Quarkonium potential in QCD medium with momentum dependent relaxation time

Sunny Kumar Singh

Based on **arXiv:2508.09108** in collaboration with *Samapan Bhadury, Ritesh Ghosh and Manu Kurian*
S. Singh, M. Kurian, V. Chandra, **Phy. Rev. D 110, 014004 (2024)**
S. Singh, S. Bhadury, M. Kurian, V. Chandra, **Phy. Rev. D 111, 114007 (2025)**
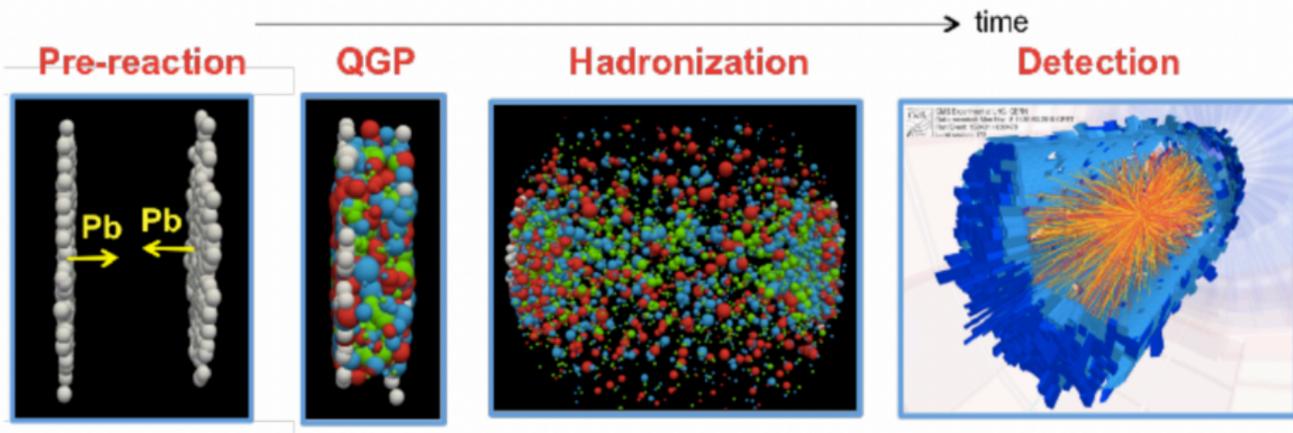
Contents

Introduction

Figure: Evolution of QGP in heavy ion collisions [1]

The lifetime of the QGP phase is approximately $\sim 0.5 - 5$ fm.
After $\sim 5 - 10$ fm, Hadronization is complete.
After chemical freezeout, all unstable hadrons decay into stable ones and then continue their journey towards the detector.

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[1] Pramana 79 (2012) 719-735

The framework of Relativistic Hydrodynamics

Conservation laws

The framework of Relativistic Hydrodynamics is uselful to explain fluid like behaviour of the QGP medium. We start with a brief recap:

- Conservation of particle number and the energy momentum tensor,

$$\partial_\mu T_f^{\mu\nu} = 0 \tag{1}$$

$$\partial_\mu N_f^\mu = 0 \tag{2}$$

- With the fluid velocity at each point being u^μ, the fluid energy momentum tensor and number current can be decomposed along and orthogonal to fluid velocity as:

$$T_f^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + 2u^{(\mu}h^{\nu)} + \pi^{\mu\nu} \tag{3}$$

$$N_f^\mu = n u^\mu + n^\mu \tag{4}$$

- Here $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the orthogonal projector to u^μ.

- In Landau frame definition,

$$u^\mu = \frac{u_\nu T^{\mu\nu}}{\epsilon}$$

And hence, the heat current $h^\mu = 0$, i.e, there is no energy dissipation.

Equations of motion

- The fluid equations of motion are now given by:

$$\dot{\varepsilon} + (\varepsilon + P)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{5}$$

$$(\epsilon + P)\dot{u}^{\mu} - \nabla^{\mu}P + \Delta^{\mu}_{\nu}\partial_{\gamma}\pi^{\gamma\nu} = 0, \tag{6}$$

$$\dot{n} + n\theta + \partial_{\mu}n^{\mu} = 0 \tag{7}$$

- ϵ, n and P are related to each other via the equation of state.

- These are not closed equations: additional relations (constitutive equations) are needed to specify $\pi^{\mu\nu}$ and n^{μ}.

- In the ideal limit, $\pi^{\mu\nu} = 0$ and $n^{\mu} = 0$:
  - The fluid behaves as a perfect fluid without dissipative effects.

Entropy-current analysis

- The evolution equations for $\pi^{\mu\nu}$ and n^{μ} which are needed can be derived from entropy-current analysis, requiring $\partial_{\mu}S^{\mu} \geq 0$.

- The evolution equation for $\pi^{\mu\nu}$ and n^{μ} that we need to ensure the second law of thermodynamics is guaranteed is:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = 2\beta_{\pi}\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + 2\pi_{\gamma}^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_{\gamma}^{\langle\mu}\sigma^{\nu\rangle\gamma}$$
$$- \tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle}. \tag{8}$$

$$\dot{n}^{\langle\mu\rangle} + \frac{n^{\mu}}{\tau_{n}} = \beta_{V}\nabla^{\mu}\alpha - \lambda_{V\pi}\pi^{\mu\lambda}\nabla_{\lambda}\alpha - \tau_{V\pi}\pi_{\lambda}^{\mu}\dot{u}^{\lambda} - \delta_{VV}n^{\mu}\theta$$
$$+ l_{V\pi}\Delta_{\alpha}^{\mu}\partial_{k}\pi^{\alpha k} - \lambda_{VV}\sigma_{\lambda}^{\mu}n^{\lambda} - \lambda_{\omega}\omega_{\lambda}^{\mu}n^{\lambda}. \tag{9}$$

- The transport coefficients must be determined from a microscopic theory.

Kinetic Theory Approach with ERTA

Kinetic Theory Approach

- To solve the resulting equation of motion, we need to find the evolution equation of $\pi^{\mu\nu}$ as well as the other dissipative quantities. We do that using kinetic theory approach.

- We start with a distribution function for the particle, $f(x, p)$ and which can then be used to define:

$$T^{\mu\nu} = \int \mathrm{dP} p^\mu p^\nu f \qquad N^\mu = \int \mathrm{dP} p^\mu f \qquad (10)$$

- With $f_0(x, p)$ being the local equilibrium distribution, the deviation of equilibrium can be written as $\delta f = f - f_0$.

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int \mathrm{dP} p^\alpha p^\beta \delta f \qquad (11)$$

$$n^\mu = \Delta^\mu_\nu \int \mathrm{dP} p^\nu \delta f \qquad (12)$$

Boltzmann Equation and the RTA collision kernel

- To get the form of δf, the evolution of $f(x, p)$ is needed via the Boltzmann equation.

$$p^\mu \partial_\mu f = \mathcal{C}[f] \qquad (13)$$

- The collision term which encodes the details about various collisional processes happening in the system can be approximated using the relaxation time approximation (RTA) as:

$$\mathcal{C}[f] = -\frac{(u \cdot p)}{\tau_R}(f - f_0) \qquad (14)$$

- Conservation laws implies that for $\mathcal{C}[f]$,

$$\int \mathrm{dP} \mathcal{C}[f] = 0 = \int \mathrm{dP} p^\mu \mathcal{C}[f] \qquad (15)$$

- The relaxation time τ_R is momentum independent for RTA since a momentum dependent $\tau_R(p)$ leads to violation of conservation laws with Landau matching condition:

$$\int \mathrm{dP} \mathcal{C}[f] \neq 0 \neq \int \mathrm{dP} p^\mu \mathcal{C}[f] \qquad (16)$$

Extended relaxation time: Motivation

- $\tau_R(x,p)$ reflects how quickly particles at that momentum equilibriate with rest of the fluid.

- According to some studies[2], various physical scenarios lead to various forms of momentum dependence for the relaxation time.

- If energy loss of particles grows linearly with momentum, $\frac{dp}{dt} \propto p$, relaxation time is expected to be independent of p

- If energy loss of particles approaches a constant value, relaxation time follows $\tau_R \propto p$

- For scalar $\lambda\phi^4$ theory, relaxation time follows $\tau_R \propto p$.

- For QCD, the momentum dependence should lie between these two cases.

- For example, in case of QCD radiation energy loss taken into account, we expect $\tau_R \propto p^{0.5}$

- This leads to the formulation of Hydrodynamics using a momentum-dependent relaxation time approach.

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[2]K. Dusling, G.D. Moore & D. Teany, **Phy Rev C 81, 034907 (2010)**

Extended relaxation time

- The collision kernel for extended relaxation time reads [3]:

$$C[f] = -\frac{(u \cdot p)}{\tau_R(x,p)}(f - f_0^*) \tag{17}$$

Where f_0^* is the equilibrium distribution function in the "thermodynamic" frame.

$$f_0^* = e^{-\beta^*(u^* \cdot p) + \alpha^*} \tag{18}$$

- Where $u^{*\mu}, \beta^*$ and α^* are related with the usual variables by:

$$u^{\mu*} = u^\mu + \delta u^\mu, \qquad \mu^* = \mu + \delta\mu, \qquad T^* = T + \delta T, \tag{19}$$

- This lets us use a momentum dependent relaxation time to determine the transport coefficients. The form of momentum dependent $\tau_R(x,p)$ is taken as:

$$\tau_R(x,p) = \frac{\kappa}{T}\left(\frac{u \cdot p}{T}\right)^\ell \tag{20}$$

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[3] D. Dash, S. Bhadury, S. Jaiswal, A. Jaiswal, *Physics Letters B, 831 (2022)*

Gradient expansion

- We use a Champan-Enskog like gradient expansion to get the form of δf upto second order.

$$\Delta f = -\frac{\tau_R}{(u \cdot p)} p^\mu \partial_\mu f_0 - \frac{\tau_R}{(u \cdot p)} p^\mu \partial_\mu \Big[- \frac{\tau_R}{(u \cdot p)} p^\mu \partial_\mu f_0 + \delta f_{(1)}^* \Big] + \Delta f_{(2)}^*. \qquad (21)$$

Where $\Delta f_{(2)}^*$ is given by $\Delta f_{(2)}^* = f_0^* - f_0$.

- This can be obtained by Taylor expanding f_0^* about T, μ and u^μ:

$$\Delta f_{(2)}^* = \left[-\frac{\Delta u \cdot p}{T} + \left(\frac{u \cdot p - \mu}{T^2} \right) \Delta T + \frac{\Delta \mu}{T} + \frac{(\Delta u \cdot p)^2}{2T^2} + \mathcal{O}(\partial^3) \right] f_0. \qquad (22)$$

- With the three conditions, viz. Landau frame condition, $\epsilon = \epsilon_0$ and $n = n_0$, we obtain the forms of Δu^μ, ΔT and $\Delta \mu$.

Shear stress and number evolution

Shear stress evolution within ERTA

- The Δf with the necessary counter terms inside $u^{*\mu}$, β^* and μ^* leads to:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}$$
$$- \tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha + l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle}. \tag{23}$$

- This evolution equation is second order in the gradient expansion of the hydrodynamic fields
- The central result of the our work[4] is that in the massless MB limit, **the evolution of the number diffusion n^μ is coupled to the evolution of the shear stress tensor** whereas they are decoupled in the RTA limit [5].
- The relaxation time for the shear mode was found to be dependent on the momentum dependence paramter ℓ as,

$$\tau_\pi = \frac{\bar{\kappa}\Gamma(5+2\ell)}{T\Gamma(5+\ell)}, \quad \ell > -\frac{5}{2} \tag{24}$$

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[4]S. Singh, M. Kurian, V. Chandra, *Phy. Rev. D 110, 014004 (2024)*
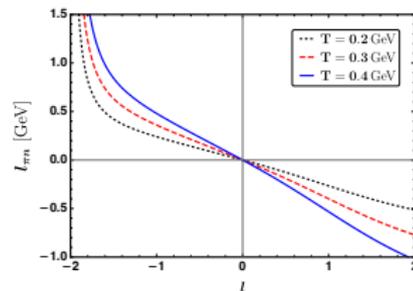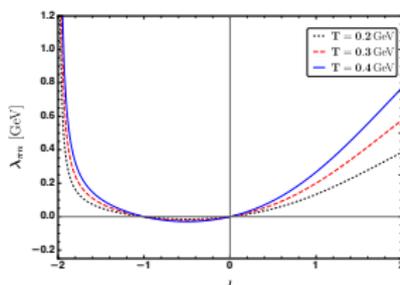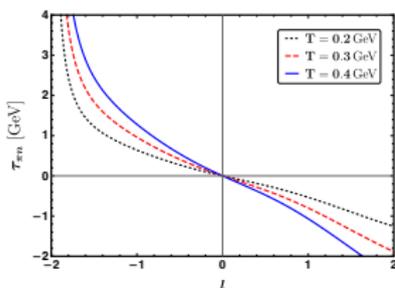[5]A. Jaiswal, B. Friman, K. Redlich, *Phy Lett B 751 (2015)*

Transport coefficients

$$l_{\pi n} = \frac{T\ell\Gamma(5+\ell)\left\{\Gamma(5+\ell)\Gamma(4+\ell) - 48\Gamma(4+2\ell)\right\}}{15(\ell^2 - \ell + 4)\Gamma(3+\ell)\Gamma(5+2\ell)}, \quad \ell > -2 \tag{25}$$

$$\tau_{\pi n} = \frac{4T\ell\Gamma(5+\ell)\left\{\Gamma(5+\ell)\Gamma(4+\ell) - 48\Gamma(4+2\ell)\right\}}{15(\ell^2 - \ell + 4)\Gamma(3+\ell)\Gamma(5+2\ell)}, \quad \ell > -2 \tag{26}$$

$$\lambda_{\pi n} = \frac{\ell(\ell+1)T T\Gamma(5+\ell)\left\{-\Gamma(4+\ell)\Gamma(5+\ell) + 48\Gamma(4+2\ell)\right\}}{60(\ell^2 - \ell + 4)\Gamma(3+\ell)\Gamma(5+2\ell)}, \quad \ell > -2 \tag{27}$$

- In the massless MB limit, the coefficients of the term $\tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle}$, $l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle}$ and $\lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha$ are plotted below [6]:

[6]S. Singh, M. Kurian, V. Chandra, *Phy. Rev. D 110, 014004 (2024)*

Number diffusion evolution within ERTA

- The above Δf was also used to determine the number diffusion evolution equation[7],

$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \beta_V \nabla^\mu \alpha - \lambda_{V\pi} \pi^{\mu\lambda} \nabla_\lambda \alpha - \tau_{V\pi} \pi^\mu_\lambda \dot{u}^\lambda - \delta_{VV} n^\mu \theta$$
$$+ l_{V\pi} \Delta^\mu_\alpha \partial_k \pi^{\alpha k} - \lambda_{VV} \sigma^\mu_\lambda n^\lambda - \lambda_\omega \omega^\mu_\lambda n^\lambda. \tag{28}$$

- The relaxation time for the number diffusion mode was also found to be dependent on the momentum dependence paramter ℓ,

$$\tau_n = -\frac{\left[192\Gamma(2(2+\ell)) + 16\Gamma^2(4+\ell) - 8\Gamma(4+\ell)\Gamma(5+\ell) + \Gamma^2(5+\ell) - 384\Gamma(3+2\ell) - 24\Gamma(5+2\ell)\right]\bar{\kappa}}{24T\left[16\Gamma(3+\ell) - 8\Gamma(4+\ell) + \Gamma(5+\ell)\right]}, \tag{29}$$

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[7] S. Singh, S. Bhadhury, V. Chandra, *Phy. Rev. D 111, 114007 (2025)*

Ratio of $\tau_n \, / \, \tau_\pi$

- The ratio of the relaxation time for number diffusion mode to the relaxation time for shear mode is then given by,

$$\frac{\tau_n}{\tau_\pi} = \frac{(4+\ell)(3+\ell)}{24(\ell^2-\ell+4)} \left(\frac{48(2\ell^2-\ell+2)}{(4+2\ell)(3+2\ell)} - \frac{\ell^2\Gamma^2(4+\ell)}{\Gamma(5+2\ell)} \right), \qquad \ell > -\frac{3}{2}. \tag{30}$$

- As $\ell \to 0$, the ratio $\frac{\tau_n}{\tau_\pi}$ approaches unity, corresponding to the RTA limit where both shear and diffusion modes decay to their Navier-Stokes limits on the same timescale

- Hence, we see that as expected in a realistic system, the relaxation time-scales for the shear and particle diffusion components are unequal.

Comparision with first principle approaches

- Some recent studies [8],[9] analytically extracted a full set of eigenvalues and eigenfunctions of the relativistic linearized Boltzman collision operator for $\lambda\phi^4$ theory.

$$\hat{L}\phi_k = \frac{g}{2} \int dK'dPdP' f_{0k'}(2\pi)^5 \delta^{(4)}(k+k'-p-p')(\phi_p + \phi_{p'} - \phi_k - \phi_{k'}). \quad (31)$$

- Where the eigenfunctions and thier eigenvalues are given by:

$$\hat{L}L_{n\mathrm{k}}^{(2m+1)}k^{\langle\mu_1}\ldots k^{\mu_\ell\rangle} = -\frac{g\mathcal{M}}{2}\left[\frac{n+m-1}{n+m+1} + \delta_{\ell 0}\delta_{n0}\right]L_{n\mathrm{k}}^{(2m+1)}k^{\langle\mu_1}\ldots k^{\mu_m\rangle}, \quad (32)$$
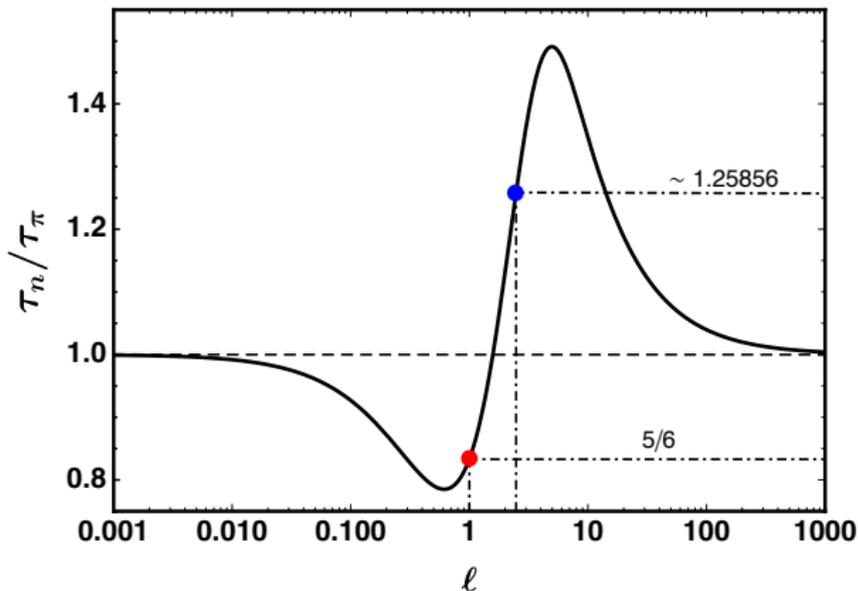
- Expanding ϕ_k in terms of these eigenfunctions and keeping the terms with zero eigenvalues leads to the collision kernel being:

$$\hat{L}\phi_k = -\frac{g\mathcal{M}}{2}\left[\phi_k - c_0 - c_1 L_{1k}^{(1)} - c_0^\mu k_{<\mu>}\right] \quad (33)$$

- Recovering the RTA limit from the exact theory leads to the form of momentum-dependent relaxation time being:

$$\tau_R(p) = \frac{2(u\cdot p)}{g\mathcal{M}}$$

Which implies $\ell = 1$ and $\kappa = 4\pi^2/(ge^\alpha)$ in the corresponding ERTA framework.

Ratio of τ_n/τ_π

- The red dot corresponds to the $\ell = 1$ case i.e, $\lambda\phi^4$ theory
- The blue dot corresponds to the case of hard sphere scattering where we see that $\tau_n/\tau_\pi > 1$

Comparision with self interacting $\lambda\phi^4$ theory for shear evolution

Coefficients	RTA $(l=0)$	ERTA $(l=1)$	$\lambda\phi^4$ results
τ_π	τ_c	$\frac{24d_g}{gn_0\beta^2}$	$\frac{72}{gn_0\beta^2}$
η	$\frac{4P\tau_c}{5}$	$\frac{16d_g}{g\beta^3}$	$\frac{48}{g\beta^3}$
κ	$\frac{n_0\tau_c}{12}$	$\frac{d_g}{g\beta^2}$	$\frac{3}{g\beta^2}$
$\delta_{\pi\pi}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
$\tau_{\pi\pi}$	$\frac{10}{7}$	2	2
$l_{\pi n}$	0	$-\frac{4}{3\beta}$	$-\frac{4}{3\beta}$
$\tau_{\pi n}$	0	$-\frac{16}{3\beta}$	$-\frac{16}{3\beta}$
$\lambda_{\pi n}$	0	$\frac{2}{3\beta}$	$\frac{5}{6\beta}$

Table: Comparison of the ERTA coefficients with exact results from $\lambda\phi^4$ theory for shear stress tensor evolution.

Comparision with self interacting $\lambda\phi^4$ theory for number diffusion

Coefficients	RTA ($\ell = 0$)	ERTA ($\ell = 1$)	$\lambda\phi^4$ results
τ_n	τ_0	$\frac{20d_g}{gn_0\beta^2}$	$\frac{60}{gn_0\beta^2}$
τ_n/τ_π	1	$\frac{5}{6}$	$\frac{5}{6}$
β_n	$\frac{n_0}{12}$	$\frac{n_0}{20}$	$\frac{n_0}{20}$
$2\lambda_{n\pi} + (\kappa/\eta)\lambda_{nn}$	$\frac{3\beta}{16}$	$\frac{19\beta}{80}$	$\frac{19\beta}{80}$
$\tau_{n\pi} + l_{n\pi}$	0	$\frac{\beta}{20}$	$\frac{\beta}{20}$
$l_{n\pi}$	0	$\frac{\beta}{20}$	$\frac{\beta}{40}$
δ_{nn}	1	1	1
λ_ω	-1	-1	-1

Table: Comparison of the ERTA coefficients with exact results from $\lambda\phi^4$ theory for number diffusion coefficients.

ERTA Phenomenology with Heavy Quarkonia

Bjorken Expansion in First-Order Hydrodynamics

- In high-energy heavy-ion collisions, the produced QGP exhibits approximate boost invariance in the longitudinal direction —modeled by the **Bjorken flow**.

- In (τ, x, y, η_s) coordinates (Milne coordinates), the four-velocity is:

$$u^\mu = (1, 0, 0, 0), \quad \text{with } \tau = \sqrt{t^2 - z^2}$$

- For the longitudinally expanding fluid, the energy-momentum conservation equation reduces to:

$$\frac{d\epsilon}{d\tau} + \frac{\epsilon + P + \Pi - \pi_\eta^\eta}{\tau} = 0 \tag{34}$$

- The $\eta - \eta$ component of shear stress tensor is given by $\pi_\eta^\eta = 4\eta\tau/3$, where η is the coefficient of shear viscosity.

- This closed set of equations governs the dissipative evolution of a longitudinally expanding QGP.

Cooling in Bjorken Expansion within ERTA

- The temperature evolution in Bjorken expansion is given by,

$$\frac{dT}{d\tau} = \frac{1}{dE/dT}\left[-\frac{\epsilon + P}{\tau} + \frac{(4\eta/3 + \zeta)}{\tau^2}\right] \tag{35}$$
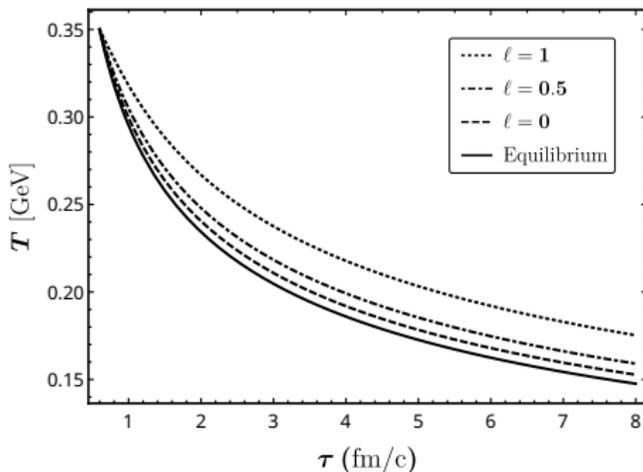
Where ζ is the coefficient of bulk viscosity.

Figure: Temperature evolution for different values of ℓ compared to the non-dissipative, ideal case

Gluon Self Energy: Real Time Formalism

- The longitudinal component of gluon self-energy with quark-loop is given as,

$$\Pi_{q,R}^L(P) = -i2N_f g_s^2 \int \frac{d^4K}{(2\pi)^4}(q_0k_0 + \mathrm{q}\cdot\mathrm{k} + m^2)\Delta_F(K)\Delta_A(Q), \tag{36}$$

- Where $\Delta_F(K)$ and $\Delta_A(Q)$ are Feynman and Advanced propagators in momentum space.
- The expression for the gluon self energy using the distribution function is given by,

$$\Pi_{q,R}^L(P) = 4\pi N_f g_s^2 \int \frac{k^2 dk d\Omega}{(2\pi)^4 E_k} f_F(x,p) \Bigg\{ \frac{2E_k^2 - p_0 E_k - \mathrm{k}\cdot\mathrm{p}}{P^2 - 2E_k p_0 + 2\mathrm{k}\cdot\mathrm{p} - i\epsilon} \tag{37}$$

$$+ \frac{2E_k^2 + p_0 E_k - \mathrm{k}\cdot\mathrm{p}}{P^2 + 2E_k p_0 + 2\mathrm{k}\cdot\mathrm{p} + i\epsilon} \Bigg\}. \tag{38}$$

- The Debye mass can then be determined by taking the limit,
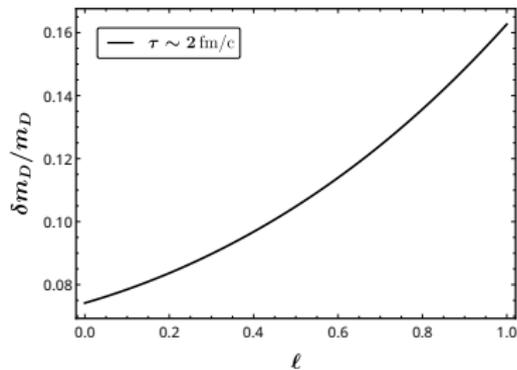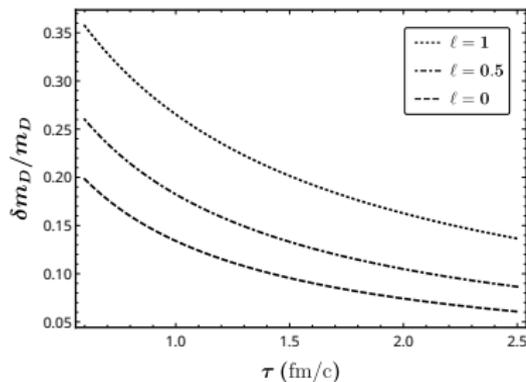
$$m_{D,q}^2 = -\Pi_R^L(p_0 = 0, p \to 0, m) \tag{39}$$

Debye mass within ERTA formalism

- In the massless limit, we determine the screening mass as,

$$
m_D^2 = \frac{g_s^2 T^2}{6}(N_f + 2N_c) + \frac{4g_s^2 t_R T^2}{3\pi^2 \tau}\left[N_c + N_f(1 - 2^{-(\ell+1)})\right]\Gamma(\ell+3)\zeta(\ell+2), \qquad \ell > -1.
\tag{40}
$$

- We find that there is a significant effect of the momentum dependence in the relaxation time, τ_R via the momentum dependence parameter, ℓ on the screening of the medium at finite temperature.

- This is summarized in the following plots,

In-Medium Quarkonium Potential

- The interaction between a heavy quark–antiquark pair $(Q\bar{Q})$ in a thermal QCD medium can be described by a complex potential:

$$V(r,T) = \Re V(r,T) + i\,\Im V(r,T) \tag{41}$$

- This is given by a modification to the Cornell potential that accurately captures the quarkonia properties in vaccum,

- The momentum-space form of the cornell potential is given by,

$$V_{\text{Cornell}}(p) = -\frac{4\pi\alpha}{p^2} - \frac{8\pi\sigma}{p^4}, \tag{42}$$

Where the first term mimics the short-range coulomb interaction and the second term mimics the long-range linear confinement term known as the **Coulomb** and **String** parts respectively.

- Taking the fourier transform of this leads us to the heavy quarkonia potential w.r.t distance r from each other.

$$V(r) = \int \frac{d^3}{(2\pi)^3}(e^{i\mathbf{p}\cdot\mathbf{r}} - 1)V_{\text{Cornell}}(p)\epsilon^{-1}(p) \tag{43}$$

Where, $\epsilon(p)$ is the dielectric permittivity in the medium which depends on the screening strength and hence, on the debye mass m_D^2

Real part of Quarkonia potential

- The real part of the potential is responsible for quarkonia "melting" and determines the dissociation temperature, T_{diss}.
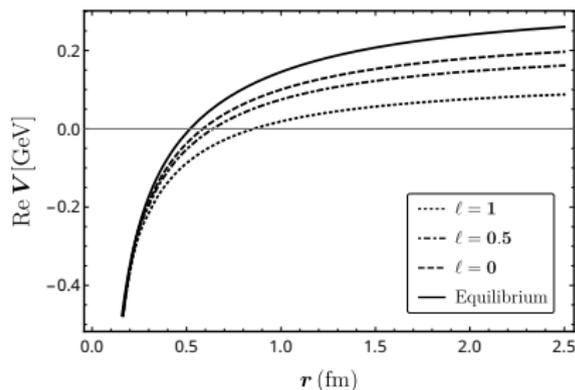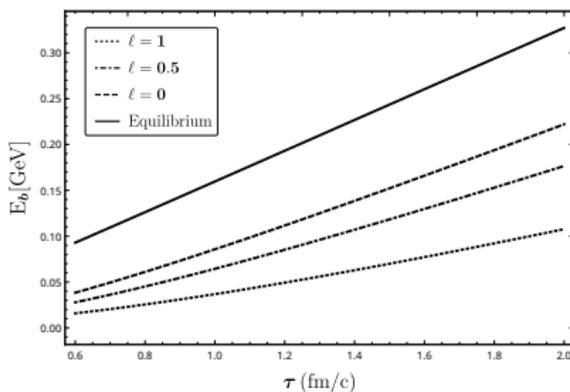
- The form of the real potential is given as,

$$
\begin{aligned}
\operatorname{Re} V(r) &= -\int \frac{d^3 p}{(2\pi)^3} \left(e^{i\boldsymbol{p}\cdot\boldsymbol{r}} - 1\right)\left(\frac{4\pi\alpha}{p^2} + \frac{8\pi\sigma}{p^4}\right)\frac{p^2}{p^2 + m_{D,R}^2} \\
&= -\alpha\, m_{D,R}\left(\frac{e^{-\hat{r}}}{\hat{r}} + 1\right) + \frac{2\sigma}{m_{D,R}}\left(1 + \frac{e^{-\hat{r}} - 1}{\hat{r}}\right),
\end{aligned}
\tag{44}
$$

Where $\hat{r} = r m_D$

- As T increases, the Debye screening suppresses the binding potential, leading to quarkonium dissociation.

Binding energy and Real Potential for J/Ψ charmonium

- Considering the ground state of $c\bar{c}$ charmonium, J/Ψ_{1s} by treating it as a hydrogen atom, we get the following results for the binding energy and the Real potential $\mathrm{Re}V(r)$,

- We observe that the introduction of momentum dependent relaxation time for the medium has a significant impact on these Quarkonia properties.

Imaginary part of Quarkonia Potential

- The imaginary part of the Quarkonia potential is responsible for in-medium decay processes and hence the **line width** of a Quarkonia state.
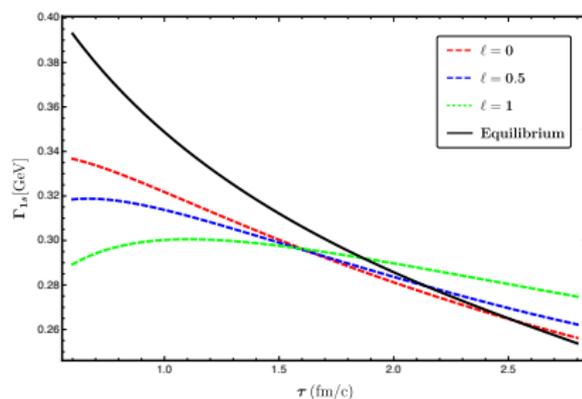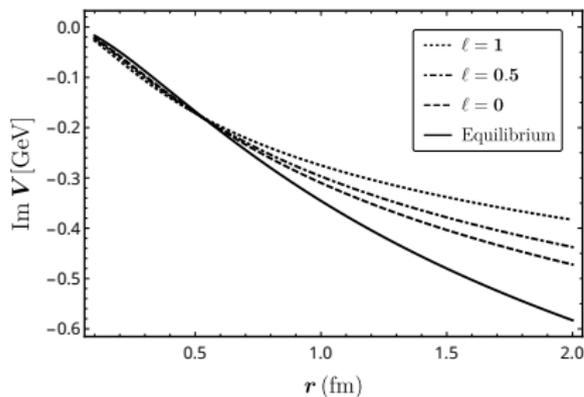
- The form of the Imaginary part is,

$$\operatorname{Im} V(r) = \int \frac{d^3 p}{(2\pi)^3} \left(e^{i\boldsymbol{p}\cdot\boldsymbol{r}} - 1 \right) \left(\frac{4\pi\alpha}{p^2} + \frac{8\pi\sigma}{p^4} \right) \frac{\pi T p, m_{D,R}^2}{(p^2 + m_{D,R}^2)^2}. \tag{45}$$

- Leads to a **thermal width** Γ of the quarkonium state: $\Gamma \sim -2\Im V$

- Even if a state is bound, a large Γ implies:
  - Fast thermal decay
  - Reduced survival probability
  - Broad peaks or disappearance in spectral functions

Result for Imaginary Part and Line width

- The Imaginary part of the potential can be split into **Coulomb** and **String** parts as,

$$\text{Im } V(r) = -T\alpha\phi_2(\hat{r}) - 2\frac{T\sigma}{m_D^2}\chi(\hat{r})$$

$$= \text{Im}_{\text{Coulomb}} + \text{Im}_{\text{String}} \tag{46}$$

- We again observe that the momentum dependence in the relaxation time has a significant impact on these properties.

Conclusion and Future prospects

- The one to one correspondence between the value of ℓ and the type of underlying theory was established for a few simple cases.

- We observed that there is significant modification to both the medium as well as heavy Quarkonia properties when a momentum dependent relaxation time is considered.

- The underlying theories for various ℓ and $\tau_0(x)$ should be explored.

- The form of $\ell(x)$ as a function of space-time in the fireball should be determined from a model-to-data analysis to understand the interaction dynamics in the QGP phase's evolution.

- Extension to spin-hydrodynamics using a momentum dependent relaxation time for the spin component has already been done for first order and hence, the second order case should be explored.

- The dilepton spectra produced in heavy ion collision experiments should be predicted using the ERTA formalism

- Heavy quark energy loss can be studied as it passes through this ERTA modified hydrodynamic evolution and its R_{AA} can be determined for various ℓ and $\tau_0(x)$.

Thank you!