Revisiting Instantons Effects in the Broken Phase

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(Gauge-Theory) Instanton

Topological soliton in non-abelian (e.g. $\mathrm{SU}(2)$) gauge theories on 4D Euclidean space.

- Classified by **winding number** $w = 0, \pm 1, \ldots$
- Local minima of the action with

$$S_{\text{Euclidean}} = \frac{8\pi^2}{g^2}|w|\,.$$

- Winding Number:

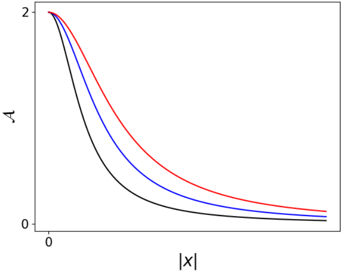
$$\begin{aligned} w &= (\text{constant}) \times \int \mathrm{d}^4x \;\mathrm{tr}\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\ &= 0,\ \pm 1,\ \pm 2,\ \ldots \qquad (\text{ for } S_{\text{Euclidean}} < \infty). \end{aligned}$$

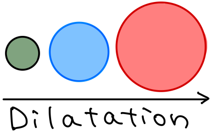
Dilatation Zero Mode

The action is classically indepenednt of the **size** ρ.

i.e. dilatation is a **zero mode**, which do not cost the action.

Instanton (in singular gauge): $A^a_\mu = \bar{\eta}^a_{\mu\nu}\frac{x_\nu}{x^2}\mathcal{A}(|x|)\,, \quad \mathcal{A}(|x|) = \frac{2\rho^2}{x^2+\rho^2}\,.$



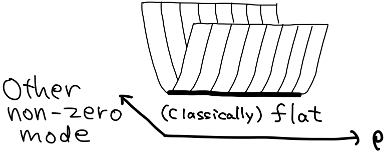


Semiclassical Approximation

Note: $\hbar$-expansion $\simeq g^2$-expansion $\rightarrow$ breakdown in $g \gg 1$.

$\hbar$-expansion of $Z = \int \mathcal{D}A \exp(-S[A]/\hbar)$:

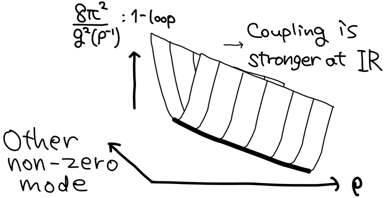
0. $S \simeq$ Classical minima.
1. $S \simeq$ Classical minima
 $+$ [Field fluctuations around the minima]^2.



Integrating out fluctuations [e.g. G. 't Hooft (1976)] ,

bare g $\rightarrow$ renormalized $g(\rho^{-1})$.

i.e. ρ is **not** a flat modulus in quantum level.



Instanton in Broken Phase

What If Gauge Symmetry is Spontaneously Broken? [I. Affleck (1980)]

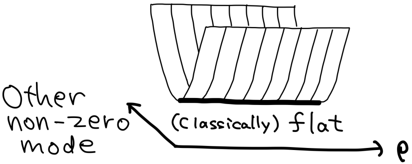
$$\mathcal{L}_{\mathrm{YMH}} = \frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu} + |D_\mu H|^2 + \frac{\lambda}{4}(H^\dagger H - v^2)^2\,. \quad \text{(SU(2) gauge theory with SU(2) doublet } H\text{.)}$$

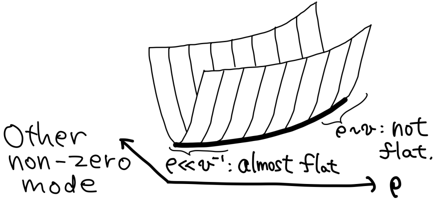
- **Instanton action increases as the size increases.**

  Why?: $|H| \sim v$ at $|x| \gtrsim v^{-1}$ $\rightarrow$ $|A_\mu H|^2$ picks up the instanton tail at $|x| \gtrsim v^{-1}$.

- There is **no strict minimum** with non-trivial winding number.

- However, small instantons ($\rho \ll v^{-1}$) “do not see” symmetry breaking, effectively.





Constrained Instanton

- Instantons with $\rho \lesssim v^{-1}$ have non-negligible effects.
- To compute their effects, those (non-minimal) configurations should be extracted.

Minimization with constraint (of size) helps us picking up $\rho \neq 0$ configurations.

Solution at $|x| \ll v^{-1}$ in singular gauge:

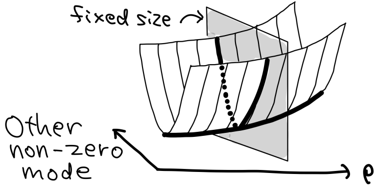
[I. Affleck (1980)]

$$A^a_\mu = \bar{\eta}^a_{\mu\nu} \frac{x_\nu}{x^2} \left[\frac{2\rho^2}{x^2 + \rho^2} + \mathcal{O}(\rho^2 v^2) \right],$$

$$H = \begin{pmatrix} 0 \\ v \end{pmatrix} \left[\sqrt{\frac{x^2}{x^2 + \rho^2}} + \mathcal{O}(\rho^2 v^2) \right].$$

$$S_{\mathrm{YMH}} = \frac{8\pi^2}{g^2} + 2\pi^2 \rho^2 v^2 + \mathcal{O}(\rho^4 v^4) .$$

Overlapping between instanton tail and $|H| \sim v$, at $|x| \gtrsim v^{-1}$.



Constrained Instanton: Procedure

Minimization with Constraint at Classical Level: **Lagrange Multiplyer Method**

$$S_{\text{total}}[A, H; \sigma] = S_{\text{YMH}}[A, H] + \sigma(S_{\text{constraint}} - f(\rho)) ,$$

$$S_{\text{constraint}} = \int \mathrm{d}^4 x \, O_{\text{constraint}} .$$

Procedure:

- **Lagrange Multiplier Method**:
  For a fixed value of $f(\rho)$, find a stationally point of $S_{\text{total}}[A, H; \sigma]$.

Constraint term:

- An example:
  $$O_{\text{constraint}} = \left(\tfrac{1}{2} \operatorname{Tr} F\tilde{F}\right)^2 , \qquad \int \mathrm{d}^4 x \, O_{\text{constraint}} = \frac{384\pi^2}{7} \rho^{-4} \Big(1 + O\big(\rho^2 v^2\big)\Big)$$

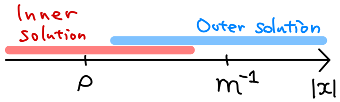
Constrained Instanton: Explicit Construction

Profile Functions $\mathcal{A}$ and $\mathcal{H}$:

$$A^a_\mu = \bar{\eta}^a_{\mu\nu} \frac{x_\nu}{x^2} \mathcal{A}(|x|)\,, \qquad H = \begin{pmatrix} 0 \\ v \end{pmatrix} \mathcal{H}(|x|)$$

Recipe for analytically expanded solution of $\mathcal{A}$ (similar for $\mathcal{H}$): (Note: $m = gv/\sqrt{2}$)

1. Rewrite EOM at inner ($|x| \ll m^{-1}$) and outer ($|x| \gg \rho$) regions in terms of dimensionless variables, $|x|/\rho$ and $m|x|$, respectively.



2. Solve at inner/outer regions, order by order of perturbation in $\rho m \ll 1$.

3. **Matching** inner/outer solutions at $\rho \ll |x| \ll m^{-1}$.

Constrained Instanton: Explicit Construction

Executing the recipe:

1. Write down the inner/outer EOM.

2. Leading-order (LO) in ρm:

$$\mathcal{A}(|x|) = \begin{cases} \dfrac{2\rho^2}{x^2+\rho^2} & |x| \ll m^{-1} \\ (\text{const}) \times K_2(m|x|) & |x| \gg \rho \end{cases} \qquad m = gv/\sqrt{2}\,.$$

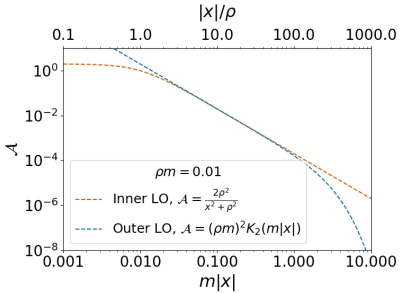
("Massive free field" solution: $K_2(m|x|) \sim \begin{cases} \dfrac{2}{m^2x^2} & |x| \ll m^{-1} \\ \sqrt{\dfrac{\pi}{2mx}}\, e^{-m|x|} & |x| \gg m^{-1} \end{cases}$)

Higher-order (in ρm) inner/outer solutions can be obtained iteratively.

3. Matching of LO (in ρm) inner/outer solutions

$$\mathcal{A}(|x|) = \begin{cases} \dfrac{2\rho^2}{x^2+\rho^2} & |x| \ll m^{-1} \\ c_0 \times K_2(m|x|) & |x| \gg \rho \end{cases}$$

Matching $\rightarrow$ $c_0 = (\rho m)^2$

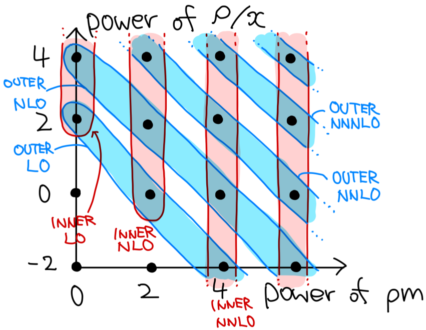


Do inner/outer solutions match consistently at higher-order, too?

- M. Nielsen and N. K. Nielsn (1999) indicated that **matching fails** at the next order ($[\rho m]^2$-order corrections), for **almost all the conventionally-used choices** of constraint.

Our Work: Matching

Matching can be done through **double expansion** in ρm and ρ/x.



Example: LO (of ρm) outer solution

Outer LO

$$\mathcal{A}(|x|) \sim (\rho m)^2 K_2(mx)$$

$$= (\rho m)^2 \left[\frac{2}{m^2 x^2} - \frac{1}{2} + \mathcal{O}\left(m^2 x^2\right) \right] \quad (x \ll m^{-1})$$

$$= \frac{2\rho^2}{x^2} - \frac{1}{2}(\rho m)^2 + \mathcal{O}\left(\rho^4 m^4\right)$$

Inner LO, Inner NLO, Inner NNLO

- **The matching can be consistently done** at the higher-order.
- Particulary, **we constructed the solution to $(\rho m)^2$-order, explicitly**.
- We also checked the consistency of our expansion, using numerical solutions.

Summary

- **Constrained instantons** are instanton-like configurations and are the minima of action on the constrained surface of a fixed size.
- Nielsen and Nielsen (1999) indicated that ρv -expanded consistent solutions do not exist for almost every constraint, due to **inner**/**outer** **mismatch**.
- We clarified that the **matching works well almost independently to the choice of the constraint**, if we treat the expansion parameters properly.
- We derived explicit $(\rho m)^2$ corrections and a framework covering higher orders.

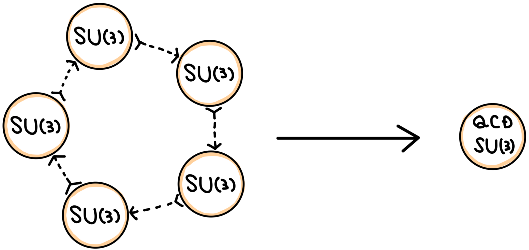
BACKUP

Axion Mass Enhancement?: [P. Agrawal & K. Howe (2017)]

Product group toy model: Gauge $[\mathrm{SU}(3)]^{n_s}$ symmetry with n_s axions.

$$\mathcal{L} = \sum_{i=1}^{n_s} \mathrm{tr}\left[-\frac{1}{2}F_iF_i + \left(\theta_i + \frac{a_i}{f_i}\right)\frac{g_i^2}{16\pi^2}F_i\tilde{F}_i\right] + (\text{scalars for symmetry breaking})$$

Bi-fundamental scalars break gauge symmetry: $[\mathrm{SU}(3)]^{n_s} \to \mathrm{SU}(3)_{\mathrm{QCD}}$



Application of Instanton in Broken Phase

Small instanton effects from each $\mathrm{SU}(3)_i$: $\Delta V(a_i) \propto -v^4 \exp\left(-\frac{8\pi^2}{g_i^2(v)}\right) \cos\left(\frac{a_i}{f_i}\right).$

(v: breaking scale)

Axion mass enhancement in a toy model. [P. Agrawal & K. Howe (2017), C. Csaki et al (2020)]

Note:

There is **a difficulty** in enhancing the axion mass by small instantons, while "naturally" solving the strong CP problem [TA, M.Ibe, S.Shirai and K.Watanabe (2024)].

Procedure of Constraint in **Quantum** Level

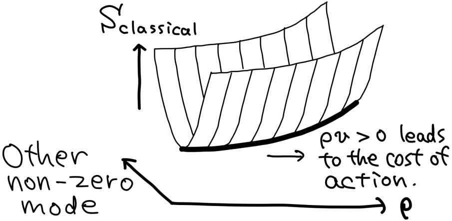
Constrained instaton procedure = "**insertion of 1**" to path integral.

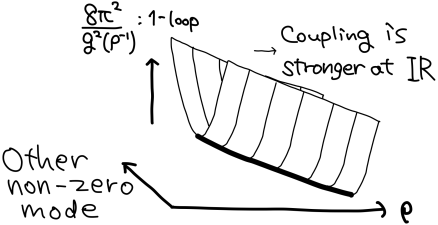
- (Entire configuration space) = $\sum_\rho$(slice of configurations with the fixed size ρ)

Two size-dependent effects compete (in asymptotically free case),

Classical: Action increases as ρv increases.

Quantum effect: Effective coupling $g(\rho^{-1})$ depends on the instanton size ρ.





Constrained Instanton: Procedure

Procedure of Constraint in **Quantum** Level [Gervais, Neveu and Virasoro (1977)]

Constrained instaton procedure can be understood as "**insertion of 1**" to Z.

$$
\begin{aligned}
Z &= \int \mathcal{D}\Phi\, \mathrm{d}f\, \delta(f - S_{\text{constraint}}) \exp(-S_{\text{YMH}}) \\
&= \int \mathcal{D}\Phi\, \mathrm{d}f\, \delta(f - S_{\text{constraint}}) \exp\Bigg(\overbrace{-S_{\text{YMH}} - \sigma(S_{\text{constraint}} - f(\rho))}^{-S_{\text{total}}}\Bigg) \quad \text{(Zero is just added.)} \\
&= \ldots = \int \overbrace{\mathcal{D}\varphi}^{\phi =: \varphi + (\text{statinary point})} \mathrm{d}f\, \frac{\mathrm{d}\mu}{2\pi} \exp\Bigg(- \overbrace{S^{\text{classical}}_{\text{total}}(f)}^{\text{stationary-point value}} - i\mu \int \frac{\delta S_{\text{constraint}}}{\delta \Phi}\varphi - \frac{1}{2}\int \varphi \frac{\delta^2 S_{\text{total}}}{\delta \Phi^2}\varphi + \mathcal{O}\big(\varphi^3\big)\Bigg) \\
&= \ldots = \int \mathcal{D}\tilde{\varphi}\, \mathrm{d}f \exp\Bigg(-S^{\text{classical}}_{\text{total}}(f) - \frac{1}{2}\int \tilde{\varphi}\frac{\delta^2 S_{\text{total}}}{\delta \Phi^2}\tilde{\varphi} + \mathcal{O}\big(\tilde{\varphi}^3\big)\Bigg) \int \frac{\mathrm{d}\mu}{2\pi} \exp\Bigg(-\frac{\mu^2}{2}\bigg(-\frac{\partial f}{\partial \sigma}\bigg)\Bigg) \\
&\simeq \int \mathrm{d}f \left(-\frac{\partial f}{\partial \sigma}\right)^{-1/2} \exp\Big(-S^{\text{classical}}_{\text{total}}(f)\Big) \int \mathcal{D}\tilde{\varphi}\, \exp\bigg(-\frac{1}{2}\int \tilde{\varphi}\frac{\delta^2 S_{\text{total}}}{\delta \Phi^2}\tilde{\varphi}\bigg)
\end{aligned}
$$

Our Work: Matching

More closely, the right thing to do is the **double expansion** in $|x|/\rho$ in addition to ρm.

Key consequence: refinement of outer ($|x| \gg \rho$) solution.

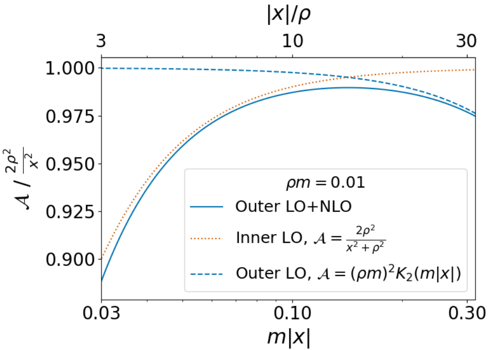
- Naive intuition: "$\mathcal{A} \propto K_2(m|x|)$ (massive **free** field), at far away from $x = 0$."
- This holds at $|x| \gg m^{-1}$, while **fails at the matching region $\boldsymbol{\rho \ll |x| \ll m^{-1}}$**.

More precisely,

- $\mathcal{A}(|x|) \propto K_2(m|x|)$ **uniformly to all orders** (of ρm) for **far-outer region, $\boldsymbol{|x| \gg v^{-1}}$**.
- Whereas for $\boldsymbol{\rho \ll |x| \ll v^{-1}}$, the behavior is **order dependent**.

Our Work: Matching

Exapmle: Outer ($x \gg \rho$) solution at LO (- - -) and LO+NLO (——)



1. NLO outer solution is merely an $\mathcal{O}(\rho^2 m^2) = \mathcal{O}(10^{-4})$ correction to LO outer solution at $|x| \gtrsim m^{-1}$.
2. However, NLO correction is **essential at matching region** $\boldsymbol{\rho \ll |x| \ll m^{-1}}$.
3. NLO outer solution is needed to match to inner solutions **already at inner LO** (·····).

The difference in order of ρm between inner and outer solutions is crucial in matching.

→ This is handled by the double expansion in $|x|/\rho$ in addition to ρm.

Our Work: Matching

Explicitly, double expansions with respect to ρm and ρ/x match as

$$\mathcal{A}^{(\mathrm{LO})}_{\mathrm{inner}}\left(\rho^2/x^2\right) + \mathcal{A}^{(\mathrm{NLO})}_{\mathrm{inner}}\left(\rho^2/x^2\right)$$

$$= \frac{2\rho^2}{x^2} - \frac{2\rho^4}{x^4} + (\rho m_A)^2\left[-\left(c_2 - \frac{1}{12}\right)\hat{r}^2 - 6c_2 + \left(12c_2 \log\frac{\rho^2}{x^2} - 36c_2 - \frac{1}{12}\right)\frac{\rho^2}{x^2} + \mathcal{O}\left(\frac{\rho^4}{x^4}\right)\right].$$

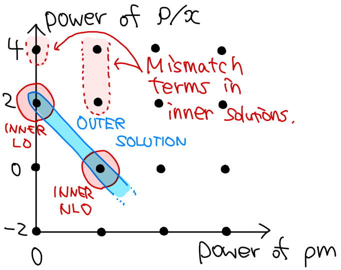
$$\mathcal{A}^{(\mathrm{LO})}_{\mathrm{outer}}(m^2x^2) + \mathcal{A}^{(\mathrm{NLO})}_{\mathrm{outer}}(m^2x^2)$$

$$= \frac{2\rho^2}{x^2} - \frac{2\rho^4}{x^4} + (\rho m)^2\left[-\frac{1}{2} + \frac{\rho^2}{x^2}\ln\frac{\rho^2}{x^2} + c_{\mathrm{out}}\frac{\rho^2}{x^2}\right] + \mathcal{O}\left(\rho^4 m^4\right).$$

- Matching is possible by adjusting c_2 and c_{out}. For example, $c_1 = 1/12$.
- Other constants are consistently determined by, for example, $S < \infty$ at $r = 0$

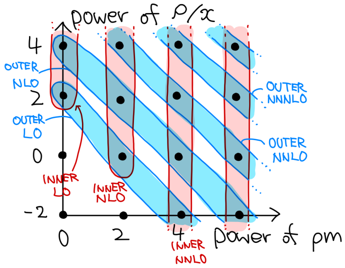
Our Work: Matching

Previous Work



- Diverse behavior of outer solution at the matching region is missing.
- $\mathcal{O}_{\text{constraint}}$ **is severely restricted** to avoid the appearing mismatch.

Our Work



- Double expansion at matching region.
- Matching works well **independently to the choice of** $\mathcal{O}_{\text{constraint}}$.