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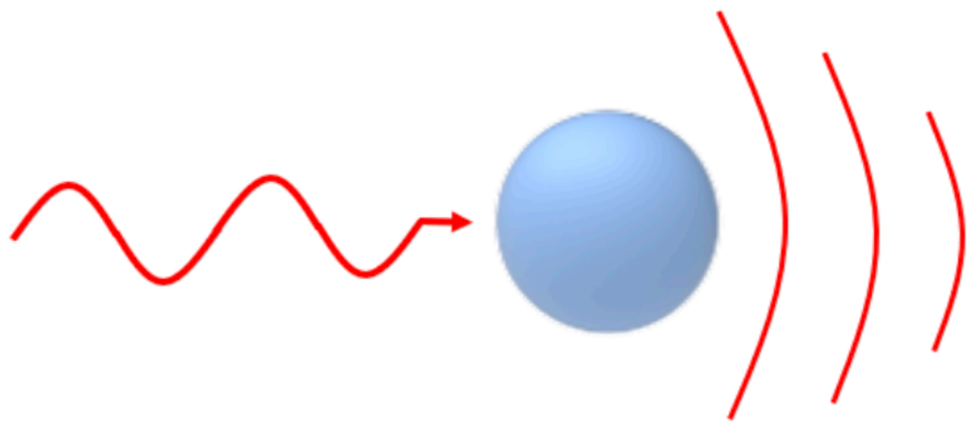

Perturbative Solutions of Einstein's Equations: Recursive Techniques and Multipole Expansions

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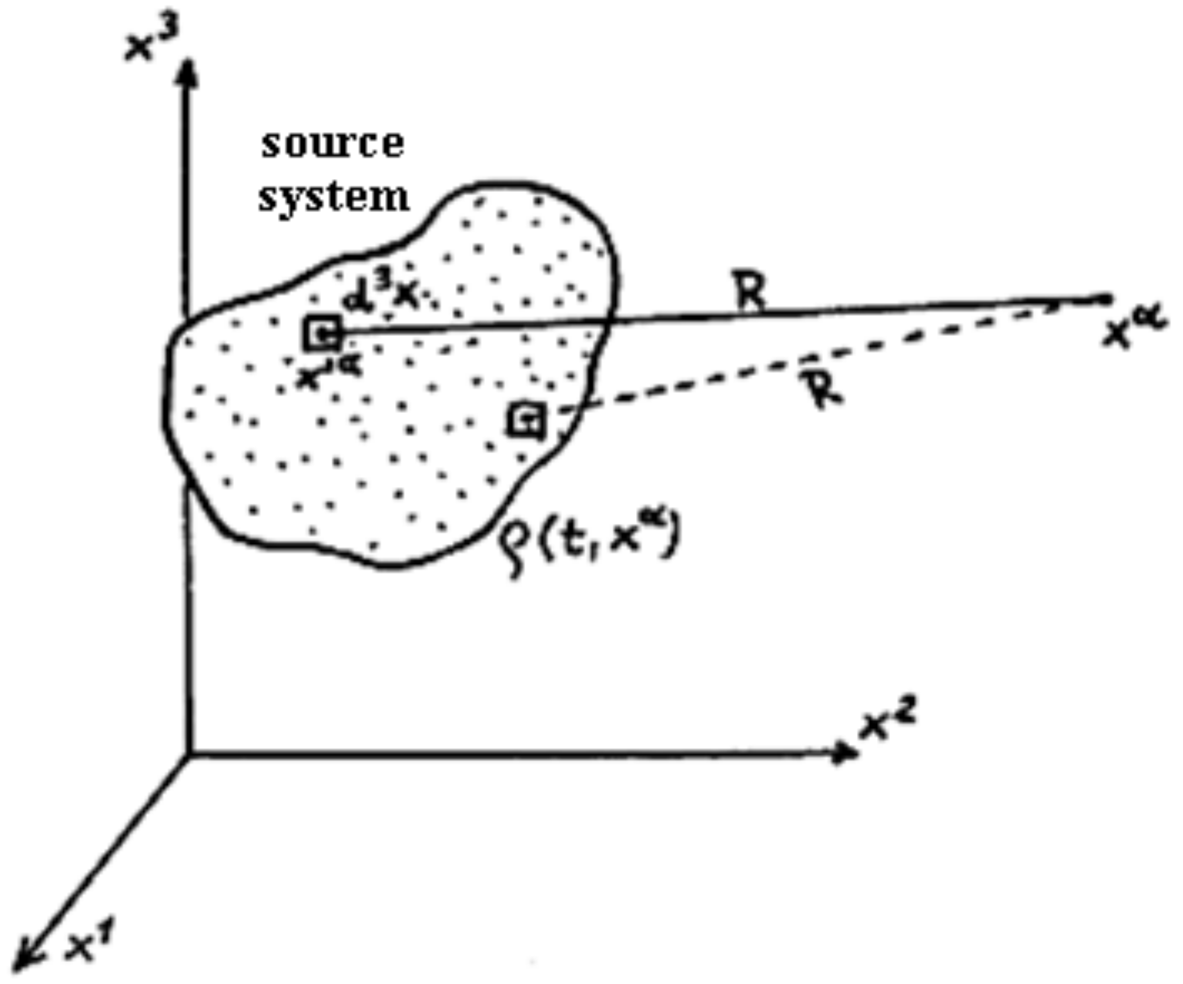

2nd INPP-Demokritos-APCTP meeting

APCPT, Pohang

28th August, 2025

Work in progress - in collaboration with Prof. Poul Henrik Damgaard and Prof. Kanghoon Lee

All gravitational waves, no matter how strong or weak, can be completely characterized by a set of multipole moments of the source.

[Thorne (1980)]

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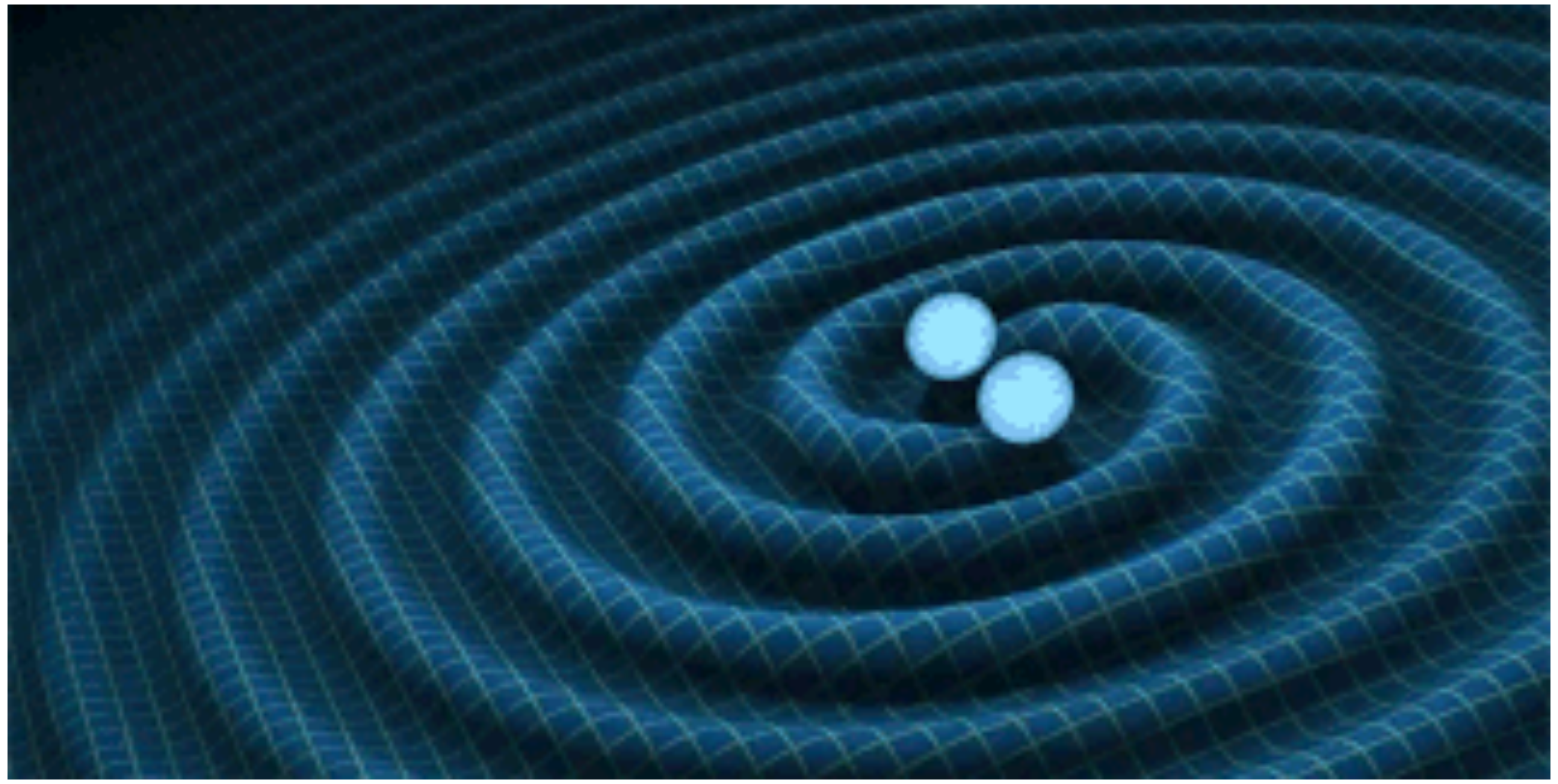
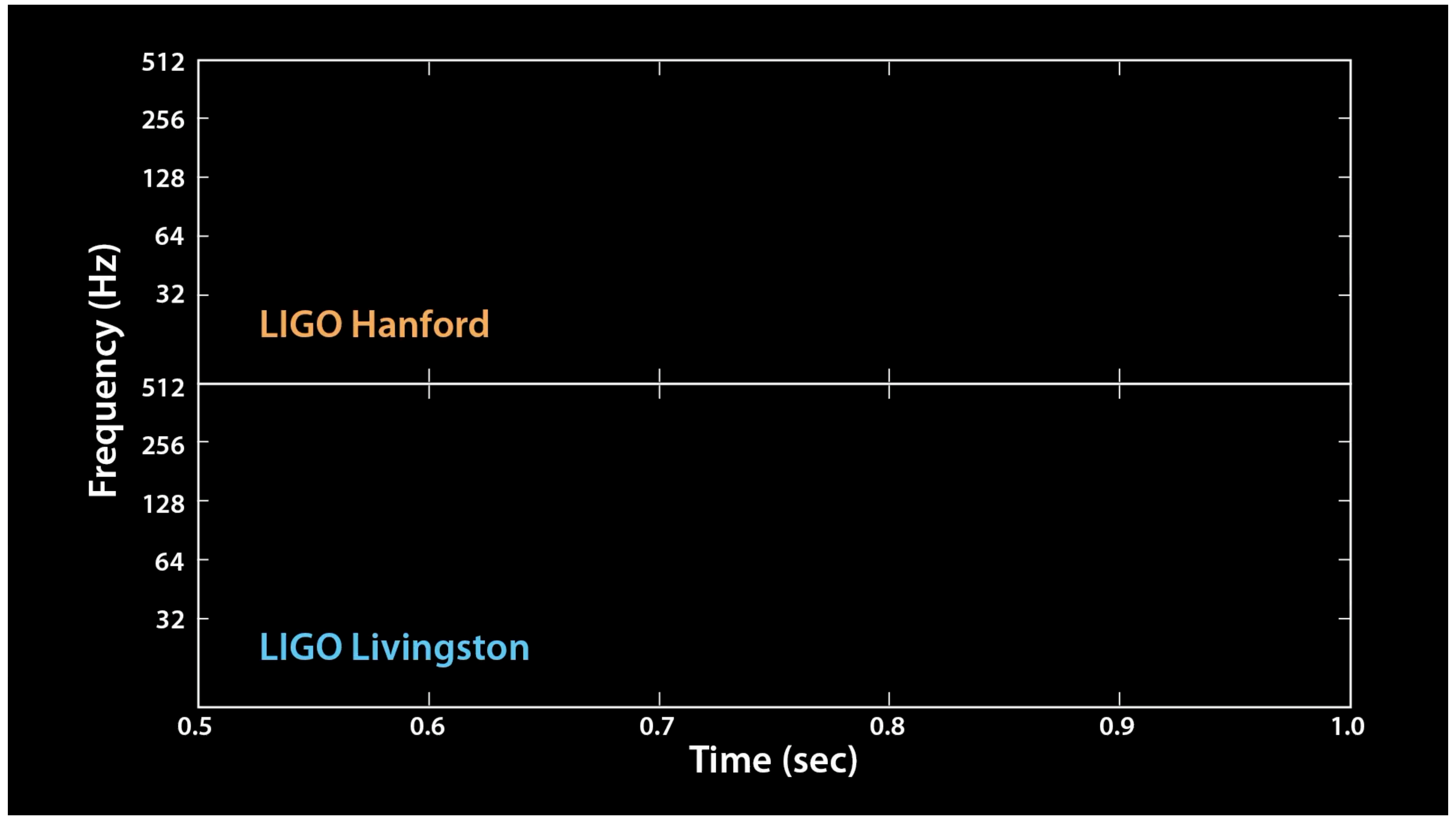
Outline

- ✦ Introduction and Motivation
- ✦ Multipolar Post-Minkowskian Expansion (MPM Formalism)
- ✦ Classical Perturbiner Techniques in solving Gravity perturbatively
- ✦ Applications to Perturbiner in MPM expansions for Stationary Sources
- ✦ Summary and Outlook

Music of the spheres to Gravitational waves ...



Animation Credit: LIGO

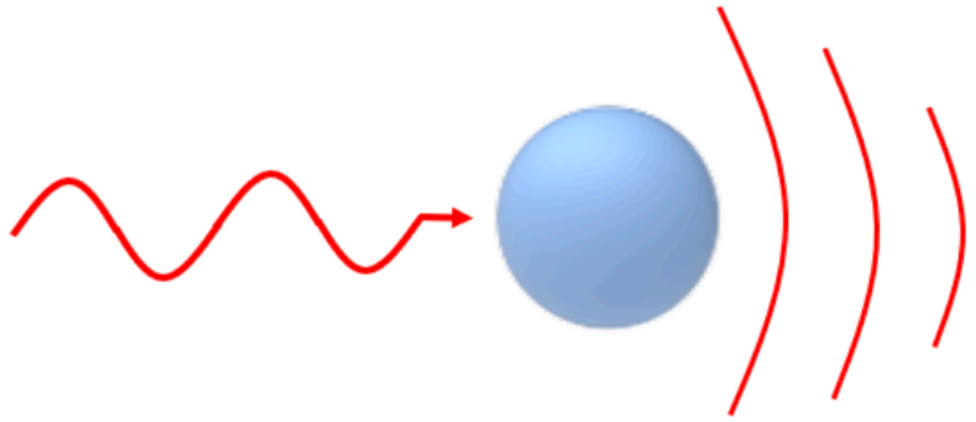


The Sound of Two Black holes colliding (On September 14, 2015)

Audio Credit: Caltech/MIT/LIGO Lab

Aristotle, *On the Heavens* II.9, 290b

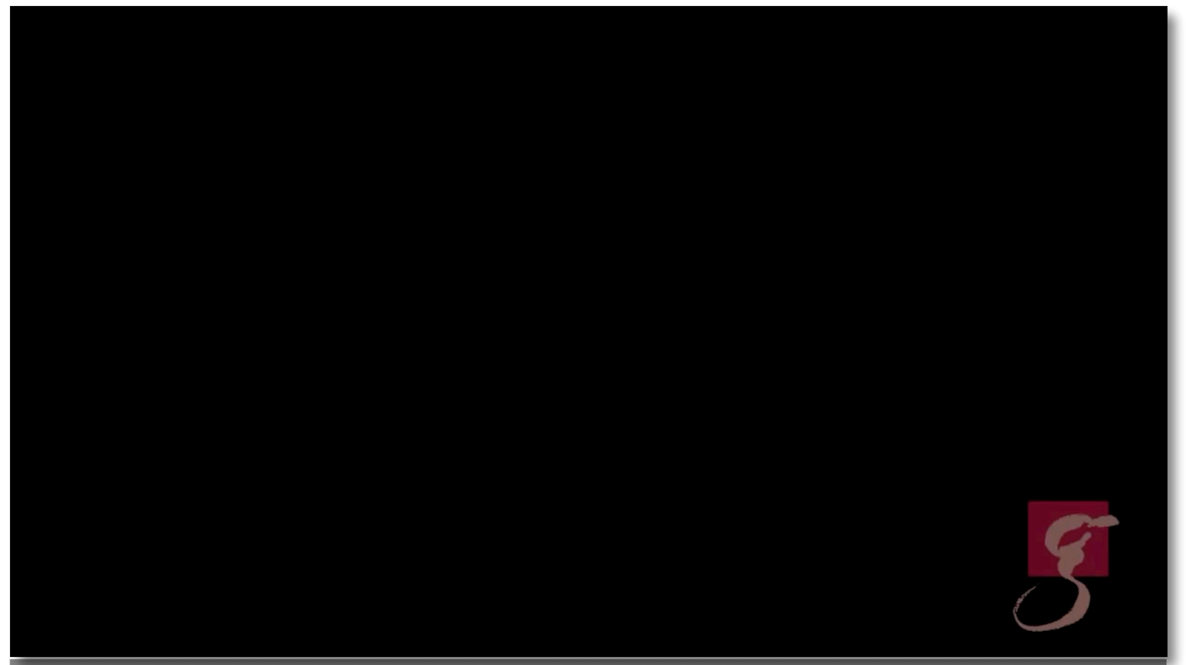
Why Multipole Expansions?



Studying gravitational multipoles gives us the language to connect source dynamics with the ripples we detect in spacetime.

- **Far-field simplification**: Einstein equations are nonlinear, but in the weak-field wave zone the field can be written as a superposition of multipole moments (spherical harmonics).
- **Source–radiation connection**: Links source dynamics to the radiation detected at infinity.
- **Universality**: Any isolated system (binaries, neutron stars, supernovae) can be described by *its multipole moments* $\{M_L, S_L\}$.
- **Observational relevance**: The detector-frame strain $h_+(t), h_\times(t)$ is a sum of (ℓ, m) modes, with higher multipoles encoding spins-induced moments, tidal deformations, and asymmetries, motivating their inclusion in LIGO/Virgo/KAGRA analyses and future LISA detectors.

$$g_{\mu\nu}^{\text{ext}} \quad \longleftrightarrow \quad \{M_L, S_L\}_{\ell=0}^{\infty}$$



Credit: AEI

Multipolar Post-Minkowskian Expansion (MPM)

[Blanchet, Damour, Iyer (1980-90)]

- Gravitational waveform is derived in expansion based on successive approximations in powers of G_N, namely **post-Minkowskian (PM)** expansion.

Weak Field + Any Velocity expansion

$$\mathfrak{g}^{\mu\nu}(x^\mu) := \sqrt{g}\,\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} - \sum_{n=1}^{\infty} G^n h^{\mu\nu}_{(n)},$$

...general gothic perturbation

Expansion: $$h^{\mu\nu} = \sum_{n=1}^{\infty} G^n\, h^{\mu\nu}_{(n)}\left[M_L, S_L\right]$$

$$M^L(t) = (-1)^\ell r^\ell \int d^3x\, T^{00}(t,\mathbf{x})\,\hat{n}^L, \qquad S^L(t) = \mathrm{STF}_L \int d^3x\, \hat{x}_{L-1}\,\epsilon_{ijk} x^j T^{0k}$$

Mass and current as multipole moments

$$h_{\mu\nu}(x) \sim \int d^4x'\, G(x-x')\, T_{\mu\nu}(x')$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell\,!}\, x'_L\,\partial_L\left(\frac{1}{r}\right)$$

$$\hat{n} = \frac{\hat{x}}{r}$$

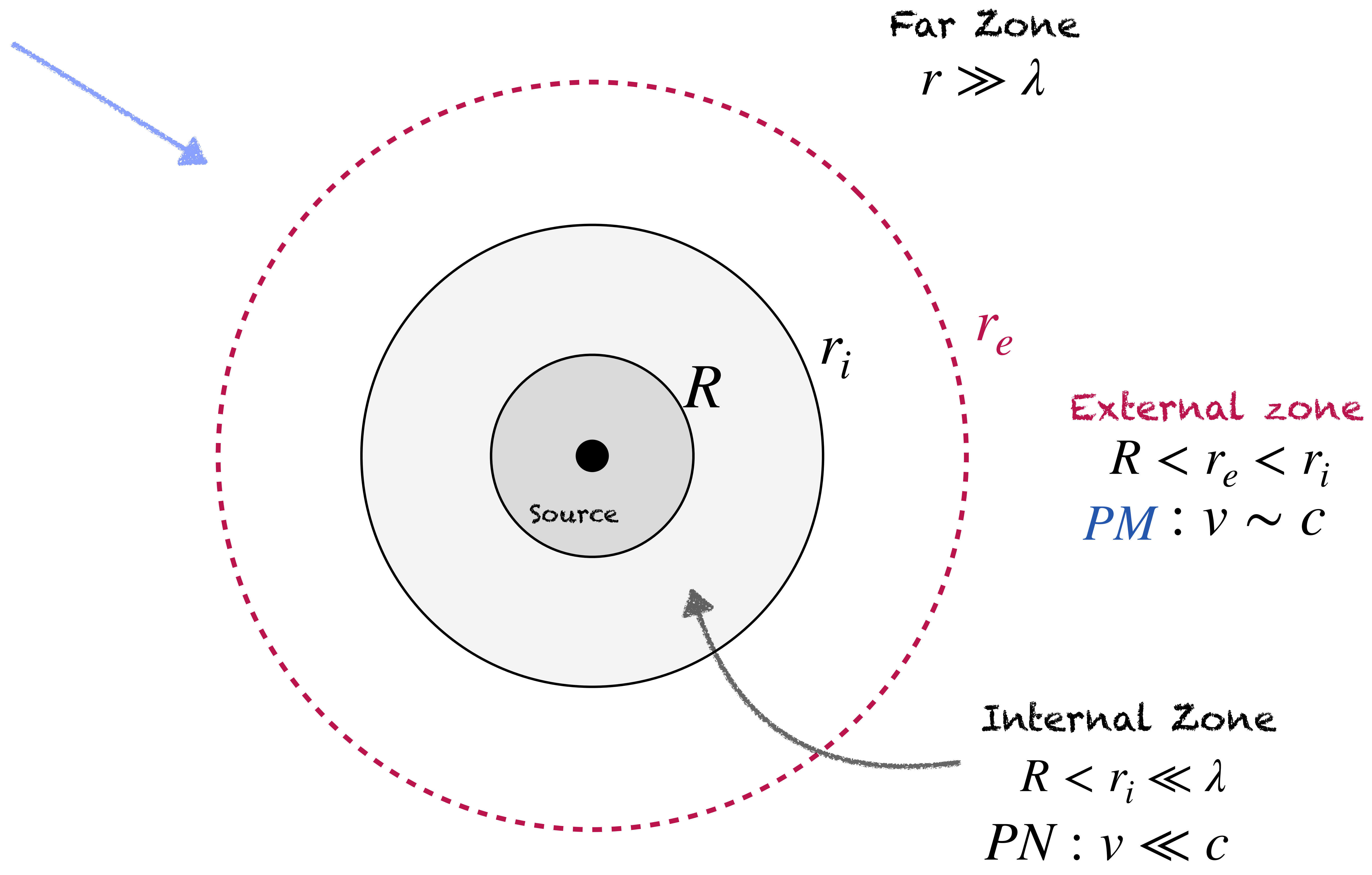
$\propto T^{\mu\nu}$ (Stress-energy tensor of matter)

Pseudo tensor for Gravitational field

- Exact Field Equation: $$\Box\, h^{\mu\nu} = -\frac{16\pi G}{c^4}\left(\tau^{\mu\nu} + t^{\mu\nu}\right)$$

...solve it perturbatively in G

MPM Zones



Far Zone $\subset$ External zone

MPM expansion for Stationary Sources

- PM expansion: $h^{\mu\nu} = \sum_{n=1}^{\infty} G^n\, h^{\mu\nu}_{(n)}\left[M_L, S_L\right], \qquad \tau^{\alpha\beta} = T^{\alpha\beta} + \sum_{n=1}^{\infty} G^n\, \tau^{\alpha\beta}_{(nPM)}, \qquad t^{\alpha\beta} = \sum_{n=1}^{\infty} G^n\, t^{\alpha\beta}_{(nPM)}$

- Expand the EOM in powers of G and solve iteratively:

$$\Box\, h^{\alpha\beta}_{(1PM)} = -\frac{16\pi}{c^4} T^{\alpha\beta},$$

$$\Box\, h^{\alpha\beta}_{(2PM)} = -\frac{16\pi}{c^4}\left(\tau^{\alpha\beta}_{(1PM)} + t^{\alpha\beta}_{(1PM)}\right),$$

$$\vdots$$

$$\Box\, h^{\alpha\beta}_{(nPM)} = -\frac{16\pi}{c^4}\left(\tau^{\alpha\beta}_{((n-1)PM)} + t^{\alpha\beta}_{((n-1)PM)}\right).$$

$\{M_L, S_L\} = const$

- **1PM** Solution in terms of multipoles of Stationary Sources:

$$h^{(1)00} = 4G \sum_{\ell=0}^{\infty} \frac{1}{\ell\,!}\, \partial_L \frac{M_L}{r}, \quad h^{(1)0i} = -\,4G \sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)\,!} \epsilon_{iab} \partial_{aL-1} \frac{S_{bL-1}}{r}, \quad h^{(1)ij} = 0$$

...will be initial conditions for Recursions

Pertubiner Expansion → Recursion Pipeline

[Rosly, Selivanov(1996-98)]

...in search of approximate methods to solve Einstein Equation by means of power series expansion

✦ Due to the complexity of gravitational Feynman rules, deriving recursion relations from diagrams is impractical; instead, the *Perturbiner expansion* provides a powerful framework for constructing off-shell recursions (**Berends Giele Recursion**).

[Berends, Giele (1988)]

Sum over ordered set of external particles

$$\Phi^A(x) = \sum_P J^A(P)\, e^{ik_P \cdot x} \; ; \quad k_P = \sum_{i \in P} k_i$$

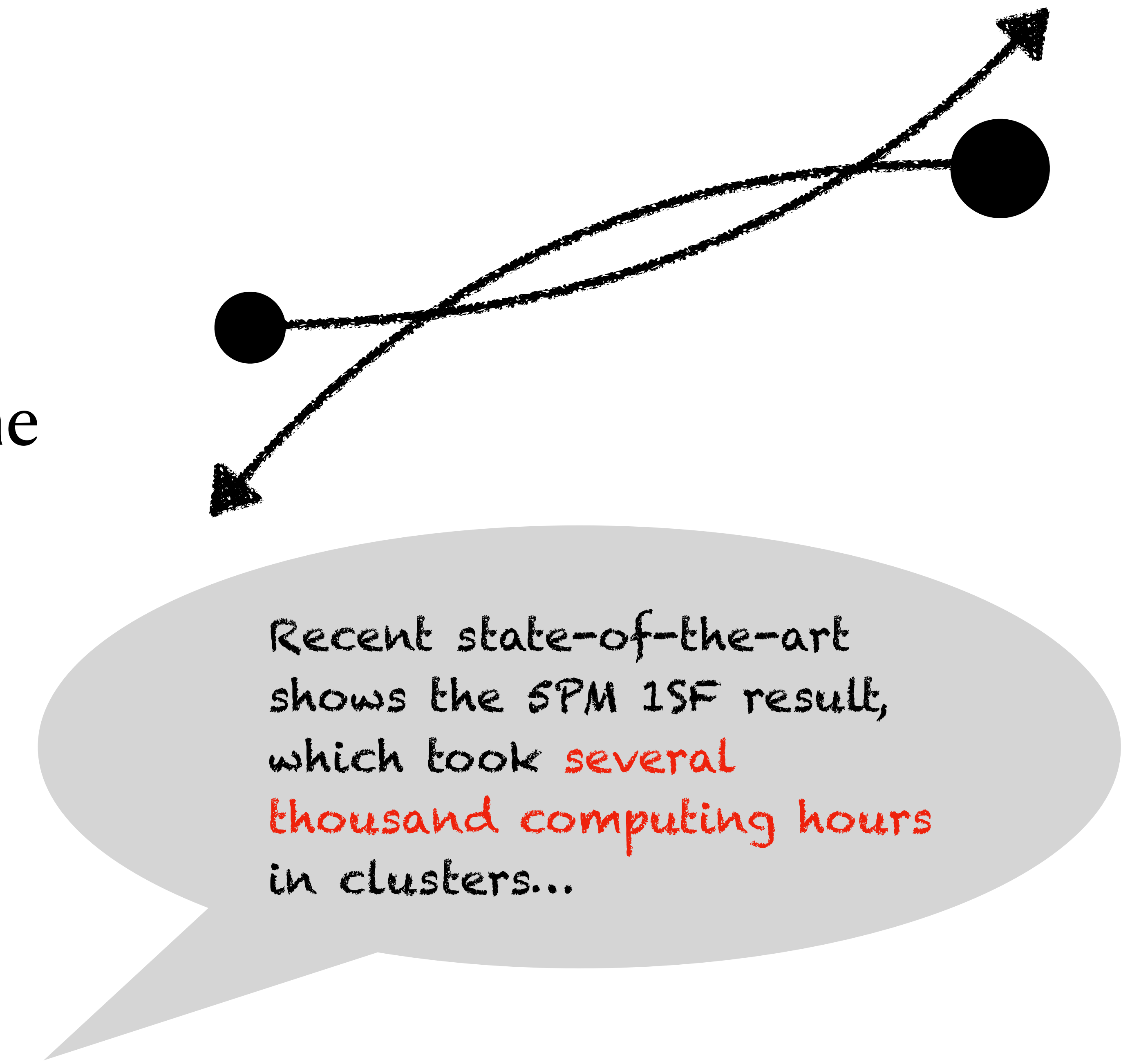
← Perturbiner Ansatz

Says, how particles combine through interactions to behave like one off-shell particle carrying momentum k_P

✦ Plug in the ansatz in the EOM: $\mathrm{K}_{AB}\Phi^B = \mathcal{N}_A(\Phi) \longrightarrow \mathrm{K}_{AB}(ik_P)J^B(P) = \sum_{P=P_1 \cup P_2 \cdots \cup P_m} \underbrace{\mathcal{V}_{A;B_1\cdots B_m}(K_{P_1}\cdots K_{P_m})}_{\text{Momentum space vertex}}\, J^{B_1}(P_1)\cdots J^{B_m}(P_m)$

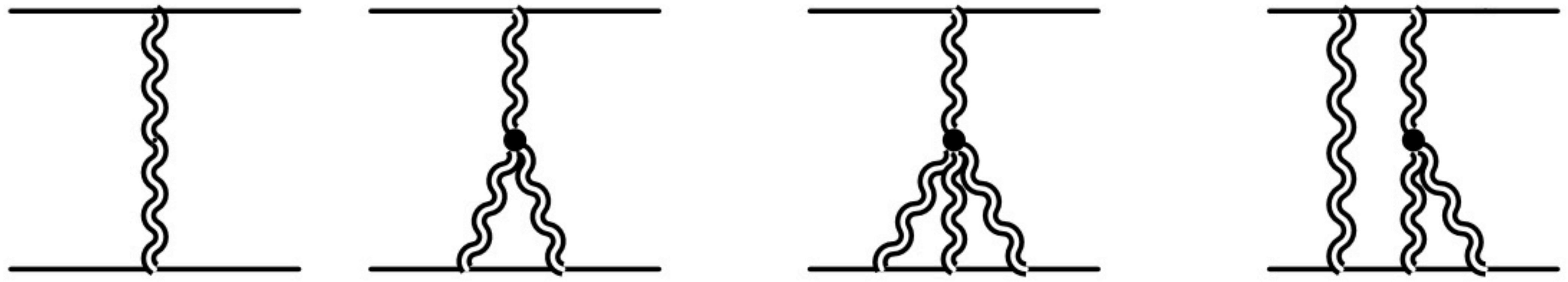
✦ Inverting K gives recursion: $J^A(P) = \underbrace{[K(ik_P)]^{-1\,AB}}_{\text{Propagator}} \sum_{P=P_1 \cup P_2 \cdots \cup P_m} \mathcal{V}_{A;B_1\cdots B_m}(K_{P_1}\cdots K_{P_m})\, J^{B_1}(P_1)\cdots J^{B_m}(P_m)$

Application to Perturbiner in Gravity



- Perturbiner fits well with the PM expansion in gravity. Complexity arises in expanding the Einstein-Hilbert action beyond leading order in coupling G_N.

$\mathcal{O}(G^n) \equiv nPM$:



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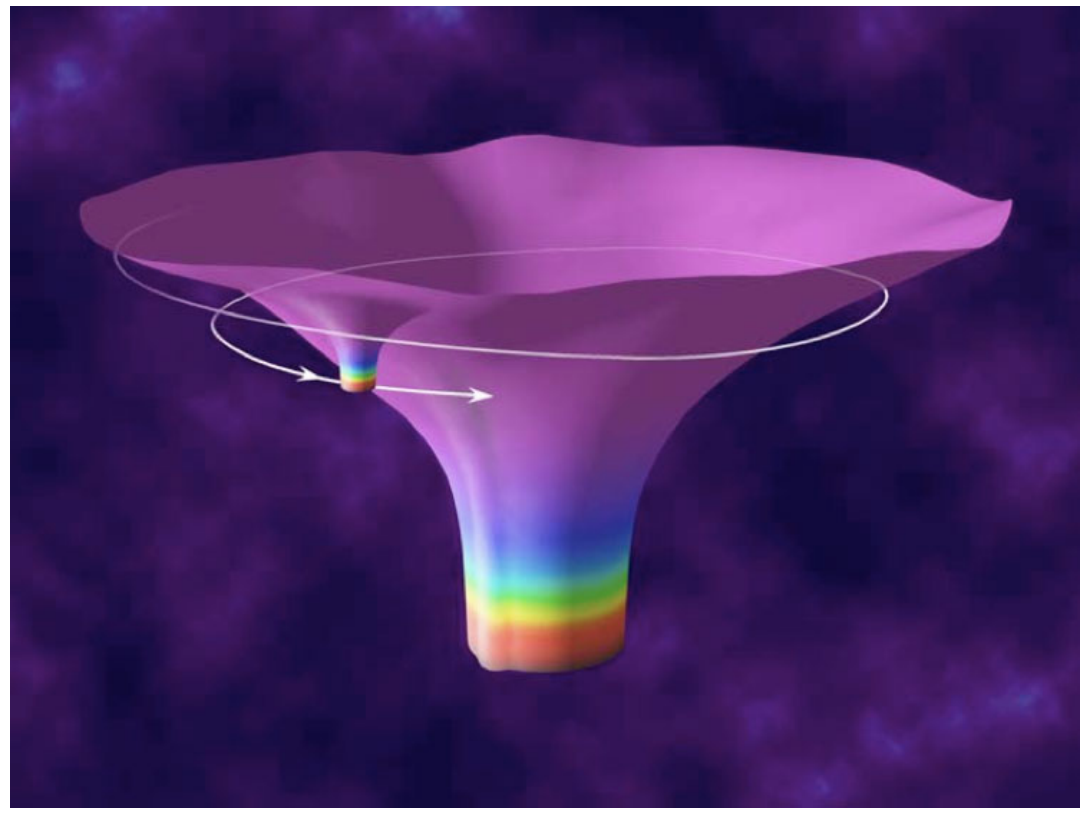
...in short, we need to compute loops efficiently!

- The overwhelming complexity of perturbative gravity is solved using the iterative systematics of perturbiner expansions of gravitational EOM combined with the *recursive structure of iterative loop integrals*.



Credit: LIGO

- Perturbiner approach provides a natural framework for EMRIs (Extreme Mass Ratio Inspirals)-key source for LISA.
- Simplification for the relativistic two-body problem.

MPM Expansion using Classical Perturbiner

Classical Perturbiner solves classical Einstein equation of motion recursively to all orders in perturbation theory!

[Damgaard, Lee (PRL 2024)]

Solutions of classical EOM ⟵ Classical Perturbiner Method ⟶ Metric for a source of matter with the full multipole structure

$$h^{\mu\nu}_{(n)}\left[M_L, S_L\right] \equiv \int_{\ell} e^{i\ell\cdot x}\,\underbrace{\mathcal{J}^{\mu\nu}_{(n)|\ell}\left[M_L, S_L\right]}_{\text{Off-shell Graviton Current}}$$

⟵ Perturbiner Ansatz

(classical) loop integrations have to be performed when we calculate the J-currents!

✦ Metric perturbation for one body at rest with full multipole structure in harmonic coordinates at **1PM** is given by,

$$h^{(1)00} = 4G\sum_{\ell=0}^{\infty}\frac{1}{\ell\,!}\,\partial_L\frac{M_L}{r}, \quad h^{(1)0i} = -4G\sum_{\ell=1}^{\infty}\frac{\ell}{(\ell+1)!}\epsilon_{iab}\partial_{aL-1}\frac{S_{bL-1}}{r}, \quad h^{(1)ij} = 0$$

$$\Box\, h^{\alpha\beta}_{(1PM)} = -16\pi T^{\alpha\beta}$$

...will be initial conditions for Recursions

MPM Expansion of stationary metric at 2PM

$$h^{\mu\nu}_{(n)}[M_L, S_L] \equiv \int_{\ell} e^{i\ell\cdot x}\mathcal{J}^{\mu\nu}_{(n)|\ell}[M_L, S_L]$$

Here, we consider expansions to the mass quadrupole term $\ell = 0{,}2$ ($\because M_a = 0$),

$$\Box h^{\alpha\beta}_{(1PM)} = -16\pi T^{\alpha\beta}$$

$$h^{(1)00} = 4\frac{GM}{r} + 6\frac{a^2 GM_{ab}}{r^3}\hat{n}_{ab} \implies \mathcal{J}^{00}_{(1)|\boldsymbol{k}} = \int_{\boldsymbol{x}}\left(4\frac{GM}{r} + 6\frac{GM_{ab}}{r^3}\hat{n}_{ab}\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = 16\pi G\left(\frac{M}{|\boldsymbol{k}|^2} - \frac{M_{ab}\hat{\boldsymbol{k}}^a\hat{\boldsymbol{k}}^b}{2|\boldsymbol{k}|^2}\right);\ \mathcal{J}^{ij}_{(1)|\boldsymbol{k}} = 0$$

$$h^{(1)0i} = -2G\epsilon_{iab}n^a\frac{S^b}{r^2} \implies \mathcal{J}^{0i}_{(1)|-\boldsymbol{k}} = -i\epsilon_{iab}\frac{8\pi G S^b\boldsymbol{k}^a}{|\boldsymbol{k}|^2}$$

✦ 2PM EOM: $$\Box h^{\alpha\beta}_{(2PM)} = -\frac{16\pi}{c^4}\left(\tau^{\alpha\beta}_{(1PM)} + t^{\alpha\beta}_{(1PM)}\right)$$

$$\partial^2 h^{00}_{(2)} = \frac{7}{8}\partial_i h^{00}_{(1)}\partial_i h^{00}_{(1)} + \frac{5}{4}h^{00}_{(1)}\partial^2 h^{00}_{(1)} - \frac{1}{2}\partial_i h^{0j}_{(1)}\partial_i h^{0j}_{(1)} - \frac{3}{2}\partial_i h^{0j}_{(1)}\partial_j h^{0i}_{(1)} - h^{0i}_{(1)}\partial^2 h^{0i}_{(1)}$$

Recursive EOM in terms of currents

$$\mathcal{J}^{00}_{(2)|-\boldsymbol{k}_1} = \frac{1}{|\boldsymbol{k}_1|^2}\int_{\boldsymbol{k}_2}\left[-\frac{7}{8}\boldsymbol{k}_{12}\cdot\boldsymbol{k}_2 + \frac{5}{4}|\boldsymbol{k}_2|^2\right]\mathcal{J}^{00}_{(1)|-\boldsymbol{k}_{12}}\mathcal{J}^{00}_{(1)|\boldsymbol{k}_2} + \left[\left(\frac{1}{2}\boldsymbol{k}_{12}\cdot\boldsymbol{k}_2 - |\boldsymbol{k}_2|^2\right)\mathcal{J}^{0i}_{(1)|-\boldsymbol{k}_{12}}\mathcal{J}^{0i}_{(1)|\boldsymbol{k}_2} + \frac{3}{2}\boldsymbol{k}^i_2\boldsymbol{k}^j_{12}\mathcal{J}^{0i}_{(1)|-\boldsymbol{k}_{12}}\mathcal{J}^{0j}_{(1)|\boldsymbol{k}_2}\right]$$

Results: Mulitpoles at 2PM and higher

✦ rank-1 current $\left(\mathcal{J}^{\mu\nu}_{(1)|-k}\right)$ ⟶ **2PM** EOM in momentum space —Recursion⟶ rank-2 current $\left(\mathcal{J}^{\mu\nu}_{(2)|-k}\right)$ —Fourier Transform⟶ **2PM** Metric With full structures for mass and current multipoles

$$h^{00}_{(2PM)} = \frac{7G^2M^2}{r^2} + 21\frac{G^2MM_{ab}}{r^4}\hat{n}_{ab} + \frac{G^2}{r^6}\left(\frac{63}{4}M_{ab}M_{cd}\hat{n}^{abcd} + 9M_{ab}M_{cb}\hat{n}^{ac} + \frac{21}{10}M_{ab}M_{ab}\right) - \frac{5G^2}{3}\frac{S^2}{r^4} + \frac{7G^2S^aS^b}{r^4}\hat{n}^{ab}$$

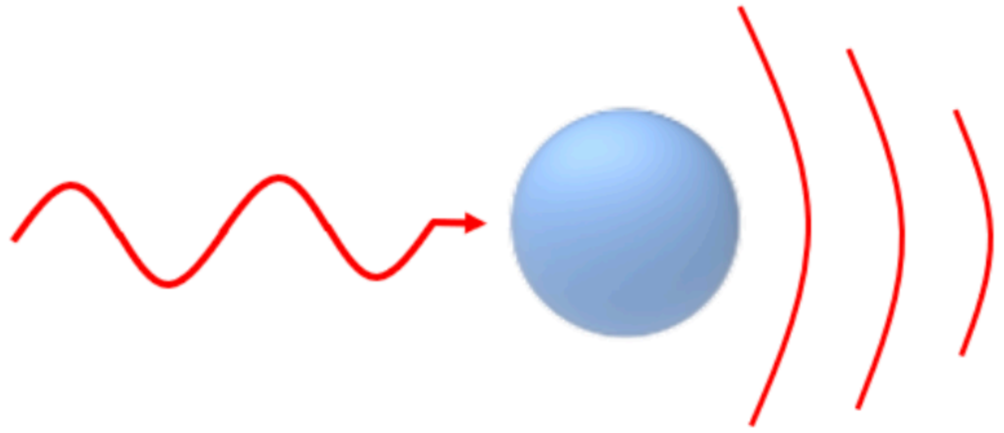
$$h^{0i}_{(2PM)} = G^2\left[8M\epsilon_{iab}S^b\frac{n^a}{r^3} + \frac{3}{2}M_{cd}\epsilon_{iab}S^b\frac{\hat{n}^{acd}}{r^5} + \frac{21}{10}M_{cd}\epsilon_{icb}S^b\frac{n^d}{r^5} + \frac{1}{2}M_{ij}\epsilon_{jab}S^b\frac{n^a}{r^5}\right]$$

$$h^{ij}_{(2PM)} = G^2\left[\frac{M^2}{r^2}\left(\hat{n}^{ij} + \frac{1}{3}\eta^{ij}\right) + \frac{M}{r^4}\left(\frac{15}{2}M^{ab}\hat{n}^{ij}{}_{ab} - \frac{12}{7}M^{a(i}\hat{n}^{j)}_{a} + \frac{11}{7}\eta^{ij}M_{ab}\hat{n}^{ab}\right) + \cdots\right.$$
$$\left. + \frac{1}{r^4}\left(\frac{9}{2}\hat{n}^{ijab}S_aS_b - \hat{n}^{cd}S^aS^b\,\epsilon^{(i}{}_{ac}\epsilon^{j)}{}_{bd} + \frac{1}{7}\eta^{ij}\hat{n}^{ab}S_aS_b + \frac{32}{7}S^a\hat{n}^{(i}{}_aS^{j)} - \frac{S^iS^j}{15} - S^2\left(\frac{20}{7}\hat{n}^{ij} + \frac{1}{5}\eta^{ij}\right)\right)\right]$$

Long expressions of Mass quadrupole terms
[Blanchet (1998)]

...can easily(?) be promoted to higher PM orders!

Towards Fully Multipolar Post-Minkowskian Gravity



New Recursive Formalism

- Solves Classical EOMs recursively
- MPM using this formalism incorporates Current-multipole structures at 2PM completely
- Valid up to all orders of multipoles
- Spin effects handled cleanly and systematically

Applications

- Lays foundation for a fully time-dependent MPM expansion
- Matching to all-order perturbed Kerr metric in harmonic gauge*
- Building a complete and systematic radiation zone metric
- Matching to EFT Wilson Coefficients.

**work in progress*

감사합니다 and Dhanyabad!