Advanced General Relativity I

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Outline

1. **Ideas of GR: General Covariance and Background Independence**

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2. **Lagrangian Formulation of GR**
3. **Hamiltonian Formulation of GR**

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2. **Lagrangian Formulation of GR**
3. **Hamiltonian Formulation of GR**
4. **Ideas of LQG and Historical Developments**

Basic Postulates of GR

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2. *The worldline of a free particle is a geodesic of the curved spacetime, satisfying*

$$U^\alpha \nabla_\alpha U^\beta = 0.$$

3. *The way that the spacetime is curved is affected by the matter distribution. The specific relation is described by the following Einstein's equations*

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}.$$

The Gauge Freedom of GR

- Let $\phi: M \to M$ be a diffeomorphsim on a spacetime $(M, g_{\alpha\beta})$. One can show that $\phi^*(R_{\alpha\beta}[g]) = R_{\alpha\beta}[\phi^* g]$, and hence
$$G_{\alpha\beta}[g] = 0 \Leftrightarrow G_{\alpha\beta}[\phi^* g] = 0.$$
Therefore, if $g_{\alpha\beta}$ is a solution of the vacuum Einstein's equations, so is $\phi^* g_{\alpha\beta}$.
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- Actually, the Bianchi identity $\nabla_{[\alpha} R_{\beta\sigma]\gamma}{}^{\delta} = 0$ implies $\nabla_\alpha G^\alpha_{\ \beta} = 0$, and hence only 6 of the 10 Einstein's equations are independent, while $g_{\alpha\beta}$ has 10 compoments.

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- *The gauge freedom of GR*: $(M, g_{\alpha\beta})$ and $(M, \phi^* g_{\alpha\beta})$ represent the same spacetime geometry.

Principle of General Covariance

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- ⋆ This principle is often rephrased in many textbooks as: *The expression of physical laws remains unchanged in any coordinate transformation*.
- ⋆ The latter requirement could be realized by expressing the physical laws in terms of the spacetime tensors and the quantities derivable from them.
- The general covariance could thus be further rephrased as: *Every physical quantity must be describable by a coordinate-free geometric object, and the laws of physics must all be expressible as geometric relationships between these geometric objects*.

Principle of General Covariance

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- ⋆ *The spacetime metric and quantities derivable from it are the only spacetime quantities that can appear in the equations of physics* [Wald 1984].
- ⋆ *In the expressions of physical laws, only the dynamical variables, including the spacetime metric and quantities derivable from it, can exert substantive effects on other physical variables* [Ma 2018, 2007].

Background Independence

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Background Independence

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- ⋆ Newtonian Mechanics: Physics depends on the absolute time and the absolute space given by the prior background of Euclidean space.
- ⋆ Special Relativity: Physics depends on the prior background of Minkowski space.
- ⋆ General Relativity: Spacetime geometry is not prior background but dynamical quantity, determined by Einstein's equations, and hence the theory is diffeomorphism invariant.

Lagrangian Theory with Finite Degrees of Freedom

- The Lagrangian is a function L of $2N$ generalized coordinates $\{q^i | i = 1, 2, ..., N\}$ and velocities $\{\dot{q}^j | j = 1, 2, ..., N\}$, i.e., a function on the tangent bundle of the configuration space $\mathcal{C}$.
- *Variational Principle*: The path in $\mathcal{C}$ given by the equation of motion corresponds to the extreme value of the action:

$$S := \int_{t_0}^{t_1} L(q^i(t), \dot{q}^i(t)) dt.$$

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- S is a functional of the paths in $\mathcal{C}$.
  Let $q^i = q^i(t, \lambda)$ be an one-paramater family of paths. The variation of S in the family of paths is defined as

$$\delta S := \frac{dS(\lambda)}{d\lambda}|_{\lambda=\lambda_0} = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt.$$

Lagrangian Theory with Finite Degrees of Freedom

- Under the condition $\delta q^i|_{t_0} = 0 = \delta q^i|_{t_1}$, one obtains

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i dt.$$

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- Therefore, $\delta S = 0$ for any one-parameter families of paths is equivalent to

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0,$$

which is the Euler-Lagrangian equation of motion.

Lagrangian Theory of Classical Fields

- The action $S = S[\phi]$ is a functional of the fields ϕ on a 4D spacetime M, i.e., a function on the set $\mathcal{F} := \{\phi\}$ of the field configurations, which can be written as

$$S = \int_U \mathcal{L}(\phi(x), \nabla\phi(x), ..., \nabla^k\phi(x))d^4x,$$

where U is an open subset of M with boundary $\dot{U}$, and $\mathcal{L}$ is called the Lagrangian density.

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  where U is an open subset of M with boundary $\dot{U}$, and $\mathcal{L}$ is called the Lagrangian density.

- Let $\phi = \phi(\lambda)$ be an one-paramater family of field configurations, corresponding to a curve in $\mathcal{F}$. The variation of S in the family of field configurations is defined as

$$\delta S := \frac{dS(\lambda)}{d\lambda}|_{\lambda=\lambda_0} = \int_U (\frac{d\mathcal{L}}{d\lambda}|_{\lambda=\lambda_0})d^4x.$$

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- *Variational Principle*: The field configuration $\phi = \phi(\lambda_0)$ in U given by the field equation corresponds to the extreme value of the action S, i.e., $\delta S = 0$, under the condition $\phi(\lambda)|_{\dot{U}} = \phi(\lambda_0)|_{\dot{U}}$.

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- *Variational Principle*: The field configuration $\phi = \phi(\lambda_0)$ in U given by the field equation corresponds to the extreme value of the action S, i.e., $\delta S = 0$, under the condition $\phi(\lambda)|_{\dot{U}} = \phi(\lambda_0)|_{\dot{U}}$.
- For example, the Lagrangian of the Klein-Gordon field in Minkowski spacetime reads

$$\mathcal{L} := -\frac{1}{2}((\partial_\alpha \psi)\partial_\alpha \psi + m^2\psi^2).$$

- Then, $\delta S = 0$ for any one-parameter families of $\psi(\lambda)$ is equivalent to the Klein-Gordon equation:

$$\partial^\alpha \partial_\alpha \psi - m^2 \psi = 0.$$

Lagrangian Formulation of GR

- The Hilbert action on an open subset U in an 4-manifold M is given by

$$S_H[g_{\alpha\beta}] = \frac{1}{2\kappa}\int_U d^4x\sqrt{-g}R[g],$$

where g denotes the determinant of $g_{\alpha\beta}$ in the coordinates $\{x^\alpha\}$, R is the Ricci scalar determined by $g_{\alpha\beta}$, and the Lagrangian density reads $\mathcal{L} = \sqrt{-g}R$.

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- The variation of S_H gives

$$\begin{aligned}\delta S_H &= \frac{1}{2\kappa}\int_U d^4x\left(\frac{d\mathcal{L}}{d\lambda}|_{\lambda=\lambda_0}\right)\\ &= \frac{1}{2\kappa}\int_U d^4x\sqrt{-g}(\nabla^\alpha v_\alpha + G_{\alpha\beta}\delta g^{\alpha\beta}).\end{aligned}$$

Lagrangian Formulation of GR

- The dual vector field v_α is given by

$$v_\alpha = g^{\beta\gamma}(\nabla_\gamma \delta g_{\alpha\beta} - \nabla_\alpha \delta g_{\beta\gamma}).$$

- Therefore, the Hilbert action can give the vacuum Einstein's equations up to a boundary term.

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- Therefore, the Hilbert action can give the vacuum Einstein's equations up to a boundary term.
- In the case that $\dot{U}$ is non-null up to points of measure zero, one gets

$$\int_U \sqrt{-g}\nabla^\alpha v_\alpha = \int_{\dot{U}} \sqrt{|q|} n^\alpha v_\alpha,$$

where n^α is the unit normal to the boundary $\dot{U}$, and $\sqrt{|q|}$ denotes the determinant of the induced metric on $\dot{U}$.

Lagrangian Formulation of GR

- The trace of the extrinsic curvature $K_{\alpha\beta}$ of $\dot{U}$ in M reads

$$K \equiv g^{\alpha\beta} K_{\alpha\beta} = q^{\alpha}_{\ \beta} \nabla_{\alpha} n^{\beta}.$$

- A straightforward calculation gives $\delta K = -(1/2) n^{\alpha} v_{\alpha}$ with the condition $\delta g_{\alpha\beta}|_{\dot{U}} = 0$.

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- A straightforward calculation gives $\delta K = -(1/2) n^{\alpha} v_{\alpha}$ with the condition $\delta g_{\alpha\beta}|_{\dot{U}} = 0$.
- Therefore, the action for the Einstein's equations should be defined as

$$S_G[g_{\alpha\beta}] := S_H[g_{\alpha\beta}] + \frac{1}{\kappa} \int_{\dot{U}} \sqrt{|q|} K,$$

such that

$$\delta S_G = 0 \ \Leftrightarrow \ G_{\alpha\beta} = 0.$$

The Lagrangian Formulation of GR

- The non-vacuum Einstein's equation with matter fields can be obtained in a very simple and natural way. The total action is just the sum of the gravitational action with those of the matter fields, e.g.,

$$S[g,\phi] = S_G[g] + S_M[g,\phi].$$

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$$S[g,\phi] = S_G[g] + S_M[g,\phi].$$

- The variation of the the total action with respect to $g^{\alpha\beta}$ gives

$$\delta S = 0 \;\Leftrightarrow\; G_{\alpha\beta} = \kappa T_{\alpha\beta},$$

where the energy-momentum tensor can be defined as

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\alpha\beta}}.$$

Hamiltonian Theory with Finite Degrees of Freedom

- The Hamiltonian is a function H of $2N$ generalized coordinates $\{q^i | i = 1, 2, ..., N\}$ and momenta $\{p^j | j = 1, 2, ..., N\}$, i.e., a function on the phase space Γ as the cotangent bundle of the configuration space $\mathcal{C}$.

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- The Hamliltonian formulation can also be derived from the Langrangian formulation by the Legendre transformation, such that

$$p_i := \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i},$$

and

$$H(q, p) := p_i \dot{q}^i - L(q, \dot{q}),$$

depending only on the canonical pairs (q, p).

The Hamiltonian Theory with Finite Degrees of Freedom

- The Lagrangian $L(q, \dot{q})$ is said to be regular if every $\dot{q}^i$ can be resolved from the definition of p^i.
- In this case, the Euler-Lagrangian equations are equivalent to the following Hamiltonian canonical equations:

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- The Lagrangian $L(q, \dot{q})$ is said to be singular if

$$\det\left(J_{ij}\right) := \det\left(\frac{\partial^2 L(q,\dot{q})}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0.$$

- For example, $L = L(q^1, q^k; \dot{q}^k)$, $k = 2, ..., N$, is singular.

Hamiltonian Theory of a Constrained System

- Let the rank of the matrix (J_{ij}) be $Z < N$. Then $M = N - Z$ of the $\dot{q}^j$ can not be resolved from the definition of p_j, so that there exist M *primary* constraints:

$$C_a(q, p) := p_a - \frac{\partial L(q, \dot{q})}{\partial \dot{q}^a} = 0, \quad a = 1, ..., M.$$

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- In this case, the Hamiltonian can be expressed as

$$H(q,p) = \tilde{H}(q,p) - \lambda^a C_j(q,p),$$

where $\tilde{H}$ is independent of $\lambda^a \equiv \dot{q}^a$ called as Lagrangian undetermined multipliers.

- The Euler-Lagrangian equations are equivalent to the Hamiltonian canonical equations together with the constraints.

Symplectic Structure on the Phase Space

- The symplectic form on the phace space Γ of a Hamiltonian theory with finite degrees of freedom can be defined by

$$\Omega_{AB} := (dp_i)_A \wedge (dq^i)_B,$$

which is a non-degenerate two-form.

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- The Poisson bracket between two functions on Γ is defined by the inverse Ω^{AB} of the symplectic from as

$$\{f, g\} := \Omega^{AB}(\nabla_A f)\nabla_B g = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

- The Hamiltonian vecter field $X^A_{(H)} \equiv \Omega^{AB}\nabla_B H$ generates the time evolution of phase space functions as .

$$\dot{f} = X^A_{(H)}\nabla_A f = \{f, H\}.$$

Analysis of Constraints

- For a Hamiltonian system with primary constraints, physical states should be confined into the constraint surface $\Gamma_1 \subset \Gamma$, determined by $C_a = 0$.
- The consistency of the time evolution requires

$$\dot{C}_a = X^A_{(H)} \nabla_A C_a = (\{C_a, \tilde{H}\} + \{C_a, C_b\}\lambda^b)|_{\Gamma_1} = 0.$$

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- Let the rank of the matrix $(\Phi_{ab}) \equiv \{C_a, C_b\}$ be Y. Then Y of the Z Lagrangian multipliers λ^b can be determined by the consistency condition.
- The remaining $M - Y$ consistency equations will become new constraints: $\tilde{C}_m = 0$, except for the automatically satisfied ones.

Analysis of Constraints

- The new produced constraints have to satisfy the consisitency condition on the further constrained surface $\Gamma_2 \subset \Gamma_1$, similar to that for the primary constraints.
- By iterating the analysis of the consisitency conditions for the new constraints, one finally gets the result that the matrix $\left(\tilde{\Phi}_{ma}\right) \equiv \{\tilde{C}_m, C_a\}$ has the maximal rank and the consistency conditions are satisfied for all constraints.

Analysis of Constraints

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- By iterating the analysis of the consisitency conditions for the new constraints, one finally gets the result that the matrix $\left(\tilde{\Phi}_{ma}\right) \equiv \{\tilde{C}_m, C_a\}$ has the maximal rank and the consistency conditions are satisfied for all constraints.
- The new produced constraints are called *secondary* constraints, except for those resolved by fixing the Lagrangian multipliers.
- A consistent Hamiltonian theory is obtained such that, the physical states are confined into the constraint surface $\bar{\Gamma} \subset \Gamma$ determined by the remaining constraints: $C_r = 0, \; r = 1, ..., R$.

Analysis of Constraints

- A function f on Γ is called of first class provided that its Hamiltonian vector field $X^A_{(f)}|_{\bar{\Gamma}}$ is tangent to $\bar{\Gamma}$, i.e.,

$$\{f, C_r\}|_{\bar{\Gamma}} = 0, \quad r = 1, ..., R,$$

  otherwise of second class.
- It is obvious that the Hamiltonian $H(q, p)$ is of first class.

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⋆ Some remarkable results:

1. The number of first-class constraints equals to the number of free Lagrangian multipliers [Dirac 1964].
2. Each gauge symmetry of theory corrresponds to a first-class constraint, and the Hamiltonian vector fields of first-class constraints generate gauge transformations of the system [Lee and Wald 1990].
3. Every second-class constraint removes one degree of freedom in Γ, while every first-class constraint removes two.

Hamiltonian Theory of Classical Fields

- From the spacetime viewpoint, the first step in producing a Hamiltonian formulation of a field theory is to choose a time function t and a vector field t^α on a 4D spacetime M, such that $t^\alpha \nabla_\alpha t = 1$ and the surfaces Σ_t of constant t are spacelike Cauchy surfaces.

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- Given a Lagrangian formulation of a classical field ϕ, one can define its instantaneous configuration q as the field ϕ evaluated on Σ_t.
- Then the Lagrangian density $\mathcal{L}$ can be viewed as a function of q, its time derivatives and space derivatives.
- Assuming that the Lagrangian density does not depend on the time derivatives of q higher than first order, the Legendre transformation gives the momentum as $p := \partial\mathcal{L}/\partial\dot{q}$.

Hamiltonian Theory of Classical Fields

- The Hamiltonian density is defined by

$$\mathcal{H}(q,p) := p\dot{q} - \mathcal{L},$$

which gives the Hamiltonian of the system as

$$H[q,p] = \int_{\Sigma_t} \mathcal{H} d^3x,$$

where a coordinate system $\{(t, x^a) | a = 1, 2, 3\}$ has been chosen to adapt to the $3+1$ decomposition of M.

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where a coordinate system $\{(t, x^a) | a = 1, 2, 3\}$ has been chosen to adapt to the $3+1$ decomposition of M.

- For a regular $\mathcal{L}$, the Lagrangian equations are equivalent to the following Hamiltonian canonical equations:

$$\dot{q}(x) = \frac{\delta H(q,p)}{\delta p(x)}, \qquad \dot{p}(x) = -\frac{\delta H(q,p)}{\delta q(x)},$$

The ADM Formalism of GR

- Consider the Hilbert action on an 4-manifold M:

$$S_H[g_{\alpha\beta}] = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}R[g].$$

- To carry out the Hamiltonian analysis, let M be topologically $\Sigma \times \mathbf{R}$ for some 3-dimensional compact manifold Σ without boundary. For noncompact Σ, we consider the case that all boundary terms can be removed by boundary conditions.

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- To carry out the Hamiltonian analysis, let M be topologically $\Sigma \times \mathbf{R}$ for some 3-dimensional compact manifold Σ without boundary. For noncompact Σ, we consider the case that all boundary terms can be removed by boundary conditions.
- By the natural $3+1$ decompotion parameterized by a time function t, each surface Σ_t of constant t is identified with Σ.
- The time-evolution vector field t^α can be decomposed with respect to the unit normal vector n^α of Σ as:

$$t^\alpha = Nn^\alpha + N^\alpha,$$

where N is called the *lapse function* and N^α the *shift vector*.

The ADM Formalism of GR

- The spacetime metric $g_{\alpha\beta}$ induces a spatial metric on Σ as

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta.$$

- In a coordinate sysytem $\{(t, x^a)|a = 1, 2, 3\}$ adapted to the $3+1$ decomposition of M, the line element of $g_{\alpha\beta}$ can be written as

$$ds^2 = (-N^2 + N^a N_a)dt^2 + 2N_a dt dx^a + h_{ab} dx^a dx^b.$$

Hence, the information of $g_{\alpha\beta}$ is contained in the fields (h_{ab}, N, N_a) as its instantaneous configuration on Σ.

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- In the adapted coordinate system, one also has

$$\sqrt{-g} = N\sqrt{h}.$$

The ADM Formalism of GR

- Evaluated on Σ, the Ricci scalar can be expressed as

$$R = {}^{(3)}R + K_{ab}K^{ab} - K^2 + 2\nabla_\alpha(n^\alpha \nabla_\beta n^\beta - n^\beta \nabla_\beta n^\alpha),$$

where ${}^{(3)}R$ denotes the Ricci scalar determined by the 3-metric h_{ab}, K_{ab} is the extrisic curvature of Σ and $K \equiv K_{ab}h^{ab}$,

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- Therefore, discarding the boundary term, the Lagrangian density can be expressed as

$$\mathcal{L}_G = N\sqrt{h}({}^{(3)}R + K_{ab}K^{ab} - K^2).$$

- K_{ab} is related to the time derivative of h_{ab} by

$$K_{ab} = \frac{1}{2}N^{-1}(\dot{h}_{ab} - 2D_{(a}N_{b)}),$$

where $\dot{h}_{ab} := h_a^\alpha h_b^\beta L_{\vec{t}} h_{\alpha\beta}$, and D_a is the covariant derivative on Σ associated with h_{ab}.

The ADM Formalism of GR

- By the Legendre transformation, the momentum conjugate to h_{ab} reads

$$\tilde{\pi}^{ab} := \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ab}} = \sqrt{h}(K^{ab} - Kh^{ab}).$$

- The momenta conjugate to N and N_a become primary constraints respectively as

$$\pi_{(N)} := \frac{\partial \mathcal{L}_G}{\partial \dot{N}} = 0, \qquad \pi^a_{(N_a)} := \frac{\partial \mathcal{L}_G}{\partial \dot{N}_a} = 0.$$

- There is no other primary constraint, since $\dot{h}_{ab}$ can be resolved from the momentum as

$$\dot{h}_{ab} = \frac{2N}{\sqrt{h}}(\tilde{\pi}_{ab} - \frac{1}{2}\tilde{\pi} h_{ab}) + 2D_{(a}N_{b)}.$$

The ADM Formalism of GR

- The Hamiltonian density is defined by

$$\begin{aligned}\mathcal{H}_G := &\ \pi_{(N)}\dot{N} + \pi^a_{(N_a)}\dot{N}_a + \tilde{\pi}^{ab}\dot{h}_{ab} - \mathcal{L}_G \\ = &\ \pi_{(N)}\dot{N} + \pi^a_{(N_a)}\dot{N}_a \\ + &\ N\sqrt{h}[-{}^{(3)}R + \frac{1}{h}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \frac{1}{2}\tilde{\pi}^2)] + 2\tilde{\pi}^{ab}D_aN_b,\end{aligned}$$

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\end{aligned}
$$

which gives the Hamiltonian of the system up to a boundary term as

$$
H_G = \int_\Sigma \mathcal{H}_G d^3x = \int_\Sigma d^3x(\pi_{(N)}\dot{N} + \pi^a_{(N_a)}\dot{N}_a + N\mathcal{S} + N_a\mathcal{V}^a),
$$

where

$$
\mathcal{S} \equiv -{}^{(3)}R\sqrt{h} + \frac{1}{\sqrt{h}}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \frac{1}{2}\tilde{\pi}^2), \qquad \mathcal{V}^a \equiv -2D_b\tilde{\pi}^{ab}.
$$

The ADM Formalism of GR

- Since the consistency condition for the primary constraints requires

$$\dot{\pi}_{(N)} = -\frac{\delta H_G}{\delta N}|_{\Gamma_1} = 0, \qquad \dot{\pi}_{(N_a)} = -\frac{\delta H_G}{\delta N_a}|_{\Gamma_1} = 0,$$

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one gets the secondary constraints: $\mathcal{S} = 0$ and $\mathcal{V}^a = 0$.

- Since $\pi_{(N)}$ and $\pi_{(N_a)}$ are contrained to be zero while N and N_a can take arbitrary values, these two canonical pairs can be descarded from the phase space.

  Then the phase space Γ is only composed of the canonical pair $(h_{ab}, \tilde{\pi}^{ab})$ with the symplectic form:

$$\Omega^{AB} := \int_\Sigma d^3x \left(\frac{\delta}{\delta h_{ab}}\right)^A \wedge \left(\frac{\delta}{\delta \tilde{\pi}^{ab}}\right)^B,$$

and the Hamiltonian:

$$H_G[h, \tilde{\pi}] = \int_\Sigma d^3x (N\mathcal{S} + N_a\mathcal{V}^a).$$

The ADM Formalism of GR

- By the basic Poisson bracket:

$$\{h_{ab}(x), \widetilde{\pi}^{cd}(y)\} = \delta^{c}_{(a}\delta^{d}_{b)}\delta^{3}(x-y),$$

one can show that the secondary constraints are of first class, with the following hypersurface deformation algebra

$$\begin{aligned}
\{\mathcal{V}(\vec{N}),\ \mathcal{V}(\vec{N}')\} &= \mathcal{V}([\vec{N},\ \vec{N}']),\\
\{\mathcal{V}(\vec{N}),\ \mathcal{S}(M)\} &= -\mathcal{S}(\mathcal{L}_{\vec{N}}M),\\
\{\mathcal{S}(N),\ \mathcal{S}(M)\} &= -\mathcal{V}((N\partial_b M - M\partial_b N)h^{ab}),
\end{aligned}$$

where $\mathcal{V}(\vec{N}) \equiv \int_\Sigma d^3x N^a \mathcal{V}_a$ and $\mathcal{S}(M) \equiv \int_\Sigma d^3x M\mathcal{S}$.

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where $\mathcal{V}(\vec{N}) \equiv \int_\Sigma d^3x N^a \mathcal{V}_a$ and $\mathcal{S}(M) \equiv \int_\Sigma d^3x M\mathcal{S}$.

- As the Hamiltonian is just a linear combination of the first-class constraints, there is no further secondary constraint.

The ADM Formalism of GR

- By direct calculations, one has

$$X^A_{(\mathcal{V}(\vec{N}))} = \int_\Sigma d^3x \left[(L_{\vec{N}} h_{ab}) \left(\frac{\delta}{\delta h_{ab}} \right)^A + (L_{\vec{N}} \tilde{\pi}^{ab}) \left(\frac{\delta}{\delta \tilde{\pi}^{ab}} \right)^A \right].$$

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- Hence the smeared diffeomorphism constraint $\mathcal{V}(\vec{N})$ generates the infinitesimal spatial diffeomorphisms by the vector field N^a on Σ;
- The smeared Hamiltonian constraint $\mathcal{S}(N)$ generates the infinitesimal bubble time evolution off Σ, since

$$X^A_{(\mathcal{S}(N))} = \int_\Sigma d^3x \left[(L_{N\vec{n}} h_{ab}) \left(\frac{\delta}{\delta h_{ab}} \right)^A + (L_{N\vec{n}} \tilde{\pi}^{ab}) \left(\frac{\delta}{\delta \tilde{\pi}^{ab}} \right)^A \right].$$

- The constraints encode the local diffeomorphism invariance.

The ADM Formalism of GR

- The Hamiltonian canonical equations given by $H_G[h, \tilde{\pi}]$, together with the constraints, are equivalent to the vacuum Einstein's equations.
- In the case of coupling to matter fields, the totol Hamiltonian is just the sum of the gravitational and matter ones as $H_{tot} = \frac{1}{2\kappa} H_G + H_M$.

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- In the case of coupling to matter fields, the totol Hamiltonian is just the sum of the gravitational and matter ones as $H_{tot} = \frac{1}{2\kappa} H_G + H_M$.
- In the case of asymptotically flat spacetimes M, one may consider the $3+1$ decomposition of M with asympototically flat spatial hypersurfaces Σ_t. Then the variations of $H[h, \tilde{\pi}]$ will contribute boundary terms.
- For a variation of h_{ab} and $\tilde{\pi}^{ab}$, which preserves asymptotic flatness, one has

$$\delta H_G[h, \tilde{\pi}] = \int_\Sigma d^3x (\dot{h}_{ab} \delta \tilde{\pi}^{ab} - \dot{\tilde{\pi}}^{ab} \delta h_{ab}) - \delta B.$$

The ADM Formalism of GR

- The boundary term reads

$$\begin{aligned}\delta B &= \lim_{r\to\infty} N \int_S r^a h^{bc}[D_c(\delta h_{ab}) - D_a(\delta h_{bc})] \\ &= \hat{N}\delta \left[\lim_{r\to\infty} \sum_{a,b=1}^{3} \int_S r^a \left(\frac{\partial h_{ab}}{\partial x^b} - \frac{\partial h_{bb}}{\partial x^a} \right) \right],\end{aligned}$$

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  where S denotes a coordinate sphere of radius r with the unit normal r^a.

- By suitably choosing $\hat{N}$, one gets $B = 2\kappa E_{ADM}$, and hence the Hamiltonian on the asymptotically flat Σ should be given by

$$H'_G := \frac{1}{2\kappa} H_G[h, \tilde{\pi}] + E_{ADM}.$$

Basic Ideas of LQG

- ⋆ GR-Notions of spacetime and causality:
  General Covariance $\Leftrightarrow$ Spacetime is dynamical.
  QM-Notions of matter and measurement: Dynamical entity is made up of quanta and in probabilistic superposition state.

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$$g_{ab} = \eta_{ab} + h_{ab}.$$

  The separation of the gravitational field from background spacetime is in contradiction with the very lesson of GR.
- ⋆ *The viewpoint of background independence:*
  GR's revolution: particle and fields are neither immersed in space nor moving in time, but live on one another.
  The quanta of the field cannot live in a prior spacetime. They should build spacetime themselves.

Holonomies

- LQG inherits the basic idea of Einstein that gravity is fundamentally spacetime geometry.
  Hence the theory of quantum gravity is a quantum theory of spacetime geometry with diffeomorphism invariance.
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- ⋆ The choice of the algebra of field functions to be quantized: Not the positive and negative components of the field modes as in conventional QFT; but the holonomies of the gravitational connection and the electric flux.
- ⋆ The physical meaning of holonomies:
  Faraday - lines of force: the relevant variables do not refer to what happens at a point, but rather refer to the relation between different points connected by a line.

$$A(c) = \mathcal{P}\exp\Big(-\int_0^1 [A_a^i \dot{c}^a \tau_i]\, dt\Big).$$

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- Combining two fundamental principles: Background independence (GR) and Quantum mechanical property(QM).
- Minimal principle: QGR.
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- Minimal principle: QGR.
- Mathematical rigor.

⋆ Being conserve and of small ambition:
No major additional physical hypothesis, no claim of being final theory of everything.

⋆ Radical and ambitious side:
To merge the conceptual insight of GR into QM.

Historical Developments of Connection Formulation

- The first canonical formalism of general relativity is the ADM (Arnowitt, Deser, Misner) formalism (Geometric dynamics) from the Hilbert action.
- Another well-known action of general relativity is the Palatini formalism, where the tetrad and the connection are regarded as independent dynamical variables.

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- In 1986, Ashtekar gave a formalism of true connection dynamics with a relatively simple Hamiltonian constraint, and thus opens the door to apply quantization techniques from gauge fields theory.
  - However the weakness of that formalism is that the canonical variables are complex variables, which needs a complicated real section condition.
  - Moreover, the quantization based on the complex connection could not be carried out rigorously, since the internal gauge group is noncompact.

Historical Developments

- In 1995, Barbero modified the Ashtekar new variables to give a system of real canonical variables for dynamical theory of connections.
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- Then Holst constructed a generalized Palatini action to support Barbero's real connection dynamics.
- Although there is a free (Barbero-Immirzi) parameter in generalized Palatini action and the Hamiltonian constraint is more complicated than the Ashtekar one, now the generalized Palatini Hamiltonian with the real connections is widely accepted by loop theorists for the quantization programme.

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