# Threading the Thimble: A Stable Route to Real-Time Path Integrals


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# Motivation

- ✳ Real-time (Lorentzian) path integrals:

$$\int \mathcal{D}x \; e^{\frac{iS[x]}{\hbar}}$$

⟶ integrand **oscillates**, does not decay ⟶ direct numerical integration fails

- ✳ Relevant for:

⟶ finite-density lattice field theory (sign problem)

⟶ quantum cosmology (e.g. no-boundary proposal)

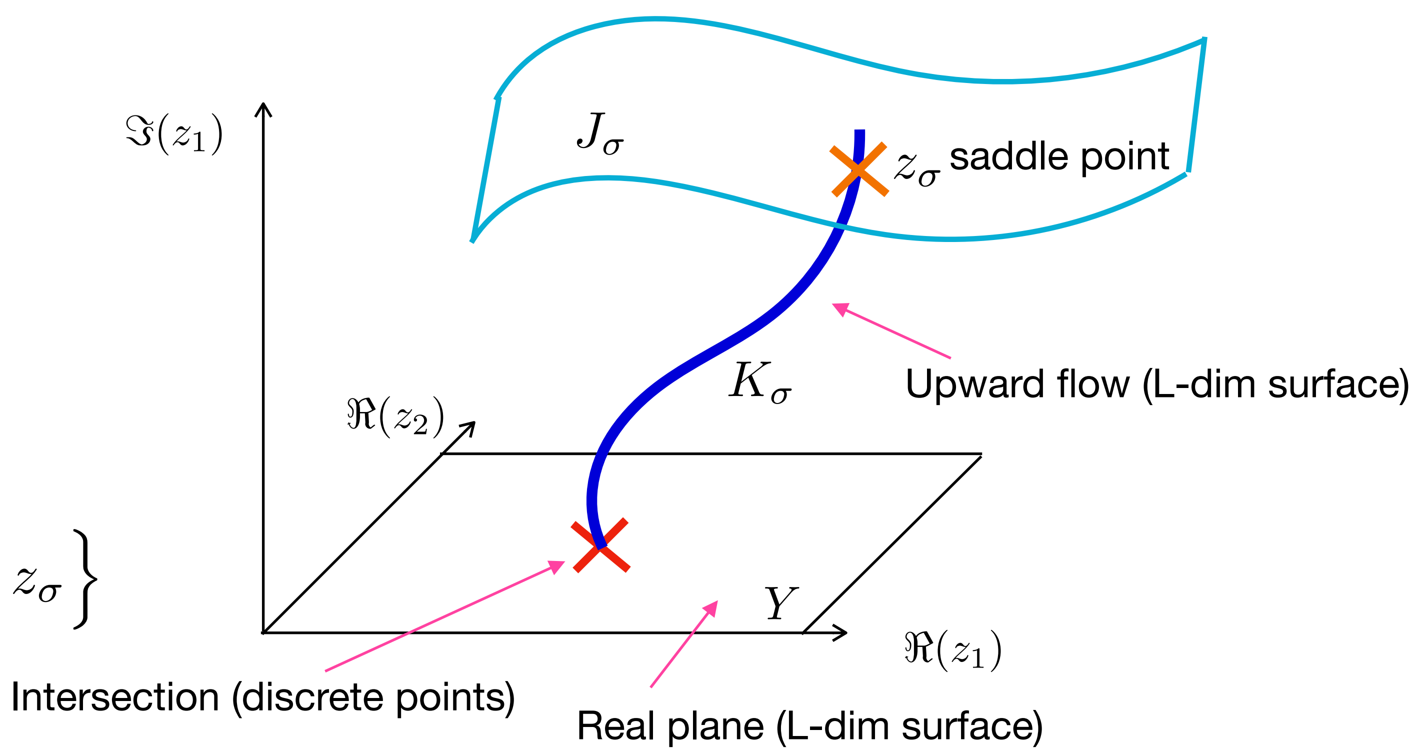
⟶ real-time field theory (e.g. vacuum decay)

- ✳ Picard-Lefschetz theory cures this: deform original contour into sum of contours with convergent integrals

⟶ **caveat:** knowing which ones contribute is hard!

# Picard-Lefschetz Theory

$$\int d^L x\, e^{\frac{I(x)}{\hbar}} = \sum_\sigma n_\sigma \int_{J_\sigma} d^L z\, e^{\frac{I(z)}{\hbar}}$$




✳ Lefschetz-Thimble:

$$J_\sigma = \left\{ z(0) \in \mathbb{C}^L \,\middle|\, \underbrace{\frac{\partial z_i}{\partial u} = -\overline{\frac{\partial I}{\partial z_i}}}_{\text{downward flow}},\ z(u=-\infty) = z_\sigma \right\}$$

✳ Along $J_\sigma$ :

$\Im[I(z)]$ remains **constant**

$\Re[I(z)]$ **decreases** monotonically

$$\left| \int_{J_\sigma} d^L z\, e^{\frac{I(z)}{\hbar}} \right| < \infty$$

✳ Intersection number: $n_\sigma = \langle Y, K_\sigma \rangle$

$$K_\sigma = \left\{ z(0) \in \mathbb{C}^L \,\middle|\, \underbrace{\frac{\partial z_i}{\partial u} = \overline{\frac{\partial I}{\partial z_i}}}_{\text{upward flow}},\ z(u=-\infty) = z_\sigma \right\}$$

# Picard-Lefschetz Theory

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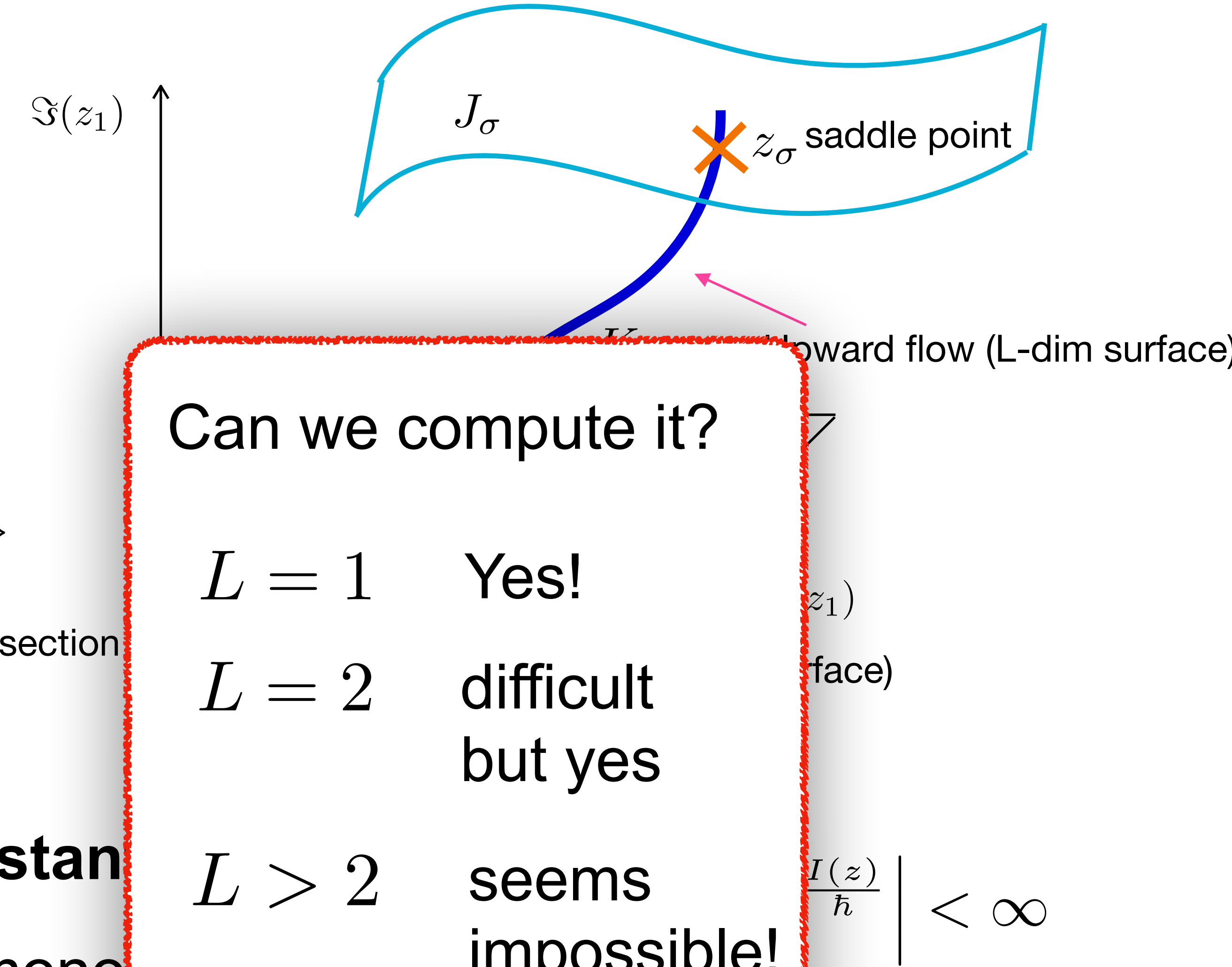


- Lefschetz-Thimble:

$$J_\sigma = \left\{ z(0) \in \mathbb{C}^L \middle| \underbrace{\frac{\partial z_i}{\partial u} = -\overline{\frac{\partial I}{\partial z_i}}}_{\text{downward flow}}, \; z(u=-\infty) = z_\sigma \right\}$$

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Can we compute it?

| | |
|---|---|
| $L = 1$ | Yes! |
| $L = 2$ | difficult but yes |
| $L > 2$ | seems impossible! |

… or not

# Solving Boundary Value Problem

Why is it difficult?

✳ Single shooting method:

1. Choose initial condition around saddle point
2. Solve upward flow equation


3. Check if solution intersects with real plane
4. If not, choose different initial condition and try again

Chaotic for $L > 1$ (extreme sensitivity to initial conditions)

iteration does not converge

difficulty increases exponentially with $L$

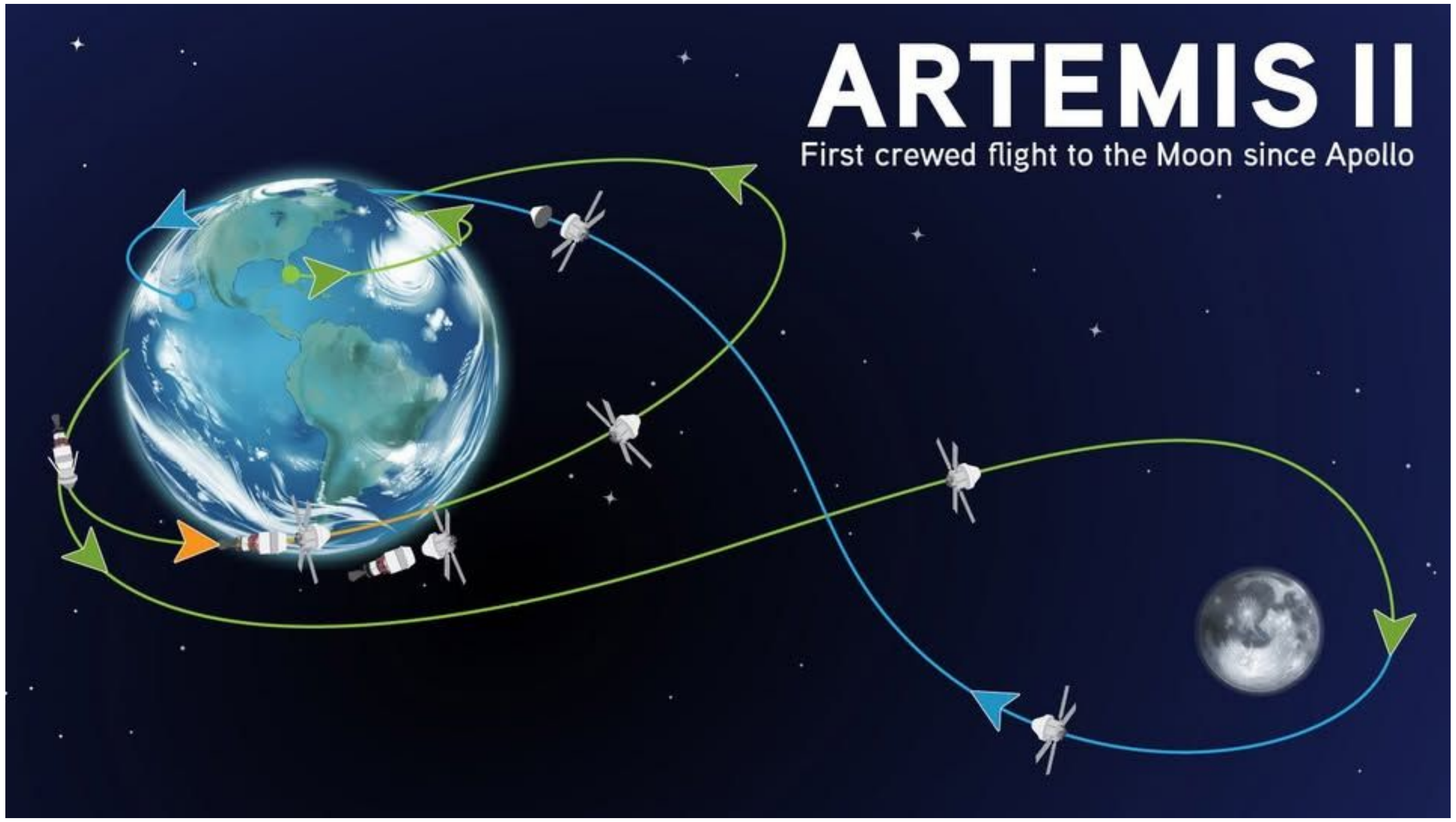
✳ Other methods:

only for additional symmetries or $L = 1, 2$

[J. Feldbrugge, U.-L. Pen, N. Turok, '23],
[A. V. Shanin, A. I. Korolkov, N. M. Artemov, R. C. Assier, '25],
[A. Weber, J. Feldbrugge, E. Pisanty, '26],
[T. Fujimori, S. Kamata, T. Misumi, M. Nitta, N. Sakai, '23]

# Solving Boundary Value Problem

Is it really difficult?




BVPs with $L = 3$ solved already decades ago!

→ Multiple shooting method

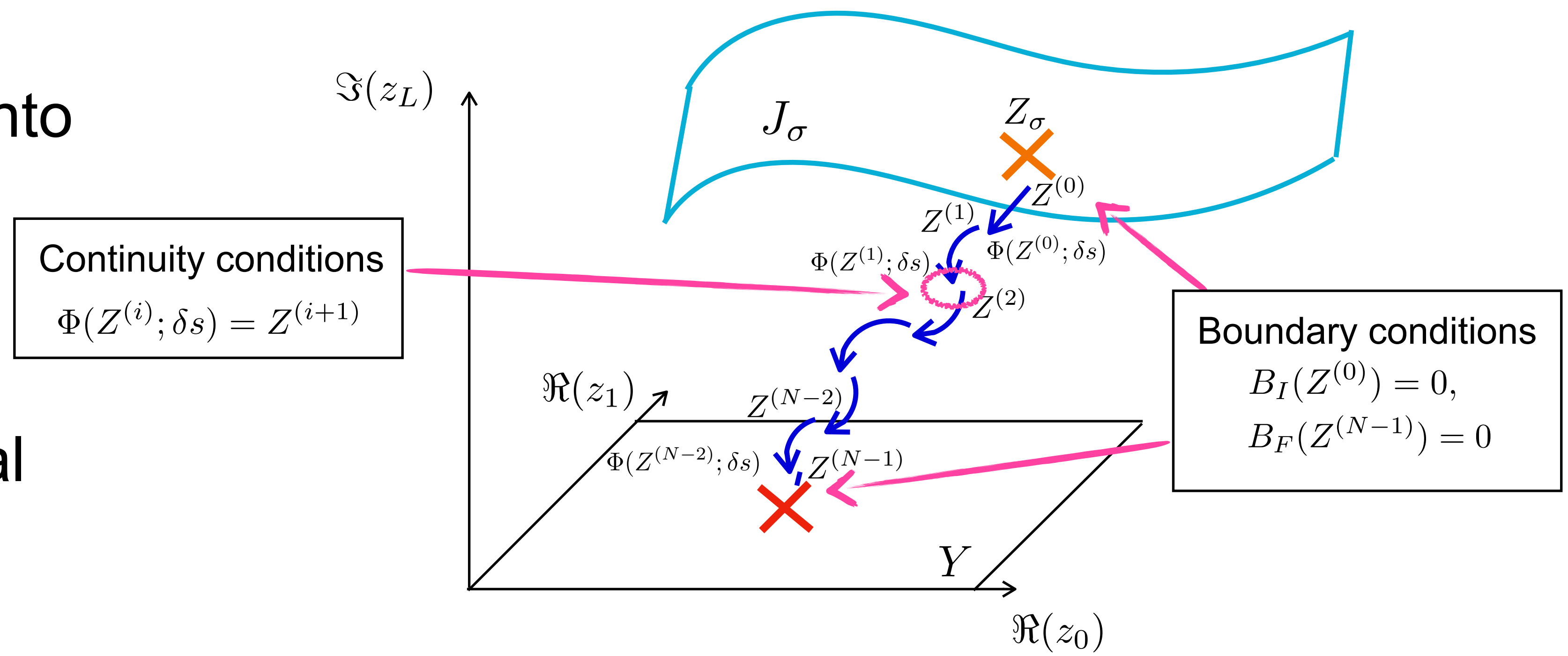
# Multiple Shooting Method

[D. D. Morrison, J. D. Riley, J. F. Zancanaro, '62], …

Standard Method for solving BVPs in celestial mechanics, chemical engineering, …

1. Divide integration domain into subintervals

2. Construct short-time solution for each subinterval (almost linear response)




3. Solve continuity conditions together with boundary conditions

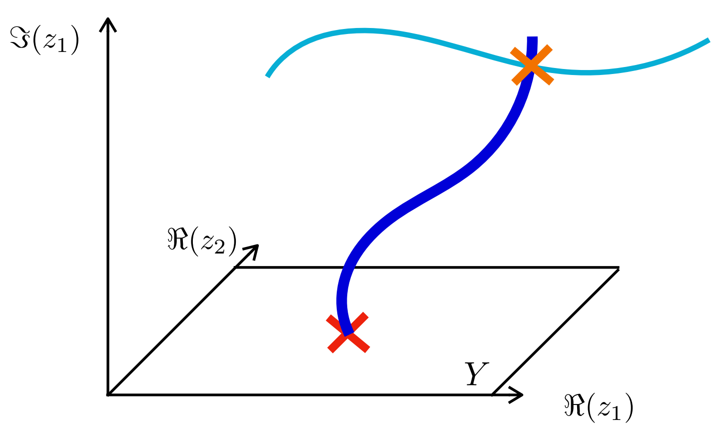
Non-linear optimization problem

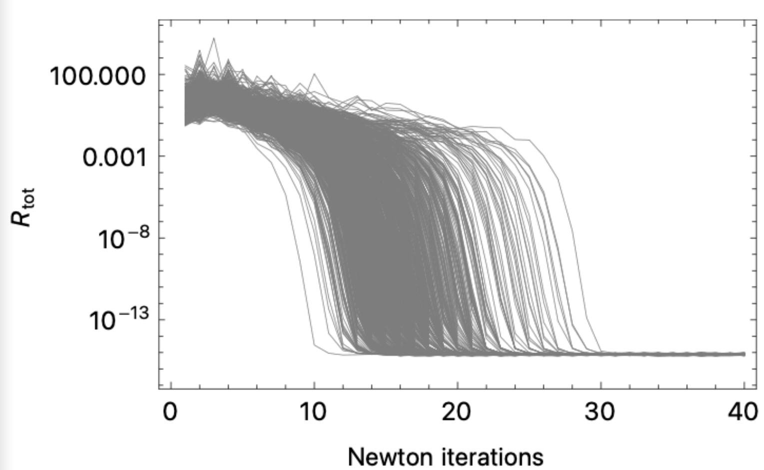
efficiently solved with Newton's method

# Intersection number

if solution exists:
rapid convergence

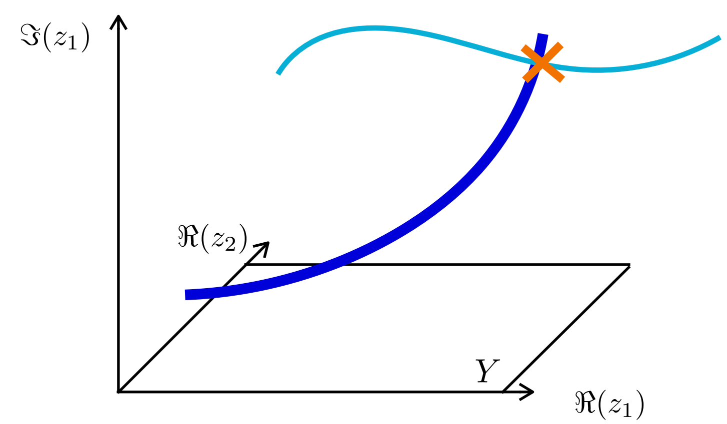
$$n_\sigma = \pm 1$$

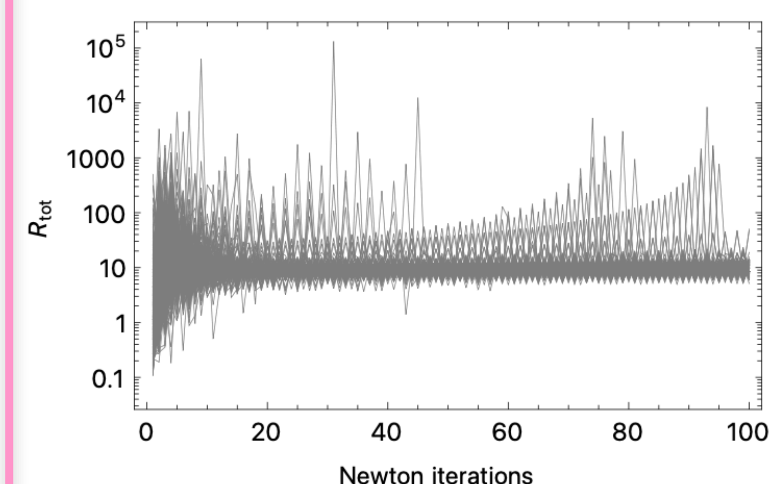



if no solution exists:
sequences oscillates

$$n_\sigma = 0$$



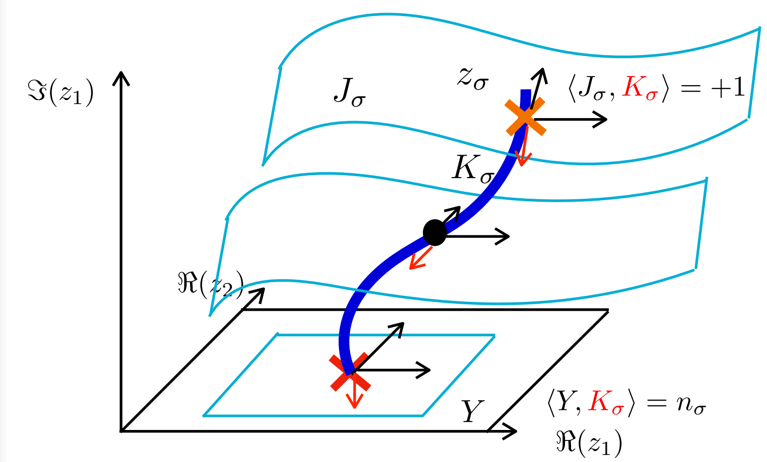




sign determination

$$n_\sigma \int_{J_\sigma} d^L z \; e^{\frac{I(z)}{\hbar}}$$

sign convention fixed by

$$\langle J_\sigma, K_\tau \rangle = \delta_{\sigma\tau}$$




tangent space of $K_\sigma$ at saddle point propagated to tangent space at intersection point

# Example: Airy-Type Integral

✳ **Three**-variable example:

$$I(x) = i\left[\frac{x_0^3 + x_1^3 + x_3^3}{3} - x_0x_1 - x_0x_2 - x_1x_2 + c_0x_0 + c_1x_1 + c_2x_2\right] \qquad c_n = 0.5e^{i(n+1)\alpha} \quad 0 < \alpha < 2\pi$$

Mixing

Airy x 3

✳ Numerical test:

$$\int d^3x\ e^{I(x)/\hbar} = \sum_\sigma n_\sigma \int_{J_\sigma} d^3z\ e^{I(x)/\hbar} = \sum_\sigma n_\sigma\, A_\sigma\, e^{I(z_\sigma)/\hbar}\,[1 + \mathcal{O}(\hbar)]$$

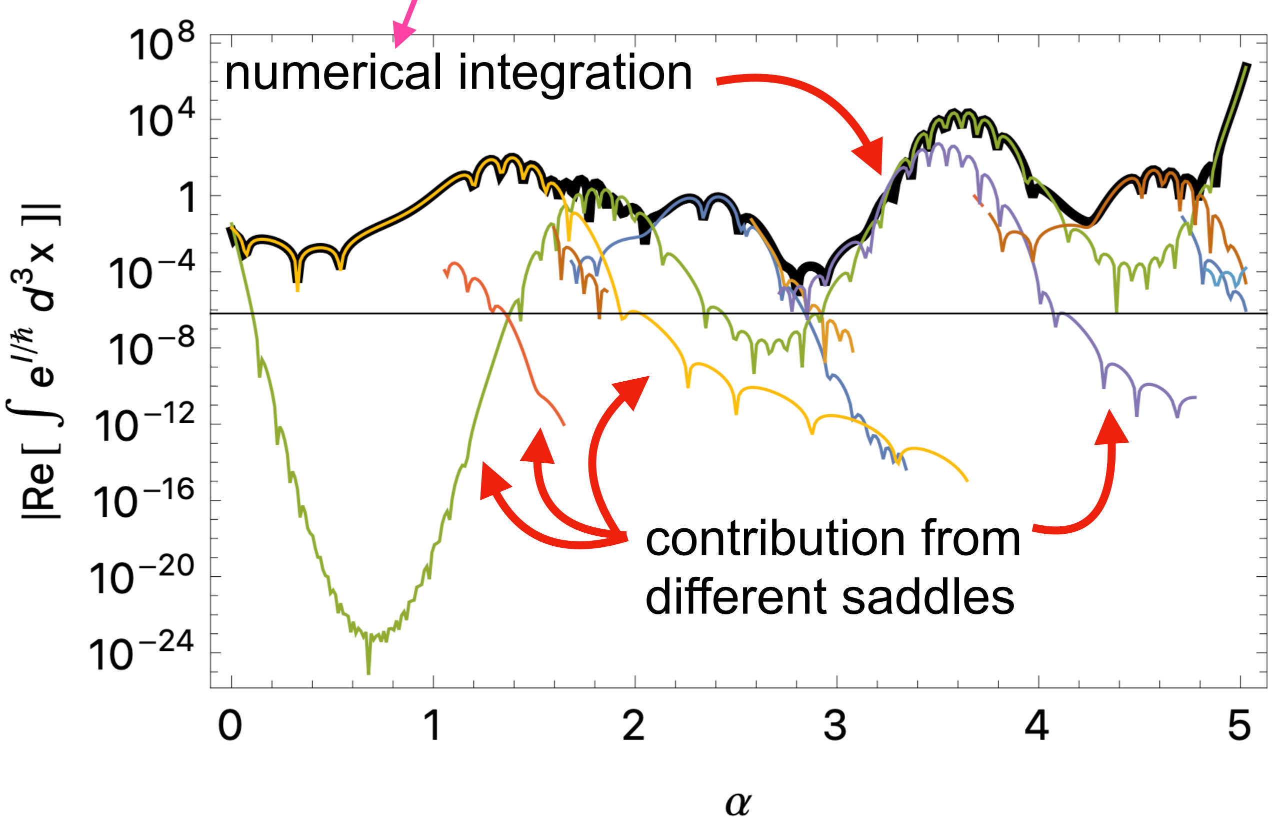
8 saddles

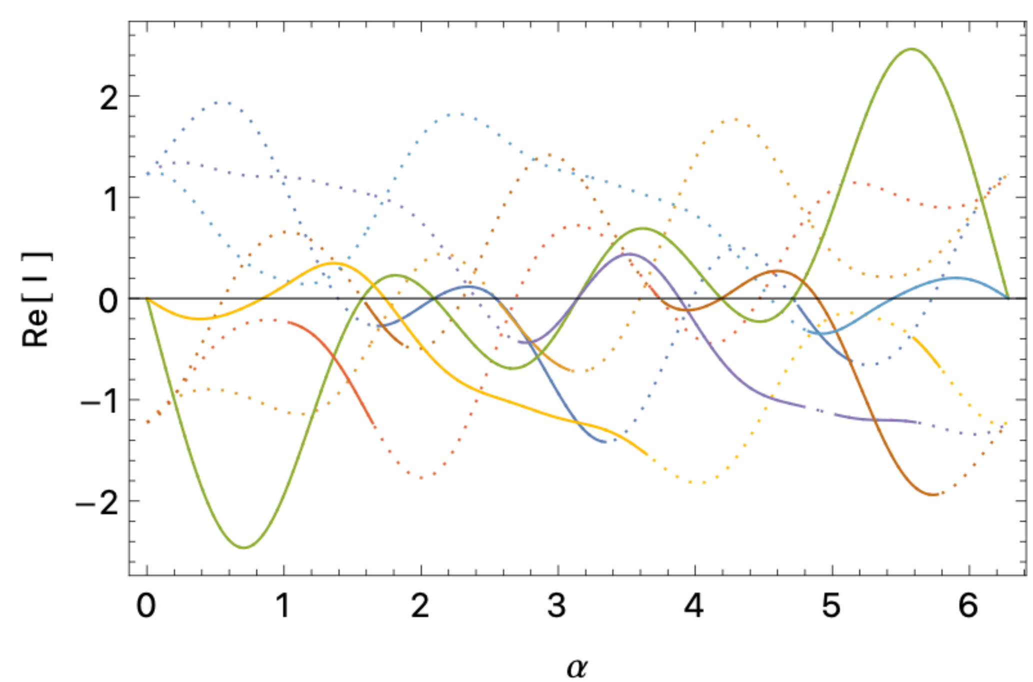
$$= \int_{\tilde{Y}} d^3z \det\frac{dx}{dz}\ e^{\frac{I(z)}{\hbar}}$$

$\tilde{Y}$ : deformed Y cycle for convergence

saddle point approximation $\hbar \sim 0$

$A_\sigma$ : Jacobian + Gaussian integral




Stokes phenomenon!

# Discrete Real-Time Path Integrals

Do complex saddles contribute to real time path integrals?

# Real-Time Path Integrals: Double Well

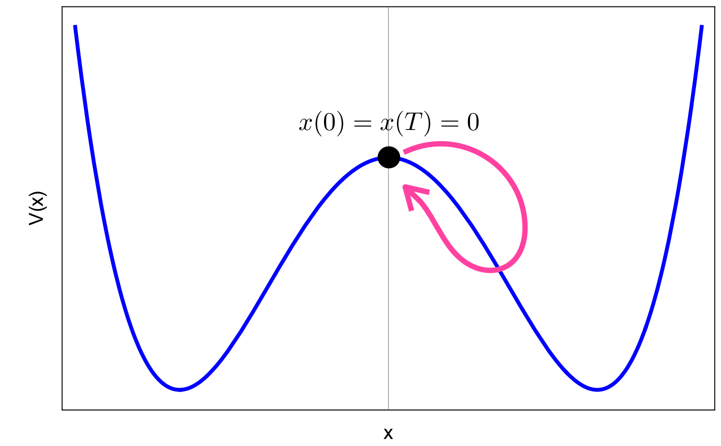
$$\int_{x(0)=x(T)=0} \mathcal{D}x \; e^{I(x)} \simeq \mathcal{N}_L \int d^L x \; e^{I(x)} = \sum_\sigma n_\sigma \mathcal{N}_L \int_{J_\sigma} d^L z \; e^{I(z)} \simeq \sum_\sigma n_\sigma \mathcal{N}_L \; A_\sigma \; e^{I(z_\sigma)}$$

Finite time path integral — discretize time — Thimble decomposition — saddle point approx.

$\mathcal{N}_L = (2\pi i \Delta t)^{-\frac{L+1}{2}}$ $\qquad \Delta t = T/(L+1)$ $\qquad x_i = x((i+1)\Delta t)$, $i = 1, \ldots, L$

$$I(x) = i\left[\frac{1}{2}\sum_{i=1}^{L-1}\left(\frac{x_i - x_{i-1}}{\Delta t}\right)^2 \Delta t + \frac{x_0^2 + x_{L-1}^2}{2\Delta t} - \frac{1}{2}\sum_{i=0}^{L-1}(x_i^2 - 1)^2 \Delta t - \Delta t\right] + \Delta I(x)$$

Morsification term (if needed)




sign of $n_\sigma$ depends on orientation of thimble $J_\sigma$

→ fix such that $\Re(\mathcal{N}_L A_\sigma) > 0$

→ sign of $n_\sigma$ remains unchanged when changing L

# Real-Time Path Integrals: Double Well

* Classification of saddle points by two integers $(n, m)$
  [Y. Tanizaki, T. Koike, 2014]

Real saddles

$\Im(z_\sigma) = 0 \qquad |n_\sigma| = 1$

Non-contributing complex saddles

$\Im(z_\sigma) \neq 0,\ \Re[I(z_\sigma)] \geq 0 \qquad n_\sigma = 0$

Non-trivial complex saddles

$\Im(z_\sigma) \neq 0,\ \Re[I(z_\sigma)] < 0 \qquad n_\sigma = ?$

First determination of $n_\sigma$ for non-trivial saddles!

$T = 5$

| | | $L \to \infty$ | finite $L$ | | |
|---|---|---|---|---|---|
| $n$ | $m$ | $\mathcal{I}_\infty[z]$ | $\mathcal{I}(z)$ | $L$ | $n_\sigma$ |
| 2 | 1 | $-1.280 + 1.427i$ | $-0.775 + 1.271i$ | 12 | +1 |
| 1 | −2 | $-1.280 + 1.427i$ | $-0.764 + 1.257i$ | 12 | +1 |
| 3 | 2 | $-7.357 - 0.759i$ | – | – | 0 |
| 2 | −3 | $-7.357 - 0.759i$ | – | – | 0 |
| 4 | 1 | $-14.926 + 19.727i$ | $-5.783 + 17.860i$ | 16 | −1 |
| 1 | −4 | $-14.926 + 19.727i$ | $-5.783 + 17.862i$ | 16 | −1 |
| 4 | 2 | $-23.946 + 4.198i$ | $-15.311 + 6.545i$ | 20 | −1 |
| 2 | −4 | $-23.946 + 4.198i$ | $-15.314 + 6.549i$ | 20 | −1 |
| 4 | 3 | $-21.025 - 18.980i$ | – | – | 0 |
| 3 | −4 | $-21.025 - 18.980i$ | – | – | 0 |

results stable against change of L

# Real-Time Path Integrals: Double Well

✳ Classification of saddle points by two integers $(n, m)$
[Y. Tanizaki, T. Koike, 2014]

**Non-trivial complex saddles**

$$\Im(z_\sigma) \neq 0,\ \Re[I(z_\sigma)] < 0 \qquad n_\sigma = ?$$

First determination of $n_\sigma$ for non-trivial saddles!

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| $n$ | $m$ | $\mathcal{I}_\infty[z]$ | $\mathcal{I}(z)$ | $L$ | $n_\sigma$ |
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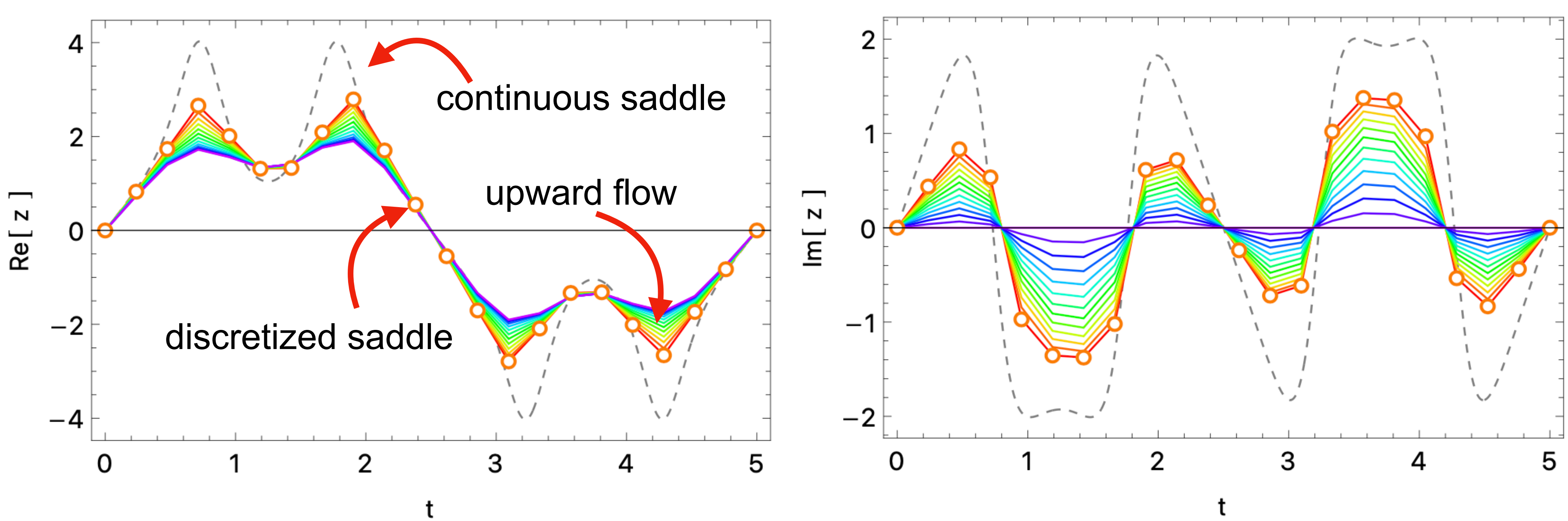
| [illegible] | [illegible] | [illegible] | [illegible] $6.549i$ | 20 | −1 |
| | | | | – | 0 |
| | | | | – | 0 |

solution with L=20!



# Conclusions

- ✳ Picard-Lefschetz theory: rigorous and powerful mathematical framework for analyzing oscillatory intergrals

  ⟶ but determining intersection numbers in multi-variante setups difficult

- ✳ We propose a robust and efficient method for computing intersection numbers

  ⟶ up to L=20 (improved algorithm L=36 [M. Honda, Y. Shoji, K.T., in preparation] )

- ✳ Airy-type example: demonstrated effectiveness of our method by comparing with direct numerical method

- ✳ Applied our method to real-time path integral with double-well potential

  ⟶ for first time determined intersection number of several non-trivial complex saddle points

  ⟶ provides compelling evidence for existence of their contributions

# Thank you!

# Multiple Shooting Method

- Normalized upward flow: $\dfrac{\partial z_i}{\partial s} = \dfrac{\overline{\dfrac{\partial I}{\partial z_i}}}{\left|\dfrac{\partial I}{\partial z}\right|}$

- Decompose complex variables: $Z = (\Re z_0 \cdots \Re z_{L-1} \Im z_0 \cdots \Im z_{L-1})$

- Fix: $N-1$ subintervals with step size $\delta s$

- Continuity condition: $R^{(k)} = Z^{(k)} - \Phi(Z^{(k-1)}; \delta s)$

$$\frac{\partial \Phi}{\partial s}(Z_{\text{init}}; s) = \frac{1}{\left|\frac{\partial I}{\partial z}(\Phi(Z_{\text{init}}; s))\right|} \begin{pmatrix} \Re \frac{\partial I}{\partial z}(\Phi(Z_{\text{init}}; s)) \\ -\Im \frac{\partial I}{\partial z}(\Phi(Z_{\text{init}}; s)) \end{pmatrix}$$

with $\Phi(Z_{\text{init}}; 0) = Z_{\text{init}}$

# Multiple Shooting Method

- Around saddle point $Z_\sigma$ compute eigenvalues $\lambda_i > 0$ and eigenvectors $W_i^\pm$ of Hessian

$$H = \begin{pmatrix} \Re \frac{d^2 I}{dz^2} & -\Im \frac{d^2 I}{dz^2} \\ -\Im \frac{d^2 I}{dz^2} & -\Re \frac{d^2 I}{dz^2} \end{pmatrix}$$

- Boundary conditions:

→ $R_A^{(0)} = |Z^{(0)} - Z_\sigma| - \delta r = 0$ fixing shift freedom along solution

→ $R_B^{(0)} = (W^-)^t (Z^{(0)} - Z_\sigma) = 0$ selecting upward flow

→ $R_B^{(N)} = (0_{L\times L}\ 1_{L\times L}) Z^{(N-1)} = 0$ endpoint on real plane

# Multiple Shooting Method

- $2NL+1$ nonlinear equations for $2NL+1$ variables

$$R = \begin{pmatrix} R_A^{(0)} \\ R_B^{(0)} \\ R^{(1)} \\ \vdots \\ R^{(N-1)} \\ R_B^{(N)} \end{pmatrix} = 0 \qquad X = \begin{pmatrix} Z^{(0)} \\ \vdots \\ Z^{(N-1)} \\ \delta s \end{pmatrix}$$

- Efficiently solved by Newton's method

linear system $\dfrac{\partial R}{\partial X}\Delta X = -R \qquad X \to X + \Delta X$

nearly block-diagonal