Optimized numerical evolution of perturbations across sharp background trajectory turns in multifield inflation

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Motivation

- Modern high-energy frameworks (supergravity and string-inspired models) predict inflation with multiple fields evolving on non-Euclidean manifolds.
- Field-space curvature modifies effective masses and induces mode mixing.
- Standard techniques rely heavily on slow-roll or assume nearly flat field space, limiting their applicability.
- Next-generation observations require theoretical tools capable of capturing the full dynamics of these models with high fidelity.

Background equations

Action governing the dynamics of the scalar fields:

$$\boxed{S = \int \mathrm{d}^4x \sqrt{-g}\left[\frac{1}{2}M_{\mathrm{Pl}}^2 R - \frac{1}{2}h_{AB}\, g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B - V(\phi)\right]}$$

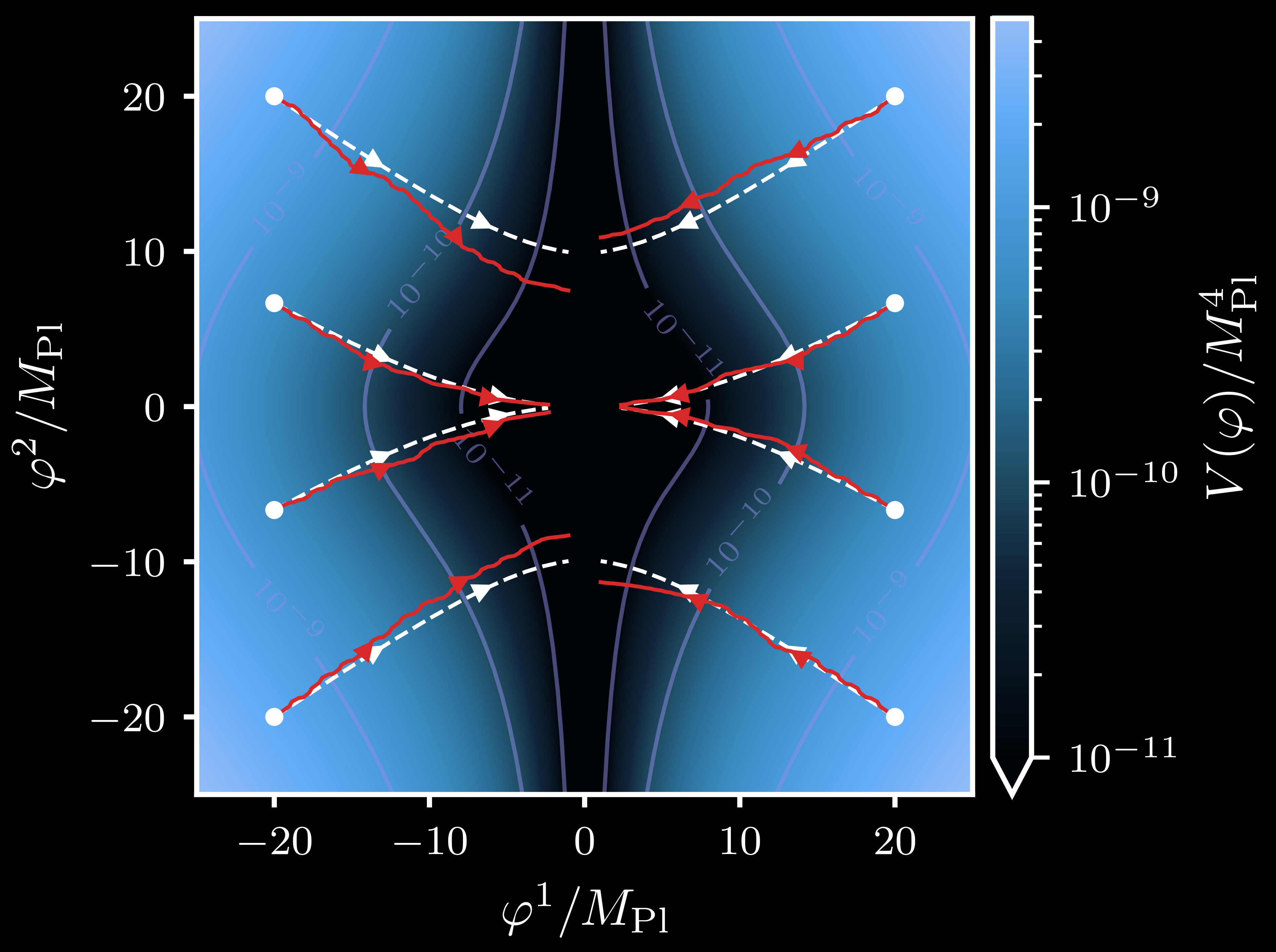
Background equation of motion:

$$\boxed{\mathscr{D}_t\dot{\varphi}^A + 3H\dot{\varphi}^A + h^{AB}\mathscr{D}_B V = 0}$$

where

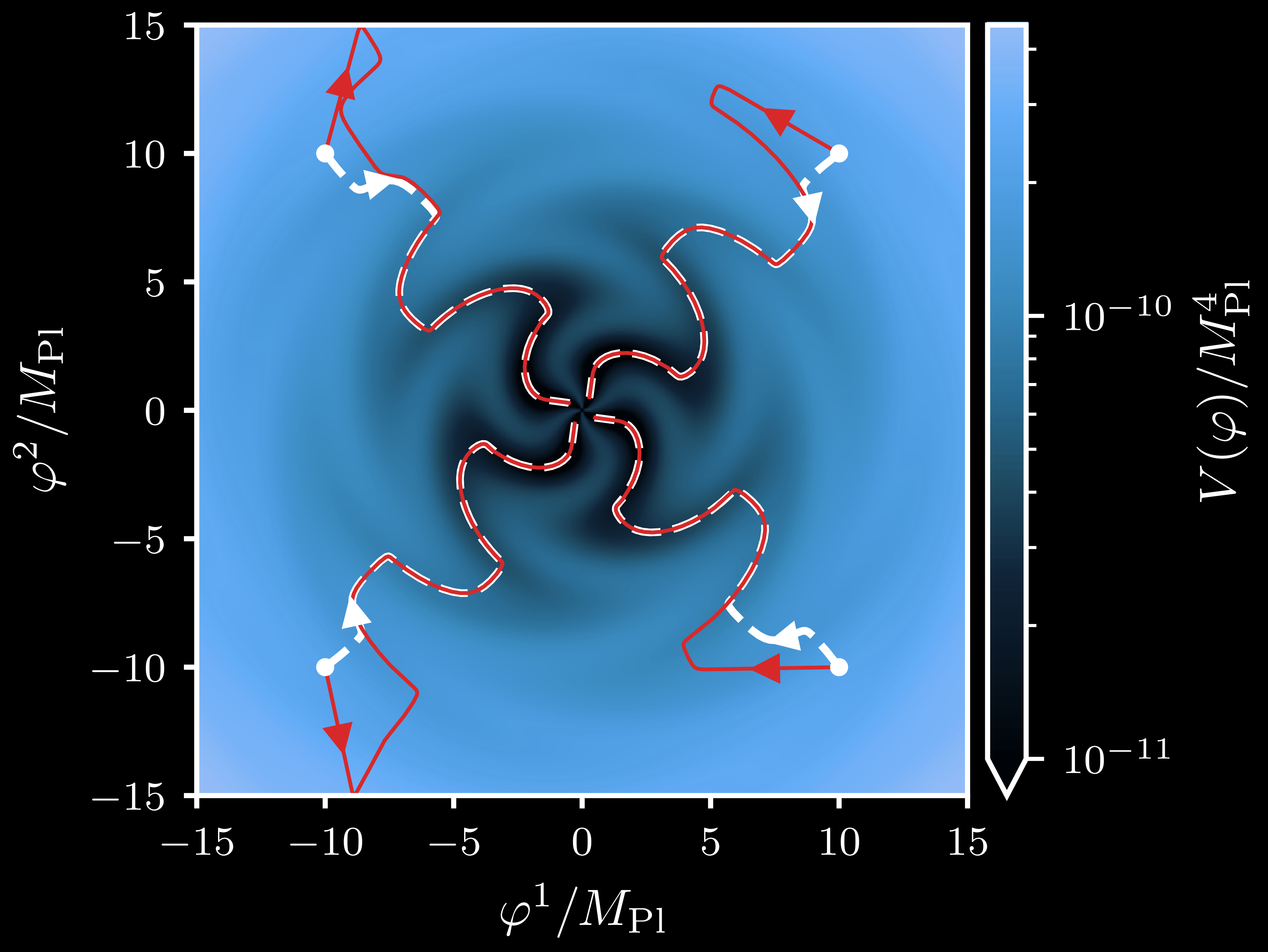
$$\mathscr{D}_t X^A \equiv \dot{\varphi}^B\mathscr{D}_B X^A = \dot{X}^A + \Gamma^A_{BC}\dot{\varphi}^C X^B \qquad \Gamma^A_{BC} = \frac{1}{2}h^{AD}(h_{DB,C} + h_{DC,B} - h_{BC,D})$$

Inflationary trajectories



$$h_{AB} = \delta_{AB} + \delta_A^2\delta_B^2 \sum_{i=1}^{N_{\rm b}} A_i \exp\left[-\frac{1}{2}\frac{(\varphi^1 - x_i)^2}{\sigma^2}\right]$$

$$V(\varphi^1, \varphi^2) = \frac{\lambda}{4}(\varphi^1)^4 + \frac{g}{2}(\varphi^2)^2(\varphi^1)^2$$



$$h_{AB} = \delta_{AB}$$

$$V(\varphi^1, \varphi^2) = \frac{1}{2}m_\varphi^2[(\varphi^1)^2 + (\varphi^2)^2]$$

$$+\lambda_{\rm V}^4 \cos^2\left[\frac{f_1}{2}\tan^{-1}(\varphi^2/\varphi^1) - \frac{f_2}{2}\sin\left(f_3\sqrt{(\varphi^1)^2 + (\varphi^2)^2}\right)\right]$$

Perturbation equations

Perturbation action in Fourier space

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^3\mathbf{k}\,a^3\left[h_{AB}\,\mathscr{D}_t\Phi^A\mathscr{D}_t\bar{\Phi}^B - \left(\frac{k^2}{a^2}h_{AB} + \mathscr{M}^2_{AB}\right)\Phi^A\bar{\Phi}^B\right]$$

where $\mathscr{M}^2_{AB} = V_{;AB} - R_{ACDB}\dot{\varphi}^C\dot{\varphi}^D - \dfrac{1}{a^3M^2_{\mathrm{Pl}}}\mathscr{D}_t\left(\dfrac{a^3}{H}\dot{\varphi}_A\dot{\varphi}_B\right)$

Equations of motion for the perturbations:

$$\begin{aligned}\Phi''^A + 2(\Gamma^A_{BC}\varphi'^C + \mathscr{H}\delta^A_B)\,\Phi'^B + \Big[\Gamma^A_{BC}\varphi''^C + \Gamma'^A_{BC}\varphi'^C + 2\mathscr{H}\Gamma^A_{BC}\varphi'^C&\\ + \Gamma^A_{CD}\Gamma^C_{BE}\varphi'^D\varphi'^E + \left(k^2\delta^A_B + a^2\mathscr{M}^A{}_B\right)\Big]\Phi^B = 0&\end{aligned}$$

Parallel transported vielbeins

We introduce the canonically rescaled fields

$$\boxed{v^a(\mathbf{k},\tau) \equiv a\,\Lambda^a_A\,\Phi^A(\mathbf{k},\tau)}$$

where $\mathscr{D}_t\Lambda^A_a = \dot{\Lambda}^A_a + \Gamma^A_{BC}\,\dot{\varphi}^C\Lambda^B_a = 0$ and $\Lambda^A_a\,\Lambda^B_b\,h_{AB} = \delta_{ab}$

$$\Downarrow$$

$$\boxed{S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3\mathbf{k}\,\left(\delta_{ab}v'^a\,\bar{v}'^b - \Omega^2_{ab}v^a\,\bar{v}^b\right)}$$

$$\Downarrow$$

$$\boxed{v''^a + \delta^{ab}\,\Omega^2_{bc}\,v^c = 0}$$

where $\Omega^2_{ab} \equiv \left(k^2 - \dfrac{a''}{a}\right)\delta_{ab} + a^2\mathscr{M}^2_{ab}$

Canonical quantization

Fourier components in terms of mode functions:

$$\hat{v}^b(\mathbf{k},\tau) = \sum_{\hat{\alpha}=1}^{N} \left[\bar{U}^b_{\hat{\alpha}}(k,\tau)\, \hat{a}_{\hat{\alpha}}(\mathbf{k}) + U^b_{\hat{\alpha}}(k,\tau)\, \hat{a}^\dagger_{\hat{\alpha}}(-\mathbf{k}) \right]$$

Canonical commutation relations

$$\left[\hat{a}_{\hat{\alpha}}(\mathbf{k}), \hat{a}^\dagger_{\hat{\beta}}(\mathbf{k}')\right] = \delta_{\hat{\alpha}\hat{\beta}}\, \delta^3(\mathbf{k}-\mathbf{k}')$$

$$\left[\hat{v}^a(\mathbf{k},\tau), \hat{\pi}^b(\mathbf{k}',\tau)\right] = i\, \delta^{ab}\, \delta^3(\mathbf{k}+\mathbf{k}')$$

The symplectic structure of the $2N$-dimensional phase space preserves the symplectic bilinear form:

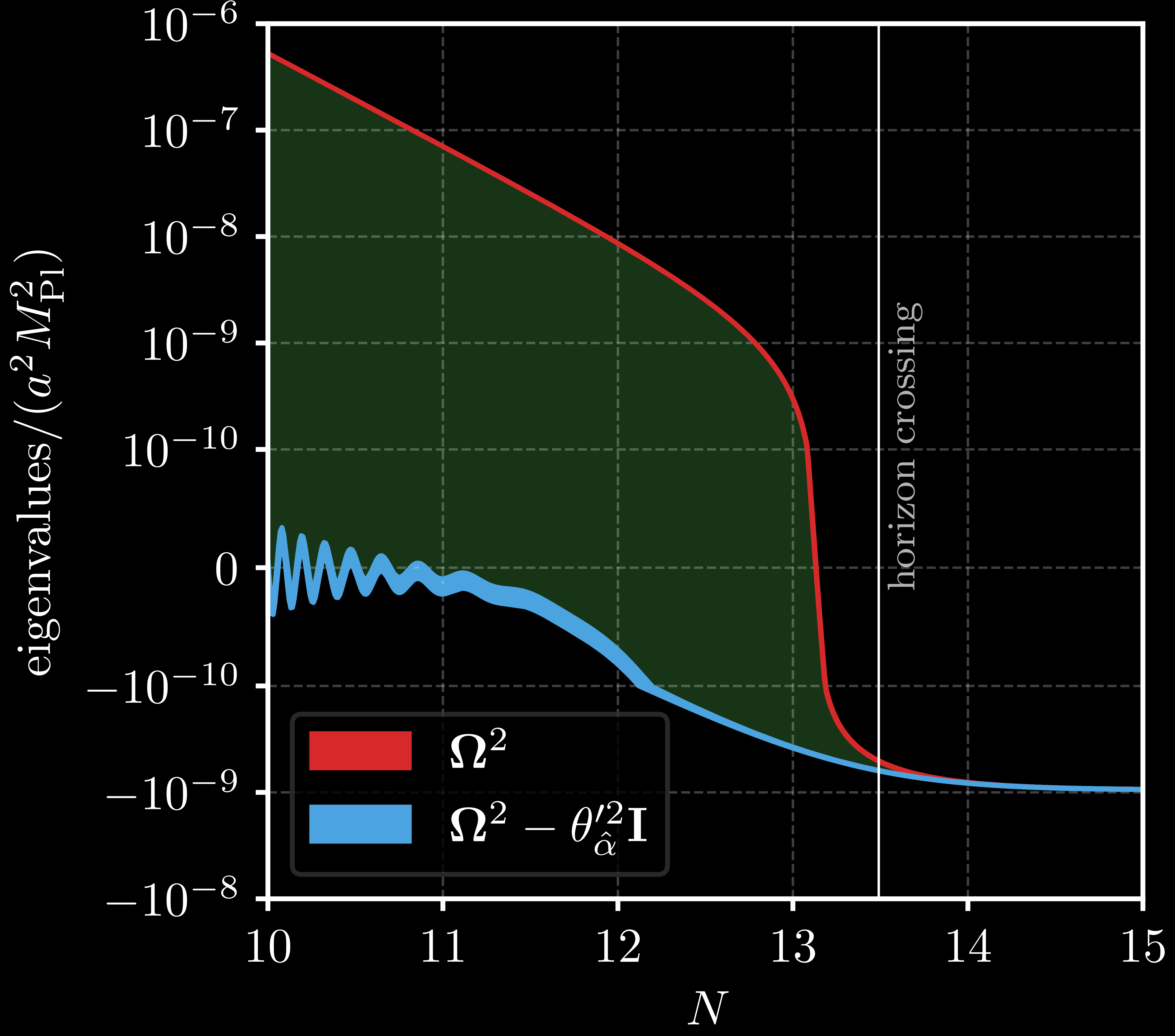
$$\begin{pmatrix} \bar{\mathbf{U}} & \mathbf{U} \\ \bar{\mathbf{U}}' & \mathbf{U}' \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{U}} & \mathbf{U} \\ \bar{\mathbf{U}}' & \mathbf{U}' \end{pmatrix}^{\intercal} = i \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

Amplitude-phase decomposition

	Single-field	**Multi-field**
EOM	$u'' + \Omega^2 u = 0$	$\mathbf{U}'' + \boldsymbol{\Omega}^2\mathbf{U} = \mathbf{0}$
Decomposition	$u = r\ \exp\left(i\theta\right)$	$\mathbf{U} = \left(\mathbf{r}_1 \exp\left(i\theta_1\right) \quad \cdots \quad \mathbf{r}_N \exp\left(i\theta_N\right)\right)$
Symplectic structure constraint	$2\theta' r^2 = 1$	$2\theta'_{\hat{\alpha}}\, r^2_{\hat{\alpha}} = 1$
Amplitude equation	$r'' + (\Omega^2 - \theta'^2)r = 0$	$\mathbf{r}''_{\hat{\alpha}} + (\boldsymbol{\Omega}^2 - \theta'^2_{\hat{\alpha}}\mathbf{I})\mathbf{r}_{\hat{\alpha}} + i(\theta''_{\hat{\alpha}}\,\mathbf{r}_{\hat{\alpha}} + 2\theta'_{\hat{\alpha}}\,\mathbf{r}'_{\hat{\alpha}}) = \mathbf{0}$

Dynamical frequency suppression

Suppressed oscillation scale in a mode injected at $k = e^{10} k_{\text{phys}}$, with $k_{\text{phys}} = 10^5 H_e$.



Curvature and isocurvature perturbations

$$\hat{\mathscr{R}}(\mathbf{k},\tau)=\frac{H}{|\dot{\varphi}|}e^A_{\mathscr{R}}\hat{\Phi}_A(\mathbf{k},\tau)$$

$$\hat{\mathscr{S}}_1(\mathbf{k},\tau)=\frac{H}{|\dot{\varphi}|}e^A_{\mathscr{S}_1}\hat{\Phi}_A(\mathbf{k},\tau)$$

$$\hat{\mathscr{S}}_2(\mathbf{k},\tau)=\frac{H}{|\dot{\varphi}|}e^A_{\mathscr{S}_2}\hat{\Phi}_A(\mathbf{k},\tau)$$

$$\vdots$$

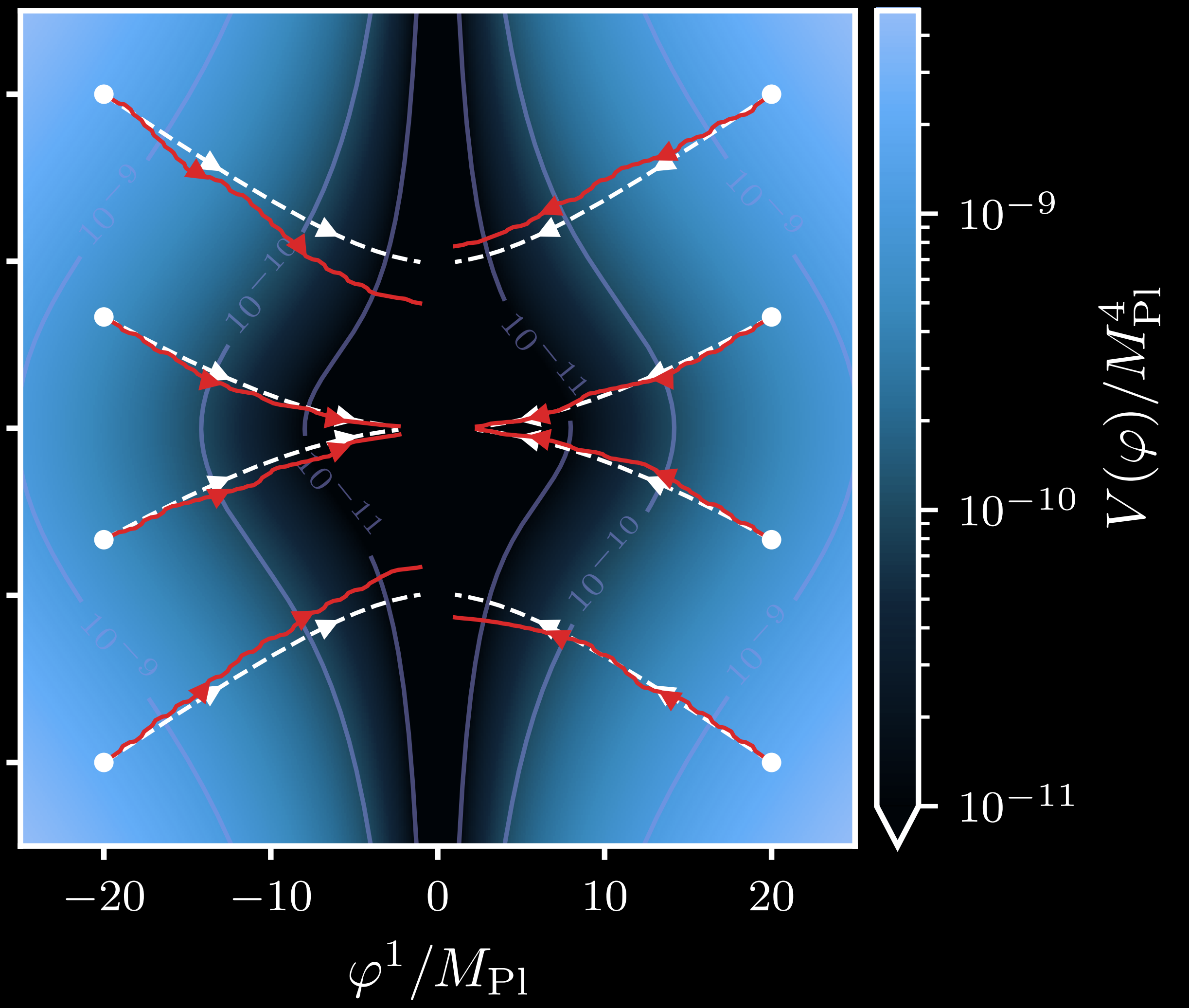
$$\hat{\mathscr{S}}_{N_\mathrm{f}-1}(\mathbf{k},\tau)=\frac{H}{|\dot{\varphi}|}e^A_{\mathscr{S}_{N_\mathrm{f}-1}}\hat{\Phi}_A(\mathbf{k},\tau)$$

$$\langle\hat{\mathscr{X}}(\mathbf{k},\tau)\hat{\mathscr{Y}}(\mathbf{k}',\tau)\rangle=\frac{2\pi^2}{k^3}\delta^3(\mathbf{k}+\mathbf{k}')\mathscr{P}_{\mathscr{X}\mathscr{Y}}(k,\tau)$$

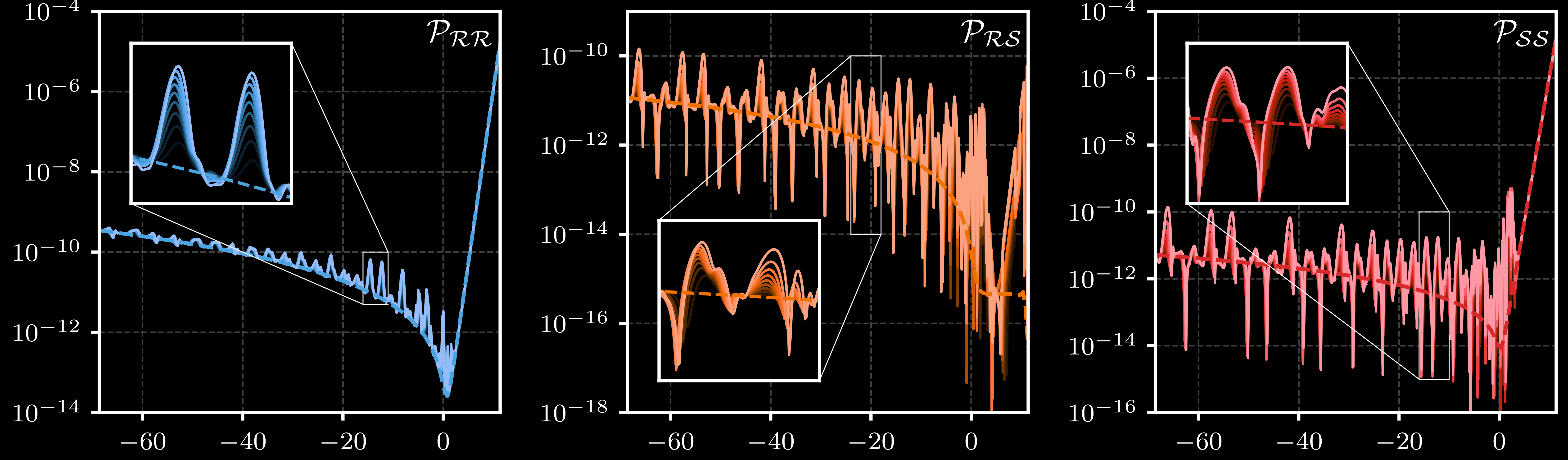
$$\Downarrow$$

$$\mathscr{P}_{\mathscr{X}\mathscr{Y}}=\frac{k^3}{2\pi^2}\left(\frac{H}{a|\dot{\varphi}|}\right)^2 e^A_{\mathscr{X}}h_{AB}\Lambda^B_i\left(\bar{\mathbf{U}}\mathbf{U}^\intercal\right)^{ij}\Lambda^D_j h_{CD}e^C_{\mathscr{Y}}$$

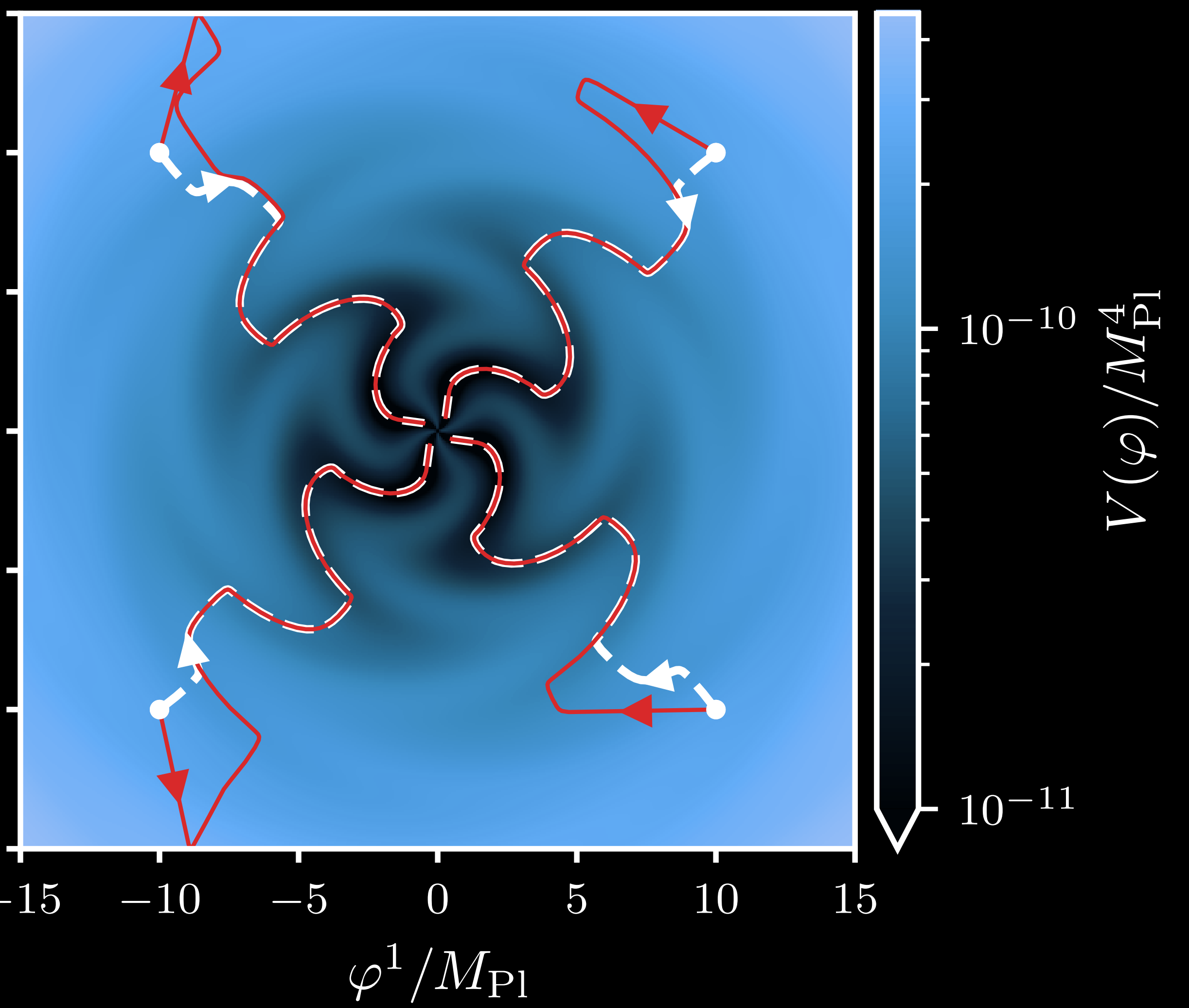
Inflationary trajectories and power spectra

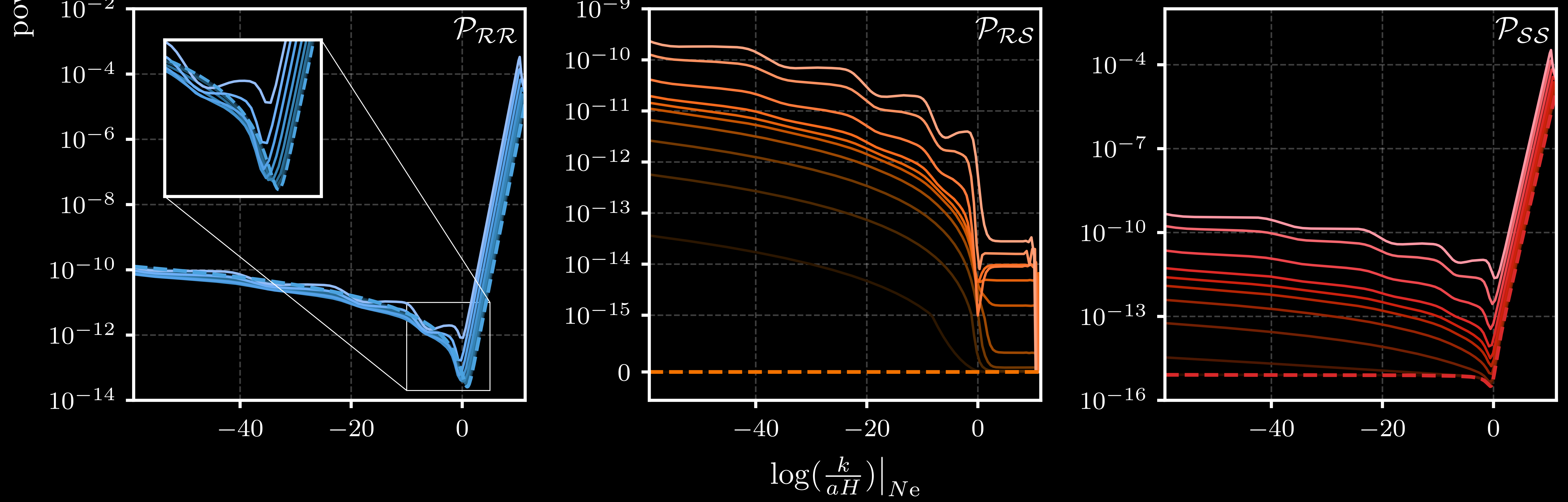


$h_{AB} = \delta_{AB} + \delta_A^2\delta_B^2 \sum_{i=1}^{N_{\rm b}} A_i \exp\left[-\frac{1}{2}\frac{(\varphi^1-x_i)^2}{\sigma^2}\right]$ and $V(\varphi^1,\varphi^2) = \frac{\lambda}{4}(\varphi^1)^4 + \frac{g}{2}(\varphi^2)^2(\varphi^1)^2$



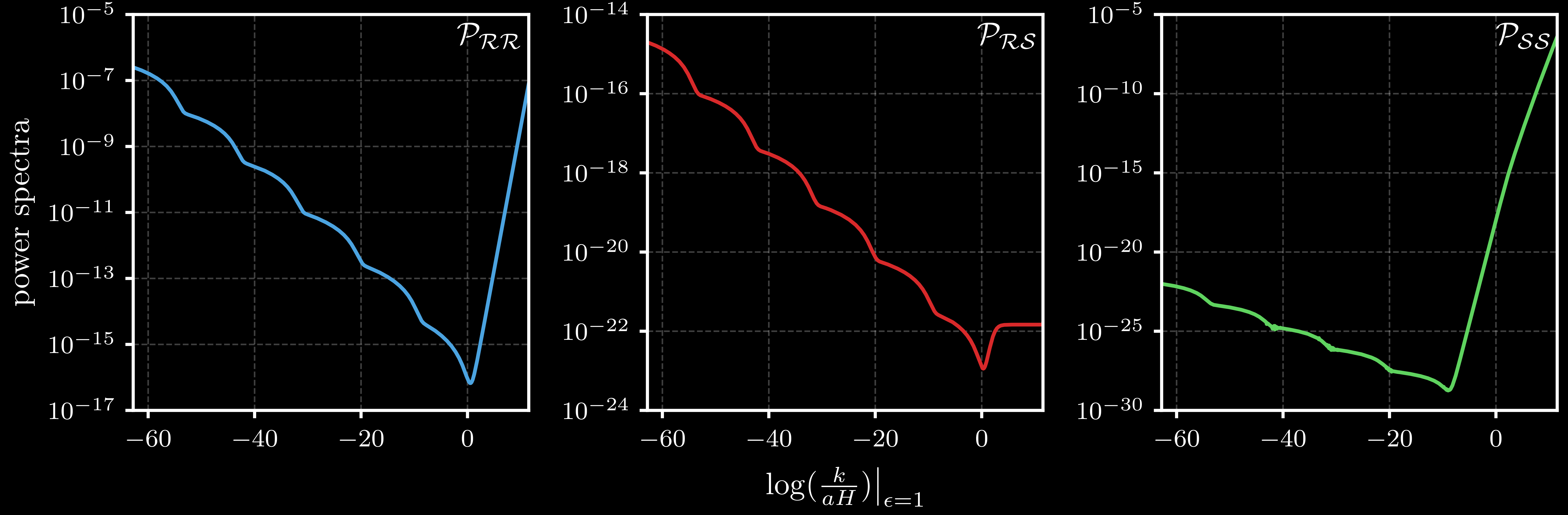
$h_{AB} = \delta_{AB}$ and $V(\varphi^1,\varphi^2) = \frac{1}{2}m_\varphi^2[(\varphi^1)^2+(\varphi^2)^2] + \lambda_{\rm V}^4\cos^2\left[\frac{f_1}{2}\tan^{-1}(\varphi^2/\varphi^1) - \frac{f_2}{2}\sin\left(f_3\sqrt{(\varphi^1)^2+(\varphi^2)^2}\right)\right]$



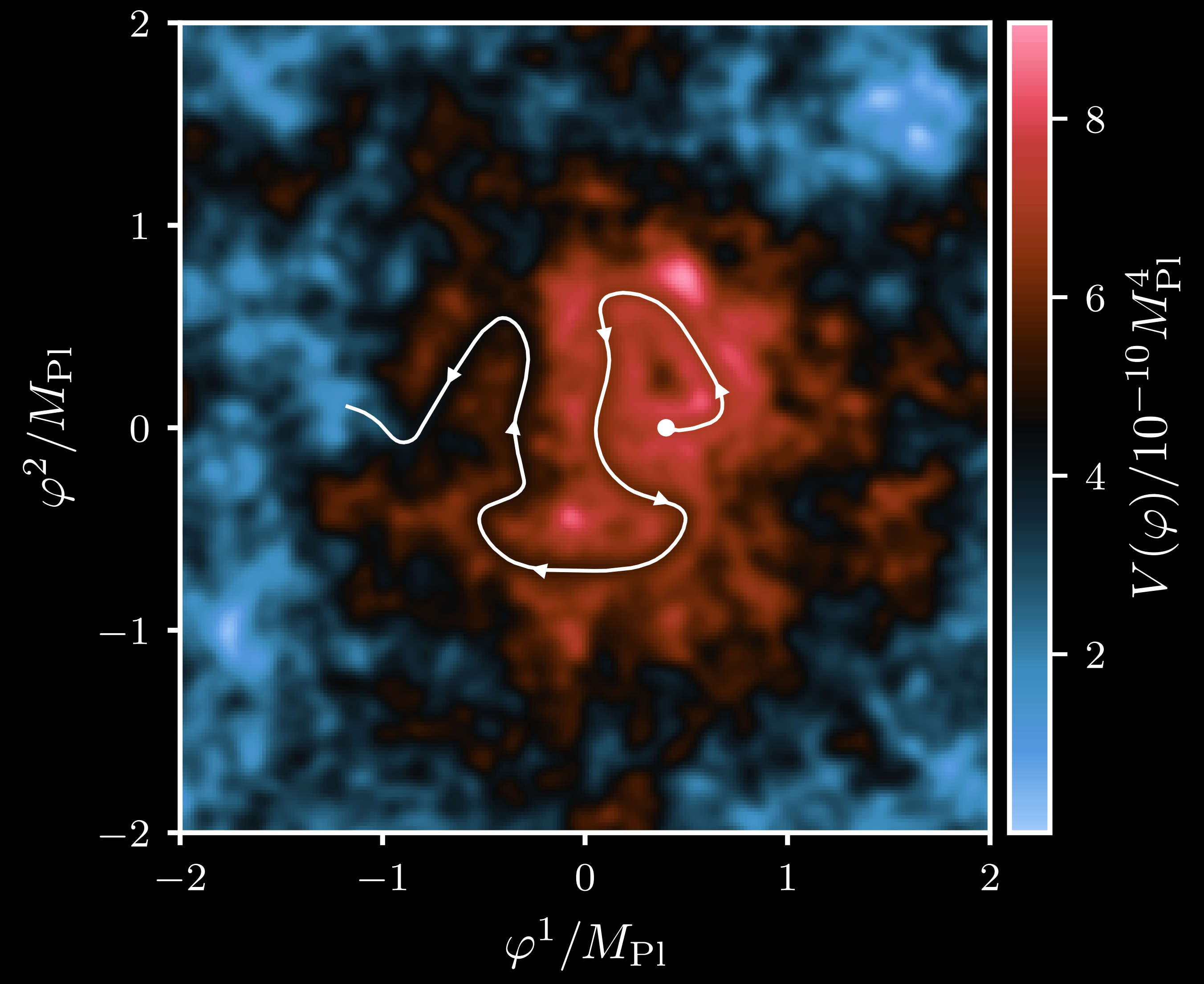


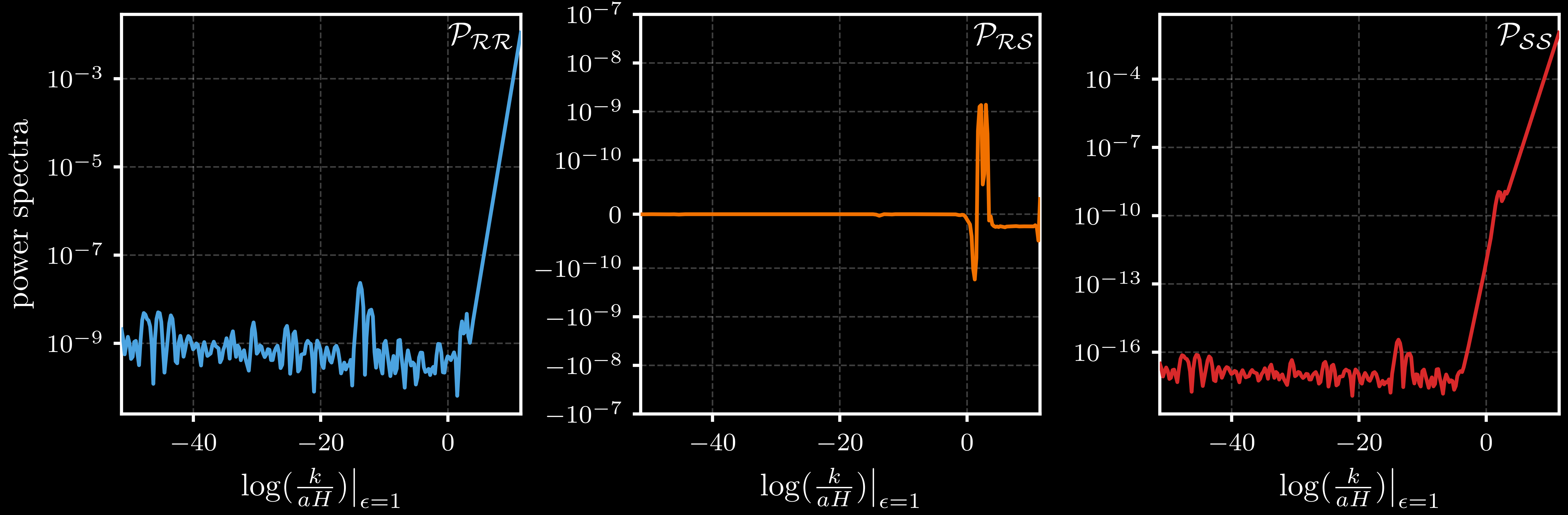
Inflationary trajectories and power spectra

$$h_{AB} = \delta_{AB} \quad \text{and} \quad V(\varphi^1, \varphi^2, \ldots, \varphi^6) = \tfrac{1}{2}\textstyle\sum_{B=1}^{6} m_B^2 (\varphi^B)^2$$



Inflationary trajectories and power spectra





Conclusions

- We presented a geometric framework that casts multi-field perturbations into a canonical form through parallel-transported vielbeins.
- Our generalized amplitude-phase method provides a robust way to separate oscillatory and slowly evolving components in the presence of field-space curvature.
- Together, these ingredients lead to a numerically stable and highly efficient evolution of mode functions in fully dynamical, curved field-space backgrounds.

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