Dimer, cluster, and supersymmetric guage theories

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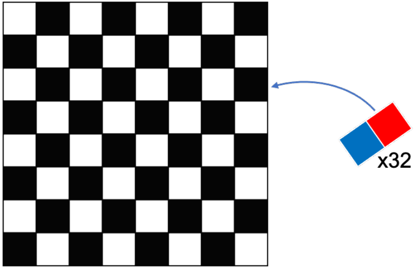
Introduction

Let me start with a simple question:

- Given an 8×8 chessboard, how many distinctive ways one can cover it with 2×1 domino (a.k.a dimer) without overlapping?



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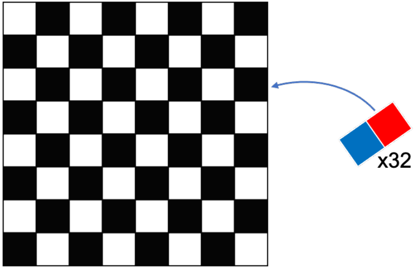

Answer: 12,988,816.

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In this talk I will give a brief introduction on the connection between dimer model, cluster algebra, and supersymmetric gauge theories.

Dimer model

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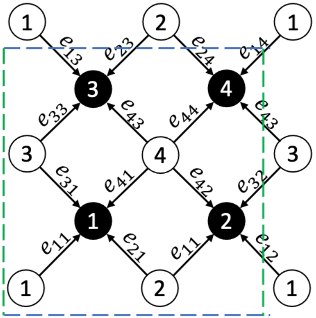
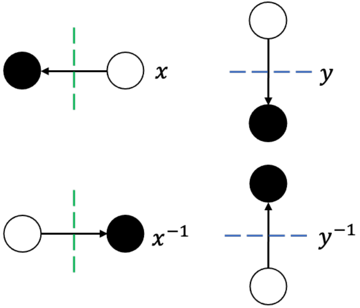
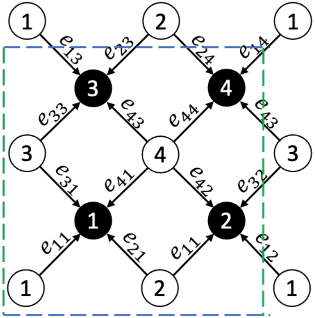
We choose the oriented surface to be T^2, represented by the *unit cell.*



Dimer model

Introduce parameters $(x, y) \in \mathbb{C}^\times \times \mathbb{C}^\times$ based on if the edge passing through the boundary of the unit cell

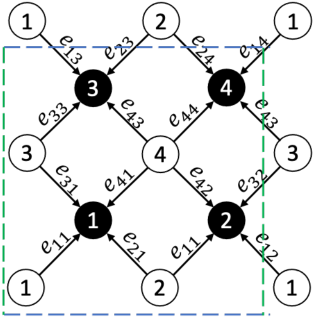




Dimer model

- The information of the dimer is encoded in the *Kasteleyn matrix*:

$$K = \left(\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline w_1 & e_{11} & e_{12}x & e_{13}y & e_{14}xy \\ w_2 & e_{21} & e_{22} & e_{23}y & e_{24}y \\ w_3 & e_{31} & e_{32}x & e_{33} & e_{34}x \\ w_4 & e_{41} & e_{42} & e_{43} & e_{44} \end{array}\right) \qquad (1)$$



Dimer model

- A prefect matching $p_\alpha \subset E$ is a collection of edges e_{ij} in the dimer model such that edges in p_α connect to all white nodes w_i and black node b_j uniquely once (the dominos).

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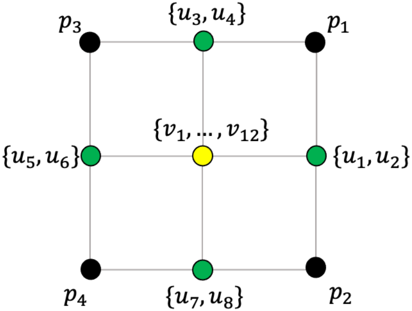
- Its weight is defined by

$$\bar{p}_\alpha = \prod_{e_{ij}\in p_\alpha} e_{ij}^{+} \ , \ (\bar{p}_\alpha)^{-1} = \prod_{e_{ij}\in p_\alpha} e_{ij}^{-} \tag{2}$$

- Here we define directed edges $e_{ij}^{+}\ :\ w_i \to b_j$, $e_{ij}^{-}\ :\ b_j \to w_i$.

Dimer model

- The Newton polynomial $P(x, y)$ is given by the permanent of the Kasteleyn matrix (shifted in the $(-1, -1)$ direction):



$$
\begin{aligned}
P(x, y) &= \frac{\text{perm } K}{xy} \\
&= \bar{p}_1 xy + \bar{p}_2 \frac{x}{y} + \bar{p}_3 \frac{y}{x} + \bar{p}_4 \frac{1}{xy} \\
&+ (\bar{u}_1 + \bar{u}_2)x + (\bar{u}_3 + \bar{u}_4)y \\
&+ (\bar{u}_5 + \bar{u}_6)\frac{1}{x} + (\bar{u}_7 + \bar{u}_8)\frac{1}{y} \\
&+ \bar{v}_1 + \cdots + \bar{v}_{12}
\end{aligned}
$$

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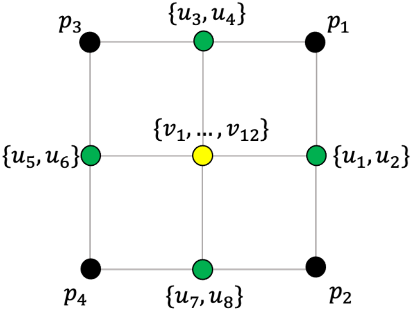

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Spectral curve: $\Sigma : \{(x, y) \mid P(x, y) = 0\}.$

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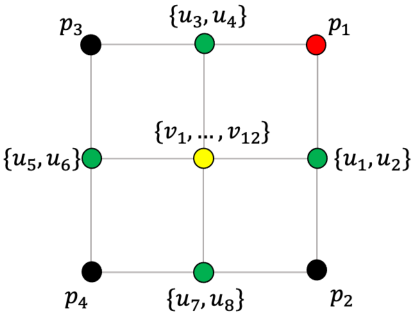

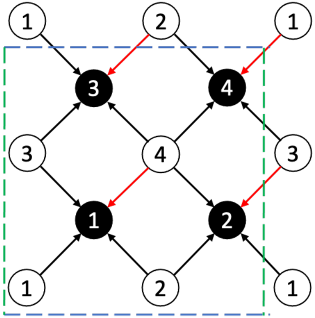

Figure: $p_1 = \{e_{14}, e_{23}, e_{32}, e_{41}\}$.

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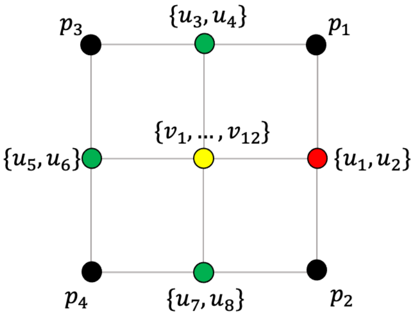

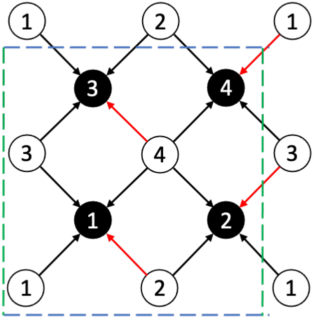

Figure: $u_1 = \{e_{14}, e_{21}, e_{32}, e_{43}\}$.

Dimer model

- Given two perfect matchings p_α and p_β, we can define the product of their weights

$$\bar{p}_\alpha(\bar{p}_\beta)^{-1} \equiv \cdots e^+_{ij} e^-_{kj} e^+_{kl} e^-_{ml} \cdots \quad (3)$$

- Difference between perfect matchings generates loops on the dimer:

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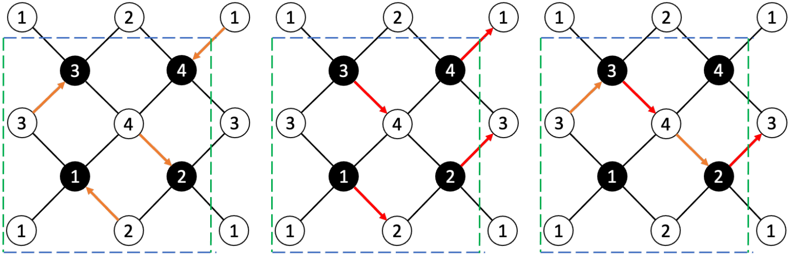

$\bar{v}_7$ $\quad$ $(\bar{u}_1)^{-1}$ $\quad$ $\bar{v}_7(\bar{u}_1)^{-1}$

Dimer model

- All loops can be decompose into product of 16 fundamental loops: 8 face loops $f_{1,\ldots,8}$ and 8 zigzag loops $z_{1,\ldots,8}$.

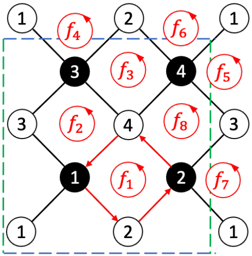

Figure: 8 face loops .

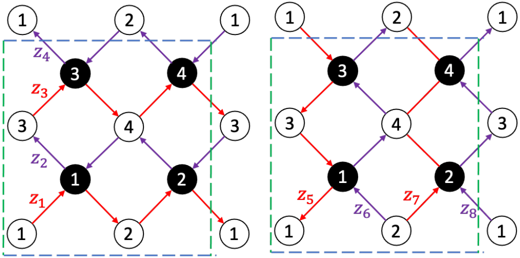

Figure: 8 zigzag loops.

Dimer model

- Factor out the reference perfect matching $\bar{u}_1 = e^+_{14} e^+_{21} e^+_{32} e^+_{43}$ from the Newton polynomial:

$$
\begin{aligned}
P(x,y) = \bar{u}_1 \Big[& \delta_{(1,1)} xy + \delta_{(1,-1)} \frac{x}{y} + \delta_{(-1,1)} \frac{y}{x} + \frac{\delta_{(-1,-1)}}{xy} \\
& + \delta_{(1,0)} x + \delta_{(0,1)} y + \delta_{(-1,0)} \frac{1}{x} + \delta_{(0,-1)} \frac{1}{y} + H \Big]
\end{aligned}
\tag{4}
$$

- $H = \sum_{k=1}^{12} \gamma_k = \sum_{k=1}^{12} \bar{v}_k (\bar{u}_1)^{-1}$.

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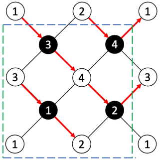
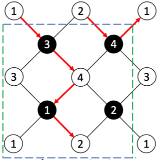
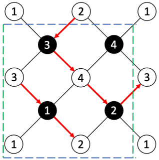
- δ's are called *Casimir loops*, built only from the zigzag loops z's.
- γ's are called *1-loops*, built from both zigzag loops z's and face loops f's.



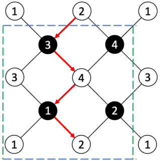
$\delta_{(-1,1)} = z_2^{-1} z_4^{-1} z_6^{-1}$



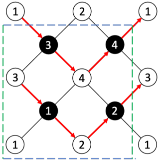
$\delta^1_{(0,1)} = z_4^{-1} z_6^{-1}$



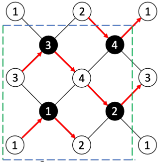
$\delta^2_{(0,1)} = z_2^{-1} z_6^{-1}$



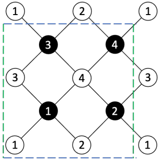
$\delta_{(1,1)} = z_6^{-1}$



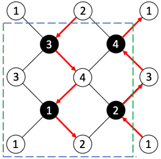
$\delta^1_{(-1,0)} = z_2^{-1} z_4^{-1} z_6^{-1} z_7^{-1}$



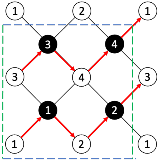
$\delta^2_{(-1,0)} = z_1 z_3 z_7 z_8$



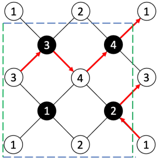
$\delta^1_{(1,0)} = 1$



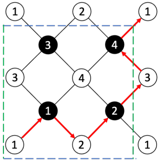
$\delta^2_{(1,0)} = z_6^{-1} z_8$



$\delta_{(-1,-1)} = z_1 z_3 z_8$



$\delta^1_{(0,-1)} = z_3 z_8$



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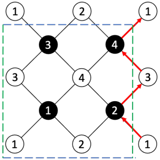
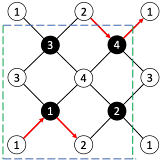
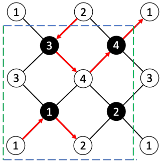


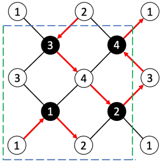
$\delta_{(1,-1)} = z_8$



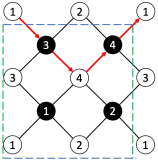
$\gamma_1 = z_1 f_6$



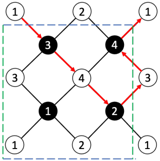
$\gamma_2 = z_1 f_3 f_6$



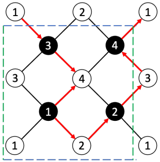
$\gamma_3 = z_1 f_3 f_6 f_8$



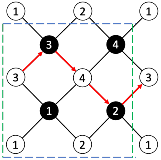
$\gamma_4 = z_1 f_3 f_4 f_6$



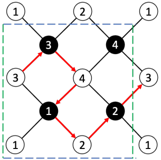
$\gamma_5 = z_1 f_3 f_4 f_6 f_8$



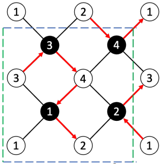
$\gamma_6 = z_1 f_1 f_3 f_4 f_6 f_8$



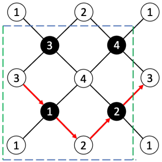
$\gamma_7 = z_1 f_3 f_4 f_5 f_6 f_8$



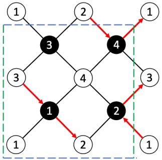
$\gamma_8 = z_1 f_1 f_3 f_4 f_5 f_6 f_8$



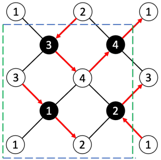
$\gamma_9 = z_1 f_1 f_3 f_4 f_5 f_6^2 f_8$



$\gamma_{10} = z_1 f_1 f_2 f_3 f_4 f_5 f_6 f_8$



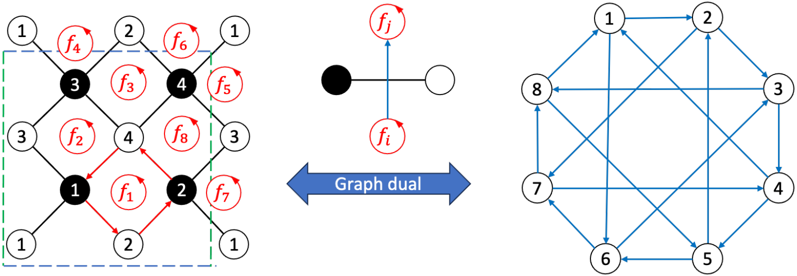
$\gamma_{11} = z_1 f_1 f_2 f_3 f_4 f_5 f_6^2 f_8$



$\gamma_{12} = z_1 f_1 f_2 f_3^2 f_4 f_5 f_6^2 f_8$

Cluster integrable system

- The dual graph of the dimer model is a planar, periodic quiver.



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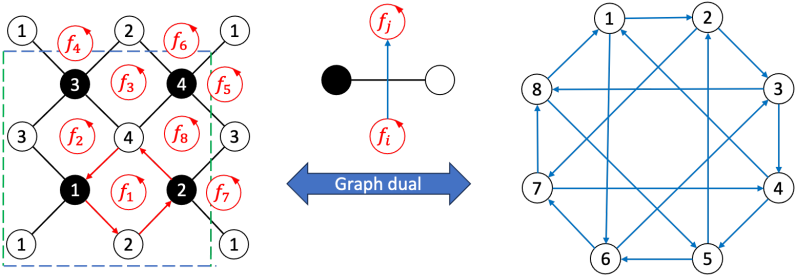
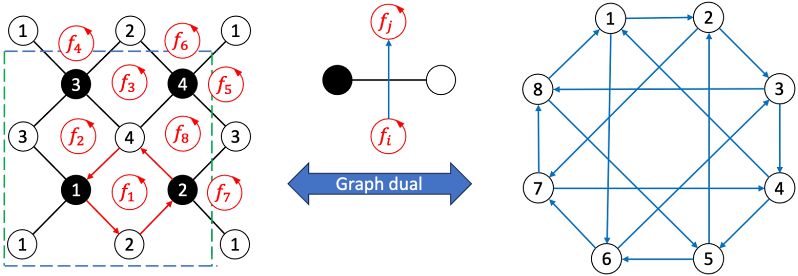


- The dual quiver has a χ-cluster structure with Poisson commutation relation: [Goncharov-Kenyon '12]

$$\{f_i, f_j\} = \epsilon_{i,j} f_i f_j \ , \ \epsilon_{i,j} = \# \text{ of arrows from } f_i \text{ to } f_j \ , \tag{5}$$

Cluster integrable system

- The dual graph of the dimer model is a planar, periodic quiver.



- The dual quiver has a χ-cluster structure with Poisson commutation relation:

  [Goncharov-Kenyon '12]

$$\{f_i, f_j\} = \epsilon_{i,j} f_i f_j \ , \ \epsilon_{i,j} = \#\text{ of arrows from } f_i \text{ to } f_j \ , \ \{z_i, f_j\} = \{z_i, z_j\} = 0. \tag{5}$$

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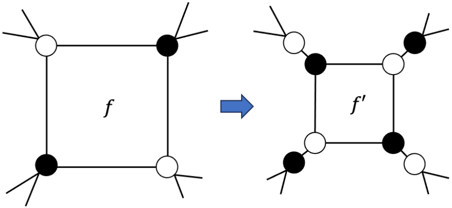

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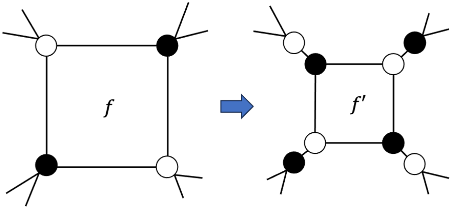

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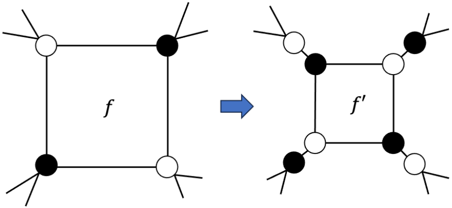

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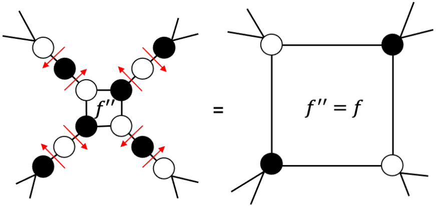

- The Casimir loops δ Poisson commute with all 1-loops and other Casimirs.
- Commutation relation for 1-loops:

$\frac{\{\gamma,\gamma'\}}{\gamma\gamma'}$	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}	γ_{12}
γ_1	0	1	1	1	1	0	0	-1	-1	-1	-1	0
γ_2	-1	0	1	1	2	1	1	0	-1	-1	-2	-1
γ_3	-1	-1	0	0	1	1	1	1	0	0	-1	-1
γ_4	-1	-1	0	0	1	1	1	1	0	0	-1	-1
γ_5	-1	-2	-1	-1	0	1	1	2	1	1	0	-1
γ_6	0	-1	-1	-1	-1	0	0	1	1	1	1	0
γ_7	0	-1	-1	-1	-1	0	0	1	1	1	1	0
γ_8	1	0	-1	-1	-2	-1	-1	0	1	1	2	1
γ_9	1	1	0	0	-1	-1	-1	-1	0	0	1	1
γ_{10}	1	1	0	0	-1	-1	-1	-1	0	0	1	1
γ_{11}	1	2	1	1	0	-1	-1	-2	-1	-1	0	1
γ_{12}	0	1	1	1	1	0	0	-1	-1	-1	-1	0

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Theorem [Goncharov-Kenyon]
Any dimer graph on a torus defines an cluster integrable system with

$$\{H_n, H_m\} = 0. \tag{6}$$

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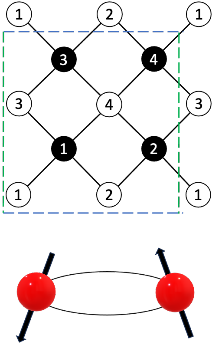

Figure: The dimer model for XXZ spin chain with 2 spins.

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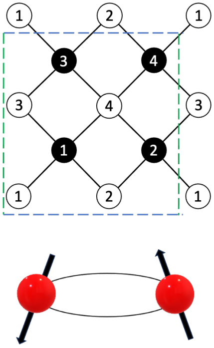
NB

The Lax matrices of the integrable system can be constructed from the Kasteleyn matrix.



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- 4d $\mathcal{N} = 1$ supersymmetric quiver gauge theories (4 supercharges).
- 5d $\mathcal{N} = 1$ supersymmetric gauge theories (8 supercharges).

Supersymmetric gauge theories

- A D3(0123) brane probes a toric Calabi-Yau 3-fold. Its worldvolume is a 4d $\mathcal{N} = 1$ supersymmetric quiver gauge theory, which is encoded in the bipartite graph known as a dimer model or a brane tiling. [Hanany-Kennaway] [Franco-Hanany-Kennaway-Vegh-Wecht]

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- T-dual along the compact 46 directions:

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	$\vdash$		Σ	$\dashv$		
D5	x	x	x	x	x		x			

- Σ: holomporphic curve embedded in the 4567 direction.

Supersymmetric gauge theories

- Dictionary between dimer model, string theory, and quiver gauge theories.

Dimer	String	Gauge
$2n$ sided face	D5 branes	Gauge group with n flavors
Edges	String between D5 through NS5	Bifundamental
Vertexes	Region string interact locally	Superpotential

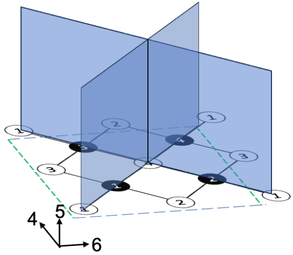


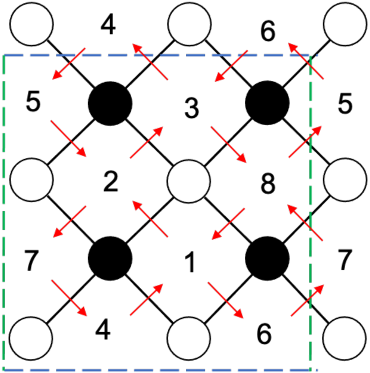
Figure: NS5-D5 system characterized by dimer model (brane tiling), obtained from Calabi-Yau 3-fold $\mathcal{C}/\mathbb{Z}_2 \times \mathbb{Z}_2$

Supersymmetric gauge theories

The white (black) nodes corresponds to positive (negative) terms in the superpotential. They have clockwise (anti-clockwise) orientation.

X_{IJ}: bifundamental between face I and face J

$$W =$$



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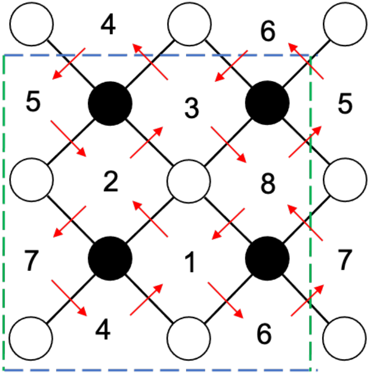

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& - X_{67}X_{78}X_{81}X_{16} - X_{41}X_{12}X_{27}X_{74}
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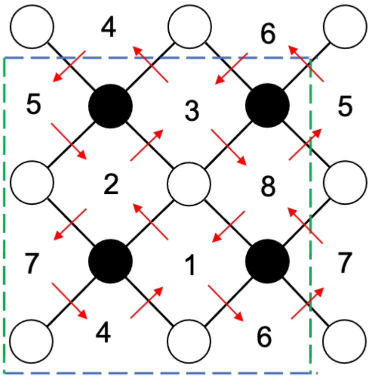

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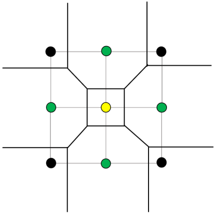

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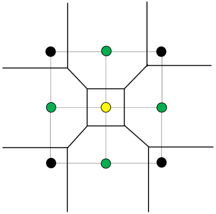

Figure: 5-brane web for 5d $\mathcal{N} = 1$ $SU(2)_0 + 4F$ theory and the toric diagram.

- The Seiberg-Witten curve of the 5d $\mathcal{N} = 1$ supersymmetric gauge theory on $\mathbb{R}^4 \times S^1$ coincides with the spectral curve of the dimer integrable system.

Quantization

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- A natural $q = e^{\hbar}$-uplift of the cluster algebra:

$$\{f_i, f_j\} = \epsilon_{i,j} f_i f_j \implies f_i f_j = q^{\epsilon_{i,j}} f_i f_j \ . \tag{7}$$

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- The wavefunction Ψ is identified as the monodromy defect (Aharonov–Bohm effect) in the gauge theory [Nekrasov '17] [Lee-Nekrasov '20] [Jeong-Lee-Nekrasov '21 '23 '24] [Jeong-Lee '25]

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4. There is a natural quantization of the Goncharov-Kenyon integrable system. The Baxter Q-operator and wavefunction can be constructed from 5d $\mathcal{N} = 1$ gauge theory with defect.

Thank you for your attention!