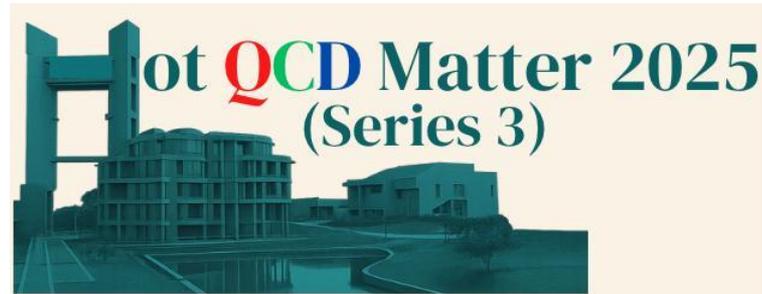# Aspects of spin hydrodynamic framework: spin chemical potential as the leading order term in the gradient expansion

Arpan Das

Birla Institute of Technology and Science Pilani (BITS Pilani), Pilani Campus

ot QCD Matter 2025 (Series 3)

Journal References: *Phys.Rev.D* 111 (2025) 7, 074037

Collaborator: Sourav Dey (NISER)

PIONEERING EDUCATION PARADIGMS

Non-central heavy ion collision: angular momentum, vorticity and spin:

Non-central heavy ion collisions can create QCD medium with a large angular momenta
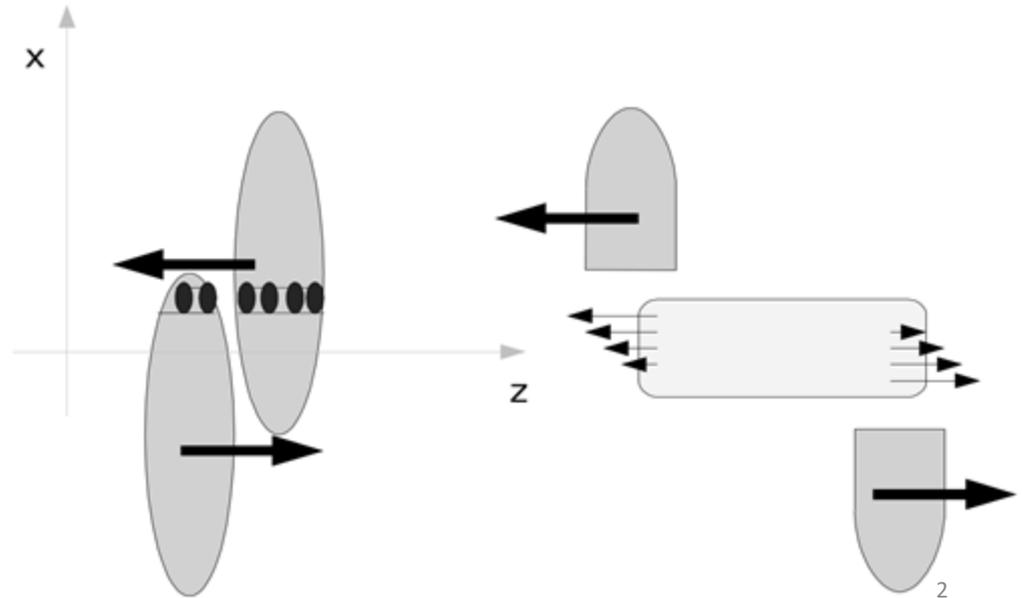
✦ Initial large orbital angular momenta

✦ Inhomogeneity of density profile in transverse plane.

F. Becattini, et.al., 0711.1253

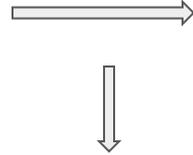F. Becattini, F. Piccinini,
J. Rizzo, PRC 77 (2008) 024906

Vorticity generation:

$$\omega_y(t=0) = -\frac{1}{2}\frac{\partial v_{z0}}{\partial x}$$

Observation of the spin polarization of hadrons and the "*Spin-sign problem*" $\implies$ New challenges in the modeling of the "*spin dynamics*" in an evolving QCD medium.

STAR collaboration, Nature 548, 62-65 (2017); Becattini, F., et. al. PRC 95, 054902 (2017)

Z.-T. Liang, et. al., PRL 94 (2005) 102301; PLB 629 (2005) 20–26 J.-H. Gao, et al., PRC 77 (2008) 044902; S.-W. Chen, et. al. , Front. Phys. China 4 (2009) 509–516 B. Betz, M. Gyulassy, G. Torrieri, PRC 76 (2007) 044901, F.Becattini, et. al. Ann.Phys.323,2452(2008)
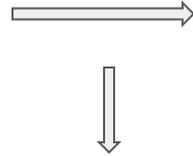
``Spin hydrodynamics frameworks''

N. Weickgenannt, et. al., Phys.Rev.D 100 (2019) 5, 056018; Phys.Rev.Lett. 127 (2021) 5, 052301 ; Phys.Rev.D 104 (2021) 1, 016022; W. Florkowski, et.al., Prog.Part.Nucl.Phys. 108 (2019) 103709; Phys.Rev.C 98 (2018) 4, 044906; S. Bhadury, et.al., Phys.Lett.B 814 (2021) 136096 ; Phys.Rev.D 103 (2021) 1, 014030; K. Hattori, et.al. PLB 795 (2019) 100-106; K. Fukushima, S. Pu, Phys.Lett.B 817 (2021) 136346, Golam Sarwar et. al. Phys.Rev.D 107 (2023) 5, 054031, R. Biswas, et.al. Phys.Rev.D 107 (2023) 9, 094022......

Macroscopic conservation equations: $\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu J^\mu = 0,$

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$\Longrightarrow$

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$\Downarrow$

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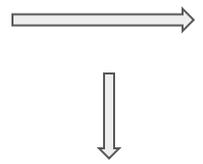
Macroscopic conservation equations: $\quad \partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu J^\mu = 0, \quad \partial_\lambda J^{\lambda\mu\nu} = 0,$

Conservation of the total angular momentum $\Longrightarrow$ Evolution of six component anti-symmetric tensor: **Spin chemical potential**

Generalized thermodynamic relations:
$$\varepsilon + P = Ts + \mu n + \omega_{\alpha\beta} S^{\alpha\beta},$$
$$d\varepsilon = Tds + \mu dn + \omega_{\alpha\beta} dS^{\alpha\beta},$$
$$dP = sdT + nd\mu + S^{\alpha\beta} d\omega_{\alpha\beta}.$$

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$$dP = sdT + nd\mu + S^{\alpha\beta} d\omega_{\alpha\beta}.$$

What about the consistency of the thermodynamic relations ??

Spin hydrodynamics and pseudo gauge

Macroscopic conserved quantity: $\partial_\mu T^{\mu\nu} = 0, \ \partial_\mu N^\mu = 0, \ \partial_\mu J^{\mu\alpha\beta} = 0$

Total angular momenta: $J^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + S^{\mu\alpha\beta}$.

Energy-momentum tensor calculated using Noether theorem is not unique

Pseudo-gauge transformation:

$$T'^{\mu\nu} = T^{\mu\nu} + \partial_\lambda \Psi^{\nu\mu\lambda}, \ \ \Psi^{\nu\mu\lambda} = -\Psi^{\nu\lambda\mu}, \ \ \partial_\mu T^{\mu\nu} = 0$$
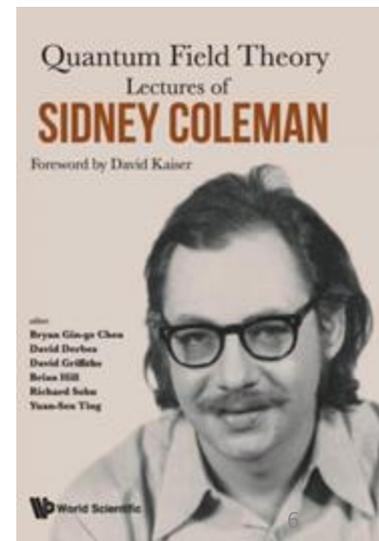
$$\hat{T}'^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\hat{\Phi}^{\lambda,\mu\nu} - \hat{\Phi}^{\mu,\lambda\nu} - \hat{\Phi}^{\nu,\lambda\mu} \right)$$

$$\hat{S}'^{\lambda,\mu\nu} = \hat{S}^{\lambda,\mu\nu} - \hat{\Phi}^{\lambda,\mu\nu}$$

42 different energy-momentum tensors that occur in the literature

Canonical framework, Belinfante-Rosenfeld framework (BR), de Groot-van Leeuwen-van Weert framework (GLW), Hilgevoord-Wouthuysen framework (HW)....

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Bryan Gin-ge Chen
David Derbes
David Griffiths
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Richard Sohn
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Spin hydrodynamics and pseudo gauge

Covariant entropy current ansatz: $\quad \mathcal{S}^\mu = T^{\mu\nu}\beta_\nu + P\beta^\mu - \alpha J^\mu - \beta\omega_{\alpha\beta}S^{\mu\alpha\beta}$.

Under the pseudo-gauge transformation: $\quad \mathcal{S}'^\mu = \mathcal{S}^\mu + \partial_\lambda A^{\lambda\mu}$ $\quad$ Entropy Gauge transformation

Under the entropy gauge transformation entropy production is unaffected

F. Becattini, A. Daher, Xin Li-Sheng, PLB 850 (2024) 138533

Thermodynamics quantities: $\quad p' = p + Tu_\mu\partial_\lambda A^{\lambda\mu}$

Thermodynamic relations: $\quad \dfrac{\partial p'}{\partial T}\Big|_{\mu,\omega} = s' + u_\mu T \dfrac{\partial}{\partial T}\partial_\lambda A^{\lambda\mu}\Big|_{\mu,\omega}$

Pseudo-gauge dependence of the spin polarization observable !!!

M. Buzzegoli, PHYSICAL REVIEW C 105, 044907 (2022)

Spin hydrodynamics and pseudo gauge

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Spin kinetic theory results to write down thermodynamic relations:

$$H^\mu = -\int \mathrm{dP}\,\mathrm{dS}\,p^\mu[f^+_{s,\mathrm{eq}}(\ln f^+_{s,\mathrm{eq}} - 1)$$

$$+ f^-_{s,\mathrm{eq}}(\ln f^-_{s,\mathrm{eq}} - 1)].$$

S Bhadury, W. Florkowski, A. Jaiswal,, A. Kumar, and R. Ryblewski, Phys. Rev. D 103, 014030 (2021)

$$H^\mu = \beta_\alpha T^{\mu\alpha}_{\mathrm{eq}} - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\mathrm{eq}} - \xi N^\mu_{\mathrm{eq}} + P\beta^\mu$$

Different gradient counting schemes

Gradient ordering:

$$T^{\mu\nu}_{(0)} \quad \text{or} \quad T, \mu, u^{\mu} \to \mathcal{O}(1) \qquad\qquad T^{\mu\nu}_{(1)} \to \mathcal{O}(\partial)$$

Gradient ordering of spin density and spin chemical potential

$$\omega^{\mu\nu} \to \mathcal{O}(\partial) \qquad S^{\alpha\beta} \to \mathcal{O}(1), \quad S^{\mu\alpha\beta}_{\mathrm{can}(1)} \to \mathcal{O}(\partial)$$

K. Hattori, M. Hongo, X –G. Huang, M. Matsuo, H. Taya, PLB 795 (2019) 100-106; K. Fukushima, S. Pu, Phys.Lett.B 817 (2021) 136346

Abhishek Tiwari, et.al., *Phys.Rev.D* 112 (2025) 3, 036014

$$\omega^{\mu\nu} \to \mathcal{O}(1) \qquad S^{\alpha\beta} \to \mathcal{O}(1), \quad S^{\mu\alpha\beta}_{\mathrm{can}(1)} \to \mathcal{O}(\partial)$$

D. She, A. Huang, D Hou, J. Liao, 2105.04060

Spin Hydrodynamics: Entropy current

Entropy current :

$$\mathcal{S}^\mu = T^{\mu\nu}\beta_\nu + P\beta^\mu - \alpha J^\mu - \beta\omega_{\alpha\beta}S^{\mu\alpha\beta}$$

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} \qquad T^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu}$$

$$J^\mu = J^\mu_{(0)} + J^\mu_{(1)}, \quad J^\mu_{(0)} = nu^\mu$$

$$S^{\mu\alpha\beta} = S^{\mu\alpha\beta}_{(0)} + S^{\mu\alpha\beta}_{(1)}, \quad S^{\mu\alpha\beta}_{(0)} = u^\mu S^{\alpha\beta}$$

K. Hattori, et.al. PLB 795 (2019) 100-106; K. Fukushima, S. Pu, Phys.Lett.B 817 (2021) 136346

A. D. Gallegos, et. al. JHEP 05 (2023) 139.

Evolution of the entropy current :

$$\partial_\mu \mathcal{S}^\mu_{(0)} = 2\beta\omega_{\alpha\beta}T^{[\alpha\beta]}_{(1)} \qquad\qquad \partial_\lambda S^{\lambda\mu\nu} = -T^{\mu\nu} + T^{\nu\mu} = -2T^{[\mu\nu]}$$

$$\partial_\mu \mathcal{S}^\mu = T^{\{\mu\nu\}}_{(1)}\partial_{\{\mu}\beta_{\nu\}} + T^{[\mu\nu]}_{(1)}\partial_{[\mu}\beta_{\nu]} - J^\mu_{(1)}\partial_\mu\alpha - S^{\mu\alpha\beta}_{(1)}\partial_\mu\Omega_{\alpha\beta} + 2\Omega_{\alpha\beta}T^{[\alpha\beta]}_{(1)}.$$

Entropy current analysis:

$$\mathcal{S}^\mu = T^{\mu\nu}\beta_\nu + P\beta^\mu - \alpha J^\mu - \beta\omega_{\alpha\beta}S^{\mu\alpha\beta}.$$

IRRep of dissipative currents:

$$T_{(1)}^{\{\alpha\beta\}} = h^\alpha u^\beta + h^\beta u^\alpha + \pi^{\alpha\beta} + \Pi\Delta^{\alpha\beta},$$

$$S_{(1)}^{\mu\alpha\beta} = 2u^{[\alpha}\Delta^{\mu\beta]}\Phi + 2u^{[\alpha}\tau_{(s)}^{\mu\beta]} + 2u^{[\alpha}\tau_{(a)}^{\mu\beta]} + \Theta^{\mu\alpha\beta}$$

K. Hattori, et.al. PLB 795 (2019) 100-106; K. Fukushima, S. Pu, Phys.Lett.B 817 (2021) 136346

R. Biswas, et.al. Phys.Rev.D 108 (2023) 1, 014024
D. She, et.al., 2105.04060

Entropy production in dissipative systems:

Using On-shell equations or spin hydrodynamic equations.

$$\partial_\mu\mathcal{S}^\mu = h^\mu\frac{S^{\alpha\beta}}{\varepsilon + P}\nabla_\mu\Omega_{\alpha\beta} - \mathcal{J}^\mu\nabla_\mu\alpha + \beta\pi^{\mu\nu}\sigma_{\mu\nu} + \beta\Pi\theta$$

$$- 2\Phi u^\alpha\nabla^\beta(\beta\omega_{\alpha\beta}) - 2\tau_{(s)}^{\mu\beta}u^\alpha\Delta_{\mu\beta}^{\gamma\rho}\nabla_\gamma(\beta\omega_{\alpha\rho}) - 2\tau_{(a)}^{\mu\beta}u^\alpha\Delta_{[\mu\beta]}^{[\gamma\rho]}\nabla_\gamma(\beta\omega_{\alpha\rho})$$

$$- \Theta_{\mu\alpha\beta}\Delta^{\alpha\delta}\Delta^{\beta\rho}\Delta^{\mu\gamma}\nabla_\gamma(\beta\omega_{\delta\rho})$$

$$\mathcal{J}^\mu = J_{(1)}^\mu - \frac{n}{\varepsilon + P}h^\mu.$$

Entropy production

$$\partial_\mu\mathcal{S}^\mu \geq 0$$

$$\Pi = \zeta\theta,$$ Bulk viscous term

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$ Shear viscous term

Entropy current analysis:

$$\partial_\mu \mathcal{S}^\mu = -\beta h^\mu X_\mu - \beta \mathcal{J}^\mu Y_\mu$$

$$h^\mu = aX^\mu + bY^\mu$$

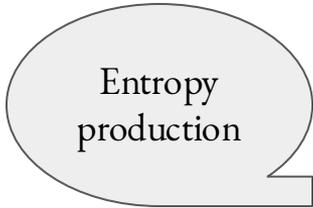$$\mathcal{J}^\mu = cX^\mu + dY^\mu$$

$$\partial_\mu \mathcal{S}^\mu = -a\beta X^\mu X_\mu - \beta d Y^\mu Y_\mu - \beta(b+c)X^\mu Y_\mu$$

$$\boxed{a\beta \geq 0, \quad \beta d \geq 0, \quad 4ad\beta^2 \geq (b+c)^2\beta^2}$$

Entropy production in dissipative systems:

$$h^\mu = -\kappa_{11}\frac{S^{\alpha\beta}}{\varepsilon + P}\nabla^\mu \Omega_{\alpha\beta} - \kappa_{12}\nabla^\mu \alpha,$$

$$\mathcal{J}^\mu = \widetilde{\kappa}_{11}\nabla^\mu \alpha + \widetilde{\kappa}_{12}\frac{S^{\alpha\beta}}{\varepsilon + P}\nabla^\mu \Omega_{\alpha\beta},$$

Entropy production

$$\partial_\mu \mathcal{S}^\mu \geq 0$$

Cross diffusion like terms

$$\boxed{\begin{array}{l}\kappa_{11} \geq 0, \quad \widetilde{\kappa}_{11} \geq 0 \\ \kappa_{12}^2 - \kappa_{11}\widetilde{\kappa}_{11} \leq 0\end{array}}$$

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Entropy current analysis:

IRRep of dissipative currents:

$$\mathcal{S}^\mu = T^{\mu\nu}\beta_\nu + P\beta^\mu - \alpha J^\mu - \beta\omega_{\alpha\beta}S^{\mu\alpha\beta}.$$

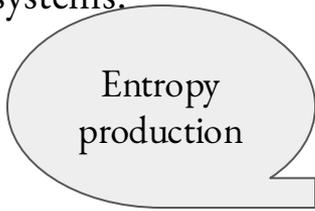$$T^{\{\alpha\beta\}}_{(1)} = h^\alpha u^\beta + h^\beta u^\alpha + \pi^{\alpha\beta} + \Pi\Delta^{\alpha\beta},$$

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Entropy production

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$$h^\mu = -\kappa_{11}\frac{S^{\alpha\beta}}{\varepsilon + P}\nabla^\mu\Omega_{\alpha\beta} - \kappa_{12}\nabla^\mu\alpha,$$

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \qquad\qquad\Longrightarrow \text{Shear viscous term}$$

$$\mathcal{J}^\mu = \tilde{\kappa}_{11}\nabla^\mu\alpha + \tilde{\kappa}_{12}\frac{S^{\alpha\beta}}{\varepsilon + P}\nabla^\mu\Omega_{\alpha\beta},$$

$$\longrightarrow \text{Cross diffusion like terms}$$

$$\Phi = -2\chi_1 u^\alpha\nabla^\beta(\beta\omega_{\alpha\beta}),$$

$$\tau^{\mu\beta}_{(s)} = -2\chi_2\Delta^{\mu\beta,\gamma\rho}\nabla_\gamma(\beta\omega_{\alpha\rho})u^\alpha,$$

$$\tau^{\mu\beta}_{(a)} = -2\chi_3\Delta^{[\mu\beta][\gamma\rho]}\nabla_\gamma(\beta\omega_{\alpha\rho})u^\alpha,$$

$$\Theta^{\mu\alpha\beta} = \chi_4\Delta^{\delta\alpha}\Delta^{\rho\beta}\Delta^{\gamma\mu}\nabla_\gamma(\beta\omega_{\delta\rho}).$$

$$\longrightarrow \text{Spin transport}$$

Non equilibrium statistical operator:

$$\widehat{\rho}(t) = \frac{1}{Q} \exp\left[-\int d^3x \ \widehat{Z}(\vec{x}, t)\right]$$

X.-G. Huang, et. al., Annals Phys. 326 (2011) 3075–3094;
D. N. Zubarev, Nonequilibrium statistical thermodynamics;
A. Hosoya, et.al., Annals Phys. 154 (1984) 229; J. Hu, Phys. Rev. D 103 no. 11, (2021) 116015.

Decomposition of the statistical operator:

$$\int d^3x \ \widehat{Z}(\vec{x}, t) = \widehat{A}(t) - \widehat{B}(t).$$

$$\widehat{A}(t) = \int d^3x \left[\beta^\nu(\vec{x}, t)\widehat{T}_{0\nu}(\vec{x}, t) - \alpha(\vec{x}, t)\widehat{J}^0(\vec{x}, t) - \Omega_{\rho\sigma}(\vec{x}, t)\widehat{S}^{0\rho\sigma}(\vec{x}, t)\right]$$

$$\widehat{B}(t) = \int d^3x \int_{-\infty}^{t} dt' \ e^{\epsilon(t'-t)}\left[\partial_\mu\beta_\nu(\vec{x}, t')\widehat{T}^{\mu\nu}(\vec{x}, t') - \partial_\mu\Omega_{\rho\sigma}(\vec{x}, t')\widehat{S}^{\mu\rho\sigma}(\vec{x}, t') - \partial_\mu\alpha(\vec{x}, t')\widehat{J}^\mu(\vec{x}, t')\right]$$

$$= \int d^3x \int_{-\infty}^{t} dt' \ e^{\epsilon(t'-t)} \ \widehat{C}(\vec{x}, t').$$

This term contains the gradients or thermodynamic forces

Local equilibrium statistical operator:

$$\widehat{\rho}_l(t) = \frac{1}{Q_l} \exp\left(-\widehat{A}\right)$$

Linear response: $e^{-\widehat{A}+\widehat{B}} \simeq e^{-\widehat{A}} + \int_0^1 d\tau \; e^{-\tau\widehat{A}}\widehat{B}e^{\tau\widehat{A}}e^{-\widehat{A}} + ...$

$$\widehat{\rho}(t) \simeq \left[1 + \int_0^1 d\tau \left\{ e^{-\tau\widehat{A}}\widehat{B}e^{\tau\widehat{A}} - \langle e^{-\tau\widehat{A}}\widehat{B}e^{\tau\widehat{A}}\rangle_l \right\} \right] \widehat{\rho}_l$$

Macroscopic conserved quantities:

$$\langle \widehat{T}^{\mu\nu}(\vec{x}, t)\rangle = \mathrm{Tr}\left(\widehat{\rho}(t)\widehat{T}^{\mu\nu}(\vec{x}, t) \right)$$

$$= \mathrm{Tr}\left(\widehat{\rho}_l(t)\widehat{T}^{\mu\nu}(\vec{x}, t) \right) + \mathrm{Tr}\left(\int_0^1 d\tau \left\{ e^{-\tau\widehat{A}}\widehat{B}e^{\tau\widehat{A}} - \langle \widehat{B}_\tau\rangle_l \right\}\widehat{\rho}_l \; \widehat{T}^{\mu\nu}(\vec{x}, t) \right)$$

$$= \langle \widehat{T}^{\mu\nu}(\vec{x}, t)\rangle_l + \delta\langle \widehat{T}^{\mu\nu}(\vec{x}, t)\rangle,$$

$$\delta\langle \widehat{T}^{\mu\nu}(\vec{x}, t)\rangle = \int d^3x' \int_{-\infty}^t dt' \; e^{\epsilon(t'-t)} \int_0^1 d\tau \left\langle \widehat{T}^{\mu\nu}(\vec{x}, t)\left\{ e^{-\tau\widehat{A}}\widehat{C}(\vec{x}', t')e^{\tau\widehat{A}} - \langle \widehat{C}(\vec{x}', t')_\tau\rangle_l \right\} \right\rangle_l$$

$$= \int d^3x' \int_{-\infty}^t dt' \; e^{\epsilon(t'-t)} \left(\widehat{T}^{\mu\nu}(\vec{x}, t), \widehat{C}(\vec{x}', t') \right)_l$$

This term contains all thermodynamic forces

$$\widehat{C}(\vec{x}\,',t') = \widehat{T}^{\mu\nu}(\vec{x}\,',t')\partial_\mu\beta_\nu(\vec{x}\,',t') - \widehat{J}^\mu(\vec{x}\,',t')\partial_\mu\alpha(\vec{x}\,',t') - \widehat{S}^{\mu\rho\sigma}(\vec{x}\,',t')\partial_\mu\Omega_{\rho\sigma}(\vec{x}\,',t')$$

1. Use the full decomposition of all the currents.
2. Use spin hydrodynamic equation of motion

$$\widehat{C}(\vec{x}\,',t') = -\widehat{P}^\star\beta\theta - \widehat{J}^\mu\nabla_\mu\alpha + \widehat{h}^\mu\frac{S^{\alpha\beta}}{\varepsilon+P}\nabla_\mu(\beta\omega_{\alpha\beta}) + \beta\widehat{\pi}^{\mu\nu}\sigma_{\mu\nu} - \widehat{S}^{\mu\alpha\beta}_{(1)}\nabla_\mu(\beta\omega_{\alpha\beta}).$$

$$\widehat{P}^\star = \left(\widehat{P} - \widehat{\Pi} - \widehat{\varepsilon}\gamma + \widehat{n}\gamma' + \widehat{S}^{\alpha\beta}\gamma_{\alpha\beta}\right)$$

$$\widehat{J}^\mu = \widehat{J}^\mu_{(1)} - \frac{n}{\varepsilon+P}\widehat{h}^\mu.$$

Identification of dissipative currents and the Kubo relations

$$\pi^{\mu\nu}(\vec{x},t) = \langle \hat{\pi}^{\mu\nu}(\vec{x},t)\rangle = \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\hat{\pi}^{\mu\nu}(\vec{x},t), \hat{C}(\vec{x}',t') \right)_l$$

$$= \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\hat{\pi}^{\mu\nu}(\vec{x},t), \hat{\pi}^{\rho\delta}(\vec{x}',t') \right)_l \beta\sigma_{\rho\delta}$$

$$\eta = \frac{\beta}{10} \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\hat{\pi}^{\mu\nu}(x), \hat{\pi}_{\mu\nu}(x') \right)_l$$

$$\zeta = \beta \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\hat{P}^{\star}(x), \hat{P}^{\star}(x') \right)_l$$

$$\tilde{\kappa}_{11} = -\frac{1}{3} \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\hat{\mathcal{J}}^{\mu}(\vec{x},t), \hat{\mathcal{J}}_{\mu}(\vec{x}',t') \right)_l$$

$$\tilde{\kappa}_{12} = \kappa_{12} = \frac{1}{3} \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\hat{\mathcal{J}}^{\mu}(\vec{x},t), \hat{h}_{\mu}(\vec{x}',t') \right)_l$$

$$\kappa_{11} = -\frac{1}{3} \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\hat{h}^{\mu}(\vec{x},t), \hat{h}_{\mu}(\vec{x}',t') \right)_l$$

Kubo relations for spin transport coefficients

$$\Phi = \langle \widehat{\Phi} \rangle = -2 \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\Phi}(\vec{x},t), \widehat{\Phi}(\vec{x}',t') \right)_l \nabla^\delta (\beta \omega_{\gamma\delta}) u^\gamma,$$

$$\tau^{\mu\alpha}_{(s)} = \langle \widehat{\tau}^{\mu\alpha}_{(s)} \rangle = -2 \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\tau}^{\mu\alpha}_{(s)}(\vec{x},t), \widehat{\tau}^{\rho\delta}_{(s)}(\vec{x}',t') \right)_l \nabla_\rho (\beta \omega_{\gamma\delta}) u^\gamma,$$

$$\tau^{\mu\alpha}_{(a)} = \langle \widehat{\tau}^{\mu\alpha}_{(a)} \rangle = -2 \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\tau}^{\mu\alpha}_{(a)}(\vec{x},t), \widehat{\tau}^{\rho\delta}_{(a)}(\vec{x}',t') \right)_l \nabla_\rho (\beta \omega_{\gamma\delta}) u^\gamma,$$

$$\Theta^{\mu\alpha\beta} = \langle \widehat{\Theta}^{\mu\alpha\beta} \rangle = - \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\Theta}^{\mu\alpha\beta}(\vec{x},t), \widehat{\Theta}^{\rho\gamma\delta}(\vec{x}',t') \right)_l \nabla_\rho (\beta \omega_{\gamma\delta})$$

Spin transport coefficients:

$$\chi_1 = \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\Phi}(\vec{x},t), \widehat{\Phi}(\vec{x}',t') \right)_l$$

$$\chi_2 = \frac{1}{5} \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\tau}^{\lambda\nu}_{(s)}(\vec{x},t), \widehat{\tau}_{(s)\,\lambda\nu}(\vec{x}',t') \right)_l$$

$$\chi_3 = \frac{1}{3} \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\tau}^{\lambda\nu}_{(a)}(\vec{x},t), \widehat{\tau}_{(a)\,\lambda\nu}(\vec{x}',t') \right)_l$$

$$\chi_4 = -\frac{1}{9} \int d^3x' \int_{-\infty}^{t} dt' \, e^{\epsilon(t'-t)} \left(\widehat{\Theta}^{\lambda\eta\zeta}(\vec{x},t), \widehat{\Theta}_{\lambda\eta\zeta}(\vec{x},t) \right)_l$$

Kubo relations for spin transport coefficients: Alternative approach

Generic decomposition: $S_{(1)}^{\mu\alpha\beta} = \Sigma^{\mu\alpha\beta\eta\gamma\delta} \nabla_\eta(\beta\omega_{\gamma\delta})$

- Orthogonality $\quad u_\mu \Sigma^{\mu\alpha\beta\eta\gamma\delta} = 0$
- Symmetry property $\quad \Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma^{\mu[\alpha\beta]\eta\gamma\delta} = \Sigma^{\mu\alpha\beta\eta[\gamma\delta]}$
- Onsager relation $\quad \Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma^{\eta\gamma\delta\mu\alpha\beta}$

$$\Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma_1 \mathcal{P}_1'^{\mu\alpha\beta\eta\gamma\delta} + \Sigma_2 \mathcal{P}_2'^{\mu\alpha\beta\eta\gamma\delta} + \Sigma_3 \mathcal{P}_3'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_1 \mathcal{Q}_1'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_2 \mathcal{Q}_2'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_3 \mathcal{Q}_3'^{\mu\alpha\beta\eta\gamma\delta}$$

Example of projectors: $\mathcal{P}_1^{\mu\alpha\beta\eta\gamma\delta} = \Delta^{\mu[\alpha}\Delta^{\beta][\gamma}\Delta^{\delta]\eta}$

$$\mathcal{Q}_1^{\mu\alpha\beta\eta\gamma\delta} = \Delta^{\mu[\alpha}u^{\beta]}\Delta^{\eta[\gamma}u^{\delta]}$$

Kubo relations for spin transport coefficients: Alternative approach

Generic decomposition: $S_{(1)}^{\mu\alpha\beta} = \Sigma^{\mu\alpha\beta\eta\gamma\delta}\nabla_\eta(\beta\omega_{\gamma\delta})$

- Orthogonality $\quad u_\mu \Sigma^{\mu\alpha\beta\eta\gamma\delta} = 0$
- Symmetry property $\quad \Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma^{\mu[\alpha\beta]\eta\gamma\delta} = \Sigma^{\mu\alpha\beta\eta[\gamma\delta]}$
- Onsager relation $\quad \Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma^{\eta\gamma\delta\mu\alpha\beta}$

$$\Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma_1 \mathcal{P}_1'^{\mu\alpha\beta\eta\gamma\delta} + \Sigma_2 \mathcal{P}_2'^{\mu\alpha\beta\eta\gamma\delta} + \Sigma_3 \mathcal{P}_3'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_1 \mathcal{Q}_1'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_2 \mathcal{Q}_2'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_3 \mathcal{Q}_3'^{\mu\alpha\beta\eta\gamma\delta}$$

$$\partial_\mu \mathcal{S}^\mu\big|_{S_{(1)}^{\mu\alpha\beta}} = -(\Sigma_1 + \Sigma_3)\nabla_\mu(\beta\omega_{\alpha\beta})\mathcal{P}_1'^{\mu\alpha\beta\eta\gamma\delta}\nabla_\eta(\beta\omega_{\gamma\delta}) - (\Sigma_2 + \Sigma_3)\nabla_\mu(\beta\omega_{\alpha\beta})\mathcal{P}_2'^{\mu\alpha\beta\eta\gamma\delta}\nabla_\eta(\beta\omega_{\gamma\delta})$$

$$-\Sigma_3\nabla_\mu(\beta\omega_{\alpha\beta})\mathcal{P}_3'^{\mu\alpha\beta\eta\gamma\delta}\nabla_\eta(\beta\omega_{\gamma\delta}) - \Lambda_1\nabla_\mu(\beta\omega_{\alpha\beta})\mathcal{Q}_1'^{\mu\alpha\beta\eta\gamma\delta}\nabla_\eta(\beta\omega_{\gamma\delta})$$

$$-\Lambda_2\nabla_\mu(\beta\omega_{\alpha\beta})\mathcal{Q}_2'^{\mu\alpha\beta\eta\gamma\delta}\nabla_\eta(\beta\omega_{\gamma\delta}) - \Lambda_3\nabla_\mu(\beta\omega_{\alpha\beta})\mathcal{Q}_3'^{\mu\alpha\beta\eta\gamma\delta}\nabla_\eta(\beta\omega_{\gamma\delta}).$$

Kubo relations for spin transport coefficients: Alternative approach

Generic decomposition: $S_{(1)}^{\mu\alpha\beta} = \Sigma^{\mu\alpha\beta\eta\gamma\delta} \nabla_\eta (\beta\omega_{\gamma\delta})$

- Orthogonality
- Symmetry property
- Onsager relation

$$u_\mu \Sigma^{\mu\alpha\beta\eta\gamma\delta} = 0$$

$$\Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma^{\mu[\alpha\beta]\eta\gamma\delta} = \Sigma^{\mu\alpha\beta\eta[\gamma\delta]}$$

$$\Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma^{\eta\gamma\delta\mu\alpha\beta}$$

$$\Sigma^{\mu\alpha\beta\eta\gamma\delta} = \Sigma_1 \mathcal{P}_1'^{\mu\alpha\beta\eta\gamma\delta} + \Sigma_2 \mathcal{P}_2'^{\mu\alpha\beta\eta\gamma\delta} + \Sigma_3 \mathcal{P}_3'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_1 \mathcal{Q}_1'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_2 \mathcal{Q}_2'^{\mu\alpha\beta\eta\gamma\delta} + \Lambda_3 \mathcal{Q}_3'^{\mu\alpha\beta\eta\gamma\delta}$$

$$\Sigma_1 + \Sigma_3 \leq 0, \ \Sigma_2 + \Sigma_3 \leq 0, \ \Sigma_3 \geq 0,$$
$$\Lambda_1 \leq 0, \ \Lambda_2 \leq 0, \ \Lambda_3 \leq 0.$$

Equivalence between different approaches

General decomposition: $\widehat{S}_{(1)}^{\mu\alpha\beta} = \widehat{\Xi}^{\mu\alpha\beta} + \widehat{\mathcal{V}}^{\mu[\alpha}u^{\beta]}$

$u_\mu\widehat{\Xi}^{\mu\alpha\beta} = 0, u_\alpha\widehat{\Xi}^{\mu\alpha\beta} = 0, u_\mu\widehat{\mathcal{V}}^{\mu\alpha} = 0, \widehat{\mathcal{V}}^{\mu\alpha}u_\alpha = 0.$

Case I

$\left\langle \widehat{\mathcal{V}}^{\mu\alpha} \right\rangle = -\left(\Delta^{\mu\alpha}\Phi + \tau_{(s)}^{\mu\alpha} + \tau_{(a)}^{\mu\alpha} \right)$

$\left\langle \widehat{\Xi}^{\mu\alpha\beta} \right\rangle = \Theta^{\mu\alpha\beta}.$

Case II

$\left\langle \widehat{\mathcal{V}}^{\mu\alpha} \right\rangle = \Lambda_1\Delta^{\mu\alpha}\Phi_{||} + \Lambda_2\Gamma_s^{\mu\alpha} + \Lambda_3\Gamma_a^{\mu\alpha}$

$\left\langle \widehat{\Xi}^{\mu\alpha\beta} \right\rangle = -(\Sigma_1 + \Sigma_3)\Delta^{\mu[\alpha}\Phi_\perp^{\beta]} + (\Sigma_2 + \Sigma_3)\varepsilon^{\mu\alpha\beta}\varphi + \Sigma_3\Phi^{\mu\alpha\beta}$

$\Lambda_1 = -4\chi_1 \,, \Lambda_2 = -2\chi_2 \,, \Lambda_3 = -2\chi_3 \,, \chi_4 = \frac{5}{9}\Sigma_3 - \frac{3}{9}\Sigma_1 - \frac{1}{9}\Sigma_2$

Open questions ?

- Pseudo-gauge choice and the spin hydrodynamic framework.

- Gradient ordering of spin hydrodynamic variables/ spin chemical potential.

- Spin transport coefficients and its phenomenological implications. Can we estimate using Yukawa interaction, NJL type of model, etc? [Follow works of Sourav Dey on Green Kubo framework of multiple conserved charges! *JHEP* 12 (2024) 192]

- Analytical solution of the spin hydrodynamic framework and its phenomenological applications? Work going on in collaboration with the NISER group!

- Numerical solution of the spin hydrodynamic equations ! [Follow the works of Sushant K. Singh Phys. Rev. C **111**, 024907]

Thank you for your attention

Local equilibrium density operator:

$$\widehat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp\left[-\int_{\Sigma} d\Sigma_{\mu} \left(\widehat{T}_{\text{can}}^{\mu\nu} \beta_{\nu} - \frac{1}{2}\omega_{\lambda\nu}\widehat{S}_{\text{can}}^{\mu\lambda\nu} - \xi \widehat{J}^{\mu} \right) \right]$$

Global equilibrium conditions:

$$\widehat{T}_{\text{can(s)}}^{\mu\nu} \partial_{\{\mu}\beta_{\nu\}} + \widehat{T}_{\text{can(a)}}^{\mu\nu} \left(\omega_{\mu\nu} + \partial_{[\mu}\beta_{\nu]} \right) - \frac{1}{2}(\partial_{\mu}\omega_{\lambda\nu})\widehat{S}_{\text{can}}^{\mu\lambda\nu} - \widehat{J}^{\mu}\partial_{\mu}\xi = 0.$$

Killing equations, thermal vorticity and spin chemical potential:

$$\partial_{\mu}\beta_{\nu} + \partial_{\nu}\beta_{\mu} = 0; \quad \omega_{\mu\nu} = -(\partial_{\mu}\beta_{\nu} - \partial_{\mu}\beta_{\nu})/2 = \varpi_{\mu\nu} \qquad \partial_{\mu}\xi = 0.$$