

Gauge invariant perturbations of $F(T, T_G)$ Cosmology

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① Introduction

② Teleparallel Gauss-Bonnet Gravity

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Why Teleparallel Gauss–Bonnet?

- **Limitations of Λ CDM:** Despite its observational success, the Λ CDM model faces unresolved issues such as the cosmological constant problem, the unknown nature of dark matter and dark energy, and persistent cosmological tensions.
- **Alternative geometric framework:** Teleparallel gravity describes gravitation through spacetime torsion instead of curvature, providing a geometrically distinct formulation with several conceptual advantages.
- **Beyond TEGR and GR:** The $f(T)$ extension of teleparallel gravity introduces modified gravitational dynamics while preserving second-order field equations, making it a viable alternative to GR-based modifications.
- **Role of the torsional Gauss–Bonnet term:** The teleparallel Gauss–Bonnet invariant T_G contributes higher-order torsional effects, leading to the generalized $F(T, T_G)$ framework with richer gravitational dynamics.
- **Cosmological viability:** $F(T, T_G)$ gravity has been shown to accommodate viable cosmic evolution and offers promising phenomenology for both late-time acceleration and early-Universe physics.

Key Questions

- What is the propagation speed of gravitational waves in $F(T, T_G)$ gravity?
- What are the stability conditions for cosmological perturbations?
- How do scalar, vector, and tensor perturbations evolve in this framework?

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Action

Action of Teleparallel Gauss-Bonnet Gravity is given by

$$S_{TGB} = \frac{1}{16\pi G} \int d^4x e F(T, T_G) + \int d^4x e \mathcal{L}_m$$

Where F is an arbitrary C^2 function, \mathcal{L}_m is matter Lagrangian and $e = \sqrt{|g|}$.
Where the Teleparallel Gauss-Bonnet Term is defined as

$$\begin{aligned} T_G = & (\mathcal{K}^{a_1}_{e a} \mathcal{K}^{e a_2}_b \mathcal{K}^{a_3}_{f c} \mathcal{K}^{f a_4}_d - 2\mathcal{K}^{a_1 a_2}_a \mathcal{K}^{a_3}_{e b} \mathcal{K}^e_{f c} \mathcal{K}^{f a_4}_d \\ & + 2\mathcal{K}^{a_1 a_2}_a \mathcal{K}^{a_3}_{e b} \mathcal{K}^{e a_4}_f \mathcal{K}^f_{c d} \\ & + 2\mathcal{K}^{a_1 a_2}_a \mathcal{K}^{a_3}_{e b} \mathcal{K}^{e a_4}_{c, d}) \delta_{a_1 a_2 a_3 a_4}^{a b c d}. \end{aligned}$$

which satisfies the relation

$$e\dot{G} = eT_G + \text{Total divergence}, \tag{1}$$

where \dot{G} is the Gauss-Bonnet term obtained from Levi-Civita connection given by

$$\dot{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

Field Equations

Field equations of $F(T, T_G)$ gravity in Weitzenböck gauge is given by

$$\begin{aligned}
 & 2(\chi^{[ac]b} + \chi^{[ba]c} - \chi^{[cb]a})_{,c} + 2(\chi^{[ac]b} + \chi^{[ba]c} - \chi^{[cb]a})C^d{}_{dc} \\
 & + (2\chi^{[ac]d} + \chi^{dca})C^b{}_{cd} + 4\chi^{[db]c}C_{(dc)}{}^a + T^a{}_{cd}\chi^{cdb} - h^{ab} \\
 & + (F - TF_T - T_GF_{T_G})\eta^{ab} = 0,
 \end{aligned}$$

where

$$\begin{aligned}
 \chi^{abc} &= F_T(\eta^{ac}\mathcal{K}^bd{}_d - \mathcal{K}^{bca}) + F_{T_G} [\\
 & \epsilon^{cprt}(2\epsilon^a{}_{dkf}\mathcal{K}^{bk}{}_p\mathcal{K}^d{}_{qr} + \epsilon_{qdkf}\mathcal{K}^{ak}{}_p\mathcal{K}^{bd}{}_r + \epsilon^{ab}{}_{kf}\mathcal{K}^k{}_{dp}\mathcal{K}^d{}_{qr})\mathcal{K}^{qf}{}_t \\
 & + \epsilon^{cprt}\epsilon^{ab}{}_{kd}\mathcal{K}^{fd}{}_p(\mathcal{K}^k{}_{fr,t} - \frac{1}{2}\mathcal{K}^k{}_{fq}C^q{}_{tr}) \\
 & + \epsilon^{cprt}\epsilon^{ak}{}_{df}\mathcal{K}^{df}{}_p(\mathcal{K}^b{}_{kr,t} - \frac{1}{2}\mathcal{K}^b{}_{kq}C^q{}_{tr})] \\
 & + \epsilon^{cprt}\epsilon^a{}_{kdf} \left[(F_{T_G}\mathcal{K}^{bk}{}_p\mathcal{K}^{df}{}_r)_{,t} + F_{T_G}C^q{}_{pt}\mathcal{K}^{bk}{}_{[q}\mathcal{K}^{df}{}_{r]} \right],
 \end{aligned}$$

Background

On the large scale we have following symmetries:

- **Homogeneity:** No special position.
- **Isotropy:** No special direction.

These two conditions along with spatial flatness allow us to write the line-element in FLRW form:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$

which means $g_{\alpha\beta} = \text{diag}(-1, a(t)^2, a(t)^2, a(t)^2)$, which is reproduced by vierbein choice

$$e^a{}_{\alpha} = \text{diag}(1, a(t), a(t), a(t))$$

For this choice we get $T = 6H^2$, $B = 6(3H^2 + \dot{H})$, $T_G = 24H^2(H^2 + \dot{H})$ etc where $H = \frac{\dot{a}}{a}$.

At cosmological scales, matter is assumed to be a perfect fluid whose energy-momentum tensor is given by:

$$\mathcal{T}_{\alpha\beta} = (\rho + p)u_{\alpha}u_{\beta} + pg_{\alpha\beta}$$

where ρ and p are the energy density and pressure of the fluid and u_{α} is its four-velocity satisfying

$$u_{\alpha}u^{\alpha} = -1$$

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Need for perturbations

The FLRW model assumes spatial homogeneity and isotropy, reflecting maximal symmetry of space on large scales. While this symmetry describes the average universe well, real structures break it. To capture these deviations, we introduce small perturbations around the symmetric FLRW background and study their evolution using perturbation theory.

Gauge Problem and Invariant Variables

- Using a homogeneous FLRW model to describe an inhomogeneous universe introduces ambiguity in the choice of coordinates, known as gauge freedom.
- In the FLRW model, hypersurfaces of uniform density and curvature coincide, but this correspondence breaks down in a real, inhomogeneous universe.
- As general relativity lacks a preferred coordinate system, mapping inhomogeneous spacetime to the homogeneous background leads to the gauge problem; the same physical situation can appear differently in different gauges.
- To address this ambiguity, perturbation theory uses **gauge-invariant variables**, which describe physical quantities independently of coordinate choices.

The vierbein can be represented as a generic 4 by 4 matrix with no symmetries, which means that it enjoys the full set of 16 dof, contrary to the metric, which can also be represented as a symmetric matrix, assuming only 10 dof. Hence, we distribute these 16 dof of the perturbation of the vierbein δe^I_α as follows¹

$$\delta e^I_\alpha := \left[\begin{array}{c} \varphi \\ \delta^I_i (\partial^i b + b^i) \quad a(\partial_i \beta + \beta_i) \\ a\delta^{Ii} \left(\psi \delta_{ij} + \partial_i \partial_j h + 2\partial_{(i} h_{j)} + \frac{1}{2} h_{ij} + \epsilon_{ijk} (\partial^k \sigma + \sigma^k) \right) \end{array} \right] \quad (2)$$

where:

- φ , ψ , β , b , and h are scalar perturbations (structure formation).
- σ is the pseudoscalar perturbation.
- b^i , β_i , and h_j are vector perturbations (divergenceless, decaying in an expanding universe).
- σ^k is the pseudovector perturbation.
- h_{ij} is the tensor perturbation (traceless & divergenceless, correspond to gravitational waves).

¹S. Bahamonde et al, Rep. Prog. Phys. 86 026901(2023)

Choosing a gauge means choosing a particular system of coordinates. This is essential in modeling physical systems, as some coefficients might have spurious degrees of freedom (e.g. x has more degrees of freedom than x^2 where x is real). So one must use gauge invariant variables.

The perturbation of a tensor field Z along the direction of a vector field Y at any order is given by

$$\tilde{Z} = e^{\mathcal{L}_Y} Z, \quad (3)$$

where \tilde{Z} is the transformed tensor field and \mathcal{L}_Y is the Lie derivative along Y . For a scalar field ϕ , a vector field X_α and a (0,2) tensor $t_{\alpha\nu}$ we have

$$\mathcal{L}_Y \phi = Y^\gamma \phi_{,\gamma}, \quad (4a)$$

$$\mathcal{L}_Y X_\alpha = X_{\alpha,\beta} Y^\beta + X_\gamma Y^\gamma_{,\alpha}, \quad (4b)$$

$$\mathcal{L}_Y t_{\alpha\beta} = t_{\alpha\beta,\gamma} Y^\gamma + t_{\alpha\gamma} Y^\gamma_{,\beta} + t_{\gamma\beta} Y^\gamma_{,\alpha}. \quad (4c)$$

Without digressing, let us state that the passive approach, counterpart to the active we just presented is just stating that an infinitesimal change of coordinates along the vector field Y^α is described by $\tilde{x}^\alpha = x^\alpha + Y_1^\alpha$ which is usually further split as $\tilde{t} = t + Y_1^0$ and $\tilde{x}^i = x^i + Y_1^i$.

Next, we will calculate the gauge transformation of the components of the perturbed tetrad. In what follows we will only determine the first order perturbations so that, we drop the subscripts $Y_1^\alpha \rightarrow Y^\alpha$. The transformed perturbed tetrad according to the active approach is

$$\tilde{\delta e}^A{}_\alpha = \delta e^A{}_\alpha + \mathcal{L}_Y e^A{}_\alpha. \quad (5)$$

We further split Y^α as $Y^\alpha = \{Y^0, \zeta(Y^i + \delta^{ij}\partial_j Y)\}$ where $\zeta \in \{1, \frac{1}{a}\}$, which incorporates both conventions (in terms of how Y^α can be split in the 3 + 1 decomposition), $\partial_i Y^i = 0$ and Y the scalar part coming from splitting Y^i

$$\tilde{\varphi} = \varphi - \dot{Y}^0, \quad \tilde{\psi} = \psi + HY^0, \quad \tilde{\beta} = \beta - \frac{1}{a}Y^0 - Y^0, \quad \tilde{\beta}_i = \beta_i, \quad (6a)$$

$$\tilde{b} = b - a(\dot{\zeta}Y + \zeta\dot{Y}), \quad \tilde{b}_i = b_i + a(\dot{\zeta}Y_i + \zeta\dot{Y}_i), \quad \tilde{\sigma} = \sigma, \quad (6b)$$

$$\tilde{\sigma}_i = \sigma^i - \frac{1}{2}\epsilon^i{}_{jk}\partial^j Y^k, \quad \tilde{h} = h - \zeta Y, \quad \tilde{h}_i = h_i + \frac{1}{2}\zeta Y_i, \quad \tilde{h}_{ij} = h_{ij}. \quad (6c)$$

These transformations suggest the grouping $\{\varphi, \psi, \beta\}$, $\{b, h\}$ and $\{b_i, h_i, \sigma_i\}$ for the fields. This grouping is generated by collecting all fields that transform in the same way. In this way, we can set only one element in each group to zero due to the similarity of their transformations. For example, we can always set the pseudovector σ_i to zero but not the pseudoscalar σ since it is gauge invariant.

It is evident from the gauge transformation that the tensor mode h_{ij} is gauge invariant. In every coordinate system, at each scale, the existence of gravitational waves is inevitable.

Introducing tensor perturbations in tetrad

$$e^I{}_\alpha := \begin{bmatrix} 1 & 0 \\ 0 & a \delta^I{}_i (1 + \frac{1}{2} h_{ij} + \frac{1}{8} h_{ij} h_{jk}) \end{bmatrix} \quad (7)$$

where the a stands for the scale factor. The associated metric becomes

$$g_{\alpha\beta} = -dt^2 + a^2 (\delta_{ij} + h_{ij}) dx^i dx^j, \quad (8)$$

where the background value of the lapse function is considered to be unity.

Expanding the action upto second order in perturbations, we get the second order action after integration by parts and using gauge conditions to be

$$\mathcal{S}_T^{(2)} = \frac{1}{2\kappa^2} \int dt d^3x \frac{a^3}{2} \mathcal{C}_{tensor} \left[\dot{h}_{ij}^2 - a^{-2} (\nabla h_{ij})^2 \right] \quad (9)$$

where $\mathcal{C}_{tensor} = -F_T + 4H\dot{F}_{T_G}$. The overdot represents the derivative with respect to cosmic time t . This action represents how tensor perturbations propagate on a flat FLRW background cosmology. one can see from the action (9) that the following condition should hold

$$\mathcal{C}_{tensor} > 0$$

in order to avoid ghost instability.

The evolution equation of tensor perturbations is:

$$\ddot{h}_{ij} + \left(3H + \frac{-\dot{F}_T + 4\dot{H}\dot{F}_{TG} + 4H\ddot{F}_{TG}}{-F_T + 4H\dot{F}_{TG}} \right) \dot{h}_{ij} + \frac{k^2}{a^2} h_{ij} = 0, \quad (10)$$

A general GWPE in most modified gravity theories usually takes the form:

$$\ddot{h}_{ij} + (3 + \alpha_M) H \dot{h}_{ij} + (1 + \alpha_T) \frac{k^2}{a^2} h_{ij} = 0, \quad (11)$$

where dots denote differentiation with respect to cosmic time, $H = \frac{\dot{a}}{a}$ is the Hubble parameter, $\alpha_M = \frac{1}{HM_*^2} \frac{dM_*^2}{dt}$ is Planck mass running rate, and $\alpha_T = c_T^2 - 1$ is the tensor excess speed. The GWPE in Eq. (11) is being considered in its Fourier domain, along with a source-free scenario.

Comparing equation (10) with (11), we have the effective mass $M_*^2 = -F_T + 4H\dot{F}_{TG}$ and the excess tensor speed $\alpha_T = 0$. Consequently, gravitational waves propagate at the speed of light, in accordance with multimessenger observational constraints derived from the binary neutron star merger event GW170817 and its associated electromagnetic counterpart, GRB170817A.

Further breaking down cosmological perturbations involves considering vector and pseudovector perturbations, which are solenoidal in nature. Typically, in an expanding universe, such vector modes decay unless maintained by anisotropic stress. Although we do not expect a different outcome here, it is still essential to scrutinise these cases to ensure accuracy. The vector and pseudovector part of perturbations is given by

$$\left[\begin{array}{c} \delta e^\ell \\ \mu \\ \text{Vector} \end{array} \right] = \left[\begin{array}{cc} 0 & a\beta_i \\ B^{\bar{l}} & a\delta^{\bar{l}\hat{i}} \left(2\partial_{(\hat{i}} h_{\hat{j})} + \epsilon_{\hat{i}\hat{j}\hat{k}} \sigma^{\hat{k}} \right) \end{array} \right]. \quad (12)$$

Mixed $(0i)$ and spatial (ij) components of antisymmetric parts of equations are:

$$\mathbb{E}_{[0\hat{i}]} \equiv \left(\dot{F}_T + 4H^2 \dot{F}_{TG} \right) \left(\Delta h_{\hat{i}} - \epsilon_{\hat{i}\hat{j}\hat{k}} \partial^{\hat{j}} \sigma^{\hat{k}} \right) + \frac{2H}{a} \dot{F}_{TG} \Delta \left(\beta_{\hat{i}} - B_{\hat{i}} + 2ah_{\hat{i}} \right) = 0, \quad (13)$$

Vector

$$\mathbb{E}_{[\hat{i}\hat{j}]} \equiv \left(\dot{F}_T + 4H^2 \dot{F}_{TG} \right) \left(\partial_{\hat{i}} \beta_{\hat{j}} - \partial_{\hat{j}} \beta_{\hat{i}} \right) = 0. \quad (14)$$

Vector

After taking divergence, (14) implies

$$\left(\dot{F}_T + 4H^2 \dot{F}_{TG} \right) \Delta \beta_{\hat{j}} = 0, \quad (15)$$

which for $\dot{F}_T + 4H^2 \dot{F}_{TG} \neq 0$ should be solved as $\beta_{\hat{i}} = 0$ in perturbation theory, due to Liouville's theorem. Considering this constraint and since $\beta_{\hat{i}}$ is gauge invariant as well as defining the gauge invariant quantities,

$$\mathcal{B}_{\hat{i}} = B_{\hat{i}} - 2a\dot{h}_{\hat{i}}, \quad (16a)$$

$$\mathcal{U}_{\hat{i}} = U_{\hat{i}} + B_{\hat{i}}, \quad (16b)$$

$$\mathcal{V}_{\hat{i}} = \sigma_{\hat{i}} + \epsilon_{\hat{i}\hat{j}\hat{k}} \partial^{\hat{j}} h^{\hat{k}}. \quad (16c)$$

In terms of the above gauge invariant quantities, (13) can be written as

$$-\left(\dot{F}_T + 4H^2 \dot{F}_{TG}\right) \epsilon_{\hat{i}\hat{j}\hat{k}} \partial^{\hat{j}} \mathcal{V}^{\hat{k}} - \frac{2H}{a} \dot{F}_{TG} \Delta \mathcal{B}_{\hat{i}} = 0. \quad (17)$$

The symmetric part of the Equations is given as,

$$\begin{array}{l} \mathbb{E}_{(0\hat{i})} \\ \text{Vector} \end{array} \equiv \Delta \mathcal{B}_{\hat{i}} \left(-F_T + 2H \dot{F}_{TG}\right) - a \left(\dot{F}_T + 4H^2 \dot{F}_{TG}\right) \epsilon_{\hat{i}\hat{j}\hat{k}} \partial^{\hat{j}} \mathcal{V}^{\hat{k}} = 2\kappa^2 a^2 \mathcal{U}_{\hat{i}} (\rho + p), \quad (18)$$

$$\begin{array}{l} \mathbb{E}_{(\hat{i}\hat{j})} \\ \text{Vector} \end{array} \equiv \left(\partial_{\hat{i}} \dot{\mathcal{B}}_{\hat{j}} + \partial_{\hat{j}} \dot{\mathcal{B}}_{\hat{i}}\right) + \left(\partial_{\hat{i}} \mathcal{B}_{\hat{j}} + \partial_{\hat{j}} \mathcal{B}_{\hat{i}}\right) \left(2H + \frac{-\dot{F}_T + 4\dot{H}\dot{F}_{TG} + 4HF_{TG}''}{-F_T + 4H\dot{F}_{TG}}\right) = 0. \quad (19)$$

Under antisymmetric part (17), Eq. (18) reduces to:

$$\Delta \mathcal{B}_i \left(-F_T + 4H \dot{F}_{TG} \right) = 2\kappa^2 a^2 \mathcal{U}_i (\rho + p), \quad (20)$$

where ρ and p can be substituted from the background equation. Additionally, Eq. (19) can be written as,

$$\dot{\mathcal{B}} + \left(2H + \frac{-\dot{F}_T + 4\dot{H}\dot{F}_{TG} + 4H\ddot{F}_{TG}}{-F_T + 4H\dot{F}_{TG}} \right) \mathcal{B} = 0. \quad (21)$$

Eqs. (20) and (21) involve only two components, the gauge invariant geometric perturbation \mathcal{B}_i and the vortical part of fluid component encoded in the gauge invariant combination \mathcal{U}_i . We immediately see that (21) is a constraint equation and (20) is a Poisson-like equation. Therefore, it is clear that vector modes are not propagating. Furthermore, it is obvious that if we can solve the system for \mathcal{B}_i , then one can solve it for \mathcal{U}_i as well. Additionally, one can read the stability condition $-F_T + 4H\dot{F}_{TG} > 0$ off from (21). We also observe that the contribution from the pseudovector is constrained by the antisymmetric part of the equations, which is logical given that the pseudovector is essentially a Lorentz variable. Thus, we get a well-behaved vector sector in the theory.

We now move to the most important sector of cosmological perturbations, the scalar part of perturbations. These represent fluctuations in density and pressure that seed the formation of large-scale structures such as galaxies and clusters. Also, these are the primary sources of the temperature anisotropies observed in the cosmic microwave background (CMB), providing a direct link to the physics of the early universe. Unlike vector modes, which decay and tensor modes, which are subdominant, scalar modes grow under gravitational instability. The scalar part of perturbations is obtained by switching off all the vector and tensor modes from the perturbed tetrad as,

$$[\delta e^a{}_{\mu}]_{Scalar} = \begin{bmatrix} \phi & a\partial_i\beta \\ \delta\tilde{l}_{\hat{i}}\partial^{\hat{i}}B & a\delta\tilde{l}^{\hat{i}}(\psi\delta_{\hat{i}\hat{j}} + \partial_{\hat{i}}\partial_{\hat{j}}E + \epsilon_{\hat{i}\hat{j}\hat{k}}\partial^{\hat{k}}\sigma) \end{bmatrix}. \quad (22)$$

All the components of the antisymmetric part vanish except for mixed components, for which we get

$$\mathbb{E}_{[0\hat{i}]}_{Scalar} \equiv \partial_{\hat{i}} \left(H\delta F_T - \psi\dot{F}_T + 4HF_{T_G} \left(H(\phi - \psi) - \dot{\psi} \right) + \frac{T_G\delta F_{T_G}}{6H} \right) = 0. \quad (23)$$

While the components of symmetric part of equations (on shell) are given by,

$$\begin{aligned} \mathbb{E}_{00}^{\text{Scalar}} &\equiv -\frac{2k^2 \left[F_T (\psi + aH(B - \beta - a\dot{E})) + 2aH^2 \dot{F}_{T_G} (\beta - 3B + 3a\dot{E}) - 2H^2 \delta F_{T_G} \right] - \frac{1}{2} T_G \delta F_{T_G} - 6H^2 \delta F_T}{a^2} \\ &+ 6H \left[2HF_{T_G} \dot{\psi} (3\dot{\psi} - 4H\phi) + F_T (H\phi - \dot{\psi}) + 2H^2 \delta F_{T_G} \right] = \kappa^2 \delta \rho, \end{aligned} \quad (24)$$

$$\begin{aligned} \int_{\hat{i}} \mathbb{E}_{(0\hat{i})}^{\text{Scalar}} &\equiv 4HF_{T_G} \dot{\psi} (\dot{\psi} - H(2\phi + \psi)) + 2F_T (H\phi - \dot{\psi}) - \psi \dot{F}_T + 4H^2 \delta F_{T_G} + 4\dot{H}H\delta F_{T_G} - H\delta F_T \\ &= a\kappa^2 (P + \rho)(B - \beta + U), \end{aligned} \quad (25)$$

$$\begin{aligned} \int_{\hat{i}\hat{j}} \mathbb{E}_{\hat{i}\hat{j}}^{\text{Scalar}} &\equiv a^2 \left[4(H\ddot{E} + \dot{E}(3H^2 + \dot{H})) \dot{F}_{T_G} + 4\dot{E}H\ddot{F}_{T_G} - F_T (\ddot{E} + 3\dot{E}H) - \dot{E}\dot{F}_T \right] \\ &+ a \left[-4BHF_{T_G} \ddot{F}_{T_G} - 4(H^2(2B - \beta) + \dot{B}H + B\dot{H}) \dot{F}_{T_G} + 2BHF_T + B\dot{F}_T + \dot{B}F_T - 2\beta HF_T - \dot{\beta}F_T \right] \\ &+ \psi (F_T - 4HF_{T_G} \dot{\psi}) + \phi F_T = 0, \end{aligned} \quad (26)$$

$$\mathbb{E}_{\hat{i}}^{\hat{i}}{}_{\text{Scalar}} \equiv F_T \left[\frac{k^2 \left[2a(a(\ddot{E} + 3\dot{E}H) + \dot{\beta} - \dot{B}) - 2(2aH(B - \beta) + \phi + \psi) \right]}{3a^2} - 2\ddot{\psi} + 2H(3H\phi + \dot{\phi} - 3\dot{\psi}) + 4\dot{H}\phi \right] \text{ and more.} \quad (27)$$

where $\delta F_T = F_{TT} \delta T + F_{TT_G} \delta T_G$, $\delta F_{T_G} = F_{TT_G} \delta T + F_{T_G T_G} \delta T_G$ and the equations are expressed in Fourier domain. In these expressions, $T_G = 24H^2 (\dot{H} + H^2)$ is background teleparallel Gauss-Bonnet term. The linearized Eqs. (23)-(27) are in gauge-ready form, that is, being most general, they are free from any gauge choices. An important point to note here is that pseudoscalar does not enter into field equations. It can therefore be morphed as a remnant symmetry. In what follows, we shall write these in gauge invariant form and also specify various gauge-choices which are useful for different purposes.

Consider following gauge invariant combinations of geometrical perturbations,

$$\mathcal{X}_1 = \phi - \psi - a\dot{\beta}, \quad (28a)$$

$$\mathcal{X}_2 = \psi - \dot{a}\beta, \quad (28b)$$

$$\mathcal{X}_3 = -B + a\dot{E}, \quad (28c)$$

and the gauge invariant matter perturbations

$$\delta\varrho = \delta\rho - a\dot{\rho}\beta, \quad (28d)$$

$$\delta\mathcal{P} = \delta p - a\dot{p}\beta, \quad (28e)$$

$$\mathcal{U}_s = B + U. \quad (28f)$$

Full expressions of field equations in terms of gauge invariant variables are given in github repository [S. K. Mishra, "TGB Perturbations."](https://github.com/ShivamKumarMishra-01/TGB_Perturbations) https://github.com/ShivamKumarMishra-01/TGB_Perturbations . For demonstration, we present the anisotropic part here

$$a \left(\mathcal{X}_3 \left(4HF_{T_G}'' + 8H^2 F_{T_G}' + 4\dot{H}F_{T_G}' - 2HF_T - \dot{F}_T \right) - \dot{\mathcal{X}}_3 \left(F_T - 4HF_{T_G}' \right) \right) + 2\mathcal{X}_2 \left(F_T - 2HF_{T_G}' \right) + \mathcal{X}_1 F_T = 0. \quad (29)$$

An important part to note here is that all these equations are on shell, i.e., the span of all background contributions is used to obtain the equations in the presented form.

The paper contains gauge examples which can be used for various purposes.

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- This propagation equation is computed through an effective Lagrangian, which immediately gives the stability condition $\mathcal{C}_{tensor} > 0$ to be satisfied, therefore setting constraints on the theory.
- The propagation speed of the gravitational waves in this theory is equal to the speed of light. This is a novel result because a similar theory with a modified Gauss-Bonnet term has a different result.
- We developed a fully gauge-invariant formulation of cosmological perturbations in $F(T, T_G)$ gravity, which consistently reduces to general relativity in the appropriate limit.
- Using the SVT decomposition of the perturbed tetrad and matter sector, we constructed gauge-invariant variables and derived evolution equations for scalar, vector, pseudoscalar, pseudovector, and tensor modes.
- The perturbation equations show nontrivial departures from GR, mainly in the scalar and tensor sectors, with potential implications for structure formation, gravitational waves, and CMB observables.
- Tensor modes propagate at the speed of light in agreement with multimessenger constraints given GW170817 and its EM counterpart GRB170817A, vector modes decay, pseudovector modes are fully constrained, and pseudoscalar modes decouple as remnant symmetries.
- Scalar perturbations were presented in gauge-ready and gauge-invariant forms, providing a framework to analyze stability and confront viable $F(T, T_G)$ models with observational data.

The study has resulted in the following publications:

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Propagating gravitational waves in teleparallel Gauss-Bonnet gravity


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Gravitational waves offer a key insight into the viability of classes of gravitational theories beyond general relativity. The observational constraints on their speed of propagation can provide strong constraints on generalized classes of broader gravitational frameworks. In this work, we reconsider the general class of Gauss-Bonnet theories in the context of teleparallel gravity, where the background geometry is expressed through torsion. We perform tensor perturbations on a flat Friedmann-Lemaître-Robertson-Walker background, and derive the gravitational wave propagation equation. We find that gravitational waves propagate at the speed of light in these classes of theories. We also derive the distance-duality relationship for radiation propagating in the gravitational wave and electromagnetic domains.

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PAPER

Gauge invariant perturbations of $F(T, T_G)$ cosmology

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Abstract

The Gauss-Bonnet invariant connects foundational aspects of geometry with physical phenomena in a variety of ways. Teleparallel gravity (TG) offers a novel direction in which to use the Gauss-Bonnet invariant to go beyond standard cosmology. In this work, we explore the cosmological perturbations of TG generalized through the Gauss-Bonnet invariant. This is crucial in understanding the viability of these models beyond background analyses. We do this by taking a gauge-invariant approach, followed by popular gauge choices. It is essential to take this approach to understand the stability and healthiness of the underlying theory. We determine the equations of motion for all perturbative modes and provide a physical interpretation of the new contributions for each mode.

