Validity of relativistic hydrodynamics beyond local equilibrium
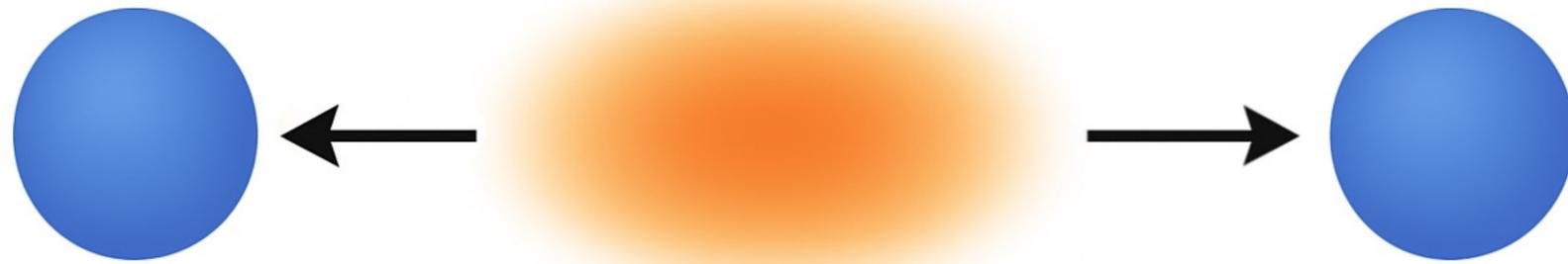
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Overview

- Heavy-Ion Collisions and Hydrodynamics

- The hydrodynamisation problem

- Hydrodynamics and Kinetic Theory

- Convergence of Gradient Expansion

- 0+1D system

- Hydrodynamics from Moments

- Summary

Quark-Gluon Plasma (QGP)

Hydrodynamic Evolution

The Hydrodynamisation Problem

- Empirical validity of Hydrodynamics at large gradients

  - Viscous Hydrodynamics is derived near equilibrium

- Divergence of the gradient expansion

  - Higher order gradient corrections are asymptotically divergent

- Hydrodynamisation

  - When is the system well defined by hydrodynamic equations?

Hydrodynamics

- A system at local thermal equilibrium can be described by macroscopic fields, energy momentum tensor and number density

- The dynamics of the system can be obtained from conservation laws and equation of state.

- Systems near equilibrium can be described by adding gradient corrections to the fields.

- For causal evolution of the system additional degrees of freedom and their evolution equations are required.

- $$T_{eq}^{\mu\nu} = \epsilon U^\mu U^\nu - P\Delta^{\mu\nu} \qquad (1)$$
$$n^\mu = nU^\mu$$

- $$\partial_\mu T^{\mu\nu} = 0$$
$$\partial_\mu n^\mu = 0, \qquad \epsilon = \epsilon(P) \qquad (2)$$

- $$T^{\mu\nu} = T_{eq}^{\mu\nu} + \Pi^{\mu\nu}$$
$$\dot{\Pi}^{\mu\nu} \sim F(\Pi^{\mu\nu}, \partial^\mu U^\nu) \qquad (3)$$

Kinetic Theory

Evolution of phase space distribution function for low density systems.

- The one particle phase space distribution function can describe a system at low densities.

- The Evolution of the distribution function is given by the Boltzmann Equation.

- We can obtain macroscopic fields from the moments of the distribution function.

- The dynamics is obtained from the Boltzmann Equation.

- $p^\mu \partial_\mu f = C(f, f)$

  (4)

- $T^{\mu\nu} = \int \dfrac{d^3 p}{p^0} p^\mu p^\nu f$

  (5)

  $n^\mu = \int \dfrac{d^3 p}{p^0} p^\mu f$

- $\dot{\Pi}^{\mu\nu} = \int \dfrac{d^3 p}{p^0} p^\mu p^\nu \dot{\delta f}$
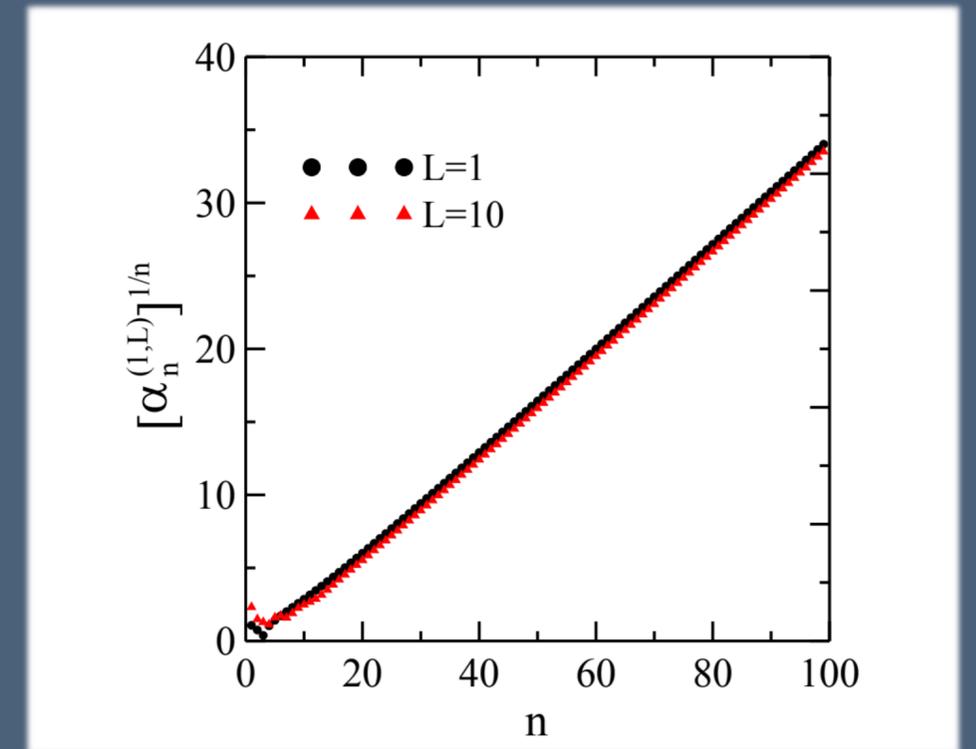
  (6)

The Gradient Expansion

Expanding the distribution function near equilibrium

$$p^\mu \partial_\mu f = -\frac{u \cdot p}{\tau_R}(f - f_{eq}).$$ （ RTA kernel ） (7)

- We can obtain an expansion (Chapman–Enskog) of the distribution function in terms of equilibrium fields

Coefficient of gradient expansion vs order. [†]

$$f = f_{eq} - \frac{\tau_R}{u \cdot p}p^\mu \partial_\mu f \longrightarrow f(p, x) = f_{eq} + \sum_{n=1}^{\infty}\left[-\frac{\tau_R}{u \cdot p}p^\mu \partial_\mu\right]^n f_{eq}(p, x)$$ (8)

- This expansion is divergent

[†]Divergence of the Chapman-Enskog expansion in relativistic kinetic theory
Gabriel S. Denicol Jorge Noronha. arXiv:1608.07869
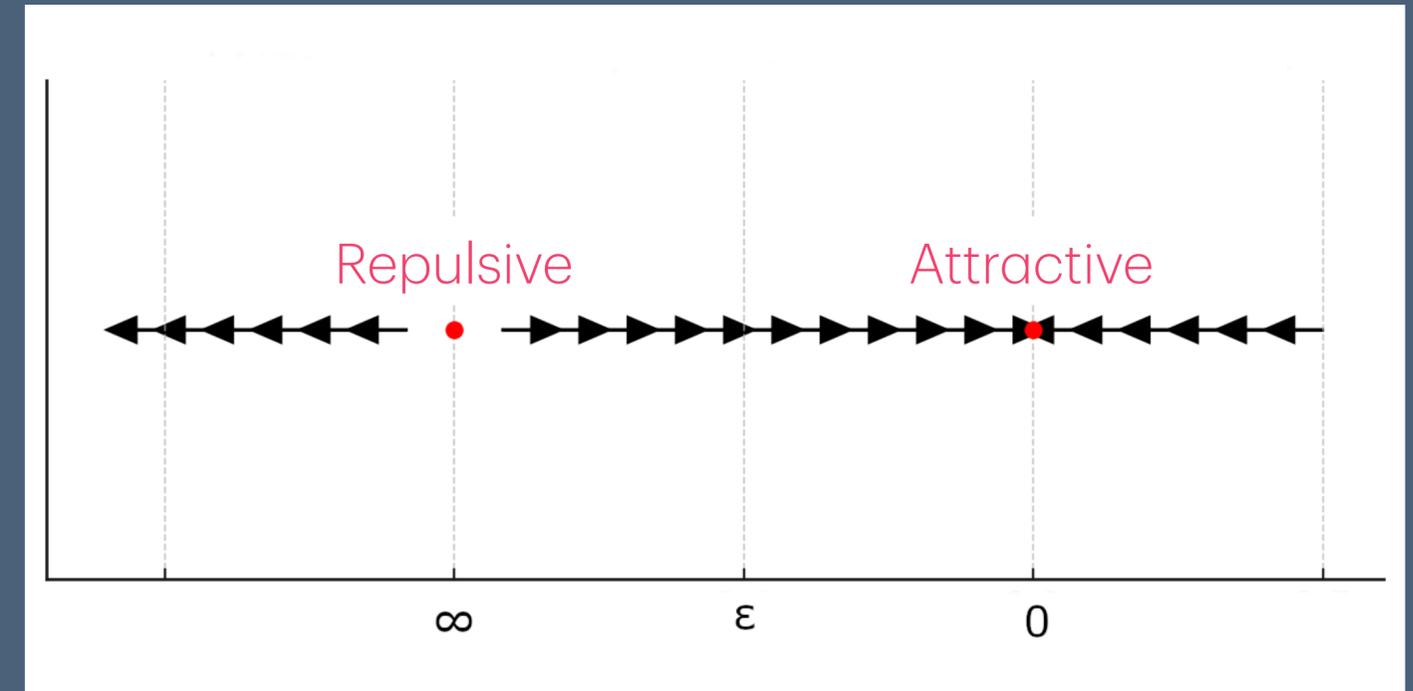
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Why is it divergent?
The Toy Model

- $\dfrac{df}{dt} = -\dfrac{1}{\epsilon}(f - g)$ (9)

- $f \sim g - \epsilon g' + \epsilon^2 g'' \ \dots$ — expansion in ϵ (10)

Fixed point schematic

- $f(x) \sim e^{-t/\epsilon} f(0) + \displaystyle\int_0^t \dfrac{e^{-s/\epsilon}}{\epsilon} g(s) ds.$ — Exact solution (11)

- $f(t) = e^{-t/\epsilon} f(0) + [\ g(t) - e^{-t/\epsilon} g(0)\ ] - \epsilon\ [g'(t) - e^{-t/\epsilon} g'(0)\ ] + \dots$ — Exact solution expansion (12)

$$f(t) = e^{-t/\delta} f(0) + \sum_{n=0}^{\infty} \left(1 - e^{-t/\epsilon} \sum_{k=0}^{n} \dfrac{(t/\epsilon)^k}{k!} \right) \dfrac{d^n g(t)}{dt^n} \qquad (13)$$

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Exact Solution to the Distribution

- $$\left(p^\mu \partial_\mu + \frac{u \cdot p}{\tau_R} \right) f = \frac{u \cdot p}{\tau_R} f_{eq.}$$ (14)

- Formal Solution

$$f(t) = e^{-\xi_0} f_0(t, t_0) + \int_{t_0}^{t} \frac{dt'}{p^0} \frac{u(t, t') \cdot p}{\tau_R(t, t')} e^{-\xi(t,t')} f_{eq}(t, t') \ ,$$ (15)

(16)

- $$\xi' = \int_{t'}^{t} \frac{u \cdot p}{p^0 \tau_R} dt''$$

- The first term is a damped free streaming term and the second term contains the contributions from collisions.

Exact Gradient Expansion

- We can obtain the exact gradient expansion*

$$f_G = \sum_{n=0}^{\infty} \left\{ 1 - e^{-\xi_0} \sum_{k=0}^{n} \frac{\xi_0^k}{k!} \right\} \left[-\frac{\tau_R}{p \cdot u} p^\mu \partial_\mu \right]^n f_{eq}(x, t)$$

(17)

- First sum gives the Chapman Enskog gradients and the second sum gives exponentially damped terms.

- This expansion is convergent.

  - Integrating the gradient expansion of the distribution function to get moment evolution is difficult.

Gradient Expansion of The Moments

Moments give bulk information

- We already know the gradient expansion of the distribution function. Why do we need to study gradient expansion of the moments?

  - Moments give bulk information.

  - The gradient expansion of moments give hydrodynamic equations.

  - Understanding their structure is key to understanding the applicability of hydrodynamics.

The 0+1 D system[†]

- The Symmetries : reduces system to 0 + 1

  - Boost invariance : no η dependence.

  - Transverse rotational and translational symmetry : no r and θ dependance

  - System can depend only on τ

  - Bjorken Flow profile

The 0+1 D solution for the Moments

Bjorken Flow

- Moment definitions

$$\rho_{n,l}(\tau) = \frac{1}{(2\pi)^3} \int d^3p (p^0)^n \left(\frac{p^z}{p^0}\right)^{2l} f(p, p^z, \tau)$$

(18)

- The evolution equations

$$\partial_\tau \rho_{n,l} + \frac{2l+1}{\tau}\rho_{n,l} - \frac{2l-n}{\tau}\rho_{n,l+1} = -\frac{1}{\tau_R}\left(\rho_{n,l} - \rho_{n,l}^{eq}\right)$$

(19)

The Operator Formalism

Vectorisation of moment equations

- $$\left[\partial_\tau + \left(\frac{2l+1}{\tau} \hat{\mathbf{I}} - \frac{2l-n}{\tau} \hat{\mathbf{s}} \right) \right] \rho_{n,l}(\tau, l) = - \frac{1}{\tau_R} \left(\rho_{n,l}(\tau) - \rho_{n,l}^{\mathrm{eq}}(\tau) \right) \quad (20)$$

- $\hat{\mathbf{s}} \; h_l = h_{l+1}$ - The l - shift operator

(21)

- $\hat{\mathbf{D}} \equiv \partial_\tau + \hat{\mathbf{F}}$, $\hat{\mathbf{F}} \equiv \frac{2l+1}{\tau} \hat{\mathbf{I}} - \frac{2l-1}{\tau} \hat{\mathbf{s}}$

The Operator Formalism

Vectorisation of moment equations

- $\hat{\mathbf{D}}\rho_{n,l}(\tau, l) = -\dfrac{1}{\tau_R}\left(\rho_{n,l}(\tau) - \rho_{n,l}^{\mathrm{eq}}(\tau)\right)$ (22)

- $\rho_n(\tau, l) = \rho_n^{\mathrm{eq}}(\tau, l) + \left[\tau_R D\right]\rho_n(\tau, l).$ - iterative structure

  (23)

- $\rho_n(\tau, l) = \rho_n^{\mathrm{eq}}(\tau, l) + \left[\tau_R D\right]\rho_n^{\mathrm{eq}} + \left[\tau_R D\right]^2 \rho_n^{\mathrm{eq}} + \dots$ - Gradient Expansion

Solution to Moment Equation

Formal Integral Solution

- $$\left[\partial_\tau + \hat{\mathbf{F}} + \frac{1}{\tau_R}\right]\rho_{n,l} = \frac{\rho_{n,l}^{eq}}{\tau_R}$$ - Relaxation equation   (24)

- $$\rho_{n,l}(\tau) = e^{-\xi_0}e^{-\hat{\mathbf{K}}(\tau,\tau_0)}\rho_{n,l}(\tau_0) + \int_{\tau_0}^{\tau} d\tau' \frac{e^{-\xi'}}{\tau_R} e^{-\hat{\mathbf{K}}(\tau,\tau')}\rho_{n,l}^{eq}(\tau')$$ (25)

$$\hat{\mathbf{K}}(\tau, \tau') = \int_{\tau'}^{\tau} \hat{\mathbf{F}}\ d\tau'$$

- $$\hat{\mathcal{K}}(\tau, \tau_0) = e^{-\xi_0}e^{-\hat{\mathbf{K}}(\tau,\tau_0)}$$ - Green's function ~ Propagator

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The Gradient Expansion of Moments

- $$\rho_{n,l}(\tau) = e^{-\xi_0} e^{-\hat{\mathbf{K}}(\tau,\tau_0)} \rho_{n,l}(\tau_0) + \sum_{k=0}^{\infty} \left(1 - e^{-\xi_0} \sum_{k=0}^{n} \frac{\xi_0^k}{k!} \right) \left[-\tau_R \hat{\mathbf{D}} \right]^k \rho_{n,l}^{\mathrm{eq}}(\tau) \qquad (26)$$

- $$\rho_{n,l}(\tau) = \sum_{k=0}^{\infty} \left[-\tau_R \hat{\mathbf{D}} \right]^k \rho_{n,l}^{\mathrm{eq}}(\tau) + e^{-\xi_0} e^{-\hat{\mathbf{K}}(\tau,\tau_0)} \left(\sum_{k=0}^{\infty} \left[-\tau_R \hat{\mathbf{D}} \right]^k \rho_{n,l}^{\mathrm{eq}}(\tau_0) + \rho_{n,l}(\tau_0) \right) \qquad (27)$$

Gradient expansion of the Moments

Non-Equilibrium components

- $\rho_{n,l}(\tau) = \rho_{n,l}^{eq}(\tau) + \pi_{nl}$ (28)   - Decomposing to Equilibrium and non-equilibrium

- $\pi_{nl} = \pi_{n,l}^{G}(\tau) + \pi_{n,l}^{T}(\tau)$ (29)   - Decomposing to Gradient and Transient terms

- $\pi_{n,l}(\tau) = \sum_{k=1}^{\infty} \left[\tau_R \hat{\mathbf{D}}\right]^k \rho_{n,l}^{\text{eq}}(\tau) - e^{-\xi_0} e^{-\hat{\mathbf{K}}(\tau,\tau_0)} \left(\sum_{k=0}^{\infty} \left[\tau_R \hat{\mathbf{D}}\right]^k \rho_{n,l}^{\text{eq}}(\tau_0) - \rho_{n,l}(\tau_0) \right)$ (30)

Non-Equilibrium Dynamics

- $\pi_{nl} = \pi_{n,l}^{G}(\tau) + \pi_{n,l}^{T}(\tau)$ (31)

- $\hat{\mathbf{D}}\pi_{n,l}^{G}(\tau) = -\dfrac{\pi_{n,l}^{G}(\tau)}{\tau_R} - \hat{\mathbf{D}}\rho_{n,l}^{eq}(\tau)$ (32)   - Dynamics of the Gradient Corrections

- $\hat{\mathbf{D}}\pi_{n,l}^{T}(\tau) = -\dfrac{\pi_{n,l}^{T}(\tau)}{\tau_R}$ (33)   - Dynamics of the Transient Corrections

- $\hat{\mathbf{D}}\pi_{n,l} = -\dfrac{\pi_{n,l}}{\tau_R} - \hat{\mathbf{D}}\rho_{n,l}^{eq}(\tau)$ (34)   - Combined dynamics - Same as Gradient

The Hydrodynamic Equations

Bulk pressure evolution

- $$\Pi = -\frac{m^2}{3}\pi_{-1,0} \tag{35}$$

- $$\hat{\mathbf{D}}\pi_{-1,0} = -\frac{\pi_{-1,0}}{\tau_R} - \hat{\mathbf{D}}\rho_{-1,0}^{eq}(\tau) \tag{36}$$

- $$\partial_\tau \Pi + \frac{\Pi}{\tau_R} = \frac{\Pi}{\tau} - \hat{\mathbf{D}}P - \frac{m^2}{3}\frac{\pi_{-1,1}}{\tau} \tag{37}$$ - Exact evolution equation for Bulk pressure

The Hydrodynamic Equations

Bulk anisotropy evolution

$$\bullet \quad \partial_\tau \Pi + \frac{\Pi}{\tau_R} = \frac{\Pi}{\tau} - \hat{\mathbf{D}} P - \frac{m^2}{3} \frac{1}{\tau} \left(\sum_{k=1}^{\infty} \frac{\gamma(k+1,\xi_0)}{k!} \left[\tau_R \hat{\mathbf{D}}\right]^k \rho_{-1,1}^{\text{eq}}(\tau) \right) - e^{-\xi_0} e^{-\hat{\mathbf{F}}(\tau,\tau_0)} \rho_{-1,1}(\tau_0)$$

$$(38)$$

- **Exact evolution ~ gradient expanded hydro + transport coefficient rescaling**

Relativistic Hydrodynamics can accommodate non-equilibrium evolution

- The evolution equation of second order hydrodynamics is structurally similar to exact evolution equation.

- Second order hydrodynamics can accommodate transient terms and therefore early time dynamics.

- The difference is between the two equations is absorbed in the transport coefficients. (Similar conclusion from fixed point analysis[†])

- Hydrodynamics works for QGP not because its at local thermal equilibrium but because hydrodynamic equations can interpolate between free streaming and equilibrium dynamics.

[†] From moments of the distribution function to hydrodynamics: The non-conformal case.
Sunil Jaiswal, Jean-Paul Blaizot, Rajeev S. Bhalerao, Zenan Chen, Amaresh Jaiswal

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Thank You