

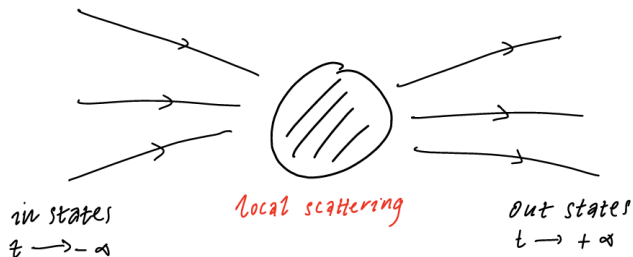
Unitarity Cuts, t -channel Divergences and the KLN Theorem for Unstable Particles

Marko Beocanin

based on [2606.26586](#) with **Michael A. Schmidt**
University of New South Wales

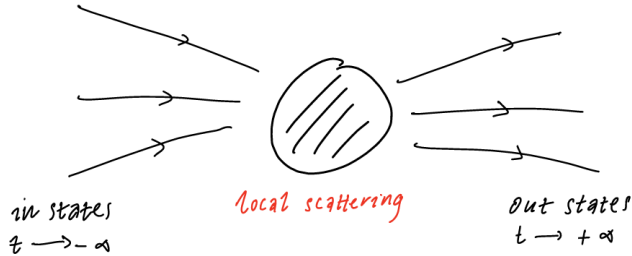
The S -Matrix

This talk will be about **perturbative calculations** involving the S -Matrix: how they can break down, and how one might hope to – at least partially – ameliorate this breakdown.



The S -Matrix

This talk will be about **perturbative calculations** involving the S -Matrix: how they can break down, and how one might hope to – at least partially – ameliorate this breakdown.



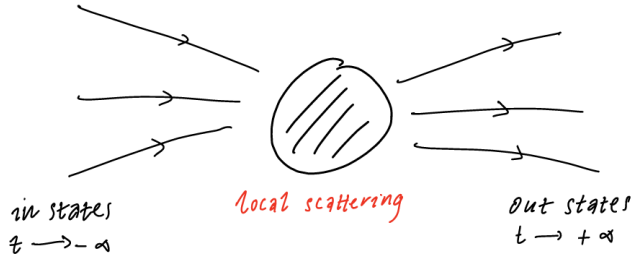
When does perturbation theory work?

Perturbative scattering calculations rely on the assumption that

$$\text{asymptotic states} \cong \text{free states}$$

The S -Matrix

This talk will be about **perturbative calculations** involving the S -Matrix: how they can break down, and how one might hope to – at least partially – ameliorate this breakdown.



When does perturbation theory work?

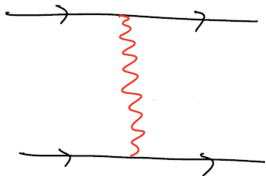
Perturbative scattering calculations rely on the assumption that

$$\text{asymptotic states} \cong \text{free states}$$

...which can fail when **massless** and **unstable** particles are involved.

t -channel divergences, or, why should you care?

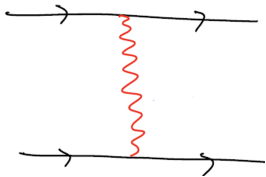
This can show up even at **tree-level** in t -channel exchange involving unstable external states.



$$\Rightarrow i\mathcal{M}_t \sim \frac{1}{t - m_t^2} \text{ can blow up when } t = m_t^2$$

t -channel divergences, or, why should you care?

This can show up even at **tree-level** in t -channel exchange involving unstable external states.



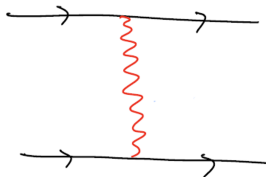
$$\Rightarrow i\mathcal{M}_t \sim \frac{1}{t - m_t^2} \text{ can blow up when } t = m_t^2$$

Shows up in **Standard Model collider physics**; see, for eg:

- pion-nucleon scattering ([Peierls, 1961](#));
- elastic Z or W boson scattering with leptons ([Nowakowski and Pilaftsis, 1993](#), [Grzadkowski et al., 2022](#));
- neutrino-mediated muon collisions ([Ginzburg, 1996](#));

t -channel divergences, or, why should you care?

This can show up even at **tree-level** in t -channel exchange involving unstable external states.



$$\Rightarrow i\mathcal{M}_t \sim \frac{1}{t - m_t^2} \text{ can blow up when } t = m_t^2$$

Shows up in **Standard Model collider physics**; see, for eg:

- pion-nucleon scattering ([Peierls, 1961](#));
- elastic Z or W boson scattering with leptons ([Nowakowski and Pilaftsis, 1993](#), [Grzadkowski et al., 2022](#));
- neutrino-mediated muon collisions ([Ginzburg, 1996](#));

and **finite-temperature cosmology** (see talk by [Lachlan E. Tobin next!](#)); see, for eg:

- dark matter annihilation ([Grzadkowski et al., 2022](#), [Beneke et al., 2014](#));
- dark matter freeze-in production ([Becker et al., 2025](#), [Coy and Schmidt, 2022](#));
- leptogenesis ([Garbrecht et al., 2013a,b](#)).

What do we usually do about it?

In practice, one usually deals with this by **resumming** mediator self-energies...

$$\text{wavy line} + \text{wavy line} \text{ (with 1 red hatched circle) } + \text{wavy line} \text{ (with 2 red hatched circles) } + \dots$$

What do we usually do about it?

In practice, one usually deals with this by **resumming** mediator self-energies...

$$\text{wavy line} + \text{wavy line with self-energy} + \text{wavy line with two self-energies} + \dots$$

which amounts to

$$\frac{1}{q^2 - m^2} \rightarrow \frac{1}{q^2 - M^2 + iM\Gamma}. \quad (1)$$

This replaces the pole with a $\sim 1/\Gamma$.

What do we usually do about it?

In practice, one usually deals with this by **resumming** mediator self-energies...

$$\text{wavy line} + \text{wavy line with self-energy} + \text{wavy line with two self-energies} + \dots$$

which amounts to

$$\frac{1}{q^2 - m^2} \rightarrow \frac{1}{q^2 - M^2 + iM\Gamma}. \quad (1)$$

This replaces the pole with a $\sim 1/\Gamma$. However, there are two disadvantages:

- by reorganizing the perturbative expansion, re-summation can* spoil **gauge invariance** or introduce **double-counting**;
- simply does not work for **stable mediators** with $\Gamma \sim 0$.

What do we usually do about it?

In practice, one usually deals with this by **resumming** mediator self-energies...

$$\text{wavy line} + \text{wavy line with self-energy} + \text{wavy line with two self-energies} + \dots$$

which amounts to

$$\frac{1}{q^2 - m^2} \rightarrow \frac{1}{q^2 - M^2 + iM\Gamma}. \quad (1)$$

This replaces the pole with a $\sim 1/\Gamma$. However, there are two disadvantages:

- by reorganizing the perturbative expansion, re-summation can* spoil **gauge invariance** or introduce **double-counting**;
- simply does not work for **stable mediators** with $\Gamma \sim 0$.

See also: **first-principles reformulations** (Kulish and Faddeev, 1970, Hirai and Sugishita, 2021, Carney et al., 2018, Giavarini and Marchesini, 1988, Contopanagos and Einhorn, 1992, Forde and Signer, 2004, Hannesdottir and Schwartz, 2023, Gonzo et al., 2021, Karamitros and Pilaftsis, 2023); **collider-geometry regularizations** (Melnikov and Serbo, 1997, 1996, Dams and Kleiss, 2003, 2004); **subtraction-schemes** (Asai et al., 2025) ...

Our question

What's the dream?

Is there a systematic procedure for constructing finite t -channel observables that is:

Our question

What's the dream?

Is there a systematic procedure for constructing finite t -channel observables that is:

- **fixed-order** in perturbation theory;

Our question

What's the dream?

Is there a systematic procedure for constructing finite t -channel observables that is:

- **fixed-order** in perturbation theory;
- applicable even with a **stable mediator**;

Our question

What's the dream?

Is there a systematic procedure for constructing finite t -channel observables that is:

- **fixed-order** in perturbation theory;
- applicable even with a **stable mediator**;
- **first-principles** and **generalizable** to other divergent processes;

Our question

What's the dream?

Is there a systematic procedure for constructing finite t -channel observables that is:

- **fixed-order** in perturbation theory;
- applicable even with a **stable mediator**;
- **first-principles** and **generalizable** to other divergent processes;
- **simple** enough for practical phenomenological calculations;

Our question

What's the dream?

Is there a systematic procedure for constructing finite t -channel observables that is:

- **fixed-order** in perturbation theory;
- applicable even with a **stable mediator**;
- **first-principles** and **generalizable** to other divergent processes;
- **simple** enough for practical phenomenological calculations;
- ...and hopefully generalizable to **finite-temperature**?

Our question

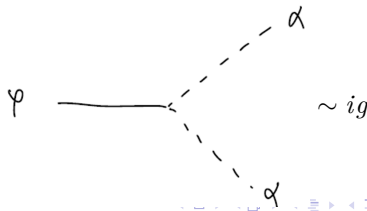
What's the dream?

Is there a systematic procedure for constructing finite t -channel observables that is:

- **fixed-order** in perturbation theory;
- applicable even with a **stable mediator**;
- **first-principles** and **generalizable** to other divergent processes;
- **simple** enough for practical phenomenological calculations;
- ...and hopefully generalizable to **finite-temperature**?

For concreteness, let's consider a simple model with:

- one **unstable** massive scalar φ ;
- one (all-orders **stable**) **massless** scalar α ;
- and a single 3-body $\varphi\alpha\alpha$ vertex.



The t -channel divergence

Let's actually have a look at the divergent t -channel cross-section:

$$\sigma_t \sim \int d\Pi_{\text{LIPS}} \left| \begin{array}{c} \text{---} \cdots \\ | \\ \text{---} \end{array} \right|^2 \approx \frac{1}{2m^2 - s} + [\delta^2 \text{ contribution for } s > 2m^2]$$

The t -channel divergence

Let's actually have a look at the divergent t -channel cross-section:

$$\sigma_t \sim \int d\Pi_{\text{LIPS}} \left| \begin{array}{c} \text{---} \text{---} \text{---} \\ | \\ \text{---} \end{array} \right|^2 \sim \frac{1}{2m^2 - s} + [\delta^2 \text{ contribution for } s > 2m^2]$$

As a **phase-space integral**, this can be written

$$\sigma_t \sim \int \frac{d^d q}{(2\pi)^d} \times [2\pi\delta^+(q^2)2\pi\delta^+((p_1 + p_2 - q)^2 - m^2)] \times (i\mathcal{M}_t)(i\mathcal{M}_t)^* \quad (2)$$

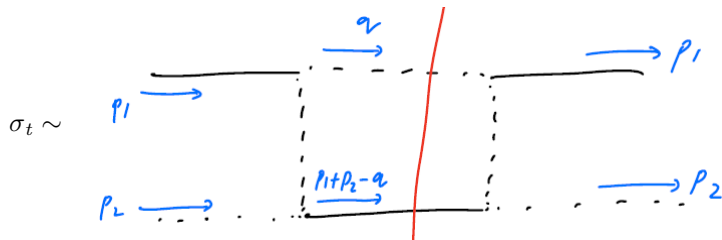
where $i\mathcal{M}_t$ is the t -channel matrix element.

Cross-Sections as Cuts

Let's rewrite this in a slightly unusual way.

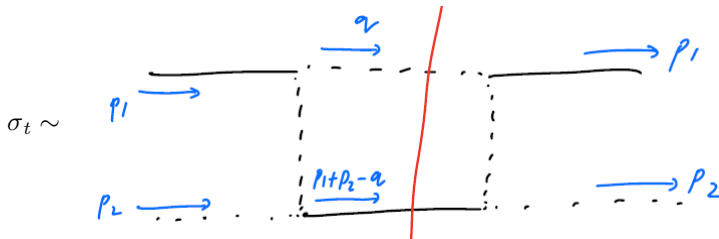
Cross-Sections as Cuts

Let's rewrite this in a slightly unusual way.



Cross-Sections as Cuts

Let's rewrite this in a slightly unusual way.



We call this a **unitarity cut** of the **box diagram**, which means:

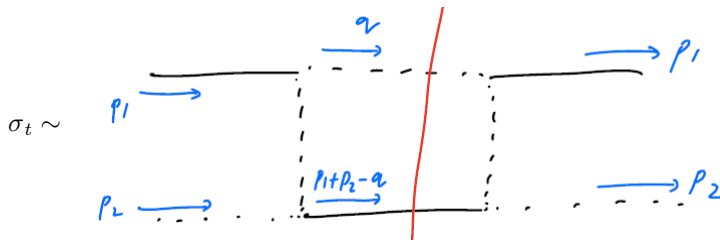
- we draw a line that cuts through the diagram;
- we take all cut internal propagators and put them **on-shell** with **positive-energy**

$$\frac{i}{q^2 - m^2 + i\epsilon} \rightarrow 2\pi\delta(q^2 - m^2)\theta(q^0) \quad (3)$$

- we **conjugate** everything on the **right-hand-side** of the cut;
- and we leave external lines unchanged.

Cross-Sections as Cuts

Let's rewrite this in a slightly unusual way.



We call this a **unitarity cut** of the **box diagram**, which means:

- we draw a line that cuts through the diagram;
- we take all cut internal propagators and put them **on-shell** with **positive-energy**

$$\frac{i}{q^2 - m^2 + i\epsilon} \rightarrow 2\pi\delta(q^2 - m^2)\theta(q^0) \quad (3)$$

- we **conjugate** everything on the **right-hand-side** of the cut;
- and we leave external lines unchanged.

NB: To interpret the cut as a cross-section, we need $\{p_i\} = \{p_f\}$.

The Cutting Rules

For any Feynman diagram $i\mathcal{M}$ with $\{p_i\} = \{p_f\}$, draw **all possible cuts** $(i\mathcal{M})_C$.

The Cutting Rules

For any Feynman diagram $i\mathcal{M}$ with $\{p_i\} = \{p_f\}$, draw **all possible cuts** $(i\mathcal{M})_C$. As before, each of these cuts can be thought of as enumerating **interference contributions** to

$$\text{Cut splitting } i\mathcal{M} \text{ into } (i\mathcal{M}_L), (i\mathcal{M}_R)^* \subset \sigma_{LR} \sim \int d\Pi_{\text{LIPS}} |i\mathcal{M}_L + i\mathcal{M}_R|^2 \quad (4)$$

The Cutting Rules

For any Feynman diagram $i\mathcal{M}$ with $\{p_i\} = \{p_f\}$, draw **all possible cuts** $(i\mathcal{M})_C$. As before, each of these cuts can be thought of as enumerating **interference contributions** to

$$\text{Cut splitting } i\mathcal{M} \text{ into } (i\mathcal{M}_L), (i\mathcal{M}_R)^* \subset \sigma_{LR} \sim \int d\Pi_{\text{LIPS}} |i\mathcal{M}_L + i\mathcal{M}_R|^2 \quad (4)$$

...but **so what?**

The Cutting Rules

For any Feynman diagram $i\mathcal{M}$ with $\{p_i\} = \{p_f\}$, draw **all possible cuts** $(i\mathcal{M})_C$. As before, each of these cuts can be thought of as enumerating **interference contributions** to

$$\text{Cut splitting } i\mathcal{M} \text{ into } (i\mathcal{M}_L), (i\mathcal{M}_R)^* \subset \sigma_{LR} \sim \int d\Pi_{\text{LIPS}} |i\mathcal{M}_L + i\mathcal{M}_R|^2 \quad (4)$$

...but **so what?**

The Largest Time Equation

A simple result following from **unitarity** known as the **Largest Time Equation** ('t Hooft and Veltman, 1974, Veltman, 2012) gives

$$\sum_C (i\mathcal{M})_C = 0 \implies \sum \sigma_C = 0 \quad (5)$$

The Cutting Rules

For any Feynman diagram $i\mathcal{M}$ with $\{p_i\} = \{p_f\}$, draw **all possible cuts** $(i\mathcal{M})_C$. As before, each of these cuts can be thought of as enumerating **interference contributions** to

$$\text{Cut splitting } i\mathcal{M} \text{ into } (i\mathcal{M}_L), (i\mathcal{M}_R)^* \subset \sigma_{LR} \sim \int d\Pi_{\text{LIPS}} |i\mathcal{M}_L + i\mathcal{M}_R|^2 \quad (4)$$

...but **so what?**

The Largest Time Equation

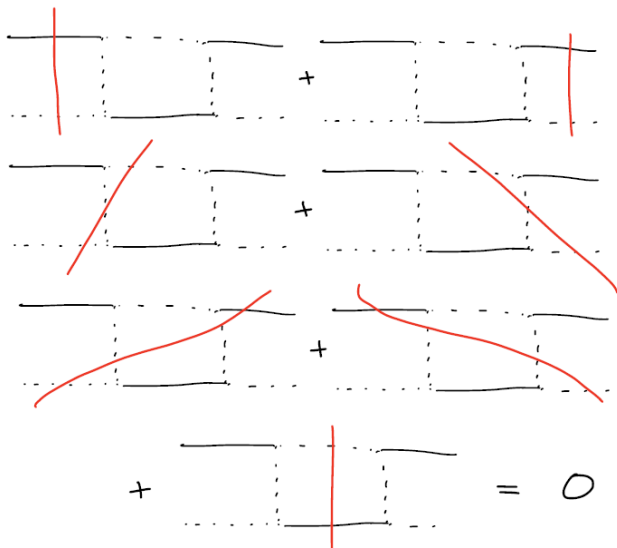
A simple result following from **unitarity** known as the **Largest Time Equation** ('t Hooft and Veltman, 1974, Veltman, 2012) gives

$$\sum_C (i\mathcal{M})_C = 0 \implies \sum \sigma_C = 0 \quad (5)$$

NB: This is more commonly written in terms of the forward cuts $-2\text{Im}\{\mathcal{M}\} = i\mathcal{M} + (i\mathcal{M})^*$.

Cuts of the Box Diagram

For our t -channel diagram, this means



The Kinoshita-Lee-Nauenberg (KLN) Theorem

Sums of Cross-Sections? What's the Physics?

The basic physical interpretation is that even though **individual** cross-sections may be divergent, one can always recover a finite **inclusive cross-section**, since $0 < \infty$.

The Kinoshita-Lee-Nauenberg (KLN) Theorem

Sums of Cross-Sections? What's the Physics?

The basic physical interpretation is that even though **individual** cross-sections may be divergent, one can always recover a finite **inclusive cross-section**, since $0 < \infty$.

The Kinoshita-Lee-Nauenberg Theorem

As (Frye et al., 2019) point out, this cancellation can be understood as an incarnation of the **Kinoshita-Lee-Nauenberg (KLN) theorem** (Kinoshita, 1962, Lee and Nauenberg, 1964), which says that there always exists such a collection $\{I\}$ of cross-sections such that

$$\sum_{i \in \{I\}} \sigma_i < \infty \quad (6)$$

although the original work took $\{I\}$ as the (rather large) set of *all initial* and *all final* states.

Issues

What are we interested in?

Elaborating on the work of ([Frye et al., 2019](#)), there were three issues we wanted to deal with:

Issues

What are we interested in?

Elaborating on the work of ([Frye et al., 2019](#)), there were three issues we wanted to deal with::

- Kinematics can be quite pathological $\implies$ how to **actually calculate** them?

Issues

What are we interested in?

Elaborating on the work of ([Frye et al., 2019](#)), there were three issues we wanted to deal with::

- Kinematics can be quite pathological $\implies$ how to **actually calculate** them?
- Some of the cuts are rather ...obscure $\implies$ how to **physically interpret** them?

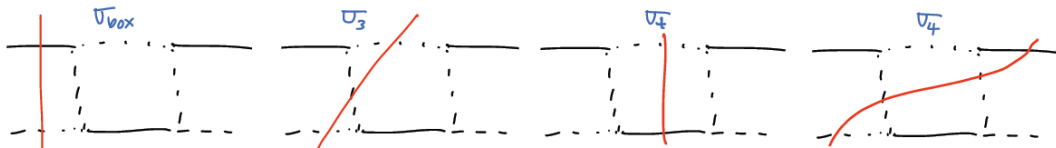
Issues

What are we interested in?

Elaborating on the work of [\(Frye et al., 2019\)](#), there were three issues we wanted to deal with::

- Kinematics can be quite pathological $\implies$ how to **actually calculate** them?
- Some of the cuts are rather ...obscure $\implies$ how to **physically interpret** them?
- We obviously do not measure "0" $\implies$ how to connect them to **collider observables**?

What do our cuts correspond to?



How do we make sense of these cuts?

The cuts enumerate interferences between **all possible scattering and sub-decay channels** at this order in perturbation theory associated with this topology. This includes **disconnected**, **forward-scattering** and **boundary-localized*** channels as well.



The Cancellation

	LM	SM	DR	NR
$\bar{\sigma}_t$	$-[s - 2m^2]^{-1}$	$-[s - 2m^2]^{-1}$	$-[s - 2m^2]^{-1}$	$-[s - 2m^2]^{-1}$
$\bar{\sigma}_{\text{box}}$	$+ [s - 2m^2]^{-1}$	0	0	$+\frac{1}{2}[s - m^2]^{-1}$
$\bar{\sigma}_3$	0	$+ [s - 2m^2]^{-1}$	$+ [s - 2m^2]^{-1}$	$+ [s - 2m^2]^{-1}$
$\bar{\sigma}_4$	0	0	0	$-\frac{1}{2}[s - m^2]^{-1}$

Table: Summary of the finite parts of squared t -channel cancellations off the pole for $s > m^2$ and $s \neq 2m^2$ with a large mass regulator $m_0 > m$ (LM), small mass regulator $0 < m_0 < m$ (SM), dimensional regularization (DR) and no regulator (NR). In each column, all results sum to zero, as expected by the KLN theorem. Results are written in terms of $\bar{\sigma}$ defined by $\text{finite}(\sigma) \equiv \frac{1}{16\pi} \frac{g^4}{m^4} \bar{\sigma}$. Note that σ_t and σ_3 have an additional squared δ -function divergent contribution for $s > 2m^2$ which also cancels.

What's the physics?

Unstable states are not asymptotic states, but colliders also do not probe asymptotic times.

What's the physics?

Unstable states are not asymptotic states, but colliders also do not probe asymptotic times. By using an S -Matrix approach, we've thrown away all information about the two competing timescales:

- how long it takes for an unstable particle to be **detected**;
- vs. how long it takes for an unstable particle to **decay**.

A priori, we have no idea which one wins out.

What's the physics?

Unstable states are not asymptotic states, but colliders also do not probe asymptotic times. By using an S -Matrix approach, we've thrown away all information about the two competing timescales:

- how long it takes for an unstable particle to be **detected**;
- vs. how long it takes for an unstable particle to **decay**.

A priori, we have no idea which one wins out.

Unstable States in Colliders

Unstable states are not asymptotic states, so if we want to treat them as such over the (finite) time-scale of our collider experiment, we need to account for **all possible decay pathways*** that can and will occur.

What's the physics?

Unstable states are not asymptotic states, but colliders also do not probe asymptotic times. By using an S -Matrix approach, we've thrown away all information about the two competing timescales:

- how long it takes for an unstable particle to be **detected**;
- vs. how long it takes for an unstable particle to **decay**.

A priori, we have no idea which one wins out.

Unstable States in Colliders

Unstable states are not asymptotic states, so if we want to treat them as such over the (finite) time-scale of our collider experiment, we need to account for **all possible decay pathways*** that can and will occur. It is only this **combined configuration** that can be treated as asymptotic.

What happened to “zero”?

All our cross-sections here sum to zero, but we obviously don't measure zero in a collider...

What happened to “zero”?

All our cross-sections here sum to zero, but we obviously don't measure zero in a collider...But notice that the KLN theorem can be rewritten

$$0 = \sum_C \sigma_C = \sum_C [\sigma_C]_{\text{degenerate}} + \sum_C [\sigma_C]_{\text{not-degenerate}} \quad (7)$$

What happened to “zero”?

All our cross-sections here sum to zero, but we obviously don't measure zero in a collider...But notice that the KLN theorem can be rewritten

$$0 = \sum_C \sigma_C = \sum_C [\sigma_C]_{\text{degenerate}} + \sum_C [\sigma_C]_{\text{not-degenerate}} \quad (7)$$

or, more suggestively,

$$\Rightarrow \sum_C [\sigma_C]_{\text{degenerate}} = - \sum_C [\sigma_C]_{\text{not-degenerate}} < \infty, \text{ but not identically } 0. \quad (8)$$

What happened to “zero”?

All our cross-sections here sum to zero, but we obviously don't measure zero in a collider...But notice that the KLN theorem can be rewritten

$$0 = \sum_C \sigma_C = \sum_C [\sigma_C]_{\text{degenerate}} + \sum_C [\sigma_C]_{\text{not-degenerate}} \quad (7)$$

or, more suggestively,

$$\Rightarrow \sum_C [\sigma_C]_{\text{degenerate}} = - \sum_C [\sigma_C]_{\text{not-degenerate}} < \infty, \text{ but not identically } 0. \quad (8)$$

Processes only need to be summed together in their region of degeneracy.

What happened to “zero”?

All our cross-sections here sum to zero, but we obviously don't measure zero in a collider...But notice that the KLN theorem can be rewritten

$$0 = \sum_C \sigma_C = \sum_C [\sigma_C]_{\text{degenerate}} + \sum_C [\sigma_C]_{\text{not-degenerate}} \quad (7)$$

or, more suggestively,

$$\Rightarrow \sum_C [\sigma_C]_{\text{degenerate}} = - \sum_C [\sigma_C]_{\text{not-degenerate}} < \infty, \text{ but not identically } 0. \quad (8)$$

Processes only need to be summed together in their region of degeneracy.

When are our processes degenerate?

A sub-decay is only physical when it occurs **on-shell**, so we choose a small window around this region as our region of degeneracy.

What happened to “zero”?

All our cross-sections here sum to zero, but we obviously don't measure zero in a collider...But notice that the KLN theorem can be rewritten

$$0 = \sum_C \sigma_C = \sum_C [\sigma_C]_{\text{degenerate}} + \sum_C [\sigma_C]_{\text{not-degenerate}} \quad (7)$$

or, more suggestively,

$$\Rightarrow \sum_C [\sigma_C]_{\text{degenerate}} = - \sum_C [\sigma_C]_{\text{not-degenerate}} < \infty, \text{ but not identically } 0. \quad (8)$$

Processes only need to be summed together in their region of degeneracy.

When are our processes degenerate?

A sub-decay is only physical when it occurs **on-shell**, so we choose a small window around this region as our region of degeneracy.

NB: More work still required to connect this to a concrete collider observable, since these configurations are not always soft/collinear. Subtleties with indistinguishability vs. degeneracy.

Computational aspects & δ -function madness

There were many mathematical issues, but the main takeaways are:

Computational aspects & δ -function madness

There were many mathematical issues, but the main takeaways are:

- **Branch point localization & regulator-dependence:** By setting $\{p_i\} = \{p_f\}$, we restricted to very specific kinematics. This is actually **branch point** of the underlying amplitude $i\mathcal{M}_{\text{box}}(s, t)$, which different regulators treat differently.

Computational aspects & δ -function madness

There were many mathematical issues, but the main takeaways are:

- **Branch point localization & regulator-dependence:** By setting $\{p_i\} = \{p_f\}$, we restricted to very specific kinematics. This is actually **branch point** of the underlying amplitude $i\mathcal{M}_{\text{box}}(s, t)$, which different regulators treat differently.
- **Non-commutativity of limits:** Sending the Feynman $i\epsilon \rightarrow 0$ does *not* commute with analytic continuation of a regulator $d \rightarrow 4, m_0 \rightarrow 0$. Always send $i\epsilon \rightarrow 0$ first.

Computational aspects & δ -function madness

There were many mathematical issues, but the main takeaways are:

- **Branch point localization & regulator-dependence:** By setting $\{p_i\} = \{p_f\}$, we restricted to very specific kinematics. This is actually **branch point** of the underlying amplitude $i\mathcal{M}_{\text{box}}(s, t)$, which different regulators treat differently.
- **Non-commutativity of limits:** Sending the Feynman $i\epsilon \rightarrow 0$ does *not* commute with analytic continuation of a regulator $d \rightarrow 4, m_0 \rightarrow 0$. Always send $i\epsilon \rightarrow 0$ first.
- **Boundary terms matter:** Even terms with apparently measure-zero support can still contribute when δ -functions are involved. Be very careful with factors of half.

Computational aspects & δ -function madness

There were many mathematical issues, but the main takeaways are:

- **Branch point localization & regulator-dependence:** By setting $\{p_i\} = \{p_f\}$, we restricted to very specific kinematics. This is actually **branch point** of the underlying amplitude $i\mathcal{M}_{\text{box}}(s, t)$, which different regulators treat differently.
- **Non-commutativity of limits:** Sending the Feynman $i\epsilon \rightarrow 0$ does *not* commute with analytic continuation of a regulator $d \rightarrow 4, m_0 \rightarrow 0$. Always send $i\epsilon \rightarrow 0$ first.
- **Boundary terms matter:** Even terms with apparently measure-zero support can still contribute when δ -functions are involved. Be very careful with factors of half.

Is there a better formalism?

Beyond our scope here, but everything can* (and should) be formalized with **Picard-Lefschetz theory** and the **multivariate residue calculus of Leray**; see [\(Abreu et al., 2017\)](#).

So, what do you get?

Following this prescription, we get **finite, fixed-order, regulator-independent*** inclusive t -channel cross-section using only the **usual S -Matrix perturbation theory**, even though our mediator is **stable**.

$$[\sigma]_{\text{degenerate}}^{(g^4)} = \frac{g^4}{\pi m^6} \begin{cases} \frac{\kappa - 2}{64(\kappa - 1)^2} + \frac{(4 - \kappa)\Lambda}{128(\kappa - 1)^3} + \mathcal{O}(\Lambda^2) & 2 - \Lambda \leq \kappa < 2 + \Lambda \\ \frac{(4 - \kappa)\Lambda}{64(\kappa - 1)^3} + \mathcal{O}(\Lambda^2) & \kappa \geq 2 + \Lambda \end{cases}. \quad (9)$$

with $\kappa \equiv s/m^2$.

So, what do you get?

Following this prescription, we get **finite, fixed-order, regulator-independent*** inclusive t -channel cross-section using only the **usual S -Matrix perturbation theory**, even though our mediator is **stable**.

$$[\sigma]_{\text{degenerate}}^{(g^4)} = \frac{g^4}{\pi m^6} \begin{cases} \frac{\kappa - 2}{64(\kappa - 1)^2} + \frac{(4 - \kappa)\Lambda}{128(\kappa - 1)^3} + \mathcal{O}(\Lambda^2) & 2 - \Lambda \leq \kappa < 2 + \Lambda \\ \frac{(4 - \kappa)\Lambda}{64(\kappa - 1)^3} + \mathcal{O}(\Lambda^2) & \kappa \geq 2 + \Lambda \end{cases}. \quad (9)$$

with $\kappa \equiv s/m^2$. Great!

So, what do you get?

Following this prescription, we get **finite, fixed-order, regulator-independent*** inclusive t -channel cross-section using only the **usual S -Matrix perturbation theory**, even though our mediator is **stable**.

$$[\sigma]_{\text{degenerate}}^{(g^4)} = \frac{g^4}{\pi m^6} \begin{cases} \frac{\kappa - 2}{64(\kappa - 1)^2} + \frac{(4 - \kappa)\Lambda}{128(\kappa - 1)^3} + \mathcal{O}(\Lambda^2) & 2 - \Lambda \leq \kappa < 2 + \Lambda \\ \frac{(4 - \kappa)\Lambda}{64(\kappa - 1)^3} + \mathcal{O}(\Lambda^2) & \kappa \geq 2 + \Lambda \end{cases} . \quad (9)$$

with $\kappa \equiv s/m^2$. Great! But there is a problem...

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

This is not necessarily a problem, since **individual topologies are not observable**...

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

This is not necessarily a problem, since **individual topologies are not observable**...but including the s -channel and $s - t$ channel interferences do not resolve this issue for sufficiently large s .

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

This is not necessarily a problem, since **individual topologies are not observable**...but including the s -channel and $s - t$ channel interferences do not resolve this issue for sufficiently large s .

Beyond our scope to fully resolve this, but we have two speculations:

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

This is not necessarily a problem, since **individual topologies are not observable**...but including the s -channel and $s - t$ channel interferences do not resolve this issue for sufficiently large s .

Beyond our scope to fully resolve this, but we have two speculations:

- **Additional degeneracies** not-yet accounted for, perhaps to do with modifying the **initial state**?

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

This is not necessarily a problem, since **individual topologies are not observable**...but including the s -channel and $s - t$ channel interferences do not resolve this issue for sufficiently large s .

Beyond our scope to fully resolve this, but we have two speculations:

- **Additional degeneracies** not-yet accounted for, perhaps to do with modifying the **initial state**? $\implies$ More in line with existing work ([Lee and Nauenberg, 1964](#), [Kinoshita, 1962](#), [Lavelle and McMullan, 2006](#), [Frye et al., 2019](#)), but harder to connect to collider observables;
- Breakdown of a fixed-order approach altogether, perhaps some sort of **ladder-resummation** required?

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

This is not necessarily a problem, since **individual topologies are not observable**...but including the s -channel and $s - t$ channel interferences do not resolve this issue for sufficiently large s .

Beyond our scope to fully resolve this, but we have two speculations:

- **Additional degeneracies** not-yet accounted for, perhaps to do with modifying the **initial state**? $\implies$ More in line with existing work ([Lee and Nauenberg, 1964](#), [Kinoshita, 1962](#), [Lavelle and McMullan, 2006](#), [Frye et al., 2019](#)), but harder to connect to collider observables;
- Breakdown of a fixed-order approach altogether, perhaps some sort of **ladder-resummation** required? $\implies$ particle decays are intrinsically non-perturbative, but is difficult to actually compute and goes against our fixed-order approach.

A Negative Cross-Section?

Even though it's finite for all $s > m^2$, our inclusive cross-section goes **negative** for $s \gtrsim 4m^2$.

This is not necessarily a problem, since **individual topologies are not observable**...but including the s -channel and $s - t$ channel interferences do not resolve this issue for sufficiently large s .

Beyond our scope to fully resolve this, but we have two speculations:

- **Additional degeneracies** not-yet accounted for, perhaps to do with modifying the **initial state**? $\implies$ More in line with existing work ([Lee and Nauenberg, 1964](#), [Kinoshita, 1962](#), [Lavelle and McMullan, 2006](#), [Frye et al., 2019](#)), but harder to connect to collider observables;
- Breakdown of a fixed-order approach altogether, perhaps some sort of **ladder-resummation** required? $\implies$ particle decays are intrinsically non-perturbative, but is difficult to actually compute and goes against our fixed-order approach.

The KLN theorem guarantees finiteness, but not positivity.

How'd we do?

Strengths

We constructed a t -channel inclusive cross-section that is:

- **finite** and **non-zero**;
- applicable with a **stable** mediator;
- completely **first-principles** & **generalizable** to arbitrary divergent processes;
- computable within the **usual perturbation theory** approach;
- and (preliminarily) quite relevant for **finite temperature** ... $\implies$ see Lachlan's talk next!

Weaknesses

More work needs to be done to:

- connect our schematic picture of degenerate physical configurations to a more **concrete collider prediction**;
- and resolve the issue of **negativity**.

Outlook & Conclusions

We studied an application of the KLN theorem (Lee and Nauenberg, 1964, Kinoshita, 1962) as formulated in (Frye et al., 2019) to t -channel divergences, and:

- explored the connection with unitarity cuts and analytic structure of the box diagram;
- developed mathematical prescriptions for handling ill-defined distributional products, regulator-dependence and boundary contributions;
- constructed a finite, fixed-order inclusive t -channel cross-section (with a stable mediator), and took some steps towards a physical interpretation in terms of collider observables.

There are obviously many open areas for future work, including:

- resolving the issue of negativity & physical interpretation;
- formulating our computations more rigorously in the language of Picard-Lefschetz theory and the multivariate residue calculus of Leray;
- uplifting our results to a finite-temperature cosmological setting $\implies$ see Lachlan E. Tobin's talk next!

References I

- Samuel Abreu, Ruth Britto, Claude Duhr, and Einar Gardi. Cuts from residues: the one-loop case. *JHEP*, 06:114, 2017. doi: 10.1007/JHEP06(2017)114.
- Kento Asai, Nagisa Hiroshima, Joe Sato, Ryusei Sato, and Masaki J. S. Yang. Analytic prescription for t-channel singularities. *Phys. Rev. D*, 112(7):076020, 2025. doi: 10.1103/wkq9-52wv.
- Mathias Becker, Emanuele Copello, Julia Harz, and Carlos Tamarit. Dark matter freeze-in from non-equilibrium QFT: towards a consistent treatment of thermal effects. *JCAP*, 03:071, 2025. doi: 10.1088/1475-7516/2025/03/071.
- Martin Beneke, Francesco Dighera, and Andrzej Hryczuk. Relic density computations at NLO: infrared finiteness and thermal correction. *JHEP*, 10:045, 2014. doi: 10.1007/JHEP10(2014)045. [Erratum: JHEP 07, 106 (2016)].
- Daniel Carney, Laurent Chaurette, Dominik Neuenfeld, and Gordon Semenoff. On the need for soft dressing. *JHEP*, 09:121, 2018. doi: 10.1007/JHEP09(2018)121.
- H. F. Contopanagos and Martin B. Einhorn. Theory of the asymptotic S matrix for massless particles. *Phys. Rev. D*, 45:1291–1321, 1992. doi: 10.1103/PhysRevD.45.1291.

References II

- Rupert Coy and Michael A. Schmidt. Freeze-in and freeze-out of sterile neutrino dark matter. *JCAP*, 08:070, 2022. doi: 10.1088/1475-7516/2022/08/070. [Erratum: JCAP 04, E01 (2023)].
- C. Dams and R. Kleiss. Muon colliders, Monte Carlo and gauge invariance. *Eur. Phys. J. C*, 36:177–181, 2004. doi: 10.1140/epjc/s2004-01892-3.
- Chris Dams and Ronald Kleiss. Singular cross-sections in muon colliders. *Eur. Phys. J. C*, 29: 11–17, 2003. doi: 10.1140/epjc/s2003-01221-6.
- Darren A. Forde and Adrian Signer. Infrared finite amplitudes for massless gauge theories. *Nucl. Phys. B*, 684:125–161, 2004. doi: 10.1016/j.nuclphysb.2004.02.024.
- Christopher Frye, Holmfridur Hannesdottir, Nisarga Paul, Matthew D. Schwartz, and Kai Yan. Infrared Finiteness and Forward Scattering. *Phys. Rev. D*, 99(5):056015, 2019. doi: 10.1103/PhysRevD.99.056015.
- Björn Garbrecht, Frank Glowna, and Matti Herranen. Right-Handed Neutrino Production at Finite Temperature: Radiative Corrections, Soft and Collinear Divergences. *JHEP*, 04:099, 2013a. doi: 10.1007/JHEP04(2013)099.

References III

- Björn Garbrecht, Frank Glowna, and Pedro Schwaller. Scattering Rates For Leptogenesis: Damping of Lepton Flavour Coherence and Production of Singlet Neutrinos. *Nucl. Phys. B*, 877:1–35, 2013b. doi: 10.1016/j.nuclphysb.2013.08.020.
- Giorgio Giavarini and Giuseppe Marchesini. IR Finite S Matrix in the QCD Coherent State Basis. *Nucl. Phys. B*, 296:546–556, 1988. doi: 10.1016/0550-3213(88)90031-4.
- I. F. Ginzburg. Initial particle instability in muon collisions. *Nucl. Phys. B Proc. Suppl.*, 51: 85–89, 1996. doi: 10.1016/0920-5632(96)00418-5.
- Riccardo Gonzo, Tristan Mc Loughlin, Diego Medrano, and Anne Spiering. Asymptotic charges and coherent states in QCD. *Phys. Rev. D*, 104(2):025019, 2021. doi: 10.1103/PhysRevD.104.025019.
- Bohdan Grzadkowski, Michał Iglicki, and Stanisław Mrówczyński. t-channel singularities in cosmology and particle physics. *Nucl. Phys. B*, 984:115967, 2022. doi: 10.1016/j.nuclphysb.2022.115967.
- Holmfridur Hannesdottir and Matthew D. Schwartz. Finite S matrix. *Phys. Rev. D*, 107(2): L021701, 2023. doi: 10.1103/PhysRevD.107.L021701.

References IV

- Hayato Hirai and Sotaro Sugishita. IR finite S-matrix by gauge invariant dressed states. *JHEP*, 02:025, 2021. doi: 10.1007/JHEP02(2021)025.
- Dimitrios Karamitros and Apostolos Pilaftsis. Toward a localized S-matrix theory. *Phys. Rev. D*, 108(3):036007, 2023. doi: 10.1103/PhysRevD.108.036007.
- T. Kinoshita. Mass singularities of Feynman amplitudes. *J. Math. Phys.*, 3:650–677, 1962. doi: 10.1063/1.1724268.
- P. P. Kulish and L. D. Faddeev. Asymptotic conditions and infrared divergences in quantum electrodynamics. *Theor. Math. Phys.*, 4:745, 1970. doi: 10.1007/BF01066485.
- Martin Lavelle and David McMullan. Collinearity, convergence and cancelling infrared divergences. *JHEP*, 03:026, 2006. doi: 10.1088/1126-6708/2006/03/026.
- T. D. Lee and M. Nauenberg. Degenerate Systems and Mass Singularities. *Phys. Rev.*, 133: B1549–B1562, 1964. doi: 10.1103/PhysRev.133.B1549.
- K. Melnikov and V. G. Serbo. New type of beam size effect and the W boson production at $\mu^+\mu^-$ colliders. *Phys. Rev. Lett.*, 76:3263–3266, 1996. doi: 10.1103/PhysRevLett.76.3263.

References V

- K. Melnikov and V. G. Serbo. Processes with the T channel singularity in the physical region: Finite beam sizes make cross-sections finite. *Nucl. Phys. B*, 483:67–82, 1997. doi: 10.1016/S0550-3213(96)00558-5. [Erratum: Nucl.Phys.B 662, 409 (2003)].
- M. Nowakowski and A. Pilaftsis. On gauge invariance of Breit-Wigner propagators. *Z. Phys. C*, 60:121–126, 1993. doi: 10.1007/BF01650437.
- Ronald F. Peierls. Possible Mechanism for the Pion-Nucleon Second Resonance. *Phys. Rev. Lett.*, 6:641–643, 1961. doi: 10.1103/PhysRevLett.6.641.
- Gerard 't Hooft and M. J. G. Veltman. DIAGRAMMAR. *NATO Sci. Ser. B*, 4:177–322, 1974. doi: 10.1007/978-1-4684-2826-1_5.
- M. J. G. Veltman. *Diagrammatica: The Path to Feynman Rules*, volume 4. Cambridge University Press, 5 2012. ISBN 978-1-139-24339-1, 978-0-521-45692-0.