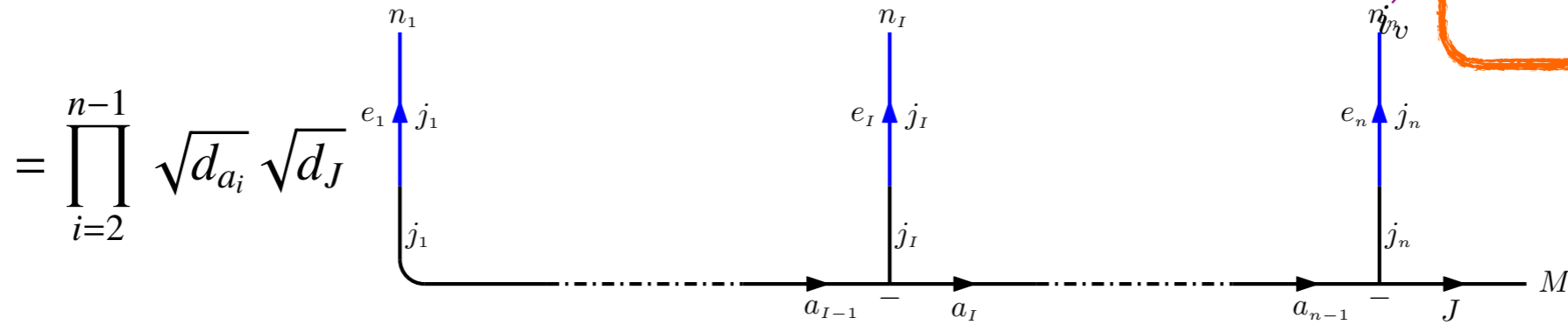


5. Graphical calculation: the actions of elementary operators

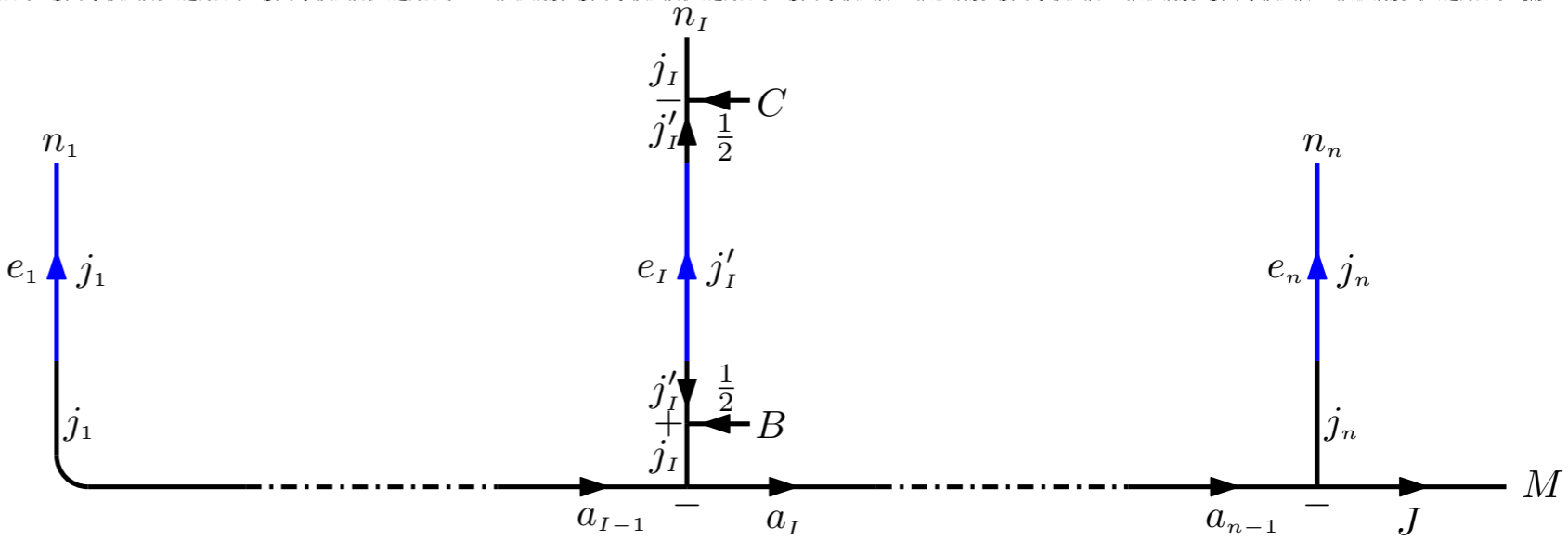
For a spin-network state $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ on a graph γ , we consider a true vertex $v \in V(\gamma)$ at which n edges e_1, \dots, e_n incident and denote $T_{\gamma, \vec{j}, \vec{i}}^v(A)$ the terms, in $T_{\gamma, \vec{j}, \vec{i}}(A)$, directly associated to v



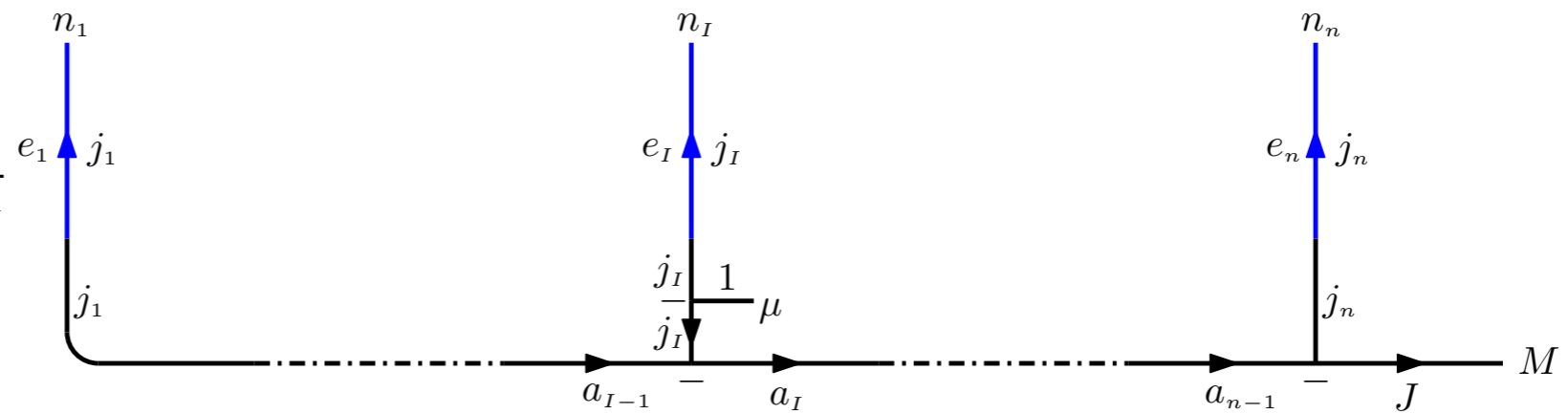
$$T_{\gamma, \vec{j}, \vec{i}}^v(A) := (i_v)_{m_1 \dots m_I \dots m_n}^M [\pi_{j_1}(h_{e_1})]_{n_1}^{m_1} \dots [\pi_{j_I}(h_{e_I})]_{n_I}^{m_I} \dots [\pi_{j_n}(h_{e_n})]_{n_n}^{m_n}$$



$$[\hat{h}_{e_I}]^B_C \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) = \prod_{i=2}^{n-1} \sqrt{d_{a_i}} \sqrt{d_J} \sum_{j'_I} d_{j'_I}$$

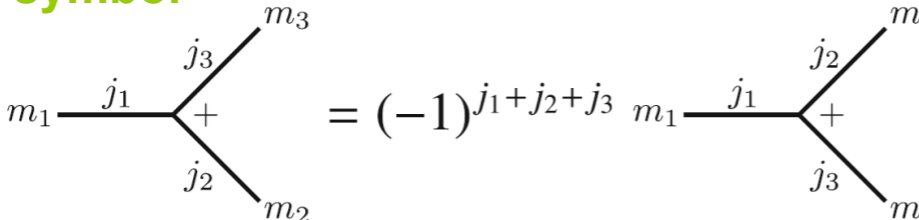
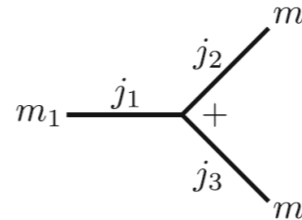


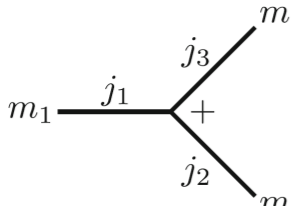
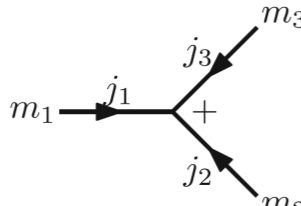
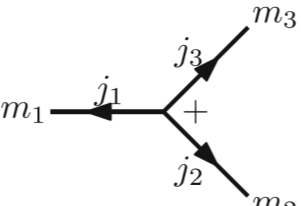
$$J_{e_I}^\mu \cdot T_{\gamma, \vec{j}, \vec{i}}^v(A) = \chi(j_I) \prod_{i=2}^{n-1} \sqrt{d_{a_i}} \sqrt{d_J}$$

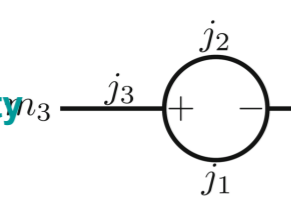
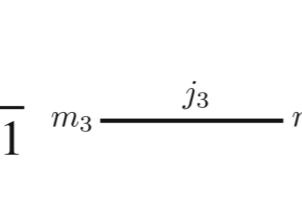


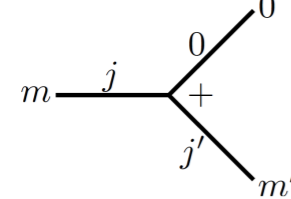
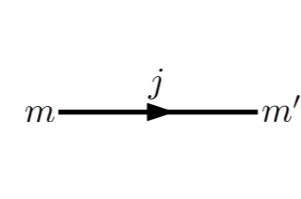
6. Graphical calculation: some useful rules of transforming graphs

Wigner 3j-symbol

Symmetry  $= (-1)^{j_1+j_2+j_3}$ 

Reality (unitary)  $=$  $=$ 

Orthogonality  $= \frac{\delta_{j_3, j'_3}}{2j_3 + 1}$ 

Special formula  $= \frac{\delta_{j, j'}}{\sqrt{2j + 1}}$ 

"Metric"

$$C_{mn}^{(j)} C_{(j)}^{nm'} = C_{(j)}^{m'n} C_{nm}^{(j)} = \delta_m^{m'}$$

$$m \xrightarrow{j} m' = m \xleftarrow{j} m' = m \xrightarrow{j} m'$$

$$C_{nm}^{(j)} C_{(j)}^{nm'} = C_{(j)}^{m'n} C_{mn}^{(j)} = (-1)^{2j} \delta_m^{m'}$$

$$m \xrightarrow{j} m' = m \xleftarrow{j} m' = (-1)^{2j} m \xrightarrow{j} m'$$

$$C_{m'm}^{(j)} = (-1)^{2j} C_{mm'}^{(j)}, \quad C_{(j)}^{mm'} = (-1)^{2j} C_{(j)}^{m'm}$$

$$m \xrightarrow{j} m' = (-1)^{2j} m \xleftarrow{j} m'$$

The rules involving the Wigner 6j-symbol

$$\begin{array}{c} m_1 \quad j_1 \quad m_4 \\ \diagdown \quad \diagup \\ + \quad \quad \quad + \\ \diagup \quad \diagdown \\ m_3 \quad j_3 \quad m_2 \end{array} \xrightarrow{j_5} \begin{array}{c} m_1 \quad j_1 \quad m_4 \\ \diagdown \quad \diagup \\ + \quad \quad \quad + \\ \diagup \quad \diagdown \\ m_2 \quad j_2 \quad m_3 \end{array} = \sum_{j_6} (2j_6 + 1) (-1)^{j_2+j_3+j_5+j_6} \begin{Bmatrix} j_1 & j_2 & j_6 \\ j_4 & j_3 & j_5 \end{Bmatrix} \begin{array}{c} m_1 \quad j_1 \quad m_4 \\ \diagdown \quad \diagup \\ + \quad \quad \quad + \\ \diagup \quad \diagdown \\ m_2 \quad j_2 \quad m_3 \end{array}$$

$$\begin{array}{c} m_3 \\ \diagdown \quad \diagup \\ j_3 \quad \quad \quad j_4 \\ \diagup \quad \diagdown \\ j_5 \quad \quad \quad j_2 \quad m_2 \\ \diagdown \quad \diagup \\ j_1 \quad \quad \quad j_6 \\ \diagup \quad \diagdown \\ m_1 \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ j_5 \quad \quad \quad j_4 \\ \diagdown \quad \diagup \\ j_3 \quad \quad \quad j_2 \\ \diagup \quad \diagdown \\ j_1 \quad \quad \quad j_6 \\ \diagdown \quad \diagup \\ m_1 \end{array} \times \begin{array}{c} m_3 \\ \diagdown \quad \diagup \\ j_3 \quad \quad \quad j_2 \\ \diagup \quad \diagdown \\ j_1 \quad \quad \quad j_2 \quad m_2 \\ \diagdown \quad \diagup \\ m_1 \end{array} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \times \begin{array}{c} m_3 \\ \diagdown \quad \diagup \\ j_3 \quad \quad \quad j_2 \\ \diagup \quad \diagdown \\ j_1 \quad \quad \quad j_2 \quad m_2 \\ \diagdown \quad \diagup \\ m_1 \end{array}$$

Integral over the product of irreducible representations

$$\int dg_e [\pi_{j_1}(ge)]^{m_1}_{n_1} [\pi_{j_2}(ge)]^{m_2}_{n_2} \cdots [\pi_{j_n}(ge)]^{m_n}_{n_n}$$

$$= \sum_{\{a_2, \dots, a_{n-2}\}} \prod_{i=2}^{n-2} d_{a_i} \begin{array}{c} n_1 \quad n_2 \quad \dots \quad n_{n-1} \quad n_n \\ j_1 \quad j_2 \quad \dots \quad j_{n-1} \quad j_n \\ \hline - a_2 \quad \dots \quad a_{n-2} \quad j_n \quad - 0 \\ + a_2 \quad \dots \quad a_{n-2} \quad j_n \quad + 0 \\ \hline j_1 \quad j_2 \quad \dots \quad j_{n-1} \quad j_n \\ m_1 \quad m_2 \quad \dots \quad m_{n-1} \quad m_n \end{array}$$

$$= \sum_{\{a_2, \dots, a_{n-2}\}} \prod_{i=2}^{n-2} d_{a_i} \begin{array}{c} n_1 \quad n_2 \quad \dots \quad n_{n-1} \quad n_n \\ j_1 \quad j_2 \quad \dots \quad j_{n-1} \quad j_n \\ \hline - a_2 \quad \dots \quad a_{n-2} \quad j_n \\ + a_2 \quad \dots \quad a_{n-2} \quad j_n \\ \hline j_1 \quad j_2 \quad \dots \quad j_{n-1} \quad j_n \\ m_1 \quad m_2 \quad \dots \quad m_{n-1} \quad m_n \end{array}$$

$$\int dg_e e^{j_1}_{m_1} e^{j_2}_{m_2} = \frac{\delta_{j_1, j_2}}{d_{j_1}} \begin{array}{c} n_1 \quad n_2 \\ j_1 \quad j_1 \\ \hline m_1 \quad m_2 \end{array}$$

$$\int dg_e e^{j_1}_{m_1} e^{j_2}_{m_2} e^{j_3}_{m_3} = \begin{array}{c} n_1 \quad n_2 \quad n_3 \\ j_1 \quad j_2 \quad j_3 \\ \hline m_1 \quad m_2 \quad m_3 \end{array}$$

Thanks for your attention!