In-Out Formalism for Effective Actions in QED and Gravity

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The Road Not Taken

(Robert Frost)

Mathematical beauty behind particle production and Stokes phenomena

[AAPPS Bulletin 23 (‘13), e-EPS (’13)]

At the James Scott Prize Lecture in 1939, P. A. M. Dirac emp hasized the theory of functions of a complex variable as an int eresting mathematical theory that fulfilled his criteria of bea uty. He found this field to be of “exceptional beauty” and henc e likely to lead to deep physical insight.

Outline

- Motivation
- Unified Picture for Spontaneous Pair Production
- Effective Actions in In-Out Formalism
- Gamma-Function Regularization for QED and/or Gravity Actions
- QED Action & Schwinger Effect in (Anti-) de Sitter
- Summary & Perspective

Motivation

Near-Horizon Geometries

- **Theorem (Static)** [Kunduri, Lucietti, Reall, CQG 24 (‘07)]

Any static near-horizon geometry is locally a warped product of AdS_2, dS_2 or $R^{1,1}$ and H. If H is simply connected, this statement is global. In this case if H is compact and the strong energy conditions holds, it must be the AdS_2 case or the direct product $R^{1,1} \times H$.

- **Theorem (Rotational)**

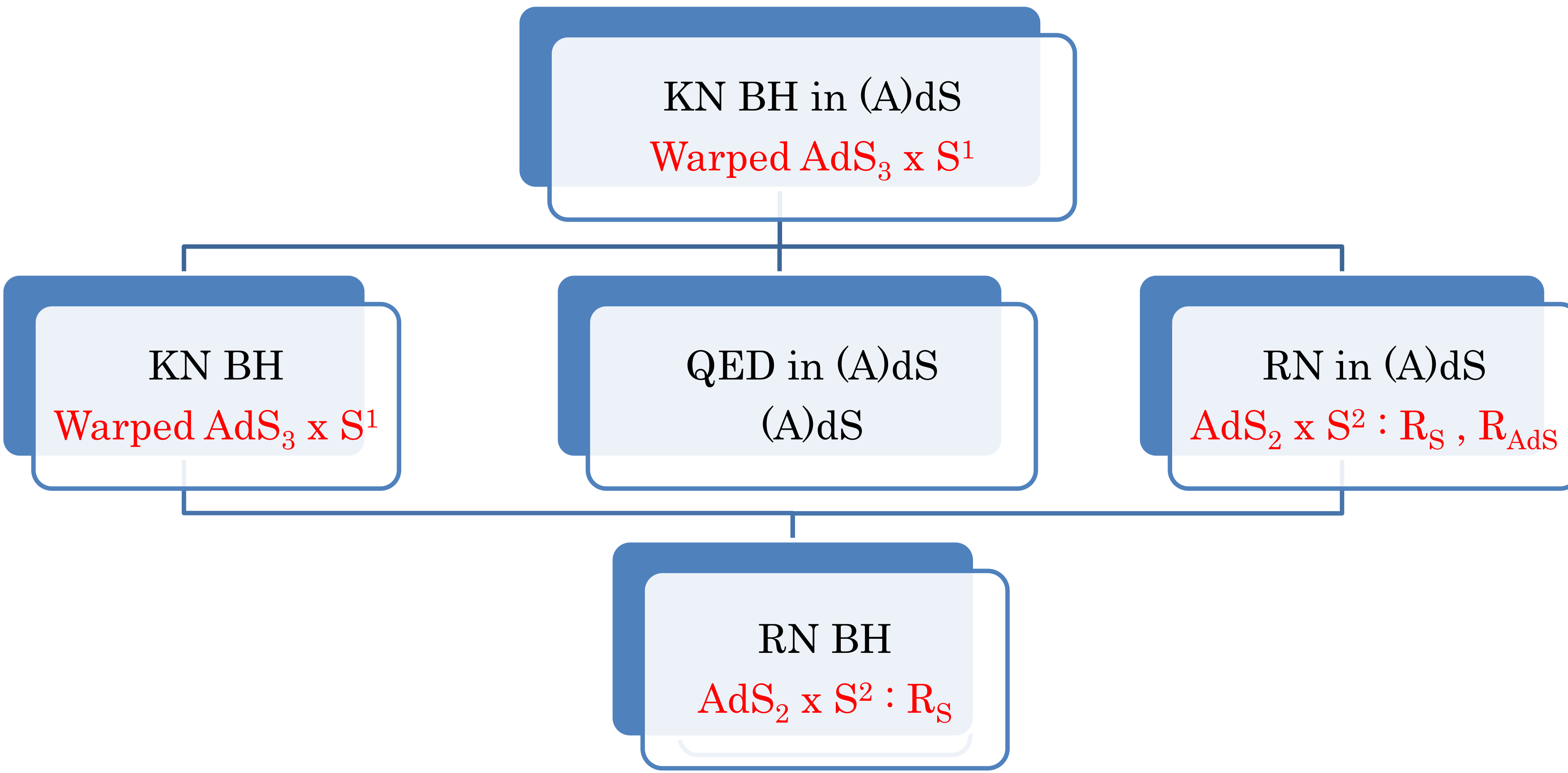
Consider a D-dimensional spacetime containing a degenerate horizon, invariant under an $R\times U(1)^{D-3}$ isometry group, satisfying the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Then the near-horizon geometry has a global $G \times U(1)^{D-3}$ symmetry, where G is either $O(2, 1)$ or the 2D Poincare group. Furthermore, if $\Lambda \leq 0$ and the near-horizon geometry is non-static, the Poincare case is excluded.

- Enhanced symmetries enable one to solve the equations for linear perturbations in terms of hypergeometric functions: bosons, fermions, EM fields, GWs, etc.

Why Loops?

- Now @ APCTP: one-loop QED actions in dS space
  - ✓ Bavarsad, SPK, Stahl, Xue, "Scalar QED effective action and renormal ization group flows in dS space," (coming soon)
  - ✓ Chen, Huang, SPK, "Saturation of Pauli blocking in near-extremal charged Nariai BHs," (coming soon)
  - ✓ Chen, SPK, Rivera Medina, "One loop effective action for pair production in near-extremal RN BHs," (maybe in September)
- Recent interest on near-extremal BHs
  - ✓ Castro, Mancilla, Ioannis Papadimitrou, "Near-extremal dynamics away from the horizon," [2507.01126] & 2nd APCTP-INPP meeting
  - ✓ Iliesiu, Turiaci, "The statistical mechanics of near-extremal BHs," [JHEP 05 (2021)]
  - ✓ Turiaci, "Les Houches lecture on two-dimensional gravity and holography," [2412.09538]
  - ✓ Debangshu Mukherjee's talk
  - ✓ Many other works on near-extremal BHs

Schwinger Emission in (Near-) Extremal BHs in (A)dS Space [Chen, et al, since 2012]



Spontaneous Pair (Particle) Production

- Proper framework for spontaneous pair (particle) production in background fields (curved spacetimes or gauge fields) is one-loop effective actions:

$$|\text{in}\rangle \quad \Leftarrow \quad \text{Bogoliubov transformation} \quad \Rightarrow \quad |\text{out}\rangle$$

$$a_{\text{k,out}} = \alpha_{\text{k,in}} a_{\text{k,in}} + \beta^{*}_{\text{k,in}} b^{+}_{\text{k,in}} = U_{\text{k}} a_{\text{k,in}} U^{+}_{\text{k}}$$

$$b_{\text{k,out}} = \alpha_{\text{k,in}} b_{\text{k,in}} + \beta^{*}_{\text{k,in}} a^{+}_{\text{k,in}} = U_{\text{k}} b_{\text{k,in}} U^{+}_{\text{k}}$$

$$|\alpha_{\text{k}}|^2 \mp |\beta_{\text{k}}|^2 = 1 \text{ (Bogoliubov relations)}$$

- Spontaneous pair production (cosmic particle production, Hawking radiation, Schwinger pair production, etc.)

$$N_{\text{k}} = |\beta_{\text{k}}|^2$$

- One-loop effective action and vacuum persistence amplitude

$$e^{iW} = e^{i\int(-g)^{1/2} d^D x L_{\text{eff}}} = \langle 0,\text{out}|0,\text{in}\rangle$$

$$|\langle 0,\text{out}|0,\text{in}\rangle|^2 = e^{-2\,\text{Im}\,W}, \qquad 2\,\text{Im}\,W = \pm VT \sum_{\text{k}} \ln(1 \pm N_k)$$

Worldline formalism for QED

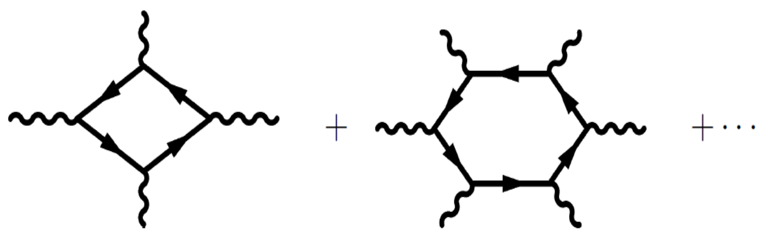
- The one-loop effective action [Feynman, PR 80 ('50); string-inspired formalism, Schubert, PR 355 ('01)]

$$\Gamma[A] = -tr\ln[(-(\partial - iqA)^2) + m^2]$$

$$= \int_0^\infty \frac{dT}{T} tr\exp[-T(-(\partial - iqA)^2 + m^2)]$$

$$= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}x(\tau) e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 - iq\dot{x}A(x(\tau))\right)}$$

- Equivalent to all one-loop N-photon amplitudes



In-Out Formalism for Effective Actions (QED and/or Gravity) at T=0 & T

- Zero-temperature effective action for scalar and spinor [SKP, Lee, Yoon, PRD 78 ('08); 82 ('10); SPK, 84 ('11)]

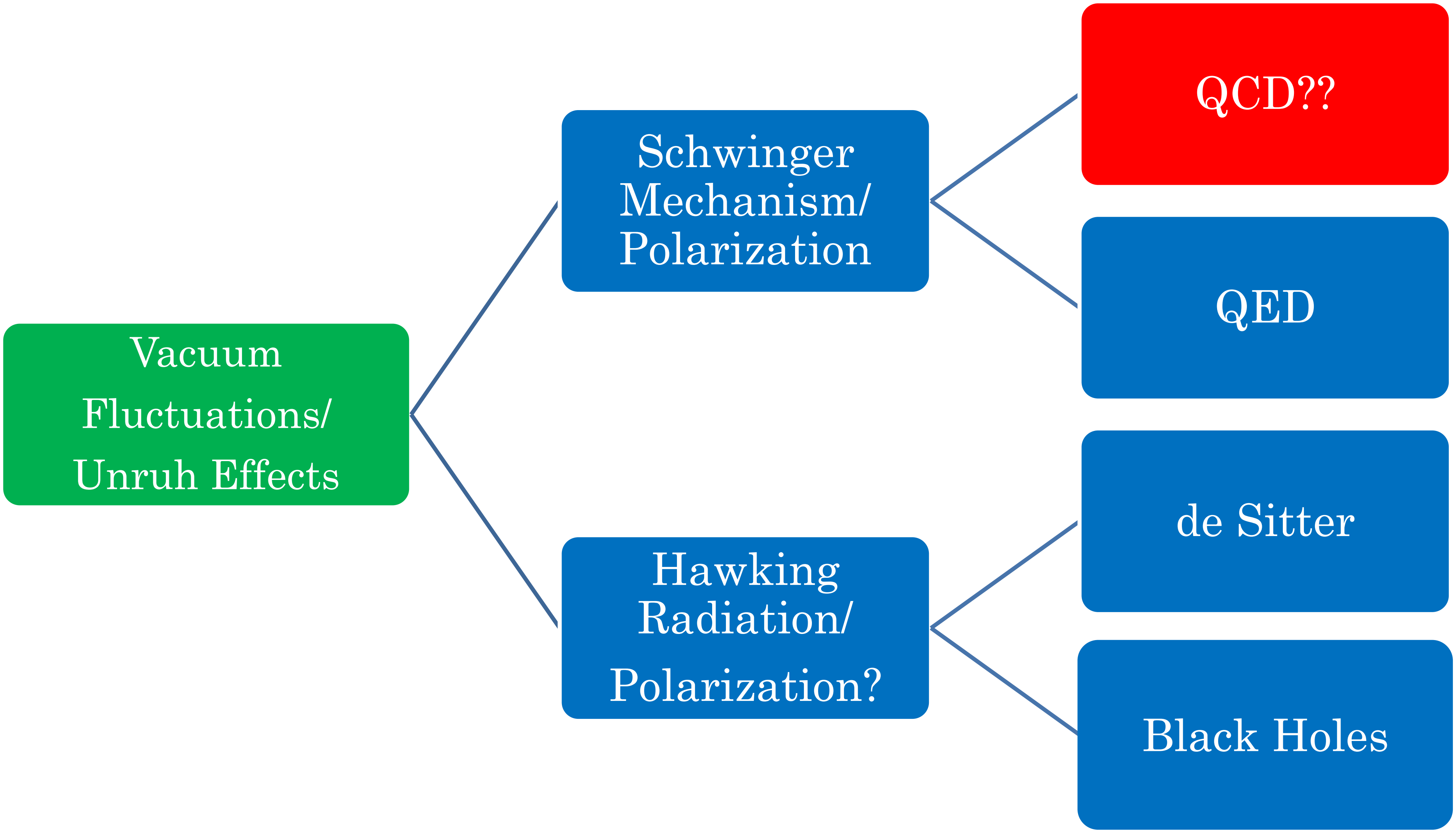
$$W = -i \ln\langle 0,\mathrm{out}|0,\mathrm{in}\rangle = \pm i \sum_{\mathrm{k}} \ln \alpha_{\mathrm{k}}^{*}$$

- finite-temperature effective action for scalar and spinor [SKP, Lee, Yoon, PRD 82 ('10)] consistent with pair production at T [SPK, Lee, PRD 76 ('07); SPK, Lee, Yoon, 79 ('09)]

$$\exp[\, i \int d^3x dt L_{\mathrm{eff}}] = \langle 0,\beta,\mathrm{in}|U^{+}|0,\beta,\mathrm{in}\rangle = \frac{\mathrm{Tr}(U^{+}\rho_{\mathrm{in}})}{\mathrm{Tr}(\rho_{\mathrm{in}})}$$

Unified Picture for Spontaneous Pair Production

Spontaneous Pair Production: Unified Picture [SPK, JHEP ('07)]

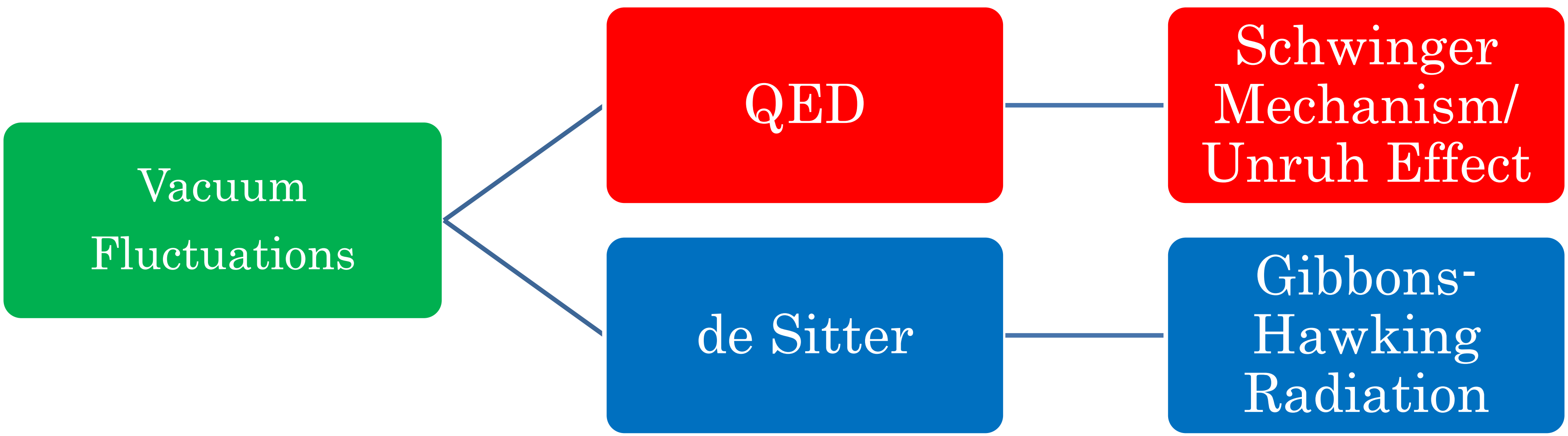


Schwinger Effect in (A)dS$_2$

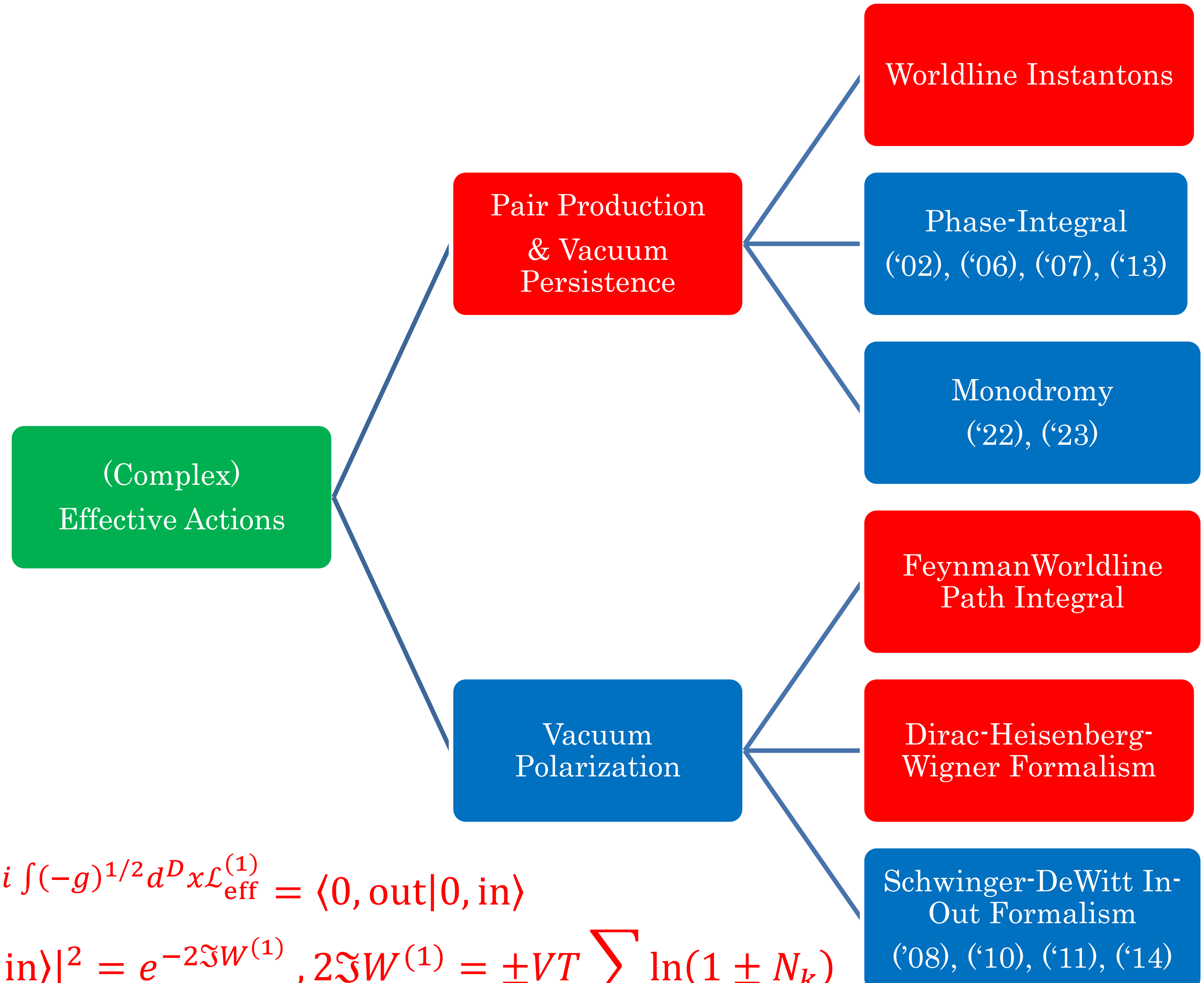
[Cai, SPK, bosons, JHEP ('14); Chen et al, fermions, ('25)]

Near Horizon Geometry of Near-Extremal Black Hole:

$AdS_2 \times S^2$



Strong Field QED: Physics and Methods



$$e^{iW^{(1)}} = e^{i\int(-g)^{1/2}d^Dx\mathcal{L}_{\text{eff}}^{(1)}} = \langle 0, \text{out}|0, \text{in}\rangle$$

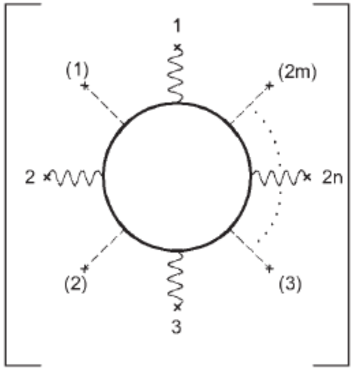
$$|\langle 0, \text{out}|0, \text{in}\rangle|^2 = e^{-2\Im W^{(1)}}, 2\Im W^{(1)} = \pm VT\sum_k \ln(1 \pm N_k)$$

Effective Actions in In-Out Formalism

In-Out Formalism for QED Actions

- In the in-out formalism, the vacuum persistence amplitude gives the effective action [Schwinger ('51); DeWitt ('75), ('03)] and is equivalent to the Feynman integral

$$e^{iW} = e^{i\int(-g)^{1/2}d^D x L_{\text{eff}}} = \langle 0,\text{out}|0,\text{in}\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}$$



- The complex effective action and the vacuum persistence for particle production

$$|\langle 0,\text{out}|0,\text{in}\rangle|^2 = e^{-2\,\mathrm{Im}\,W}, 2\,\mathrm{Im}\,W = \pm VT\sum_{\mathrm{k}}\ln(1 \pm N_k)$$

Bogoliubov Transformation & In-Out Formalism

- The Bogoliubov transformation between the in-state and the out-state, equivalent to the S-matrix,

$$a_{\mathrm{k,out}} = \alpha_{\mathrm{k,in}} a_{\mathrm{k,in}} + \beta^*_{\mathrm{k,in}} b^+_{\mathrm{k,in}} = U_{\mathrm{k}} a_{\mathrm{k,in}} U^+_{\mathrm{k}}$$

$$b_{\mathrm{k,out}} = \alpha_{\mathrm{k,in}} b_{\mathrm{k,in}} + \beta^*_{\mathrm{k,in}} a^+_{\mathrm{k,in}} = U_{\mathrm{k}} b_{\mathrm{k,in}} U^+_{\mathrm{k}}$$

- Commutation relations from quantization rule (CTP):

$$\left[a_{\mathrm{k,out}}, a^+_{\mathrm{p,out}}\right] = \delta(\mathrm{k}-\mathrm{p}), \left[b_{\mathrm{k,out}}, b^+_{\mathrm{p,out}}\right] = \delta(\mathrm{k}-\mathrm{p});$$

$$\left\{a_{\mathrm{k,out}}, a^+_{\mathrm{p,out}}\right\} = \delta(\mathrm{k}-\mathrm{p}), \left\{b_{\mathrm{k,out}}, b^+_{\mathrm{p,out}}\right\} = \delta(\mathrm{k}-\mathrm{p})$$

- Particle (pair) production

$$N_{\mathrm{k}} = |\beta_{\mathrm{k}}|^2; |\alpha_{\mathrm{k}}|^2 \mp |\beta_{\mathrm{k}}|^2 = 1$$

Out-Vacuum from In-Vacuum

- For bosons, the out-vacuum is the multi-particle states of but may be unitary inequivalent $\langle 0; \mathrm{out} | 0; \mathrm{in} \rangle = 0$ to the in-vacuum in infinite spacetime volume:

$$|0; \mathrm{out}\rangle = \prod_{\mathrm{k}} U_{\mathrm{k}} |0; \mathrm{in}\rangle = \prod_{\mathrm{k}} \frac{1}{\alpha_{\mathrm{k,in}}} \sum_{n_k} \left(- \frac{\beta^*_{\mathrm{k,in}}}{\alpha_{\mathrm{k,in}}} \right)^{n_k} |n_{\mathrm{k}}, \bar{n}_{\mathrm{k}}; \mathrm{in}\rangle$$

- The out-vacuum for fermions (Pauli blocking):

$$|0; \mathrm{out}\rangle = \prod_{\mathrm{k}} U_{\mathrm{k}} |0; \mathrm{in}\rangle = \prod_{\mathrm{k}} \left(-\beta^*_{\mathrm{k,in}} |1_{\mathrm{k}}, \bar{1}_{\mathrm{k}}; \mathrm{in}\rangle + \alpha_{\mathrm{k,in}} |0_{\mathrm{k}}, \bar{0}_{\mathrm{k}}; \mathrm{in}\rangle \right)$$

Effective Actions at T=0 & T

- Zero-temperature effective actions in proper-time integral via the gamma-function regularization [SPK, Lee, Yoon ('08), ('10); SPK ('11)]; gamma-function & zeta-function regularization [SPK, Lee ('14)]; quantum kinematic approach

$$W = \pm i \sum_{\mathrm{k}} \ln \alpha_{\mathrm{k}}^{*} = \pm i \sum_{l} \sum_{\mathrm{k}} \ln \Gamma\left(a_l + i b_l(\mathrm{k})\right)$$

- finite-temperature effective action [SPK, Lee, Yoon ('09), ('10)]

$$\exp\left[i \int d^3x dt L_{\mathrm{eff}}\right] = \langle 0,\beta,\mathrm{in}|U^{+}|0,\beta,\mathrm{in}\rangle = \frac{\mathrm{Tr}(U^{+}\rho_{\mathrm{in}})}{\mathrm{Tr}(\rho_{\mathrm{in}})}$$

Effective Action at T

- Expectation value of U in thermal vacuum

$$\exp[\, i \int d^3x dt L_{\text{eff}}] = \langle 0,\beta,\text{in}|U^+|0,\beta,\text{in}\rangle = \frac{\text{Tr}(U^+\rho_{\text{in}})}{\text{Tr}(\rho_{\text{in}})}$$

- Effective action per unit volume and time

$$L_{\text{eff}} = \mp i \sum_{k,\sigma} \left[\ln\left(1 \pm e^{-\beta(\omega_k - z_k)}\right) \underbrace{-\beta z_k}_{\text{vacuum effective action}} \underbrace{-\ln\left(1 \pm e^{-\beta\omega_k}\right)}_{\text{zero field subtraction}} \right]$$

$$\frac{1}{\alpha_k} = e^{\beta z_k}, \qquad z_k = z_r(k) + i z_i(k)$$

Vacuum Polarization & Persistence

- Purely thermal part of the effective action

$$\Delta L_{\text{eff}}(T,E) = L_{\text{eff}}(T,E) - L_{\text{eff}}(T=0,E)$$
$$= \mp i \sum_{k,\sigma} \left[\ln\left(1 \pm e^{-\beta(\omega_k - z_k)}\right) - \ln\left(1 \pm e^{-\beta\omega_k}\right)\right]$$

- Imaginary part of the effective action

$$\text{Im}(\Delta L_{\text{eff}}) = \pm \frac{1}{2} i \sum_{k,\sigma} \sum_{j=1} \frac{(\mp n_{FD/BE}(k))^j}{j} \left[(e^{\beta z_k} - 1)^j + (e^{\beta z_k^*} - 1)^j\right]$$

- Real part of the effective action (vacuum polarization)

$$\text{Re}(\Delta L_{\text{eff}}(T)) = \mp \textstyle\sum_{k,\sigma} \arctan\left[\frac{\sin(\text{Re}\, L_{\text{eff}}(T=0,k))}{e^{\beta\omega_k(1+|\beta_k|^2)^{(1+2|\sigma|)/2}} \pm \cos(\text{Re}\, L_{\text{eff}}(T=0,k))}\right]$$

Pair Production at T

- Imaginary part of the effective action (the limit of small mean number of produced pairs)

$$2\,\mathrm{Im}\,\Delta L_{\mathrm{eff}}(T) \approx \mp \sum_{k,\sigma} |\beta_k|^2 n_{FD/BE}(k)$$

- Consistent with the pair-production rate at T [SPK, Lee ('07); SPK, Lee, Yoon ('09)]

$$N^{\mathrm{sp/sc}}(T) = \begin{cases} \sum_k |\beta_k|^2 \tanh(\beta\omega_k/2) \\ \sum_k |\beta_k|^2 \coth(\beta\omega_k/2) \end{cases}$$

Gamma-Function Regularization for QED and/or Gravity Actions

Γ-Function Regularization

- Most of soluble models (probably related to symmetry algebras, SUSY QM, PT symmetry, or broken symmetries, and Trieste connection formula for (confluent) Heun equations) in QED and gravity have the Bogoliubov coefficients of the form

$$\alpha_k = A_k \prod \frac{\Gamma(a \pm ib)}{\Gamma(c \pm id)}, \qquad \beta_k = B_k \prod \frac{\Gamma(f \pm ig)}{\Gamma(h \pm ik)}$$

  - a, …, h: integers or half-integers depending on spins
  - Constants A_k and B_k to be regulated away

- Integral representation for gamma-function

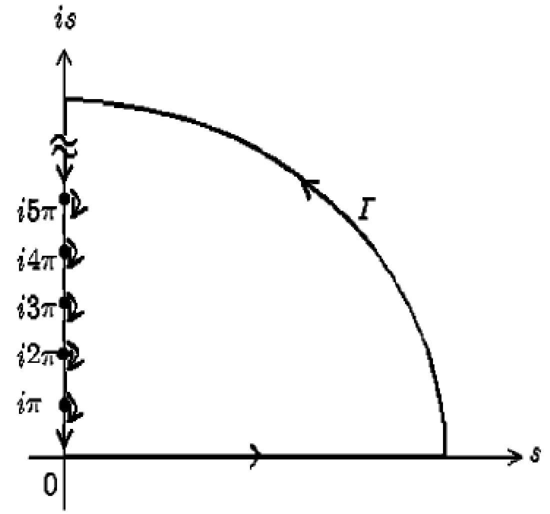
$$\ln \Gamma\,(a \pm ib) = \int_0^\infty \frac{dz}{z}\left[\frac{e^{-(a \pm ib)z}}{1-e^{-z}} - \frac{e^{-z}}{1-e^{-z}} + (a \pm ib - 1)e^{-z}\right]$$

Γ-Function Regularization

- Γ -regularization [SPK, Lee, Yoon ('08), ('10); SPK ('12)]

$$\int_0^\infty \frac{dz}{z}\frac{e^{-(a\pm ib)z}}{1-e^{-z}} = P\int_0^\infty \frac{ds}{s}\frac{e^{-(a\pm ib)(\mp is)}}{1-e^{\pm is}} \mp \pi i \sum_{n=1}^{\infty}\frac{e^{-(a\pm ib)(\mp 2n\pi i)}}{\mp 2n\pi i}$$

- Cauchy residue theorem



- Other expressions: Ramanujan's series (weak field expansion) and Whittaker-Watson formula (Hurwitz zeta function).

QED Action in Constant E

- Bogoliubov coefficient for scalar and spinor in constant E-field [SPK, Lee, Yoon ('08)]

$$\alpha_{\mathrm{k}} = \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-i(p+1)\pi/2}, p = -\frac{1}{2} + \sigma - i\frac{m^2 + \mathrm{k}^2}{2(qE)}$$

- Effective action for scalar/volume and time

$$L_{\mathrm{sc}}(E) = -i\frac{qE}{4\pi}\int \frac{d^2\mathrm{k}_{\perp}}{(2\pi)^2}\int_0^{\infty} ds \frac{e^{(p^*+1/2)s}}{s}\left[\frac{1}{\sinh(s/2)} - \underbrace{\left(\frac{2}{s} - \frac{s}{12}\right)}_{\text{Schwinger subtraction}}\right]$$

- Contour integral in 1st quadrant and residue theorem

QED Vacuum Polarization

- Scalar QED: renormalized effective action per volume and per time for a constant E-field

$$L_{\text{eff}}^{\text{sc}}(E) = -\frac{(qE)^2}{16\pi^2} P\int_0^\infty ds \frac{e^{-m^2 s/qE}}{s^2}\left[\frac{1}{\sin s} - \frac{1}{s} - \frac{s}{6}\right]$$

- Spinor QED: renormalized effective action per volume and per time for a constant E-field

$$L_{\text{eff}}^{\text{sp}}(E) = \frac{(qE)^2}{8\pi^2} P\int_0^\infty ds \frac{e^{-m^2 s/qE}}{s^2}\left[\cot(s) - \frac{1}{s} + \frac{s}{3}\right]$$

QED Vacuum Persistence & Schwinger Pair Production

- Spinor QED: Schwinger pair production in a constant E-field

$$2\,\mathrm{Im}(L_{\mathrm{eff}}^{\mathrm{sp}}) = \frac{(qE)^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left[-\frac{\pi m^2 n}{qE}\right] = -\frac{2qE}{2\pi}\int\frac{dk_\perp^2}{(2\pi)^2}\ln(1-N_k)$$

- Scalar QED: Schwinger pair production

$$2\,\mathrm{Im}(L_{\mathrm{eff}}^{\mathrm{sc}}) = \frac{(qE)^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left[-\frac{\pi m^2 n}{qE}\right] = \frac{qE}{2\pi}\int\frac{dk_\perp^2}{(2\pi)^2}\ln(1+N_{\mathrm{k}}),$$

$$N_{\mathrm{k}} = e^{-\frac{\pi(m^2+\mathrm{k}_\perp^2)}{qE}}$$

QED in (A)dS

Schwinger formula in (A)dS$_2$

- (A)dS metric and the gauge potential for E

$$ds^2 = -dt^2 + e^{2Ht} dx^2, A_1 = -(E/H)(e^{Ht} - 1)$$
$$ds^2 = -e^{2Kx} dt^2 + dx^2, A_0 = -(E/K)(e^{Kx} - 1)$$

- Schwinger formula (mean number) for scalars in dS$_2$ [Garriga ('94); SPK, Page ('08)] and in AdS$_2$ [Pioline, Troost ('05); SPK, Page ('08)]

$$N = e^{-S}, S = \frac{\pi m^2}{qE} \left(\frac{2 - \frac{R}{4m^2}}{1 + \sqrt{1 + \frac{m^2 R}{2(qE)^2} - \frac{R^2}{16(qE)^2}}} \right)$$

Effective Temperature for Schwinger formula

- Effective temperature for accelerating observer in (A)dS [Narnhofer, Peter, Thirring ('96); Deser, Levin ('97)]

$$N = e^{-m/T_{\rm eff}}, \qquad T_{\rm eff} = \sqrt{T_{\rm U}^2 + \frac{R}{8\pi^2}}, \qquad R = 2H^2, (-2K^2)$$

- Effective temperature for Schwinger formula in (A)dS [Cai, SPK ('14)]

$$N = e^{-\bar{m}/T_{\rm eff}}, \qquad \bar{m} = \sqrt{m^2 - \frac{R}{8}}, \qquad T_{\rm U} = \frac{qE/\bar{m}}{2\pi}, \qquad T_{\rm GH} = \frac{H}{2\pi}$$

$$T_{\rm dS} = \sqrt{T_{\rm U}^2 + T_{\rm GH}^2} + T_{\rm U}, \qquad T_{\rm AdS} = \sqrt{T_{\rm U}^2 + \frac{R}{8\pi^2}} + T_{\rm U}$$

Scalar QED Action in dS_2

- Mean number for pair production and vacuum polarization from the in-out formalism [Cai, SPK ('14)]

$$N_{\mathrm{dS}} = \frac{e^{-(S_\mu - S_\lambda)} + e^{-2S_\mu}}{1 - e^{-2S_\mu}}, \qquad 2\,\mathrm{Im}\, W_{\mathrm{dS}}^{(1)} = \ln\left(1 + N_{\mathrm{dS}}\right)$$

$$L_{\mathrm{dS}}^{(1)} = \frac{H^2 S_\mu}{2(2\pi)} P \int_0^\infty \frac{ds}{s} \left[e^{-(S_\mu - S_\lambda)s/2\pi} \left(\frac{1}{\sin(s/2)} - \overbrace{\left(\frac{2}{s} + \frac{s}{12} \right)}^{\text{Schwinger subtraction}} \right) - e^{-S_\mu s/\pi} \left(\frac{\cos(s/2)}{\sin(s/2)} - \left(\frac{2}{s} - \frac{s}{6} \right) \right) \right]$$

$$S_\mu = 2\pi \sqrt{\left(\frac{qE}{H^2}\right)^2 + \left(\frac{m}{H}\right)^2 - \frac{1}{4}}, \qquad S_\lambda = 2\pi \frac{qE}{H^2}$$

Scalar QED Action in AdS_2

- Mean number for pair production and vacuum polarization

$$N_{\mathrm{AdS}} = \frac{e^{-(S_\kappa - S_\nu)} - e^{-(S_\kappa + S_\nu)}}{1 + e^{-(S_\kappa + S_\nu)}}, \qquad 2\,\mathrm{Im}\,W^{(1)}_{\mathrm{AdS}} = \ln\left(1 + N_{\mathrm{AdS}}\right)$$

$$L^{(1)}_{\mathrm{AdS}} = -\frac{K^2 S_\nu}{2(2\pi)} P \int_0^\infty \frac{ds}{s} e^{-S_\kappa s/2\pi} \cosh(S_\nu s/2\pi) \left[\frac{1}{\sin(s/2)} - \frac{2}{s} - \frac{s}{12}\right]$$

$$S_\nu = 2\pi \sqrt{\left(\frac{qE}{K^2}\right)^2 - \left(\frac{m}{K}\right)^2 - \frac{1}{4}}, \qquad S_\kappa = 2\pi \frac{qE}{K^2}$$

Spinor QED Action in dS_2

- Mean number for pairs and vacuum polarization [Chen et al, 2507.09914; in preparation]

$$N_{\mathrm{ds}}^{\mathrm{sp}} = \frac{e^{-(S_\mu - S_\lambda)} - e^{-2S_\mu}}{1 - e^{-2S_\mu}}, \qquad 2\,\mathrm{Im}\, W_{\mathrm{dS}}^{(1)} = -\ln\left(1 - N_{\mathrm{ds}}^{\mathrm{sp}}\right)$$

$$L_{\mathrm{dS}}^{\mathrm{sp}} = -\frac{H^2 S_\mu}{2\pi} P \int_0^\infty \frac{ds}{s} \left(e^{-(S_\mu - S_\lambda)s/2\pi} - e^{-S_\mu s/\pi}\right)\left(\cot(\frac{s}{2}) - \frac{2}{s} + \frac{s}{6}\right)$$

$$S_\mu = 2\pi\sqrt{\left(\frac{qE}{H^2}\right)^2 + \left(\frac{m}{H}\right)^2}, \qquad S_\lambda = 2\pi \frac{qE}{H^2}$$

Spinor QED Action in AdS_2

- Mean number for pairs and vacuum polarization

$$N_{\text{AdS}}^{\text{sp}} = \frac{e^{-(S_\kappa - S_\nu)} - e^{-(S_\kappa + S_\nu)}}{1 - e^{-(S_\kappa + S_\nu)}}, \qquad 2\,\text{Im}\,W_{\text{AdS}}^{\text{sp}} = -\ln\left(1 - N_{\text{AdS}}^{\text{sp}}\right)$$

$$L_{\text{AdS}}^{\text{sp}} = -\frac{K^2 S_\nu}{2\pi} P \int_0^\infty \frac{ds}{s} \left(e^{-(S_\kappa - S_\nu)s/2\pi} - e^{-(S_\kappa + S_\nu)s/2\pi}\right)\left(\cot(\frac{s}{2}) - \frac{2}{s} + \frac{s}{6}\right)$$

$$S_\nu = 2\pi\sqrt{\left(\frac{qE}{K^2}\right)^2 - \left(\frac{m}{K}\right)^2}, \qquad S_\kappa = 2\pi\frac{qE}{K^2}$$

Summary & Perspective

- Introduce the Schwinger-DeWitt in-out formalism for effective actions for QED and gravity.
- Gamma-function regularization for strong fields
  - Heisenberg-Euler and Schwinger proper-time integral
  - QED in (A)dS spaces
  - Consistent with the vacuum persistence (Schwinger effect or Hawking radiation, etc)*
  - Higher loops?
- One-loop (QED) actions in near-extremal BHs, loop corrections to near-extremal BHs, and BH thermodynamics [future works].
- Seminar on BHs: September 18 (Thursday)