Volume as Time:
Boundary Dynamics Near Black Hole Horizons

Farbod-Sayyed Rassouli

School of Physics and Astronomy and Nottingham Centre of Gravity,
University of Nottingham, University Park, Nottingham, NG7 2RD, UK

PASCOS, 2026

In collaboration with **Altay Etkin**, based on [2601.07911].
And **Bruno Alexandre**, based on [2501.17213].

What I want to convey in this talk

1. Observers, Volume as time and its dual: the cosmological constant
2. Near Horizon BH, the rise of JT gravity and its Time.

1. **Observers, Volume as time and its dual: the cosmological constant**
2. Near Horizon BH, the rise of JT gravity and its Time.

A massive particle as an observer

- Start with a relativistic particle:

$$S_{\rm p} = -m \int d\tau \, \sqrt{-h_{\tau\tau}}, \qquad h_{\tau\tau} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu .$$

- Promote the mass to an off-shell variable $Q(\tau)$ with conjugate momentum P_Q:

$$S_{\rm obs} = \int d\tau \left(Q\sqrt{-h_{\tau\tau}} - Q\,\partial_t P_Q \right) .$$

- Equations of motion:

$$\dot{Q} = 0, \qquad \dot{P}_Q = \sqrt{-h_{\tau\tau}} .$$

P_Q measures accumulated proper length along the worldline.
Q is the observer total energy.
Quantum Mechanically they are related by $[Q, P_Q] = i$.

Witten, *A Background-Independent Algebra in Quantum Gravity*, [2308.03663].

What is the gravitational analogue?

- A particle has a fixed parameter: **mass**.
- Gravity has a fixed parameter: **cosmological constant**.
- The cosmological constant is special:

$$S_\Lambda = -\int_{\mathcal{M}} d^d x \sqrt{|g|}\,\Lambda, \qquad Z \sim \exp(\cdots \Lambda\,\mathrm{Vol}(\mathcal{M}))\,.$$

- So Λ is naturally conjugate to spacetime volume.

Volume time is gravity

We can add a clock and promote Λ to be a function off shell:

$$S_{\mathrm{HT}} = \frac{1}{2}\int_{\mathcal{M}} d^d x\left[\sqrt{|g|}R - 2\Lambda\left(\sqrt{|g|} - \partial_\mu \mathcal{T}^\mu\right)\right], \tag{1}$$

$$S_{\mathrm{tot}} = \int d\tau \left[\mathcal{L}_{\mathrm{bulk}} - Q\left(\sqrt{-h_{\tau\tau}} - \partial_\tau P_Q\right)\right] \tag{2}$$

- Equations in the clock sector:

$$\partial_\mu \Lambda = 0, \qquad \sqrt{|g|} = \partial_\mu \mathcal{T}^\mu.$$

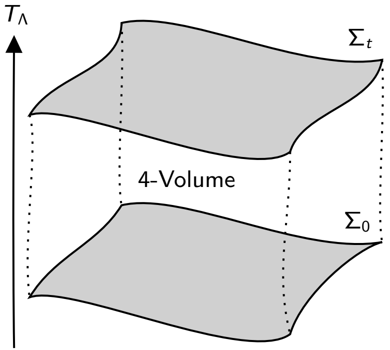
- Define the volume clock on a slice Σ:

$$T_\Lambda(\Sigma) := \int_\Sigma d\Sigma\, \mathcal{T}^0, \qquad \{T_\Lambda, \Lambda\}_{\mathrm{P.B.}} = 1.$$

- The Einstein equations are unchanged on shell.

This is the Henneaux–Teitelboim form of unimodular gravity. [Henneaux–Teitelboim 1989]

Volume time is literally accumulated volume



Integrating the HT condition between two slices gives

$$\sqrt{|g|} = \partial_\mu \mathcal{T}^\mu$$
$$\int_{\Sigma_0}^{\Sigma_t} d^d x \sqrt{|g|} = \int_{\Sigma_0}^{\Sigma_t} d^d x \, \partial_\mu \mathcal{T}^\mu .$$
$$= \int_{\Sigma_t} d\Sigma_t \, \mathcal{T}^0 - \int_{\Sigma_0} d\Sigma_0 \, \mathcal{T}^0$$

$$\mathrm{Vol}(\Sigma_0 \to \Sigma_t) = T_\Lambda(\Sigma_t) - T_\Lambda(\Sigma_0).$$

This is why I will call it **volume time**.

Why it matters: the problem of time

- In canonical GR the Hamiltonian constraint freezes the wavefunction:

$$\widehat{\mathcal{C}}_{\mathrm{GR}}\Psi = 0.$$

- With the volume time clock pair, the constraint becomes

$$\mathcal{C}_{\mathrm{obs}} = \mathcal{C}_{\mathrm{GR}} + p_{T_\Lambda} \approx 0.$$

- Upon quantisation:

$$i\,\partial_{T_\Lambda}\Psi(T_\Lambda) = \widehat{\mathcal{C}}_{\mathrm{GR}}\Psi(T_\Lambda). \tag{3}$$

p_{T_Λ} is the momentum conjugate to the volume clock, i.e. the cosmological constant sector. $p_{T_\Lambda} = \partial\mathcal{L}/\partial\dot{T}_\Lambda = \Lambda$

Top forms and the cosmological constant

Hawking's observation in four dimensions:

$$H_{\mu\nu\rho\sigma} = h\,\epsilon_{\mu\nu\rho\sigma}$$

so that

$$S_H = \frac{1}{2\cdot 4!}\int_{\mathcal{M}} d^4x\,\sqrt{g}\,H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma} = \frac{h^2}{2}\int_{\mathcal{M}} d^4x\,\sqrt{g}.$$

- A top-rank antisymmetric tensor has no local propagating degrees of freedom.
- Its on-shell value contributes like a vacuum cosmological constant.

Kaloper found a field redefinition that makes the volume-time structure explicit:

$$\frac{1}{2\cdot 4!}\int_{\mathcal{M}} d^4x\,\sqrt{g}\,H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma} \quad\Longleftrightarrow\quad \int_{\mathcal{M}} d^4x\,\Lambda\,(\sqrt{g}-\partial_\mu T^\mu)\,. \tag{4}$$

- In two dimensions, Maxwell theory is already a top-rank antisymmetric tensor theory.

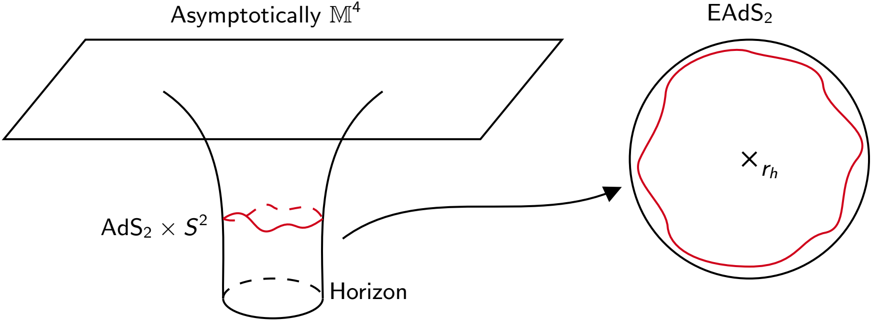
[Hawking 1984; Duff 1989; Kaloper 2023]

1. Observers, Volume as time and its dual: the cosmological constant
2. **Near Horizon BH, the rise of JT gravity and its Time.**

For 4d Reissner-Nordström near horizon:

$$ds^2 = \frac{R_2^2}{\zeta^2} \underbrace{\left[\left(1 - 4\pi^2 T^2 \zeta^2\right) d\tau^2 + \frac{d\zeta^2}{\left(1 - 4\pi^2 T^2 \zeta^2\right)} \right]}_{\mathrm{EAdS}_2} + r_h^2 d\Omega_2^2. \tag{5}$$

This is exactly $\mathrm{AdS}_2 \times S^2$ at extremal ($T = 0$) or near extremal.



Sketch from 4d to 2d: what we get

- Start from 4d Euclidean Einstein–Maxwell, schematically

$$S^{(E)}_{4d} \;\sim\; \frac{1}{G_4}\int d^4x\,\sqrt{g^{(4)}}\left(-R^{(4)} + F^{(4)}_{MN}F^{MN}_{(4)}\right).$$

- Near-horizon Euclidean RN:

$$ds^2 = R^2\left(\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2}\right) + R^2 d\Omega_2^2,$$

- Dimensional reduction on the S^2 gives a 2d Euclidean dilaton gravity:

$$S^{(E)}_{2d} \;\sim\; \int d^2x\,\sqrt{g}\Big[\Phi R + (\text{dilaton kinetic}) - \tfrac{1}{4}\,\Phi\, F^{(2)}_{\mu\nu}F^{\mu\nu}_{(2)} - V(\Phi; Q_e, Q_m)\Big],$$

- In the strict near-horizon limit we can take $\Phi \simeq \Phi_0$ constant.
- Similar structure for $\mathrm{AdS}_5 \times S^5$.

versions of AdS/CFT holography have recently generated [6–9], we consider the simplest non-trivial gravitational theory of two-dimensional flat spacetimes, introduced by Cangemi and Jackiw (CJ) in [10]. Its action reads

$$I_{\mathrm{CJ}} = S_0\chi + \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{-g}\big[\Phi R + 2\Psi(1 - \varepsilon^{\mu\nu}\partial_\mu A_\nu)\big] + \frac{1}{2}\int_{\partial\mathcal{M}} du\sqrt{-h}\big[2\Phi K - n^\mu\nabla_\mu\Phi\big]\,, \tag{1}$$

with $S_0 \in \mathbb{R}_+$, χ the Euler characteristic of the manifold $\mathcal{M}$, and $\varepsilon^{\mu\nu}$ the Levi-Civita tensor. The boundary terms are chosen to ensure the finiteness of the on-shell action,

FIGURE: Paper by Afshar, González, Grumiller, and Vassilevich on 2d flat holography and the complex SYK , [1911.05739] [2205.02240].

This structures are found and used in JT gravity unrecognized by the authors!
How do they found this formulation? ⇒ [2501.17213]

The second-order action is

$$S_{\mathrm{HT}_2} = \frac{1}{2}\int d^2x\left[\sqrt{-g}\phi(R-2\lambda) - 2\Lambda(\sqrt{-g} - \partial_\mu \mathcal{A}^\mu)\right]. \tag{6}$$

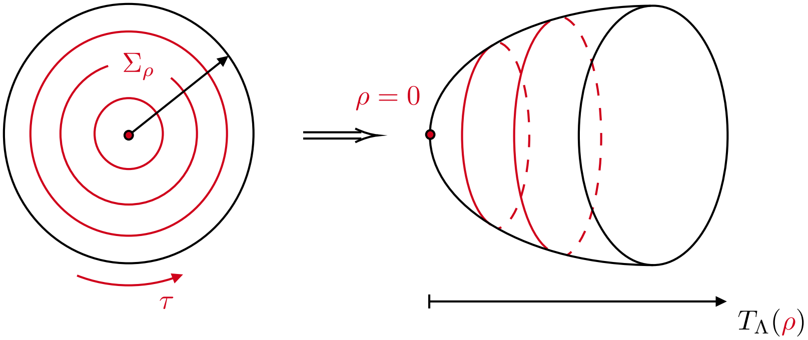
- Clock equations:

$$\partial_\mu \Lambda = 0, \qquad \sqrt{-g} = \partial_\mu \mathcal{A}^\mu.$$

- With $\mathcal{A}^\mu = \varepsilon^{\mu\nu} A_\nu$,

$$T_\Lambda(\Sigma) = \int_\Sigma d\Sigma\, \mathcal{A}^0 = -\int_\Sigma d\Sigma\, A_1.$$

In BF language this sector is the extra abelian generator in the centrally extended JT algebra.



Constant-ρ slices give a geometric way to read the bulk clock.

The bulk action reads:

$$I_{\text{bulk}} = -\frac{1}{2}\int d^2x\left[\sqrt{-g}\phi(R+2) - 2\Lambda(\sqrt{-g} - \partial_\mu \mathcal{A}^\mu)\right] + I^{\text{mixed}}_{\text{bdy}} \tag{7}$$

The useful holographic ensemble relates the boundary gauge field and the boundary value of Λ:

$$A_u\big|_{\partial\mathcal{M}} = (-\kappa\epsilon\Lambda_b + 1)\sqrt{h}.$$

This follows from the boundary action

$$I^{\text{mixed}}_{\text{bdy}} = -\int_{\partial\mathcal{M}} du\,\sqrt{h}\,\phi_b(K-1) - \int_{\partial\mathcal{M}} du\,\Lambda_b(A_u - \sqrt{h}) - \frac{\kappa\epsilon}{2}\int_{\partial\mathcal{M}} du\,\sqrt{h}\,\Lambda_b^2.$$

This is the boundary condition that lets the $U(1)$ clock become dynamical.

The dual theory

Focus on the $U(1)$ part of the boundary theory:

$$I_{\rm bdy} \supset -\int_{\partial\mathcal{M}} du\,\Lambda_b(A_u - \sqrt{h}) - \frac{\kappa\epsilon}{2}\int_{\partial\mathcal{M}} du\,\sqrt{h}\,\Lambda_b^2.$$

Now plug in the solution for the gauge field $A_u = 1/\epsilon + i\varphi'(u) \leftrightarrow \sqrt{g} = \partial_\mu \mathcal{A}^\mu$

$$I_{\rm bdy} \supset -\int_{\partial\mathcal{M}} du\left(i\Lambda_b\,\varphi' + \frac{\kappa\epsilon}{2}\Lambda_b^2\sqrt{h}\right), \tag{8}$$

Integrating out Λ_b gives

$$I_{\rm bdy} \supset -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} du\,\varphi'^2.$$

This is the universal low-energy action of complex SYK:
Schwarzian mode + particle on $U(1)$.

[Davison et al.; Mertens et al.]

The boundary Kaloper trick

Lets restart with:

$$I_{\rm bdy} \supset -\int_{\partial\mathcal{M}} du\left(i\Lambda_b\,\varphi' + \frac{\kappa\epsilon}{2}\Lambda_b^2\sqrt{h}\right),$$

Now repeat the top-form field redefinition in one lower dimension:

$$\Psi(u) = -\frac{\varphi(u)}{\kappa\epsilon\Lambda_b(u)}, \qquad \Theta(u) = -\frac{\kappa\epsilon\Lambda_b^2(u)}{2}.$$

Then the $U(1)$ particle becomes a boundary HT action:

$$I_{\rm bdy} \supset -\int_0^\beta du\,\Theta\left(i\Psi' - \sqrt{h}\right) - i\Theta\Psi\Big|_0^\beta.$$

This is the observer action that we started.

Boundary volume time

The boundary HT constraint is

$$\sqrt{h} = i\Psi'.$$

Integrating around the thermal circle gives

$$\int_0^\beta du\,\sqrt{h} = i\Psi(\beta) - i\Psi(0).$$

So define

$$T_\Theta(u) := i\Psi(u), \qquad \mathrm{Vol}(S^1_\beta) = \Delta T_\Theta.$$

T_Θ is the **boundary volume time**; Θ is its intrinsic boundary vacuum energy.

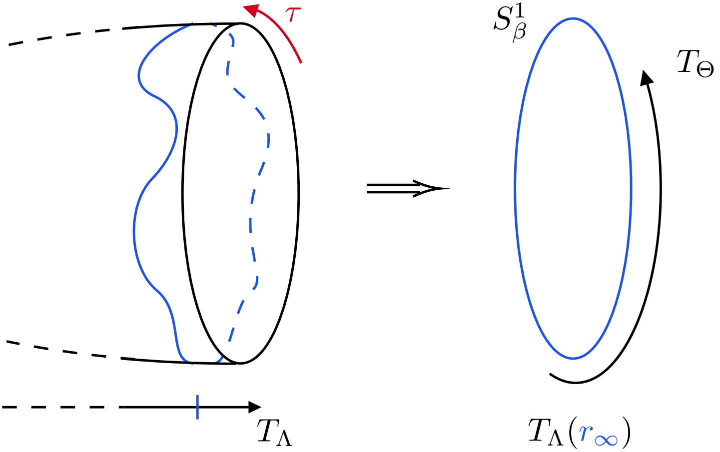


Figure 3: the thermal boundary clock is read at r_∞.

Main lessons

1. The cosmological constant is naturally conjugate to volume.
2. We can promote the CC giving a physical volume clock.
3. Holographically, the clock becomes the $U(1)$ particle of complex SYK.
4. After a boundary field redefinition, this is Witten's observer action.

Thank you!

Questions?

We integrate the bulk away enforcing $R=-2$:

$$I_{\mathrm{JT}} = \underbrace{-\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\Phi(R+2)}_{=0} - \int_{\partial\mathcal{M}} du\sqrt{h}\,\phi_b(K-1) \tag{9}$$

The only remaining mode is the way the cutoff curve sits in the disk:

$$X^\mu(u) = (t(u), z(u)).$$

Fix the induced metric at the boundary:

$$ds^2 = \frac{dt^2+dz^2}{z^2}, \qquad ds^2\big|_{\partial\mathcal{M}} = \frac{du^2}{\epsilon^2} \quad \Longrightarrow \quad z(u) = \epsilon\, t'(u) + \cdots . \tag{10}$$

The metric equation fixes the dilaton profile, behaves near $z=0$ as:

$$\Phi(t,z) \sim \frac{\phi_r(u)}{z} + \cdots \qquad \phi_b = \frac{\phi_r}{\epsilon} \tag{11}$$

How the Schwarzian appears

The boundary mode enters through the extrinsic curvature of the cutoff curve:

$$K - 1 = \epsilon^2 \, \mathrm{Sch}\{t(u), u\} + O(\epsilon^4). \tag{12}$$

Since the bulk term vanishes on shell, the renormalized GHY term gives:

$$I_{\mathrm{Sch}} = -\phi_r \int du \, \mathrm{Sch}\{t(u), u\}. \tag{13}$$

The Schwarzian derivative is:

$$\mathrm{Sch}\{t, u\} = \left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2. \tag{14}$$

The Schwarzian is the effective action for wiggling the boundary curve while the bulk stays AdS_2.

The clock shows up at the asymptotic boundary

In Poincare coordinates,

$$ds^2 = \frac{dt^2 + dz^2}{z^2}, \qquad \sqrt{g} = \frac{1}{z^2} = \partial_\mu \mathcal{A}^\mu..$$

Solving the HT constraint in axial gauge:

$$A_z = 0, \qquad A_t = \frac{1}{z} + i\partial_t \varphi(t).$$

On the boundary curve $(t(u), z(u))$,

$$A_u = \frac{t'}{z} + i\varphi'(u) \quad \Longrightarrow \quad A_u\big|_{\partial\mathcal{M}} = \frac{1}{\epsilon} + i\varphi'(u).$$

- The divergent piece is the volume; the finite piece is the boundary phase.
- The HT constraint fixes the electric-field sector: $\star F = 1$.

Boundary dynamics: Schwarzian plus a particle

Using

$$K - 1 = \epsilon^2 \mathrm{Sch}\{t, u\} + \mathcal{O}(\epsilon^4), \qquad A_u = \frac{1}{\epsilon} + i\varphi',$$

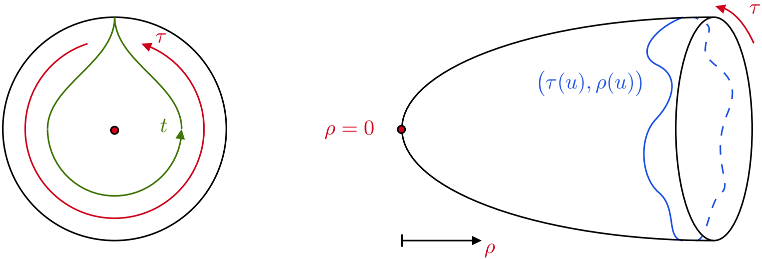
the on-shell action becomes

$$I_{\mathrm{bdy}} = -\overline{\phi}_r \int_{\partial \mathcal{M}} du \, \mathrm{Sch}\{t, u\} - \frac{1}{2\kappa} \int_{\partial \mathcal{M}} du \, \varphi'^2.$$

This is the universal low-energy action of complex SYK:
Schwarzian mode + particle on $U(1)$.

[Davison et al.; Mertens et al.]

Poincare disk versus thermal disk



(a) (b)

Poincare coordinates are natural at zero temperature;
thermal coordinates make the disk a cigar.

- **Poincare:** $$ds^2 = \frac{dt^2 + dz^2}{z^2}.$$
- **Thermal:** $$ds^2 = r_h^2 \sinh^2 \rho \, d\tau^2 + d\rho^2, \qquad r_h = \frac{2\pi}{\beta}.$$
- **Boundary map:** $$t \in (-\infty, \infty) \;\Rightarrow\; f \in \left(-\frac{\beta}{2}, \frac{\beta}{2}\right)$$

Thermal complex SYK action in Standard Literature

At finite temperature, the boundary time is

$$t(u) = \tan\left(\frac{\pi}{\beta} f(u)\right), \qquad f(u+\beta) = f(u) + \beta.$$

The thermal on-shell action is

$$I_{\mathrm{bdy}} = -\overline{\phi}_r \int_0^\beta du \left(\frac{r_h^2}{2} f'^2 + \mathrm{Sch}\{f, u\}\right) - \frac{1}{2\kappa} \int_0^\beta du \big(\varphi' + i r_h f'\big)^2.$$

A volume clock at the boundary

Take the constant slice Σ to be the asymptotic boundary:

$$T_\Lambda(\partial\mathcal{M}) := \int_{\partial\mathcal{M}} du\, A_u\big|_{\partial\mathcal{M}}.$$

For the Poincare boundary,

$$T_\Lambda(z) = \int dt\, A_t = \frac{\Delta t}{z} + i\Delta\varphi.$$

At $z = \epsilon$ and trivial winding,

$$T_\Lambda(\partial\mathcal{M}) = \frac{L}{\epsilon} = \mathrm{Vol}(\mathcal{M}).$$

The bulk volume clock can be read as a boundary flux.

1. Zero temperature and Finite temperature:

$$I_{\mathrm{HT}_2}^{T=0} = -\varphi_r \int du \, \mathrm{Sch}\{t, u\} - \frac{1}{2\kappa} \int du \, \phi'^2.$$

$$I_{\mathrm{HT}_2}^{\beta} = -\varphi_r \int_0^{\beta} du \left(\mathrm{Sch}\{f, u\} + \frac{r_h^2}{2} f'^2 \right) - \frac{1}{2\kappa} \int_0^{\beta} du \, \left(\phi' + i r_h f' \right)^2 .$$

2. Boundary Λ_b^2 form

$$I_{\Lambda_b^2}^{\beta} = -\varphi_r \int_0^{\beta} du \left(\mathrm{Sch}\{f, u\} + \frac{r_h^2}{2} f'^2 \right) - \int_0^{\beta} du \left[i\Lambda_b \left(\phi' + i r_h f' \right) + \frac{\kappa\epsilon}{2} \sqrt{h} \, \Lambda_b^2 \right] .$$

3. Boundary HT_1 form

$$I_{\mathrm{HT}_1}^{\beta} = -\varphi_r \int_0^{\beta} du \left(\mathrm{Sch}\{f, u\} + \frac{r_h^2}{2} f'^2 \right) - \int_0^{\beta} du \, \Theta \left(i\Psi' - \sqrt{h} \right) - i\Theta\Psi \Big|_0^{\beta} .$$

We have returned to the observer

Worldline observer

$$S_{\mathrm{obs}} = \int d\tau \left(P_Q \dot{Q} - Q\sqrt{-h_{\tau\tau}} \right).$$

$$\dot{P}_Q = \sqrt{-h_{\tau\tau}}.$$

Boundary observer

$$I^{\mathsf{JT}}_{\mathsf{bdy}} \supset -\int du\, \Theta(i\Psi' - \sqrt{h}).$$

$$i\Psi' = \sqrt{h}.$$

The boundary phase mode is not just a spectator.
It is the clock of a $(0+1)$-dimensional HT observer.

Thank you!

Questions?

BF-theory for JT gravity

A first-order formulation of JT gravity goes through a gauge-theoretic construction. This is achieved by taking the isometry group of (A)dS$_2$ spacetime and defining a BF-theory.

- JT gravity is based on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$:

$$[P_a, P_b] = 2\lambda J_{ab} \equiv -\lambda\epsilon_{ab}\widetilde{J}, \qquad [P_a, \widetilde{J}] = -\epsilon_a{}^b P_b. \tag{15}$$

  We further define a gauge connection and its associated curvature two-form:

$$\mathbf{A} = e^a P_a + \widetilde{\omega}\widetilde{J} \qquad \text{and} \qquad \mathbf{F} = T^a P_a + \left(\widetilde{R} - \lambda\widetilde{\Sigma}\right)\widetilde{J}. \tag{16}$$

- Dynamics is obtained by considering a BF-like action with Lagrange multipliers $\mathbf{B} = (B^a, \phi)$:

$$S = \int_{\mathcal{M}_2} tr(\mathbf{BF}) = \int_{\mathcal{M}_2} B_a T^a + \phi\big(\widetilde{R} - \lambda\widetilde{\Sigma}\big) \quad \xRightarrow{T^a=0} \quad S_{\mathrm{JT}} = \int_{\mathcal{M}_2} \phi\big(\widetilde{R} - \lambda\widetilde{\Sigma}\big).$$

  We have on-shell

$$\widetilde{R} = \lambda\widetilde{\Sigma} \qquad \text{and} \qquad D\mathbf{B} \equiv d\mathbf{B} + \big[\mathbf{A}, \mathbf{B}\big] = 0. \tag{17}$$

Extended de Sitter algebra and HT_2

We consider a central extension of the de Sitter group by $U(1)$. The corresponding algebra $\widehat{\mathfrak{sl}}(2,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$ is:

$$\left[P_a, P_b\right] = -\lambda\epsilon_{ab}\widetilde{J} + \epsilon_{ab} I, \qquad \left[P_a, \widetilde{J}\right] = -\epsilon_a{}^b P_b, \qquad \left[\widetilde{J}, \widetilde{J}\right] = 0, \tag{18}$$

- We can then consider a connection and curvature valued in the extended de Sitter algebra:

$$\mathbf{A} = e^a P_a + \widetilde{\omega}\widetilde{J} + AI \quad \text{and} \quad \mathbf{F} = T^a P_a + \left(\widetilde{R} - \lambda\widetilde{\Sigma}\right)\widetilde{J} + \left(F + \widetilde{\Sigma}\right) I. \tag{19}$$

  Here, we used the abelian connection A with associated curvature $F = dA$.

- Dynamics is obtained via Lagrange multiplier fields (B^a, ϕ, Λ) to give a BF-like action:

$$S = \int_{\mathcal{M}_2} B_a T^a + \phi(\widetilde{R} - \lambda\widetilde{\Sigma}) - \Lambda(F + \widetilde{\Sigma})) \xRightarrow{T^a = 0} S_{\mathrm{HT}_2} = \int_{\mathcal{M}_2} \phi(\widetilde{R} - \lambda\widetilde{\Sigma}) - \Lambda(F + \widetilde{\Sigma}).$$

  Using Einstein-Cartan variables and going to 2nd-order formalism gives the HT_2 action:

$$S_{\mathrm{HT}_2} = \frac{1}{2}\int_{\mathcal{M}_2} d^2x\,\sqrt{-g}\left[\phi(R - 2\lambda) - 2\Lambda\left(1 - \epsilon^{\mu\nu}\partial_\mu A_\nu\right)\right]. \tag{20}$$

Closed universes in de Sitter HT_2

We consider the mini-superspace approximation (MSS), where all the fields depend on global time coordinate t. The metric reads:

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\theta^2.$$

The HT_2 action in mini-superspace becomes

$$S_{\mathrm{HT}_2} = -\int dt \left(\frac{\dot{\phi}\dot{a}}{N} + Na(\phi + \Lambda) - \Lambda \dot{T}_\Lambda\right) \equiv S_{\mathrm{JT}} + \int dt\, \Lambda \dot{T}_\Lambda \tag{21}$$

$$T_\Lambda = \int_{S^1} A\Big|_{S^1} = -A_\theta \tag{22}$$

For global de Sitter we have this metric:

$$ds^2 = -dt^2 + \cosh^2(t)d\theta^2.$$

Geometrically, this describes a contracting closed universe at early times and an expanding closed universe at later times, with a minimal length $a(0) = 1$ reached at $t = 0$.

Unimodular time in HT_2 MSS

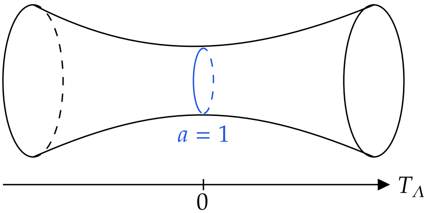
We can see that on-shell, we have:

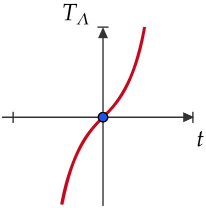
$$\sqrt{-g} = \dot{\mathcal{T}}^0 \quad \Longrightarrow \quad a(t) = \dot{T}_\Lambda, \tag{23}$$

That is unimodular time simply is

$$T_\Lambda(t) = \int_{\Sigma_t} dt \; a(t) = \sinh(t), \qquad \forall t \in \mathbb{R}. \tag{24}$$

Here a schematic drawing for the on-shell unitary time-evolution for global de Sitter geometry in unimodular time:





Lorentzian Cobordism and Morse function

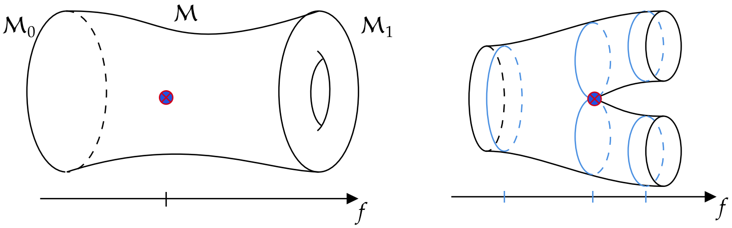


Figure: A cobordism between two manifold that have different genus, where the critical point is in the bulk manifold and is marked in the morse function direction.

Definition: Let M be a smooth n-manifold. A smooth map $f : M \to \mathbb{R}$ is a *Morse function* if every critical point $p \in M$ (i.e. $df_p = 0$) is *non-degenerate*, meaning that the Hessian quadratic form at p is nonsingular:

$$\det\big(\mathrm{Hess}_p f\big) \neq 0. \tag{25}$$

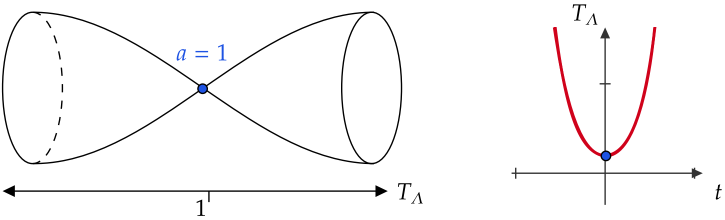
Remark on classical Topology change and T_Λ

As an example consider a milne dS closed universe which such metric:

$$ds^2 = -dt^2 + \sinh^2(t)d\theta^2.$$

Therefore in this case unimodular time is:

$$a(t) = \dot{T}_\Lambda \implies T_\Lambda = \cosh(t) \tag{26}$$



It is clear that unimodular time is storing topology change information, in this case lorentzian cobordism where T_Λ can seen as a **Morse function**.

The mini-superspace variables being a, ϕ and T_Λ have associated conjugate momenta:

$$p_\phi = -\frac{\dot{a}}{N}, \qquad p_a = -\frac{\dot{\phi}}{N} \qquad p_{T_\Lambda} = \Lambda.$$

We can find the Hamiltonian for mini-superspace HT_2:

$$\begin{aligned} H &= \sum_i p_i \dot{q}_i - L && (27) \\ &= Na(-p_a(a^{-1})p_\phi + \lambda\phi + p_{T_\Lambda}) && (28) \\ &= Na(\mathcal{C}_{\mathrm{JT}} + p_{T_\Lambda}) \equiv Na\,\mathcal{C}_{\mathrm{HT}_2} && (29) \end{aligned}$$

the Wheeler-DeWitt equation emerges as following:

$$\frac{\delta S_{\mathrm{HT}_2}}{\delta N} \Rightarrow \;\; \mathcal{C}_{\mathrm{HT}_2} \approx 0 \;\; \xrightarrow{\text{quantisation}} \;\; \widehat{\mathcal{C}}_{\mathrm{HT}_2}\Psi = 0 \tag{30}$$

We promote the phase space variables to quantum operators

$$p_\phi \to \widehat{p}_\phi = -i\frac{\partial}{\partial\phi}, \qquad p_a \to \widehat{p}_a = -i\frac{\partial}{\partial a} \qquad p_{T_\Lambda} \to \widehat{p}_{T_\Lambda} = -i\frac{\partial}{\partial T_\Lambda},$$

The **Wheeler-DeWitt equation transforms into a Schrödinger-like equation**, that is

$$\left[\partial_a\left(\frac{1}{a}\partial_\phi\right) + \phi - i\frac{\partial}{\partial T_\Lambda}\right]\Psi(a,\phi;T_\Lambda) = 0 \tag{31}$$

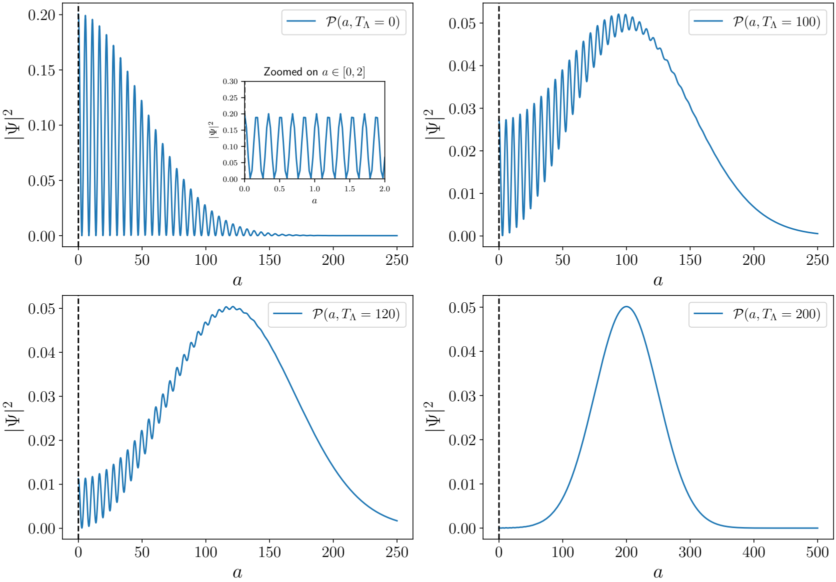
$$\left[\widehat{\mathcal{C}}_{\mathrm{JT}} - i\frac{\partial}{\partial T_\Lambda}\right]\Psi(a,\phi;T_\Lambda) = 0 \tag{32}$$

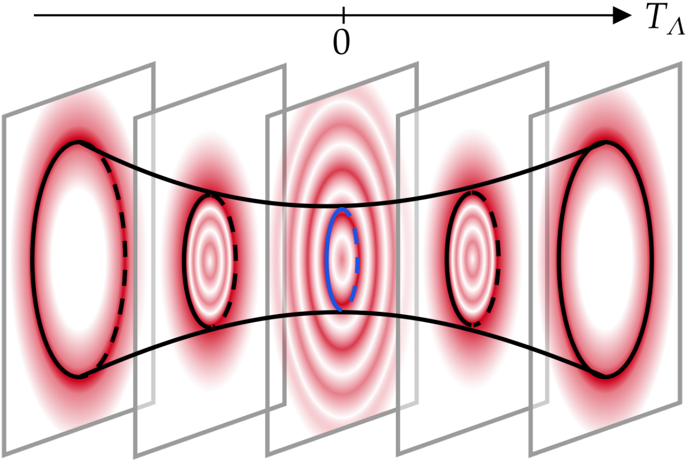
$$i\frac{\partial}{\partial T_\Lambda}\Psi(a,\phi;T_\Lambda) = \widehat{\mathcal{C}}_{\mathrm{JT}}\Psi(a,\phi;T_\Lambda). \tag{33}$$

Next we want to find the wavefunction that governs this model universe.

Quantum interference

Here we illustrate $\left|\Psi(a, \phi; T_\Lambda)\right|^2$, plotted against a, the size of the universe:




T_Λ
0