

Rapidity spectra from analytical solutions of Relativistic Hydrodynamics

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Hydrodynamics and spectra

- Landau first formulated ideal hydrodynamics in a non-boost invariant framework. [Izv. Akad. Nauk Ser. Fiz. 17, 51 \(1953\)](#)
- Within the Bjorken framework, a boost-invariant scenario produces a plateau-like rapidity distribution. [PRD 27, 140–151 \(1983\)](#)
- C.Y Wong modified the Landau solution, introducing beam-rapidity. [PRC 78, 054902 \(2008\)](#)
- s/n conserved for an ideal evolution and rapidity spectra dN/dy is proportional to entropy production.
- These formulations have been performed with an equation of state in the conformal limit $P = \epsilon/3$.

Non-conformal Landau Hydrodynamics

- The boost invariance is broken as, $y = \ln f + \eta_s$.
- With a generic *e.o.s*, $p = c_s^2 \epsilon$ and shear viscosity, the hydrodynamic equations can be exactly solved for 1d case.
- In the ideal limit, the solution is,

$$\epsilon_{id} = \epsilon_0 \exp \left[-\frac{c_+^2}{1 + c_s^2} (y_+ + y_-) + \frac{c_+ c_-}{2c_s^2} \sqrt{y_+ y_-} \right]$$

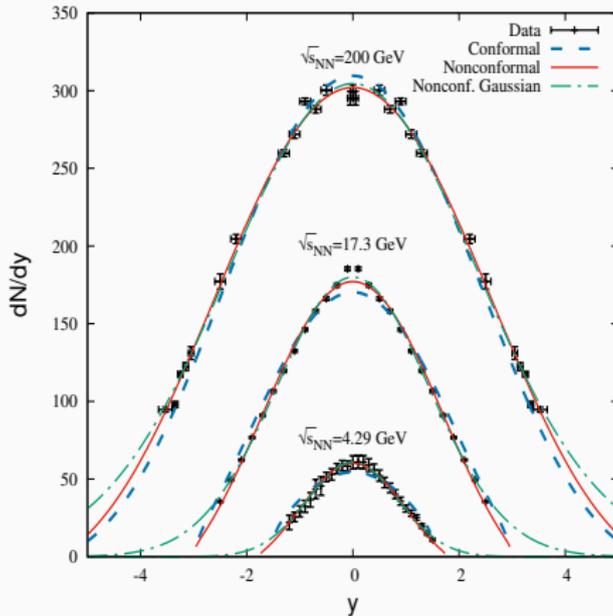
- $y_{\pm} \equiv \ln(t_{\pm}/\Delta)$ and $c_{\pm} = 1 \pm c_s^2$
- The rapidity spectra for ideal non-conformal Landau,

$$\frac{dN}{dy} \sim \exp \left(\frac{c_-}{2c_s^2} \sqrt{y_b'^2 - y^2} \right).$$

- $y_b' \equiv \frac{1}{2} \ln[c_+/(4c_s^2)] + y_b$ and $y_b \equiv \ln(\sqrt{s_{NN}}/m_p)$ is the beam rapidity.

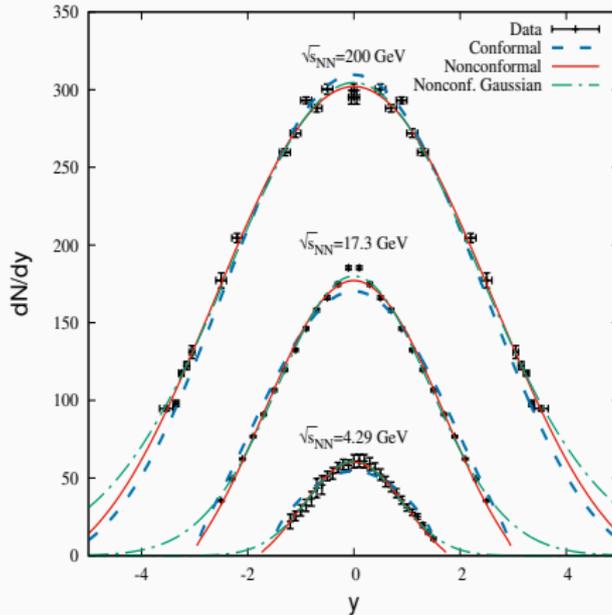
Data vs model for dN/dy

- Non-conformal solutions have better agreement with the data
Jaiswal et.al. PRC 102, 014912.



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- Can we generate dN/dy for 1 + 3D evolution analytically?

Framework

- Conservation of energy-momentum tensor. $\partial_\mu T^{\mu\nu} = 0$.
- For a non-dissipative system,

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu},$$

- u^μ is the fluid four-velocity and $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.
- Use $Ts = \varepsilon + p$ and $dp = s dT$.
- Equations orthogonal to u^μ : $u^\mu \partial_\mu (Tu^\nu) - \partial^\nu T = 0$

$$\frac{\partial}{\partial t}(T\gamma\vec{v}) + \vec{\nabla}(T\gamma) = \vec{v} \times \vec{\nabla} \times (T\gamma\vec{v}).$$

- Along the u^μ : the entropy conservation $\partial_\mu (su^\mu) = 0$

$$\frac{\partial}{\partial t}(s\gamma) + \vec{\nabla} \cdot (s\gamma\vec{v}) = 0.$$

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$$\frac{\partial}{\partial t}(T\gamma\vec{v}) + \vec{\nabla}(T\gamma) = \vec{v} \times \vec{\nabla} \times (T\gamma\vec{v}). \quad \longrightarrow \quad \text{Eq.(T)}$$

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Spherically symmetric case (t, r, θ, ϕ)

- The hydrodynamic variables depend only on time t and radial distance r from the origin.
- With fluid velocity vector $\vec{v} = (v_r, 0, 0)$, Eq.(T) and Eq.(s):

$$\frac{\partial}{\partial t}(T\gamma v_r) + \frac{\partial}{\partial r}(T\gamma) = 0 \quad ; \quad \frac{\partial}{\partial t}(s\gamma) + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 s\gamma v_r) = 0.$$

- With a Hubble-like fluid flow velocity $v_r = r/t$:

$$r \frac{\partial T}{\partial t} + t \frac{\partial T}{\partial r} = 0 \quad ; \quad t \frac{\partial s}{\partial t} + r \frac{\partial s}{\partial r} + 3s = 0.$$

Solving equations:

- System of equations solved with e.o.s $p = c_s^2 \epsilon$, c_s^2 is constant.
- For a barotropic system $s \propto T^{1/c_s^2}$. The temperature equation becomes:

$$t \frac{\partial T}{\partial t} + r \frac{\partial T}{\partial r} + 3 c_s^2 T = 0.$$

- Solving above equation with method of characteristics:

$$T(r, t) \sim t^{-3c_s^2} f\left(\frac{r}{t}\right)$$

- $f(r/t)$ is fixed from the entropy equation.

$$f(r/t) \sim \left(\frac{t^2}{t^2 - r^2}\right)^{3c_s^2/2}$$

Final solutions for spherical geometry

- With the flow profile $v_r = \frac{r}{t}$, temperature and entropy evolves as,

$$T(r, t) = T_0 \left[\frac{t_0^2 - r_0^2}{t^2 - r^2} \right]^{3c_s^2/2}$$

$$s(r, t) = s_0 \left[\frac{t_0^2 - r_0^2}{t^2 - r^2} \right]^{3/2}$$

- T_0 and s_0 are the initial temperature and entropy density, respectively, at r_0 and t_0 .
- Solutions are in agreement with earlier results [PLB 565, 107–115 \(2003\)](#) .

Cylindrical geometry (t, ρ, ϕ, z)

- With the cylindrical symmetry, $\bar{v} = (v_\rho, 0, v_z)$.
- Separating Eq.(T) component wise: for the radial part we get,

$$\frac{\partial}{\partial t} [T\gamma v_\rho] + \frac{\partial}{\partial \rho} (T\gamma) - v_z \left[\frac{\partial}{\partial \rho} (T\gamma v_z) - \frac{\partial}{\partial z} (T\gamma v_\rho) \right] = 0$$

- Similarly for the \hat{z} part,

$$\frac{\partial}{\partial t} [T\gamma v_z] + \frac{\partial}{\partial z} [T\gamma] - v_\rho \left[\frac{\partial}{\partial z} (T\gamma v_\rho) - \frac{\partial}{\partial \rho} (T\gamma v_z) \right] = 0$$

- The entropy equation Eq.(s):

$$\frac{\partial}{\partial t} (s\gamma) + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho s\gamma v_\rho) + \frac{\partial}{\partial z} (s\gamma v_z) = 0$$

Flow profile

- We consider boost-invariance for the longitudinal expansion, $v_z = \frac{z}{t}$.
- Implies the expansion geometry to be identical in all longitudinally boosted frames.
- For the radial part we consider, $v_\rho = \frac{\rho}{t}$.
- First, we shall solve only the $z = 0$ part and then generalize the solution to the $z \neq 0$ plane.

Solutions for $z = 0$:

- The temperature equations:

$$\frac{\partial}{\partial t}(T\gamma v_\rho) + \frac{\partial}{\partial \rho}(T\gamma) = 0 \quad ; \quad \frac{\partial T}{\partial z} = 0$$

- The entropy equation:

$$t \frac{\partial s}{\partial t} + \rho \frac{\partial s}{\partial \rho} + 3s = 0.$$

- The temperature equation simplifies to,

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- Similar equations like the spherical case, solutions:

$$T(\rho, t) = T_0 \left[\frac{t_0^2 - \rho_0^2}{t^2 - \rho^2} \right]^{3c_s^2/2}$$

$$s(\rho, t) = s_0 \left[\frac{t_0^2 - \rho_0^2}{t^2 - \rho^2} \right]^{3/2}$$

General solutions for $z \neq 0$:

- For $z \neq 0$ the generalized form will be,

$$T(\rho, z, t) = T_0 \left[\frac{t_0^2 - \rho_0^2 - z_0^2}{t^2 - \rho^2 - z^2} \right]^{3c_s^2/2},$$

$$s(\rho, z, t) = s_0 \left[\frac{t_0^2 - \rho_0^2 - z_0^2}{t^2 - \rho^2 - z^2} \right]^{3/2},$$

- These are scaling solutions, with $\tau_3 = \sqrt{t^2 - r^2} = \sqrt{t^2 - \rho^2 - z^2}$.
- The initial conditions corresponds to $T = T_0$, $s = s_0$ at initial proper time $\tau_3(t_0, r_0)$ and $\tau_3(t_0, \rho_0, z_0)$.
- Similar geometries were earlier solved **numerically**, and without a specific flow profile [Nucl.Phys.A 407 \(1983\) 541-570](#).
- These analytic solutions are elegant due to simpler structure, offer a valuable baseline for further numerical studies.

- The Cooper-Frye prescription

$$E \frac{dN}{d^3p} = \frac{g}{(2\pi)^3} \int p_\mu d\Sigma^\mu f(x, p),$$

- Σ^μ is the freeze-out hypersurface for spherical and cylindrical expansion.
- A constant temperature freeze-out, $T = T_{3f} \rightarrow \tau_3 = \tau_{3f}$.

$$T(\tau_{3f}) = T_0 \left(\frac{\tau_{30}}{\tau_{3f}} \right)^{3c_s^2}.$$

Spherical case

- The surface element of the hypersurface can be obtained as,

$$d\Sigma^\mu = \left(\frac{dr}{d\zeta}, \frac{dt}{d\zeta} \sin \theta \cos \phi, \frac{dt}{d\zeta} \sin \theta \sin \phi, \frac{dt}{d\zeta} \cos \theta \right) r^2 \sin \theta d\zeta d\theta d\phi$$

- $r = r(\zeta)$ with $0 < \zeta < 1$, $r(0) = 0$ and $r(1) = R$. R is freeze-out radius.
- The flow velocity profile is,

$$u^\mu = \gamma(\zeta) [1, v_r(\zeta) \sin \theta \cos \phi, v_r(\zeta) \sin \theta \sin \phi, v_r(\zeta) \cos \theta],$$

- In this case, a Hubble-like fluid flow velocity, $v_r(\zeta) = \frac{r(\zeta)}{t}$.
- Assuming spherical symmetry in momentum space $\vec{p} = (0, 0, p)$,

$$u \cdot p = \gamma(\zeta) E - \gamma(\zeta) v_r(\zeta) p \cos \theta,$$

$$p \cdot d\Sigma = \left(E \frac{dr}{d\zeta} - p \cos \theta \frac{dt}{d\zeta} \right) r^2(\zeta) \sin \theta d\zeta d\theta d\phi.$$

$$E \frac{dN}{d^3p} \approx \sum_{n=1}^{\infty} \epsilon_n \int_0^1 d\zeta r^2 \int_0^\pi d\theta \sin\theta \left(E \frac{dr}{d\zeta} - p \cos\theta \frac{dt}{d\zeta} \right) e^{-n\beta(u \cdot p)}$$

- A constant temperature freezeout hypersurface leads to,

$$\frac{dt}{d\zeta} = \frac{r}{t} \frac{dr}{d\zeta}.$$

- The final form of the particle momentum distribution,

$$E \frac{dN}{d^3p} \approx R^3 \sum_{n=1}^{\infty} \epsilon_n \int_0^1 e^{-n\beta E \sqrt{\nu^2 + \chi^2} / \nu} \left[E \frac{\sinh(na\chi)}{na\chi} + T\nu \left(\frac{\sinh(na\chi) - na\chi \cosh(na\chi)}{n^2 a\chi \sqrt{\nu^2 + \chi^2}} \right) \right] \chi^2 d\chi,$$

- $\chi \equiv r/R$, $\nu \equiv \tau_{3f}/R$, and $a \equiv p/(T\nu)$.

Cylindrical case

- The hypersurface is,

$$d\Sigma^\mu = \left(\frac{d\rho}{d\zeta} \cosh \eta_s, \frac{d\tau}{d\zeta} \cos \phi, \frac{d\tau}{d\zeta} \sin \phi, \frac{d\rho}{d\zeta} \sinh \eta_s \right) \tau(\zeta) \rho(\zeta) d\zeta d\phi$$

- $\rho = \sqrt{x^2 + y^2}$ and $\tau = \sqrt{t^2 - z^2}$.
- $0 < \zeta < 1$ such that $\rho(0) = 0$ and $\rho(1) = R$
- With longitudinal and transverse fluid flow rapidity, η_f, η_T as

$$\eta_f = \frac{1}{2} \log \left(\frac{1 + v_z}{1 - v_z} \right), \quad \eta_T = \tanh^{-1} \left(\frac{v_\rho}{\sqrt{1 - v_z^2}} \right)$$

- The fluid four-velocity and momentum is given as,

$$u^\mu = (\cosh \eta_T \cosh \eta_f, \sinh \eta_T \cos \phi, \sinh \eta_T \sin \phi, \cosh \eta_T \sinh \eta_f).$$

$$p^\mu = (m_T \cosh y, p_T \cos \phi_p, p_T \sin \phi_p, m_T \sinh y),$$

- For $v_z = z/t$, $\eta_f = \eta_s$.

$$u \cdot p = m_T \cosh(\eta_T) \cosh(y - \eta_s) - p_T \sinh(\eta_T) \cos(\phi - \phi_p)$$

$$p \cdot d\Sigma = \left[m_T \cosh(y - \eta_s) \frac{d\rho}{d\zeta} - p_T \cos(\phi - \phi_p) \frac{d\tau}{d\zeta} \right] \tau(\zeta) \rho(\zeta) d\zeta d\phi$$

- First we integrate over ϕ and η_s .
- With a constant temperature freeze-out, $\tau_{3f}^2 = \tau^2 - \rho^2$:

$$\frac{d\tau}{d\zeta} = \frac{\rho}{\tau} \frac{d\rho}{d\zeta}.$$

- For a Hubble-like transverse flow, $v_\rho = \rho/t$ we have $\rho = \tau_{3f} \sinh \eta_T$, $\tau = \tau_{3f} \cosh \eta_T$

Final expression

$$E \frac{dN}{d^3p} \approx R^3 \sum_{n=1}^{\infty} \epsilon_n \int_0^1 \left[m_T \sqrt{\nu^2 + \chi^2} I_0 \left(\frac{n\beta p_T \chi}{\nu} \right) K_1 \left(n\beta m_T \sqrt{1 + \frac{\chi^2}{\nu^2}} \right) - \chi p_T I_1 \left(\frac{n\beta p_T \chi}{\nu} \right) K_0 \left(n\beta m_T \sqrt{1 + \frac{\chi^2}{\nu^2}} \right) \right] \chi d\chi$$

$\chi \equiv \rho/R$ and $\nu \equiv \tau_{3f}/R$.

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➤ For the spherical geometry,

$$E \frac{dN}{d^3p} \approx R^3 \sum_{n=1}^{\infty} \epsilon_n \int_0^1 e^{-n\beta E \sqrt{\nu^2 + \chi^2}/\nu} \left[E \frac{\sinh(na\chi)}{na\chi} + T\nu \left(\frac{\sinh(na\chi) - na\chi \cosh(na\chi)}{n^2 a\chi \sqrt{\nu^2 + \chi^2}} \right) \right] \chi^2 d\chi$$

with $\chi \equiv r/R$, $\nu \equiv \tau_{3f}/R$, and $a \equiv p/(T\nu)$.

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with $\chi \equiv r/R$, $\nu \equiv \tau_{3f}/R$, and $a \equiv p/(T\nu)$.

- These are exact analytical form of spectra with a constant temperature freeze-out.

Average radial and transverse velocity

- We further calculate average radial and transverse velocity for spherical and cylindrical case.

$$\langle v_{r,T} \rangle = \frac{\int v_{r,\rho} \sqrt{d\Sigma_\mu d\Sigma^\mu}}{\int \sqrt{d\Sigma_\mu d\Sigma^\mu}}.$$

- For $v_r = r/t$,

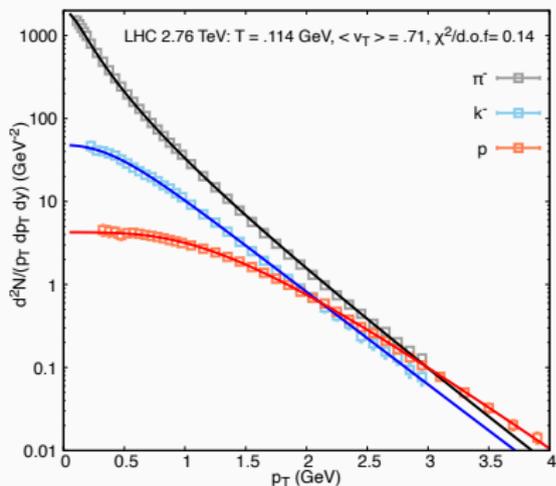
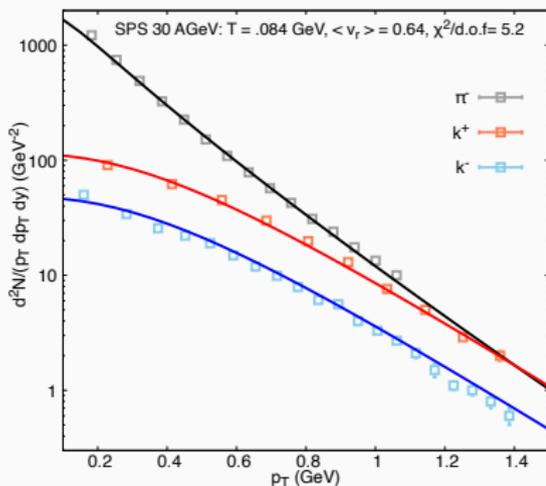
$$\langle v_r \rangle = \frac{1 + \nu^2 \log \left(\frac{\nu^2}{1 + \nu^2} \right)}{\sqrt{1 + \nu^2} - \nu^2 \log \left[\frac{\nu}{\sqrt{1 + \nu^2} - 1} \right]}.$$

- With $v_\rho = \rho/t$,

$$\langle v_T \rangle = \sqrt{1 + \nu^2} - \nu^2 \log \left[\frac{\nu}{\sqrt{1 + \nu^2} - 1} \right].$$

Results

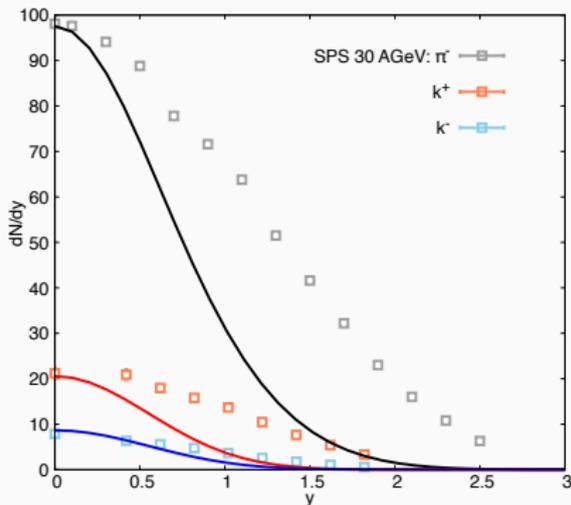
- ▶ We have used the exact analytical form to fit the spectra
- ▶ SPS 30 GeV and LHC 2.76 TeV.



- ▶ Extracted freeze-out T is similar to available results.

Rapidity spectra at SPS

- ⇒ p_T distribution from the spherical case depends on p_T and y via $a = p/(T\nu)$.
 - ⇒ $p = \sqrt{p_T^2 + m_T^2 \sinh^2 y}$.
- Integrating the distribution over p_T leads to the rapidity spectrum dN/dy .
- Disagreement at higher y as p_T fitting is performed for mid-rapidity data only.



Conclusion

- ⇒ We present **Exact** analytical solutions for spherical and cylindrical geometry.
- ⇒ For **first time** these results connect the hydrodynamic solutions to p_T spectra.
- ⇒ For spherical system, we extract the **Gaussian** distribution integrating p_T .
- ⇒ A higher value of $\langle v_T \rangle$ than $\langle v_r \rangle$ indicates later time freeze-out (in unit of τ) in LHC.
- ⇒ Provides a necessary middle ground: exact solvable cases that still give physical spectra.



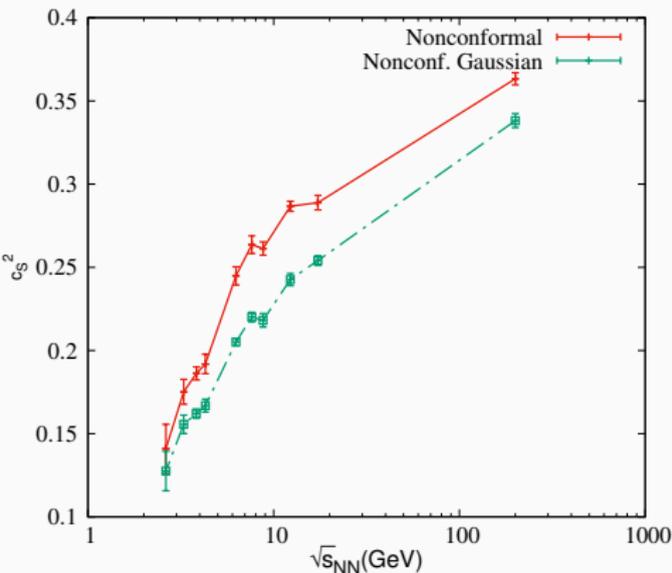
Collaborators:

Amaresh Jaiswal, Mahammad Sabir Ali, Sushant K Singh

References:

[Phys.Rev.C 102, 014912](#) and [Eur.Phys.J.C 85, 30](#)

Backup (c_S^2)



- The extracted value of c_S^2 is seen to increase with energy.
- Gazdzicki et.al proposed a minimum in this variation. [Acta Phys. Polon. B 42, 307 \(2011\)](#)
- We have not obtained a signature for minima, this difference originates from the definition of y'_b .
- We have used a constant velocity of sound to derive the analytical expression, hence the extracted c_S^2 is an approximate time-averaged value for that particular $\sqrt{s_{NN}}$.

Average v

