Black hole scattering, Feynman graph expansions and Calabi-Yau geometries

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University of Sheffield
LEVERHULME TRUST
MODERN FOUNDATIONS of SCATTERING AMPLITUDES

Perturbative QFT and its limitations

$$Z[J] = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\int \mathrm{d}^D x(\mathcal{L} + J\phi)\right] \ .$$

In ϕ^4 with Lagrangian $\mathcal{L} = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_u \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$.

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Generally the connected physical correlators are obtained by functional derivates w.r.t to the sources $J(x)$ ($W[J] = -i \log Z[J]$)

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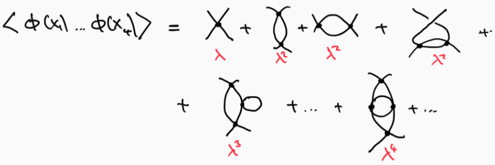
In interacting theories $\lambda \neq 0$ this is expanded asymptotically in Feynman graphs



Examples of physical processes where high loop orders pay off

$\langle\phi(x_1)..\phi(x_n)\rangle$ is an asymptotic series in the coupling constant λ. Contributions to ℓ'th loop order diverge with $CA^\ell \ell!$ in QFT. One can conclude that the order of optimal truncation is $1/(A\lambda)$.

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Examples for physically demanded higher loop calculations:

1.) The electromagnetic coupling $\alpha \sim 1/137$ is small and the anomalous magnetic moment of the electron

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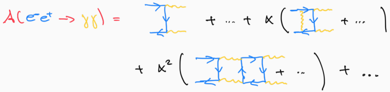
2.) In the post Minkowskian approximation to black hole scattering the expansion parameter is the gravitational coupling $(GM/bc^2) \ll 1$ and perturbation theory makes likewise very precise predictions e.g. for the wave-forms .

Collider process: Probability for e^- e^+ to annihilate to two photons $P(e^- e^+ \to \gamma\gamma) \sim |\mathcal{A}(e^- e^+ \to \gamma\gamma)|^2$, $\alpha \sim \frac{1}{137}$



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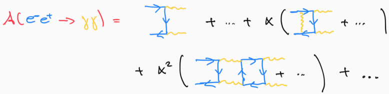
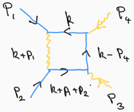


The scalar integral relevant e.g. for the one loop (box) integral $I(\epsilon, m^2, p_i \cdot p_j)$ (Propagators $1/D_l$ with $D_l = \frac{1}{k_l^2 - m^2 + i \cdot 0}$) is:



$\sum_{i=1}^{4} p_i = 0$ momentum conservation

$$\sim \int \frac{d^D k}{(k^2-m^2)\,(k+p_1)^2\,((k+p_1+p_2)^2-m^2)((k-p_4)^2-m^2)}$$

Dimensional regularisations and the functions we need to know

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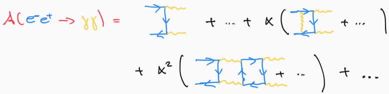

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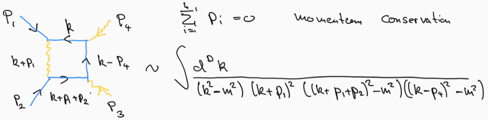

Dimensional regularisation: $D = D_0 - 2\epsilon \to I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with $I_k(m^2, p_i \cdot p_j)$ functions of (dim' less ratios of) masses and Lorentz invariant products of the external momenta that we need to know!

Feynman Integrals and Periods: What are these functions?

Close relation between Feynman integrals and periods and chain integrals in algebraic varieties in particular Calabi-Yau varieties: I. Gel'fand, S. Bloch, P. Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. Mc Leod, M. Hippel, M. Wilhelm, J. Broedel, L Trancredi, S. Müller-Stach, ... + 248 cits. in [3]

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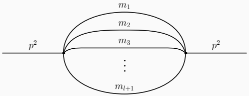

with props on-shell $1/D_k \to 2\pi i \delta^+(D_k)$ in $D_{cr} = 2$.

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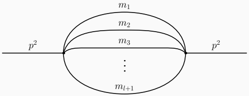

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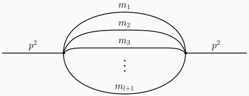

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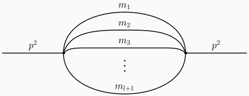

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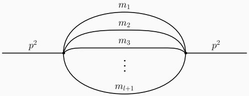

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A direct way to see that:

Write Banana integral in Feynman-Symanzik parametrisation:

$$I(z_i) = \int_\Gamma \oint_{S^1} \frac{\mu_\ell}{\mathcal{F}(t,\xi_i;x)}$$

with $\mathcal{F}(t,\xi;\underline{x}) = \left(t - \left(\sum_{i=1}^{\ell+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{\ell+1} x_i^{-1}\right)\right)$ and $\mu_\ell = \sum_{k=1}^{\ell} \mathrm{d}\, x_1 \wedge \ldots \widehat{dx_k} \ldots \wedge dx_{\ell+1} / \prod_{i\neq k}^{\ell+1} x_i$ an ℓ dimensional measure. $t = p^2/\mu^2$, $\xi_i = m_i/\mu$ and $z_i = m_i^2/p^2$.

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Comparison with Batyrevs construction of CY n-folds reveals that

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is an $(n = \ell - 1)$ dimensional CY hyper surface in the toric ambient space $\mathbb{P}^\ell_{\Delta^*}$ defined by the polyhedra Δ^*. (Δ^*, Δ) is a reflexive pair with Δ the Newton polyhedron of $\mathcal{F}$ and $\Omega = \oint_{S^1} \frac{\mu_\ell}{\mathcal{F}(t,\xi_i;x)}$ is Griffiths residuum expression for the $(n,0)$ form.

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What is the key advantage of the geometric reformulation?

The periods $\Pi_\Gamma(\underline{z})$ fulfil Picard-Fuchs differential equations. For the equal mass case $z_1 = \ldots = z_{\ell+1} = z = m^2/p^2$ and $\theta = z\frac{d}{dz}$ a single ℓ-th order one parameter ordinary differential equations

$$\mathcal{L}^{(\ell)}\Pi_{\Gamma_k}(z) = 0, \qquad k = 1, \ldots, \ell = \dim(H_{\ell-1}(M_{\ell-1}, \mathbb{Z})) = h_{\ell-1} \ .$$

The ℓ solution define a extended classes of higher transcendental (period) functions generalising algebraic and elliptic functions that will govern higher order perturbations theory in great generality.

ℓ=1	$\mathcal{L}^{(1)} = 1 - 2z - (1-4z)\theta,$
2	$\mathcal{L}^{(2)} = 1 - 3z - (2-10z)\theta + (1-z)(1-9z)\theta^2,$
3	$\mathcal{L}^{(3)} = 1 - 4z - (3-18z)\theta + (3-30z)\theta^2 - (1-4z)(1-16z)\theta^3,$
4	$\mathcal{L}^{(4)} = 1 - 5z - (4-28z)\theta + \left(6 - 63z + 26z^2 - 225z^3\right)\theta^2 -$
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Geometric reformulation

The fact $\vec{\Pi}(\underline{z})$ fulfil the Picard-Fuchs differential $\{\mathcal{L}_s^{(\ell)}\}\Pi_{\Gamma_k} = 0$ is equivalent to fact that they are flat sections of the Hodgebundle:

$$\nabla_k \vec{\Pi} = (\partial_{z_k} - A_k(\underline{z}))\vec{\Pi} = 0, \quad k = 1, \ldots, h^{n-1,1}(M),$$

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- The right section –Feynman integral– is found by integral global monodromy structure $\mathfrak{M}_i \in \begin{cases} \mathrm{Sp}(h_n, \mathbb{Z}) & n \text{ odd} \\ \mathrm{O}(h_n, \mathbb{Z}) & n \text{ even} \end{cases}$ and its singular behaviour at ultra violett, onshell, infrared limits, which correspond to degenerations of the Hodge structure of the CY: Maximal unipotent points, conifolds etc .

Solution to the full non-maximal cut Feynman integral

We saw that maximal cut Feynman integral is identified with the periods over closed cycles $\Gamma_k \in H_n(X, \mathbb{Z})$. The actual Feynman Integral $I_\ell^{full}(z)$ is rather a chain integral of Ω over open cycles in relative cohomology, so called Griffiths normal functions. They fulfil the same Picard-Fuchs equation with an inhomogeneity, which has a rather straightforward physical interpretation:

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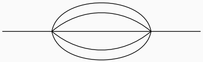

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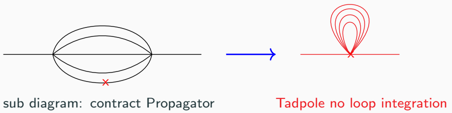

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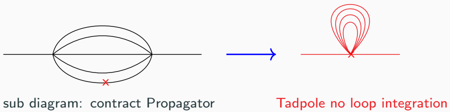

Leading to an inhomogeneous diff eq for the Banana integral

$$\mathcal{L}^{(l)} I_\ell^{full}(z) = -(l+1)!z$$

Solution to the inhomogeneous solution [3]

,

- The analytic structure of $I_\ell^{full}(z)$ is hence by fast convergence and analytic continuation extremely well under control:

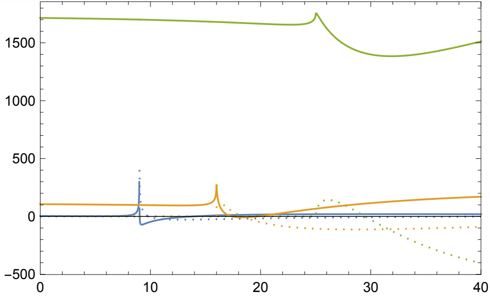

Figure 1: Loop order $\ell = (2,3,4) \sim$ (blue, orange, green); solid= real value; dashed imaginary value; $t = 1/z = p^2/m^2$; $t = (\ell+1)^2$ nearest conifold, $t = \infty$ maximal unipotent mondromy point. 10 digits in μs

Why are Banana graphs relevant? [1,2]

It looks that the Banana graphs with its vertices of arbitrary valence are not very relevant to the calculation of standard model processes. However this is wrong.

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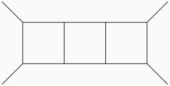

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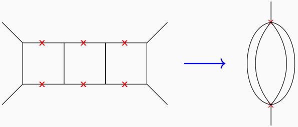

sub diagram: contract Propagators

Banana creates one inhomogeneity of douple box

The iterated integrals of banana integrals (K3-periods) appear generically in 3-loop standard model graphs, e.g. in the calculation of the anomalous mag. mom. a_e Forner, Nega, Tancredi (04).

Summary: Dictionary Feynman integrals and geometry

Perturbative QFT	Geometry X	Differential eq.
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ-deformed)	Homogeneous Gauss Manin $(d - A(z))\underline{\Pi} = 0$
	$\circlearrowleft$ Monodromy group $\in \Gamma(\mathbb{Z})$; irreducible ?	
actual Feynman integral	Chain integral (ϵ-deformed)	Inhomogeneous Gauss Manin connection $(d - A(z))\underline{\Pi} = B(z)$

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CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with $\Omega = \frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.

Periods on CY n-folds are generalisations of elliptic integrals

Periods integrals:

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z})$$

define a non-degenerate pairing $\Gamma \in H_n(M_n, \mathbb{Z})$ and $H^n(M_n, \mathbb{Q})$. On $H_n(M_n, \mathbb{Z})$ we have an intersection pairing $\Sigma : H_n(M_n, \mathbb{Z}) \times H_n(M_n, \mathbb{Z}) \to \mathbb{Z}$. If n is odd Σ is antisymmetric and can be made symplectic.

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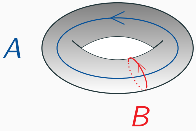

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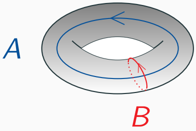

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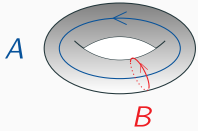

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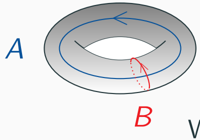

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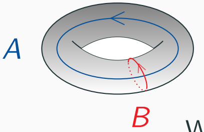

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$$\mathcal{L}^{(2)} \int_\Gamma \Omega = \left[(1-z)\partial_z^2 + (1-2z)\partial_z - \frac{1}{4}\right] \int_\Gamma \Omega = 0 \ .$$

The curve $p_3 = 0$ is known as the Legendre curve.

Generalised Gauss Manin connection and sub-sectors

We have seen that important maximal cut Feynman integrals are CY periods and that the full Feynman graphs are determined by by inhomogenous Gauss-Manin connections, whose inhomogeneities are determined by subgraphs.

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The key way to systematize this and obtain results in any dimension is to introduce so called master integrals for ℓ-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ Johannes Henn (13)

$$I_{\underline{\nu}}(\underline{z}, D) := \int \prod_{r=1}^{\ell} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \tag{1}$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the j^{th} momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses. Subject to momentum conservation the q_j are linear in the external momenta and the internal loop momenta k_r. We define $\epsilon := \frac{D_{cr} - D}{2}$.

Master Integrals and integration by parts relations

Note that the propagator exponents in the master integrals are arbitrary semi positive and together with $D_{cr} \in \mathbb{Z}$ span a semi positive cone in a lattice $(\underline{\nu}, D_{cr}) \in \mathbb{Z}_{\geq 0}^{p+1}$.

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$$\int \frac{\partial}{\partial x_k} \left(\frac{Q(x,z)\mu}{\mathcal{F}^r} \right) = 0 \to \int \frac{\mu r Q(x,a) \partial_k \mathcal{F}}{\mathcal{F}^{r+1}} = \int \frac{\mu \partial_k Q(x,a)}{\mathcal{F}^r} \ .$$

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The master integrands define a finite cohomology, which sometimes extends the cohomology $H^{\ell}(M_{\ell}, \mathbb{Z})$ of a CY M_{ℓ}.

Master Integrals and integration by parts relations

In particular there is a finite region in $\mathbb{Z}_{\geq 0}^{p+1}$ that contains all relevant master integrals. I.e. in those master integrals one can express all derivatives w.r.t. the z_k as a linear combination rational coefficients by the IBP relations.

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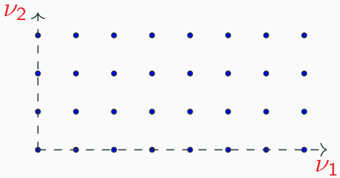

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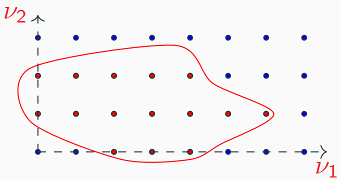

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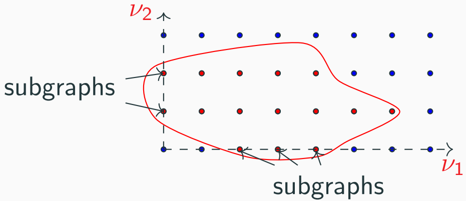

A complete set of IBP relations corresponds to the complete Picard Fuchs ideal or Gauss-Manin connection for the period integrals.

IBP relation summary:

Hence the IBP relations characterise a suitable finite set of master integrals

$$\underline{I}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \ ,$$

in a finite region in $\mathbb{Z}_{\geq 0}^{p+1}$, by a generalised first order Gauss Manin connection

$$\boxed{d\underline{I}(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon)\underline{I}(\underline{x}, \epsilon)}$$

$\epsilon = (D_{cr} - D)/2.$

Basis change on master integrals ϵ-factorized form

$$\underline{I}(\underline{x},\epsilon) \rightarrow \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

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The use of the ϵ factorized form

$$\nabla_\epsilon I(z,\epsilon) = (d_z - \epsilon \mathbf{A}(\mathbf{z}))I(z,\epsilon) = 0$$

Suppose we have boundary conditions $I(z_0, 0) = I_0$ then we can write the solution vector $I(z,\epsilon)$ as path ordered exponential

$$I(z,\epsilon) = \mathbb{P}\exp[\epsilon \int_{z_0}^{z} \mathbf{A}(\mathbf{z})dz]I_0$$

in terms of iterated integrals in the form

$$I(z,\epsilon) = \left[\mathbf{1} + \epsilon \int_{z_0}^{z} \mathbf{A}(\mathbf{z}')dz' + \epsilon^2 \int_{z_0}^{z}\int_{z_0}^{z'} \mathbf{A}(\mathbf{z}')dz'\mathbf{A}(\mathbf{z}'')dz'' + \ldots\right] I_0$$

For example for the 1 loop diagrams this formalism gives rise to the *function class* of multiple poly logarithms. For higher loop amplitudes this class generalizes to iterated Calabi-Yau periods.

Applications to General Relativity

"Emergence of Calabi Yau manifolds in high precision black hole scattering"

Driesse, Jakobsen, AK, Mogull, Nega, Plefka, Sauer

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"Calabi-Yau periods for black hole scattering in classical general relativity"
AK, Nega,Plefka,Sauer

Motivation: Experimental gravitational wave detection: LIGO, Virgo, KAGRA, ...

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Exact approaches to General Relativity

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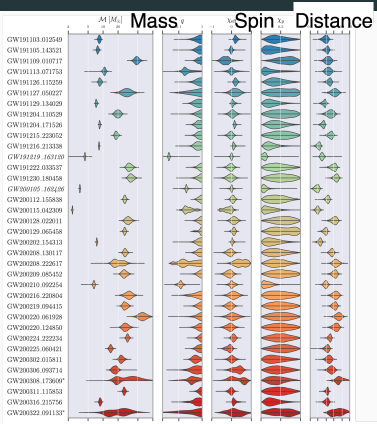


Parameter estimation [Phys.Rev.X 13 (2023) 4, 041039]

Aim: Analytical description in an effective Field theory approach.

Effective field theory

Quantum Gravity as an effective Theory: Donoghue (1993)

Classical Gravity from Amplitudes: Cheung, Rothstein, Solon (2018); Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove (2018); Bern, Cheung,Roiban, Shen, Solon, Zeng (2019)

Frameworks for classical gravity from amplitudes: KMOC: Kosower, Maybee, O'Connell (2018)

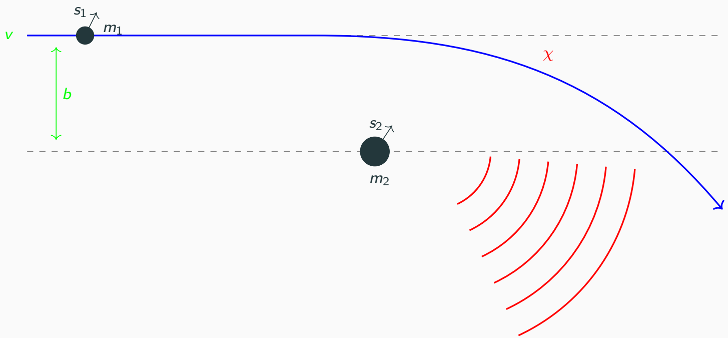
Heavy Black Hole Effective Theory: Damgaard, Haddad, Helset (2019)

Eikonal Operator: Di Vecchia, Heissenberg, Russo, Veneziano (2021)

Heavy Mass EFT: Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini (2023)

Treat scattering of two black holes (BH) as starter to the description of BH mergers in a perturbative expansion effective QFT long range decribtion to gain some analytic control.

Worldline QFT approach to General Relativity



Worldline Quantum Field Theory approach to General Relativity

The action for the scattering process Kälin, Porto '06, Dlapa, Kälin, Liu, Porto '12, Jakobsen, Mogull, Plefka, Steinhoff '21,'23,'24

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^{\mu} \dot{x}_i^{\nu}\right] + \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} R(g) + S_{\mathrm{g.f.}}^{\mathrm{Donder}}$$

is expanded in Post Minkowskian (PM) approximation in the Worldline Quantum Field Theory (WQFT) approach around the non-interacting background configurations

$$x_i^{\mu} = b_i^{\mu} + v_i^{\mu}\tau + z_i^{\mu}(\tau)\,, \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}(x)\,.$$

The dynamical fields z_i^{μ} and $h_{\mu\nu}$ are integrated out in the path integral. $\langle x_i^{\mu} \rangle$ fulfills the classical e.o.m.

Worldline QFT approach to General Relativity

The goal is to calculate from the initial data: the impact parameter $b^\mu = b_1^\mu - b_2^\mu$ and the incoming velocities v_1, v_2 the physical quantity of interest, which is the radiation induces change in the momentum say $\Delta p_1^\mu = m_1 \int \mathrm{d}\tau \ddot{x}(\tau)$ of the first particle.

In the PM approximation the latter can be expanded in the gravitational coupling G

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\,\mu}(x) \ .$$

At each order the contributions $\Delta p^{(n)\,\mu}(x)$ are calculated in the WQFT approach in the Swinger-Keldysh in-in formalism in terms of a Feynman graph expansion with retarded propagators. Here $x = \gamma - \sqrt{\gamma^2 - 1}$ with γ the Lorentz factor of the relative velocities is the only parameter.

In the 4PM approximation the Feynman integral in the 1SF sector, contributing to the $\frac{m_1}{m_2}$ order in the corresponding expansion

(K3.0)

involve symmetric products of elliptic function, which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x)\,.$$

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In the 5PM approximation we find in [8] that in the 5PM approximation the following graphs in the 1SF sector

(CY3)

Worldline QFT approach to General Relativity

The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0, \ j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$.

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The periods of the above $K3$ and CY threefold fullfill the differential equation and determine all special functions that are necessary to solve for $\Delta p^{(5)\,\mu}(x)$ and the radiative energy in the 1SF sector. Note that all solutions are either symmetric or Hadamard products of the second order Legendre differential equation.

ϵ-factorisation for the CY blocks in 5PM 1SF

To derive the ϵ-factorization we split the periods matrix into its semi-simple and unipotent part. The unipotent part satisfies

$$(\mathrm{d} - A^{\mathsf{u}}(x))W^{\mathsf{u}}(x) = 0\,, \quad A^{\mathsf{u}}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^{\mathsf{u}}(x)$ is nilpotent, i.e. $(A^{\mathsf{u}}(x))^4 = 0$.

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$$G_1(\mathrm{x}) = -\int_0^{\mathrm{x}} \frac{24576\mathrm{x}'\left(1+256\mathrm{x}'^4\right)}{(1-256\mathrm{x}'^4)^2}\frac{\varpi_0(\mathrm{x}')^2}{\alpha_1(\mathrm{x}')}\mathrm{dx}'\,,$$

$$G_3(\mathrm{x}) = \int_0^{\mathrm{x}} \frac{\mathrm{x}'}{1-256\mathrm{x}'^4}\frac{G_1(\mathrm{x}')\alpha_1(\mathrm{x}')^2}{\varpi_0(\mathrm{x}')^2}\mathrm{dx}'\,.$$

Here $\alpha_1 = (\theta\varpi_1/\varpi_0)^{-1}$ and $t = \varpi_1/\varpi_0$ is the affine coordinate related to the mirror map.

ϵ factorized Gauss-Manin connection for the 5PM 1SF [8][9]

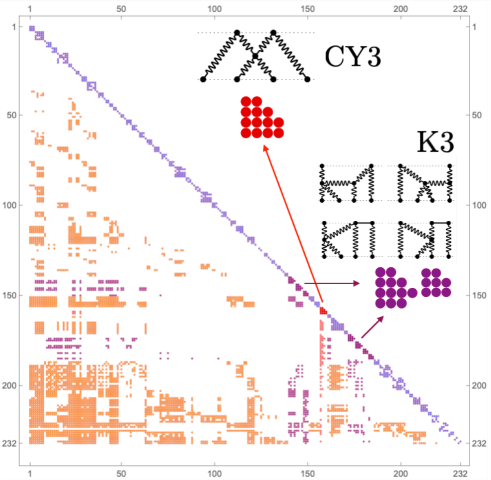

Figure 2: Non-zero entries of the 232×232 extended Gauss-Manin connection matrix $A^{best}(x)$ of odd parity. The not-magnified diagonal sectors give rise to polylogarithms.

Results in the 5PM 1SF Sector

After imposing the physical boundary conditions, determining the I_0 we can extract all regular physical contribution to the deflections angle and the radiation loss and eventually the *gravitational waveform*.

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The latter are *analytically* given in terms of iterated integrals of Calabi-Yau 3-fold periods, K3-periods and polylogarithms available online.

Scattering angle versus impact parameter up to 5PM 1SF order

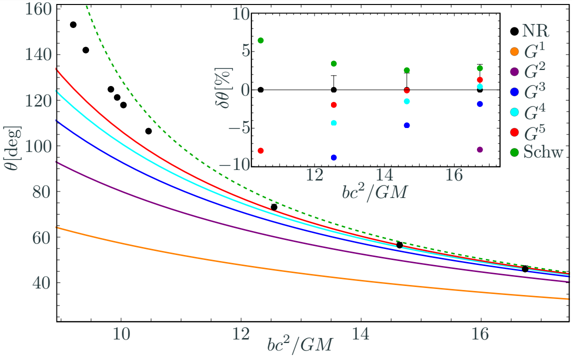


Figure 3: Scattering angle versus impact parameter

Radiation loss versus velocity

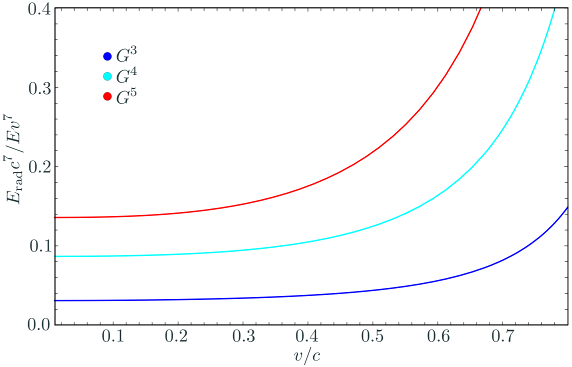
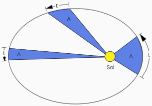
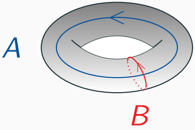


Figure 4: Radiation loss versus velocity

Conclusion and Outlook





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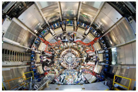
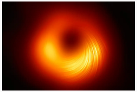
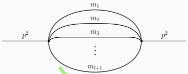
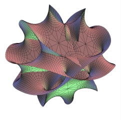


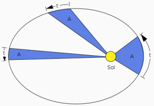


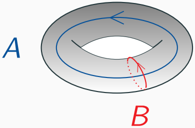




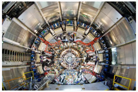
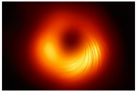
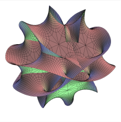
Physics

Mathematics





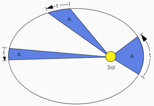
Conclusion and Outlook

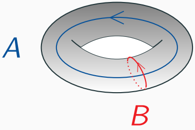




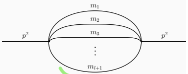
String Theory

Physics

Mathematics



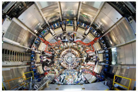


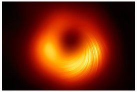


Enumerative Geometry

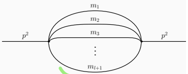
Conclusion and Outlook

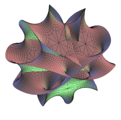
Black Hole Physics





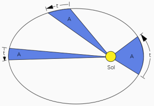
String Theory



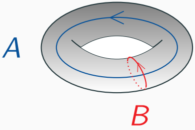


Physics

Mathematics



Arithmetic Geometry



Enumerative Geometry

Based on work with

Kilian Bönisch, Claude Duhr, Fabian Fischbach, Christoph Nega, Reza Safari, Lorenzo Tancredi:

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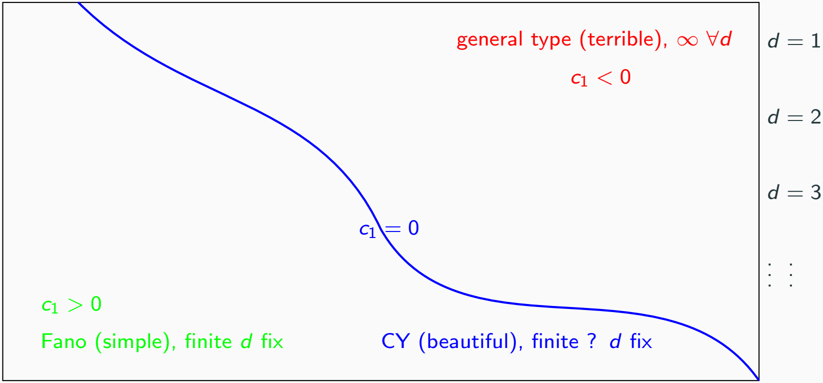
Philosophical Remarks: Kodaira map of algebraic varieties



Philosophical Remarks: Kodaira map of algebraic varieties

$\ell = 0 \quad \ell = 1 \quad \ell = 2 \quad \ell = 3 \quad \cdots$
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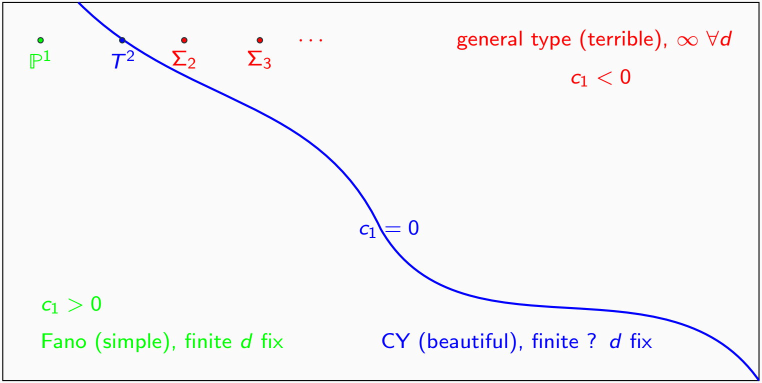

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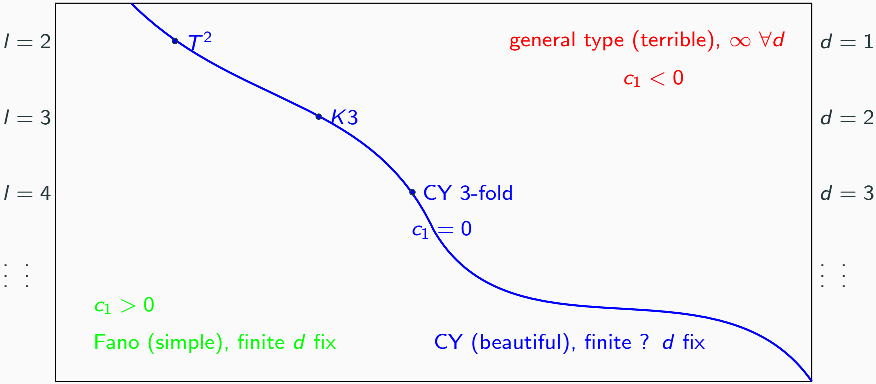

Monodromy and iteration over dimensions

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Call $\mathrm{Sym}^k(\mathcal{L}^{(n+1)})$ the operator which are solve by the kth symmetric product of periods and $\wedge^k(\mathcal{L}^{(n+1)})$ the operator that is solved by the k minor W_{I_k,J_k} of W.

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One parameter K3 operators $\mathcal{L}^3(K3)$ of type $(1,1,1)$ are always symmetric squares the ones of elliptic curves E

$$\mathcal{L}^3(K3) = \mathrm{Sym}^2(\mathcal{L}^2(E))$$

Monodromy and iteration over dimensions

One parameter CY 4-fold operators $\mathcal{L}^5(M_4)$ of type $(1,1,1,1,1)$ up to an $\mathbb{Z}_2$ choice related to picking a spin representation given by

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Slightly more complicated one has that quite generally

$$\mathcal{L}^4(M_3) = \mathrm{Hadamard}^{(2)}\mathcal{L}^2(E)\ .$$

E.g. in the 5PM 1SF calculation all CY operators are obtainable from $\mathcal{L}^2(y^2 = x(x-1)(x-z))$ by the above constrs! In which sense do we need Higher dimensional CY period geometries?

A hint from Topological string theory

CY 3-folds are special for topological field theory as $F_g \neq 0 \; \forall g$ due to $\dim_{vir} \overline{M}_g(hol.\ maps, M_3) = 0$ for all g.

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Using direct integration the F_g are solved in terms of rational functions $F_g(S^{zz}, \tilde{S}^z, S, \mathbb{Q}[\underline{z}])$ in terms of the propagators. The latter are given by $\wedge^2\mathcal{L}^4(M_3)$, i.e periods of CY 4folds as

$$\begin{aligned} S^{zz} &= -\frac{1}{C}\left(\frac{w_{0,2}}{w_{0,1}} - q^z_{zz}\right), \qquad \tilde{S}^z = -\frac{1}{C}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right), \\ \tilde{S} &= -\frac{1}{2C}\left(\frac{w_{1,3}}{w_{0,1}} - \partial_z q_{zz} - q_{zz}q^z_{zz}\right) + \frac{\partial_z C}{2C^2}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right) - \frac{q^z_z}{2}\ . \end{aligned} \tag{2}$$

with $w_{i,j}$ sub determinants of the Wronskian and $C, q_* \in \mathbb{Q}[\underline{z}]$ Gu, Kashani-Poor, Marino, AK, e-Print: 2305.19916.

A hint from Topological string theory

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Maybe the topological string is a toy explanation for the conundrum. Why do higher dim CY n-folds instead of higher genus Riemann surfaces appear.