Varieties of four-dimensional gauge theories

Khoi Le Nguyen Nguyen

DAMTP, University of Cambridge

PASCOS 2026 talk, 23 June 2026

2409.15430, 2501.09860, 2508.11583 with Ben Gripaios

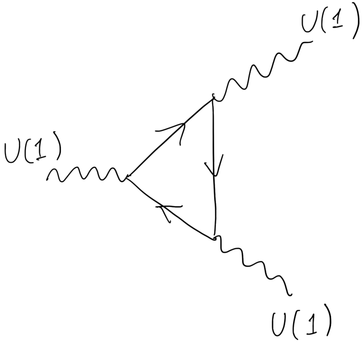
A problem

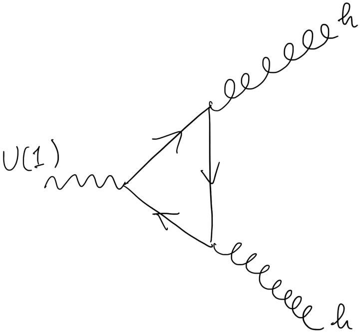
Write down all anomaly-free representations ("reps") of $\mathfrak{u}(1)$ for four-dimensional spacetime fermions.

A problem

Write down all anomaly-free representations ("reps") of $\mathfrak{u}(1)$ for four-dimensional spacetime fermions.
Equivalently: List all permissible sets of charges $(Q_1, \ldots, Q_n) \in \mathbb{Z}^n$ of n Weyl fermions in four dimensions.

$\mathfrak{u}(1)$ anomalies





Anomaly $\propto \sum_{i=1}^{n} Q_i^3$

Anomaly $\propto \sum_{i=1}^{n} Q_i$

A harder problem

Write down all anomaly-free reps of an arbitrary gauge Lie algebra for four-dimensional spacetime fermions.

A harder problem

Write down all anomaly-free reps of an arbitrary gauge Lie algebra for four-dimensional spacetime fermions.
Concession: Consider only irreducible representations ("irreps").

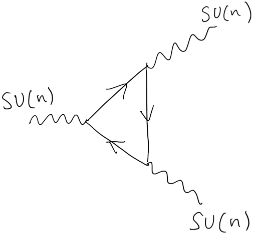
A harder problem

Write down all anomaly-free reps of an arbitrary gauge Lie algebra for four-dimensional spacetime fermions.
Concession: Consider only irreducible representations ("irreps").
Only need to worry about $\mathfrak{su}(n)$ for $n \geq 3$.

$\mathfrak{su}(n)$ irreps

- Dynkin labels: $(m_1, \ldots, m_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$.
  Example: $(1, 0, 1)$ is adjoint rep of $\mathfrak{su}(4)$.
- $(q_1, \ldots, q_{n-1}) := (m_1 + 1, \ldots, m_{n-1} + 1) \in \mathbb{Z}_{>0}^{n-1}$.
- Dual rep of $(q_1, \ldots, q_{n-1})$ is $(q_{n-1}, \ldots, q_1)$ and has negative the anomaly.
- Self-dual reps are anomaly-free. Reps that are non-self-dual are called chiral.

$\mathfrak{su}(n)$ anomalies



Banks, Georgi '76: Anomaly := $A_n = \sum a_{ijk} q_i q_j q_k$.

Which irreps are anomaly-free (ie. have $A_n = 0$)?
Partial answer: All self-dual irreps.
$\mathfrak{su}(3) : A_3 = (q_1 - q_2)(q_1 + 2q_2)(2q_1 + q_2)$
$\Rightarrow$ Only self-dual irreps of the form (a, a).
$\mathfrak{su}(4) : A_4 = (q_1 - q_3)(q_1 + q_3)(q_1 + 2q_2 + q_3)$
$\Rightarrow$ Only self-dual irreps of the form (a, b, a).

$\mathfrak{su}(n)$ anomalies

$$\mathfrak{su}(5): A_5 = 4q_1^3 + 9q_1^2q_2 + 3q_1q_2^2 + 2q_2^3 + 6q_1^2q_3 + 4q_1q_2q_3 + 4q_2^2q_3 - 2q_1q_3^2 - 4q_2q_3^2 - 2q_3^3 + 3q_1^2q_4 + 2q_1q_2q_4 + 2q_2^2q_4 - 2q_1q_3q_4 - 4q_2q_3q_4 - 3q_3^2q_4 - 3q_1q_4^2 - 6q_2q_4^2 - 9q_3q_4^2 - 4q_4^3.$$

$\mathfrak{su}(5)$ has some chiral anomaly-free irreps

Eichten, Kang, Koh '82: Dimension below 4×10^9

(m_1, m_2, m_3, m_4)	Dimension
$(0, 7, 3, 3)$	1×10^6
$(1, 8, 1, 5)$	3×10^6
$(7, 7, 15, 1)$	1×10^9

Okubo's new variables

Okubo '77: Transform the $n-1$ positive integers $q_1, \ldots, q_{n-1}$ to the n integers $\sigma_1, \ldots, \sigma_n$ according to

$$\sigma_i := -\sum_{k=1}^{i-1} k q_k + \sum_{k=i}^{n-1} (n-k) q_k.$$

Okubo's new variables

Okubo '77: Transform the $n-1$ positive integers $q_1, \ldots, q_{n-1}$ to the n integers $\sigma_1, \ldots, \sigma_n$ according to

$$\sigma_i := -\sum_{k=1}^{i-1} k q_k + \sum_{k=i}^{n-1} (n-k) q_k.$$

Example: Self-dual irreps of $\mathfrak{su}(5)$ have $(\sigma_1, \ldots, \sigma_5) = (a, b, 0, -b, -a)$ with $a > b > 0$.

Okubo's new variables

Okubo '77: Transform the $n-1$ positive integers $q_1, \ldots, q_{n-1}$ to the n integers $\sigma_1, \ldots, \sigma_n$ according to

$$\sigma_i := -\sum_{k=1}^{i-1} k q_k + \sum_{k=i}^{n-1} (n-k) q_k.$$

Example: Self-dual irreps of $\mathfrak{su}(5)$ have
$(\sigma_1, \ldots, \sigma_5) = (a, b, 0, -b, -a)$ with $a > b > 0$.
Constraints:

- $\sum_{i=1}^{n} \sigma_i = 0$.
- $\sigma_1 > \sigma_2 > \cdots > \sigma_n$ (because $\sigma_i - \sigma_{i+1} = n q_i$).

Okubo's new variables

Okubo '77: Transform the $n-1$ positive integers $q_1, \ldots, q_{n-1}$ to the n integers $\sigma_1, \ldots, \sigma_n$ according to

$$\sigma_i := -\sum_{k=1}^{i-1} kq_k + \sum_{k=i}^{n-1} (n-k)q_k.$$

Example: Self-dual irreps of $\mathfrak{su}(5)$ have $(\sigma_1, \ldots, \sigma_5) = (a, b, 0, -b, -a)$ with $a > b > 0$.
Constraints:

- $\sum_{i=1}^{n} \sigma_i = 0$.
- $\sigma_1 > \sigma_2 > \cdots > \sigma_n$ (because $\sigma_i - \sigma_{i+1} = nq_i$).

Anomaly of an irrep ρ with dimension $D(\rho)$:

$$A(\rho) \propto D(\rho) \sum_{i=1}^{n} \sigma_i^3.$$

The $\mathfrak{su}(n)$ anomaly cancellation conditions are almost the same as the $\mathfrak{u}(1)$ ones

Khoi Le Nguyen Nguyen

DAMTP, University of Cambridge

PASCOS 2026 talk, 23 June 2026
2409.15430, 2501.09860, 2508.11583 with Ben Gripaios

Anomaly-free irreps of $\mathfrak{su}(5)$

For $n = 5$, we have to solve a homogeneous cubic equation in four variables, writing $\sigma_5 = -(\sigma_1 + \cdots + \sigma_4)$,

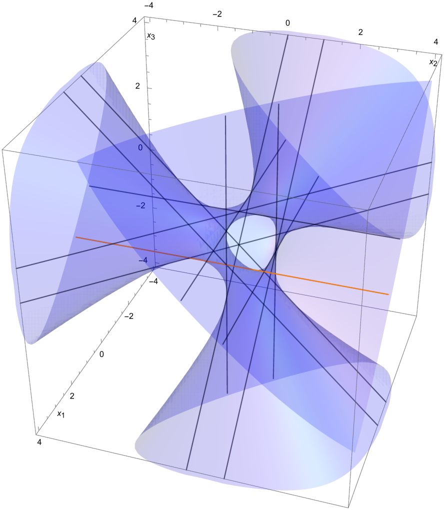
$$\sum_{i=1}^{4} \sigma_i^3 - \left(\sum_{i=1}^{4} \sigma_i \right)^3 = 0.$$

In $\mathbb{kP}^3$ ($\mathbb{k} \in \{\mathbb{C}, \mathbb{R}, \mathbb{Q}\}$), this defines a projective variety called the Clebsch diagonal cubic surface.

27 lines

Fact 1: Every smooth cubic surface has 27 lines over $\mathbb{C}$.
Fact 2: The Clebsch diagonal cubic surface is (up to isomorphisms) the only one with 27 lines over $\mathbb{R}$.
Fact 3: In the σ_i variables, 15 of these lines exist over $\mathbb{Q}$.

Rational lines on the Clebsch cubic



The method of secants

1. Pick two skew lines on the Clebsch cubic.
2. Draw all secants between them.
3. Get all points from intersections between secants and Clebsch cubic.

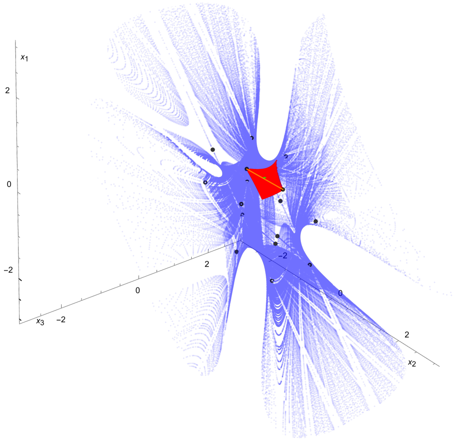
Rational variety

- The method of secants can be shown to be a birational map between any cubic surface with two skew lines and $\mathbb{kP}^1 \times \mathbb{kP}^1$ (ie. it is an isomorphism from a non-empty Zariski-open subset of the former to the latter).
- Such varieties where these maps exist are called rational.
- Practically, it means that we can parameterise all points on rational varieties using rational functions (ie. ratios of homogeneous polynomials) over the field $\mathbb{k}$.
- For $\mathbb{k} = \mathbb{Q}$, this means we have almost solved our problem.

Caveat?

How about the pesky condition $\sigma_1 > \cdots > \sigma_5$?
Answer: S_5 symmetry, plus the fortunate fact that we essentially never run into trouble.

The solution for $n = 5$



Eichten, Kang, Koh revisited

$A_5 = \sum a_{ijk} q_i q_j q_k$

(m_1, m_2, m_3, m_4)	(q_1, q_2, q_3, q_4)	Dimension
$(0, 7, 3, 3)$	$(1, 8, 4, 4)$	1×10^6
$(1, 8, 1, 5)$	$(2, 9, 2, 6)$	3×10^6
$(7, 7, 15, 1)$	$(8, 8, 16, 2)$	1×10^9
$(0, 17, 12, 5)$	$(1, 18, 13, 6)$	2.5×10^9
$(8, 16, 0, 15)$	$(9, 17, 1, 16)$	2.7×10^9
$(3, 17, 3, 11)$	$(4, 18, 4, 12)$	3×10^9

Eichten, Kang, Koh revisited

$A_5 = \sum a_{ijk} q_i q_j q_k$

(m_1, m_2, m_3, m_4)	(q_1, q_2, q_3, q_4)	Dimension
$(0, 7, 3, 3)$	$(1, 8, 4, 4)$	1×10^6
$(1, 8, 1, 5)$	$(2, 9, 2, 6)$	3×10^6
$(7, 7, 15, 1)$	$(8, 8, 16, 2)$	1×10^9
$(0, 17, 12, 5)$	$(1, 18, 13, 6)$	2.5×10^9
$(8, 16, 0, 15)$	$(9, 17, 1, 16)$	2.7×10^9
$(3, 17, 3, 11)$	$(4, 18, 4, 12)$	3×10^9

This method generalizes to all higher n.

Beyond irreps of $\mathfrak{su}(n)$

- The anomaly cancellation equations for a tensor product of two irreps $\rho_1 \otimes \rho_2$ of $\mathfrak{su}(n)$ also define a rational projective cubic hypersurface.

Beyond irreps of $\mathfrak{su}(n)$

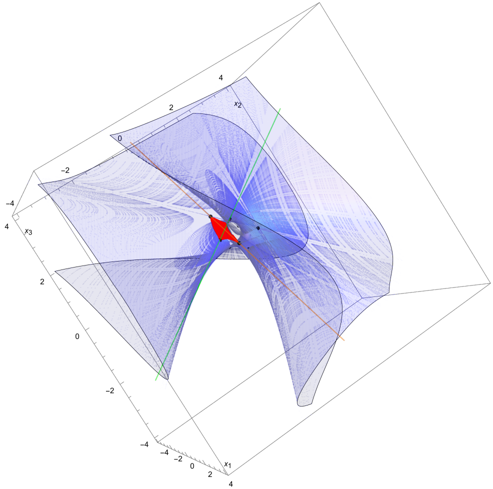
- The anomaly cancellation equations for a tensor product of two irreps $\rho_1 \otimes \rho_2$ of $\mathfrak{su}(n)$ also define a rational projective cubic hypersurface.
- With an extra $\mathfrak{u}(1)$ summand, solving for anomaly-free sets of charges under this summand gives a cubic hypersurface that is unirational.

Beyond irreps of $\mathfrak{su}(n)$

- The anomaly cancellation equations for a tensor product of two irreps $\rho_1 \otimes \rho_2$ of $\mathfrak{su}(n)$ also define a rational projective cubic hypersurface.
- With an extra $\mathfrak{u}(1)$ summand, solving for anomaly-free sets of charges under this summand gives a cubic hypersurface that is unirational.
- With more $\mathfrak{u}(1)$ summands, need to look at linear subspaces on the cubic hypersurface...

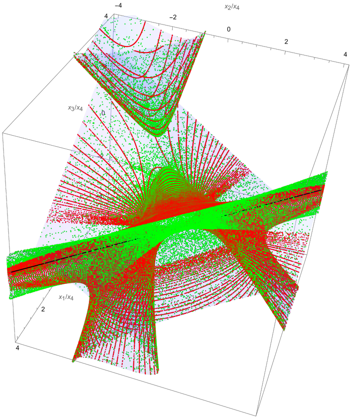
Binary product representations of $\mathfrak{su}(3)$

In this case, we get a cubic surface with two skew rational lines, so all rational points can be found using the method of secants and tangents since the surface is rational.



Anomaly-free charges of six fermions under $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$

Consider six fermions transforming in the direct sum of six irreps of $\mathfrak{su}(2)$ with dimensions $2 \oplus 2 \oplus 3 \oplus 3 \oplus 7 \oplus 8$. The resultant cubic surface has only one rational line and is thus unirational. Only an (infinite) subset of points can be found (red), from which more points can be obtained using the secant construction (green).



Concluding remarks

- Solved the anomaly-cancellation problem for irreps of $\mathfrak{su}(n)$ using arithmetic geometry.
- These methods can be used to address the case of product reps of $\mathfrak{su}(n)$ as well as reps of $\mathfrak{su}(n) \oplus \mathfrak{u}(1)$.
- Ongoing project: Anomaly-free reps of $\mathfrak{su}(n) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \dots$.

Thank you for listening!

Backups

The method of secants in action

With homogeneous coordinates $[\sigma_1 : \cdots : \sigma_4] \in \Bbbk\mathbb{P}^3$ and $\sigma_5 := -(\sigma_1 + \cdots + \sigma_4)$,

1. Two skew lines on the Clebsch cubic surface are $L_1 = [k_1 : k_2 : 0 : -k_2]$ ($\sigma_5 = -k_1$: the "non-chiral" line) and $L_2 = [0 : l_1 : l_2 : -l_2]$.
2. If $p_1 \in L_1$ and $p_2 \in L_2$, then a point on the projective line L_3 through them is $p_3 = \alpha_1 p_1 + \alpha_2 p_2$.
3. p_3 lies on the Clebsch cubic if

$$\sum_{i=1}^{5} p_{3i}^3 = 0 \Leftrightarrow \sum_{i=1}^{5} \alpha_1\alpha_2(\alpha_1 p_{1i}^2 p_{2i} + \alpha_2 p_{1i} p_{2i}^2) = 0.$$

4. "Generically"

$$[\alpha_1 : \alpha_2] = \left[\sum_{i=1}^{5} p_{1i} p_{2i}^2 : -\sum_{i=1}^{5} p_{1i}^2 p_{2i}\right].$$

The case of products

Recall that for an irrep ρ of $\mathfrak{su}(n)$ with Okubo labels $(\sigma_1, \ldots, \sigma_n)$ and dimension $D(\rho)$, the anomaly is given by

$$A(\rho) \propto D(\rho) \sum_{i=1}^{n} \sigma_i^3.$$

For a tensor product of two irreps $\rho_1 \otimes \rho_2$, the anomaly is given by

$$\begin{aligned} A(\rho_1 \otimes \rho_2) &= D(\rho_2)A(\rho_1) + D(\rho_1)A(\rho_2) \\ &= D(\rho_1)D(\rho_2)\left[\sum_{i=1}^{n}(\sigma_{1,i}^3 + \sigma_{2,i}^3)\right], \end{aligned}$$

so anomaly-free iff $\sum_{i=1}^{n}(\sigma_{1,i}^3 + \sigma_{2,i}^3) = 0$: projective cubic variety!

The case *for* products

Product reps are more attractive because the anomaly-free ones turn out to generally have smaller dimensions than the anomaly-free irreps.
For the case of binary product representations of $\mathfrak{su}(3)$, the variety is a projective surface given by

$$\sigma_{1,1}\sigma_{1,2}(\sigma_{1,1}+\sigma_{1,2})+\sigma_{2,1}\sigma_{2,2}(\sigma_{2,1}+\sigma_{2,2})=0.$$

Anomaly-free product reps

Table: Some anomaly-free chiral products of m irreducible representations of $\mathfrak{su}(n)$ for some values of n and m, with dimension below 10^5.

n	m	$\bigotimes_{\alpha=1}^{m}(q_{1,\alpha},\ldots,q_{n-1,\alpha})$			dim
3	2	$(1,7)\otimes(7,4)$	28×154	$=$	4 312
		$(2,7)\otimes(8,6)$	63×336	$=$	21 168
		$(1,8)\otimes(12,11)$	36×1518	$=$	54 468