Supersymmetric localization of $\mathcal{N}$= (2, 2) theories on a spindle

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Introduction

- **Supersymmetric localization**: powerful tool to compute supersymmetric observables exactly. [Witten '88 ; Nekrasov '03, Pestun '07,…]
  - discovery of dualities, providing topological invariants of manifold
  - precision test of AdS/CFT, applying the method both in field theory and gravity

- Supergravities have many interesting SUSY backgrounds **including orbifolds**
  e.g. AdS_4 x S^7 / Z_k , (AdS_2 x S^2)/ Z_k etc…

- Recently, new class of orbifold solutions were discovered in the form of AdS x Σ, where Σ referred as **spindle geometry.**
  - $AdS_{2,3,4,5}$ x Σ ← M2, D3, D4, M5 branes

    [Ferrero, Gauntlett, Ipiña, Martelli, Sparks '20 ' 21] [Faedo, Martelli, '21]
  - AdS_2 x Σ corresponds to the near horizon of accelerating BH in AdS_4

    [Cassani, Gauntlett, Martelli, Sparks '21; Kim, Kim, Lee, Poole '23]

- The localization was carried out for the dual field theory, 3d $\mathcal{N}$=2 on Σ x S^1

  [Inglese-Martelli-Pitteli '23]

  Large N limit has shown to provide the microscopic origin the entropy of accelerating BH in AdS_4 [Colombo, Hosseini, Martelli, Pittelli, Zafffaroni, '24]

- This example has led to recent interest in applications to various field theories :

  - 4d $\mathcal{N}$ =1 on Σ x T^2 [Pittelli '24], 4d $\mathcal{N}$=2 on Σ x Σ [Ruggeri '25] etc..

- Our aim is to extend this application to 2d $\mathcal{N}$ = (2, 2) theory on Σ .

  - extending the gauge theory on S^2

    [Benini, Cremonesi '12;Doroud,Gomis, le Floch, Lee '12]

  - N=(2,2) is in connection with the worldsheet string theory, allows us to study Calabi-Yau geometry [Jockers-Jumar-Lapan-Morrison-Romo '12] [Gomis-Lee '12]

  - Interesting to examine whether such application exist on Σ .

Outline

- Spindle geometry :
  smooth uplift, Supersymmetry, Explicit solution from 5d SUGRA

- 2d $\mathcal{N}$ = (2, 2) theories on spindle.

- Supersymmetric localization

  Exact 1-loop determinant via index theorem
  1. Unpaired eigenmode.
  2. (Atiyah-Bott fixed point formula)

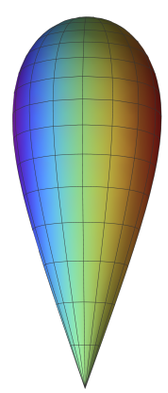
- Partition function for abelian theory

- Discussion

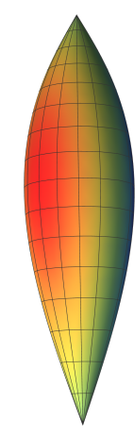
Spindle

- An example of weighted complex projective space:
  $\Sigma = \mathbb{WCP}^1_{[n_1,n_2]}, \quad n_{1,2} \in \mathbb{Z}$ and gcd(n_1, n_2) =1 $\quad (n_1 < n_2)$
  $(z_1\,, z_2) \sim (\lambda^{n_1} z_1\,, \lambda^{n_2} z_2)\,, \quad \lambda \in \mathbb{C}^*\,.$

- 'Topologically' 2-sphere but with conical deficit angles, $2\pi(1 - 1/n_{1,2})$ at the two poles



$\cong \mathbb{WCP}^1_{[1,3]}$



$\cong \mathbb{WCP}^1_{[2,3]}$

- The Euler number is

$$\frac{1}{4\pi}\int_{\Sigma} R_{\Sigma}\,\mathrm{vol}_{\Sigma} = \frac{1}{n_1} + \frac{1}{n_2} = \chi$$

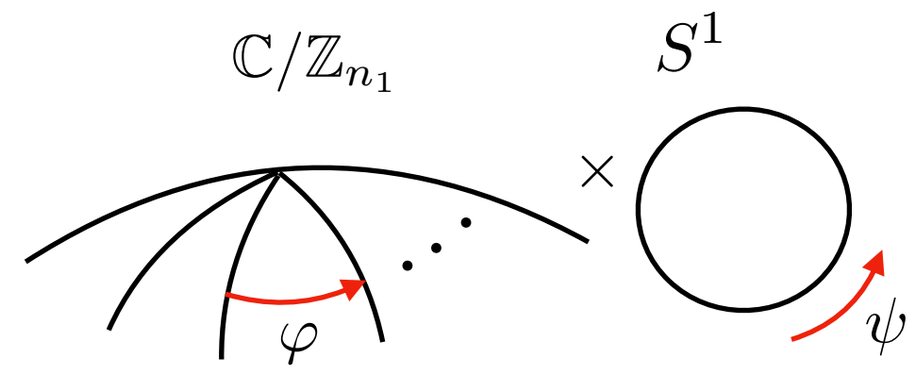
Smooth uplift

- Principal U(1) bundle over the spindle has smooth geometric description in higher dimension

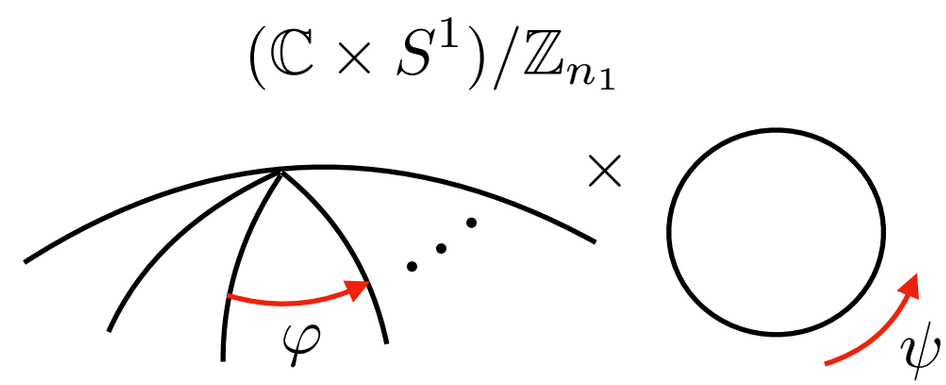
$$ds^2 \approx (d\psi + A_{1,2})^2 + d\rho^2 + \frac{\rho^2}{n_{1,2}^2} d\varphi^2$$

The 'orbi-bundle' has non-trivial holonomy

$$e^{i \oint A_{1,2}} = e^{i2\pi \frac{\mathfrak{m}_{1,2}}{n_{1,2}}} \in \mathbb{Z}_{n_{1,2}}$$



- An equivalent view is given by large gauge transformation of A through SL(2,Z) of (ψ, φ) such that there is no nontrivial holonomy.



Then the Z_n action does not have fixed point.

- Global description is given by gluing two orbi-bundle patches by gauge transformation $A_1 = A_2 + pd\varphi\,, \;\; p \in \mathbb{Z}$

- The resulting total flux is given by

$$\frac{1}{2\pi}\int dA = \frac{\mathfrak{m}}{n_1 n_2} \qquad\qquad \mathfrak{m} = pn_1n_2 - \mathfrak{m}_1 n_2 + \mathfrak{m}_2 n_1$$

- The integer $\mathfrak{m}$ uniquely specifies the orbi-bundle data up to mod n_1 and n_2 .
  - We can set $p = 0$ by Z_{n_1} or Z_{n_2} identification of $\mathfrak{m}_1$ or $\mathfrak{m}_2$
  - With the dual integer vector $(\mathfrak{a}_1, \mathfrak{a}_2)$ satisfying $1 = \mathfrak{a}_2 n_1 - \mathfrak{a}_1 n_2$ (.Bezout's Lemma)
    we can set $\mathfrak{m}_1 = \mathfrak{m}\mathfrak{a}_1\,, \qquad \mathfrak{m}_2 = \mathfrak{m}\mathfrak{a}_2\,,$

Supersymmetry

- In the standard construction of SUSY background,
  SUSY on a Riemann surface is realized by a "**topological twist**" [Maldacena-Nunez '00]

$$0 = D_\mu \epsilon \sim (\partial_\mu + \cancel{\omega_\mu} - \cancel{A_\mu})\epsilon$$

$\Rightarrow \dfrac{1}{2\pi}\displaystyle\int F = \chi$, and Killing spinor is constant along the Riemann surface.

- In contrast, SUSY on spindle can be realized by two ways: [Ferrero, Gaultelett, Sparks '21]
  Regularity condition, on the Killing spinor at both poles requires

$$\frac{1}{2\pi}\int_{\Sigma} F_R = \frac{n_2 + \sigma n_1}{2 n_1 n_2} = \chi_\sigma \qquad \sigma = \pm 1$$

"twist"

"anti-twist"

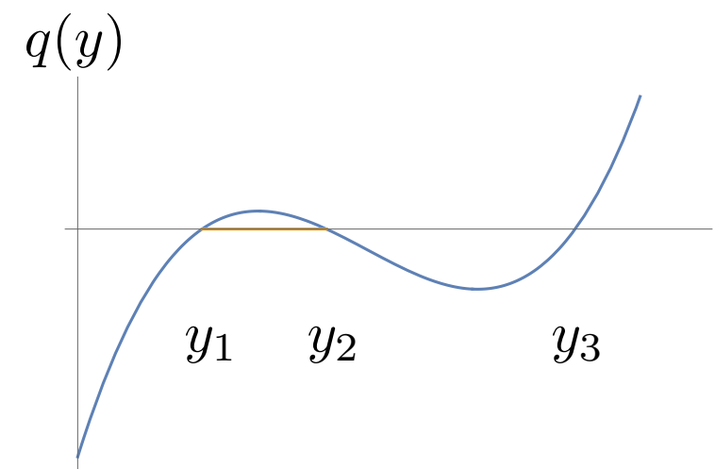
For the twist case, the spinor has the same chirality at both poles.
For anti-twist, Killing spinnor has opposite chirality at each pole

Supergravity solution

- The supersymmetric solution of $D = 5$ minimal gauged SUGRA [Ferrero et al. '20]

$$ds^2 = \frac{4y}{9} ds^2_{AdS_3} + \underbrace{\frac{y}{q} dy^2 + \frac{q}{36y^2} dz^2}_{ds^2_\Sigma}, \qquad A^R = \frac{1}{4}\left(1 - \frac{a}{y}\right) dz$$

$$q = 4y^3 - 9y^2 + 6ay - a^2.$$



- Assuming $a \in [0,1]$, the three roots of $q(y_{1,2,3}) = 0$ are positive.

  We pick $y \in [y_1, y_2]$ for positive definite metic

- For the choice

$$a = \frac{(n_1 - n_2)^2 (2n_1 + n_2)^2 (n_1 + 2n_2)^2}{4(n_1^2 + n_1 n_2 + n_2^2)^3} \qquad \Delta z = 4\pi \frac{n_1^2 + n_1 n_2 + n_2^2}{3 n_1 n_2 (n_1 + n_2)}$$

ds^2_Σ is the metric of spindle, with deficit angle $2\pi(1 - 1/n_{1,2})$ at y_1 and y_2 .

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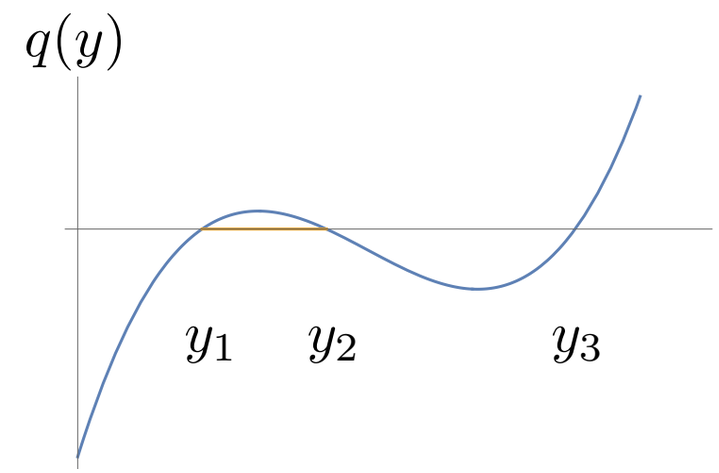

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- Note that the R-symmetry gauge flux : "**Anti-twist**"

$$\frac{1}{2\pi}\int_{\Sigma} F = \frac{n_2 - n_1}{2n_1 n_2}$$

Cf. twist supersymmetry can be realized in STU gauged SUGRA

Supergravity solution

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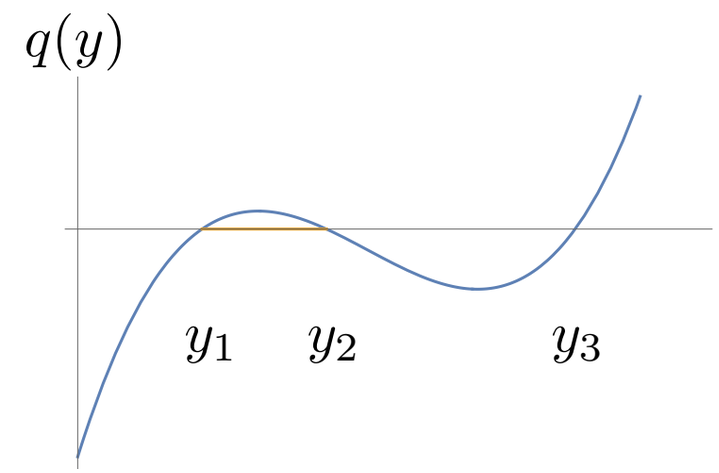

- Assuming $a \in [0,1]$, the three roots of $q(y_{1,2,3}) = 0$ are positive.
  We pick $y \in [y_1, y_2]$ for positive definite metic

- The gauge field is singular at the poles

$$A^R|_{y_{1,2}} = \frac{1}{2n_{1,2}} d\varphi \neq 0\,, \qquad \left(\varphi \equiv \frac{2\pi}{\Delta z} z\right)$$

which can lift the geometry to be smooth in higher dimension.

$\mathcal{N}$=(2, 2) theory on Spindle

$\mathcal{N}$ = (2, 2) theory

- One can define a SUSY theory on a curved manifold following the approach of [Festucia-Seiberg '11]

- Performed in detail for $\mathcal{N}$ = (2,2) by [Closset-Cremonesi '14]

  The off-shell Killing spinor equations take the form

$$D_\mu \epsilon = -\frac{1}{2} H \gamma_\mu \epsilon - \frac{1}{2} G \gamma_\mu \gamma_{12} \epsilon, \quad D_\mu \tilde{\epsilon} = -\frac{1}{2} H \gamma_\mu \tilde{\epsilon} + \frac{1}{2} G \gamma_\mu \gamma_{12} \tilde{\epsilon}$$

  characterized by the value of fields G, H , and A_μ^R :

  From the 5D Killing spinor equation of minimal supergravity, we read

$$G = -1, \qquad H = iay^{-3/2}.$$

- These values are different from all classes studied so far e.g. :

  - S^2 : $H = \pm i \, , A_\mu^R = G = 0$ (or $G \leftrightarrow H$) [Closset-Cremonesi '14]

  - $\mathbb{H}_2$: $H = \pm 1 \, , A_\mu^R = G = 0$ (or $G \leftrightarrow H$) [Lezcano - IJ - Ray '23]

$\mathcal{N}$ = (2, 2) theory

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  characterized by the value of fields G, H , and A_μ^R :

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- There are 2 supercharges, and the corresponding Killing vector is

$$\xi^\mu \partial_\mu \equiv i\tilde{\epsilon}\gamma^\mu \epsilon \, \partial_\mu = \partial_z$$

$\mathcal{N}$ = (2, 2) theory

- Among 4 SUSY of generic N=(2,2) theory, 2 supercharges are preserved

$$\epsilon = \sqrt{\frac{k_0}{12}} \begin{pmatrix} -\mathrm{i}\sqrt{\frac{q_1(y)}{y}} \\ \sqrt{\frac{q_2(y)}{y}} \end{pmatrix} \qquad \tilde{\epsilon} = \sqrt{\frac{k_0}{12}} \begin{pmatrix} \sqrt{\frac{q_2(y)}{y}} \\ -\mathrm{i}\sqrt{\frac{q_1(y)}{y}} \end{pmatrix} , \qquad k_0 = L_0 \frac{\Delta z}{2\pi}$$

$$q_1(y) = -a + 2y^{3/2} + 3y\,, \qquad q_2(y) = a + 2y^{3/2} - 3y$$

$$q_1(y_1) \;=\; q_2(y_2) \;=\; 0,$$

- And the corresponding Killing vector is

$$v^\mu \partial_\mu \;\equiv\; \mathrm{i}\tilde{\epsilon}\gamma^\mu \epsilon\, \partial_\mu = \frac{L_0}{L}\frac{\Delta z}{2\pi}\partial_z$$

$\mathcal{N}$ = (2, 2) multiplets

- Vector multiplet

$$\text{Vector} : \{\mathcal{A}_\mu, \sigma, \rho, \lambda, \overline{\lambda}, \hat{D}\} \qquad\qquad \text{Ghost} \ : \quad \{c\,, \overline{c}\,, b\}\,,$$

- Chiral multiplet

$$\text{Chiral} \ : \quad \{\phi\,, \overline{\phi}\,, \psi\,, \overline{\psi}\,, F\,, \overline{F}\}$$

with gauge charge $(1\,, -1\,, 1\,, -1\,, 1\,, -1)$

R-charge $(r, -r, r-1, -r+1, r-2, -r+2)$

- R - quantization: $r \in 2\,\mathbb{Z}$ for well-defined section on line orbibundle.

Supersymmetry algebra

- We define the equivariant supercharge

$$Q_{eq} \equiv Q + Q_{\mathrm{brst}}$$

This charge will be used for SUSY localization.

- Supersymmetry algebra is equivariantly closed.

$$Q_{eq}^2 = \mathcal{L}_v + \delta_R(\Lambda^R) + \delta_G(\Lambda_0^{\mathcal{G}})$$

$$\Lambda^R = -v^\mu A_\mu - \frac{1}{2}(H\widetilde{\epsilon}\epsilon + \mathrm{i}G\widetilde{\epsilon}\gamma_3\epsilon) \qquad \Lambda^{\mathcal{G}} \equiv -\mathrm{i}\widetilde{\epsilon}\gamma^\mu\epsilon\mathcal{A}_\mu - \widetilde{\epsilon}\epsilon\sigma - \mathrm{i}\widetilde{\epsilon}\gamma_3\epsilon\rho$$

the background values of field dependent gauge parameters.

$$Q_{eq}c = -\Lambda^{\mathcal{G}} + \Lambda_0^{\mathcal{G}}$$

BPS background (Coulomb branch)

- Using the reality condition $\sigma = \sigma^*, \quad \rho = \rho^*, \quad \mathcal{A}_\mu = \mathcal{A}_\mu^*, \quad D = D^*$
  BPS equations are solved by

$$\rho = \rho_0\,, \qquad \mathcal{F} = -\frac{\rho_0}{L}\frac{a}{y^{3/2}}\,, \quad \sigma = \sigma_0 \frac{3}{\sqrt{y}}\frac{2\pi}{\Delta z}\,, \qquad D = 3\frac{\sigma_0}{L}\frac{3y-a}{2y^2}\frac{2\pi}{\Delta z}\,,$$

- The gauge flux is quantized

$$\frac{1}{2\pi}\int_\Sigma \mathcal{F} = \frac{1}{2\pi}\int_\Sigma \mathcal{F}_{12}\sqrt{g_\Sigma}\,dydz = \frac{\mathfrak{m}}{n_1 n_2}, \qquad \mathfrak{m} \in \mathbb{Z}$$

$$\Rightarrow \quad \rho_0 L = \frac{3\mathfrak{m}}{n_1 - n_2}\,,$$

- BPS configuration is parametrized by $\mathbf{m}$ and one constant σ_0.

BPS background (Coulomb branch)

- The corresponding gauge field is given by

$$\mathcal{A}_z = \frac{\mathfrak{m}}{n_2 - n_1}\frac{1}{2}\left(1 - \frac{a}{y}\right) + \frac{2\pi}{\Delta z}\frac{\mathfrak{m}_2 - \mathfrak{m}_1}{n_2 - n_1}$$

which has singular non-trivial flat connection at north and south poles,

$$\mathcal{A}_z(y_1) = \frac{2\pi}{\Delta z}\frac{\mathfrak{m}_1}{n_1}, \qquad \mathcal{A}_z(y_2) = \frac{2\pi}{\Delta z}\frac{\mathfrak{m}_2}{n_2}, \qquad \mathfrak{m}_{1,2} \in \mathbb{Z},$$

The flux is $\mathfrak{m} = \mathfrak{m}_2 n_1 - \mathfrak{m}_1 n_2,$

- The flux $\mathbf{m}$ charaterizes uniquely the $\mathcal{O}(\mathbf{m})$ orbibundle.

Supersymmetric localization

SUSY localization

$$Z = \int D\phi\, e^{-S} = \int D\phi\, e^{-S-\lambda Q_{\rm eq}\mathcal{V}} \underset{\lambda\to\infty}{\longrightarrow} \int [d\phi_{\rm loc}] e^{S(\phi_{\rm loc})} Z_{\text{1-loop}}^{Q_{\rm eq}\mathcal{V}}$$

- Adding the Q_{eq}- exact term does not change the partition function.

- Since the partition function is independent of λ, we can take the limit $\lambda \to \infty$.

- Then, the new saddle point appears at $Q_{\rm eq}\mathcal{V} = 0$ "localisation saddle'

- Around the new localization saddle, the method of saddle point evaluation gives exact answer.

SUSY localization

$$Z = \int D\phi\, e^{-S} = \int D\phi\, e^{-S-\lambda Q_{\rm eq}\mathcal{V}} \underset{\lambda\to\infty}{\longrightarrow} \int [d\phi_{\rm loc}] e^{S(\phi_{\rm loc})} Z_{\text{1-loop}}^{Q_{\rm eq}\mathcal{V}}$$

- The choice of the QV can be arbitrary since it doesn't change the result.

- There is a canonical choice,

$$Q_{eq}\mathcal{V} = \int d^2x\sqrt{g}\, Q_{eq} \sum_{\psi} (Q_{eq}\psi)^\dagger \psi$$

- the localization locus $Q_{eq}\mathcal{V} = 0$ is given by the BPS equation $Q_{eq}\psi = 0$

- The functional integration reduces to the BPS configuration as

$$\int D\phi \to \int [d\phi_{loc}] = \sum_{\mathbf{m}} \int d\sigma_0$$

SUSY localization

$$Z = \int D\phi\, e^{-S} = \int D\phi\, e^{-S-\lambda Q_{\rm eq}\mathcal{V}} \underset{\lambda\to\infty}{\longrightarrow} \int [d\phi_{\rm loc}] e^{S(\phi_{\rm loc})} Z_{\text{1-loop}}^{Q_{\rm eq}\mathcal{V}}$$

- The action on the localization saddle $S(\phi_{loc})$ is only from the FI term and topological term:

$$S_{\rm FI}|_{\rm BPS} = -{\rm i}\xi \int_{\Sigma} {\rm d}^2x\, \sqrt{g_{\Sigma}}\widehat{D} = 4\pi\xi\gamma_{\mathcal{G}}$$

$$\gamma_{\mathcal{G}} \equiv i\sigma_0 L + \frac{\mathfrak{m}(n_2-n_1)}{6n_1n_2(n_1+n_2)}$$

$$S_{\rm top}|_{\rm BPS} = {\rm i}\frac{\theta}{2\pi}\int \sqrt{g_{\Sigma}}\mathcal{F}_{12} = {\rm i}\theta\frac{\mathfrak{m}}{n_1n_2}$$

- Aim is now to evaluate the $Z_{1-loop} = \mathrm{SDet}(\ldots)$.

  → We use index theory. We don't need to know full spectrum..

1-loop via computing index

1-loop determinant via computing index

- For 1-loop determinant, we use index formulation

  [Pestun '07; Hama, Hosomichi '12]

  We organise all the variables into the cohomological variables,

$$(\Phi\,, Q_{\rm eq}\Phi\,, \Psi\,, Q_{\rm eq}\Psi)$$

  i.e. representation of the cohomological chain

$$\begin{tikzcd} \Phi \arrow[r] \arrow[d, "D_{10}"'] & Q_{\rm eq}\Phi \arrow[d] \\ \Psi \arrow[r] & Q_{\rm eq}\Psi \end{tikzcd}$$

  Φ "elementarv" boson

  Ψ "elementarv" fermion

- Field redefinition should not have singular Jacobian. [IJ, Murthy '18]

  For chiral multiplet:

$$\Phi = \{\phi, \tilde{\phi}\}, \quad \Psi = \{\epsilon\psi, \tilde{\epsilon}\tilde{\psi}\}$$

1-loop determinant via computing index

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[Pestun '07; Hama, Hosomichi '12]

We organise all the variables into the cohomological variables,

$$(\Phi\,, Q_{\rm eq}\Phi\,, \Psi\,, Q_{\rm eq}\Psi)$$

i.e. representation of the cohomological chain

$$\begin{array}{ccc} \Phi & \longrightarrow & Q_{\rm eq}\Phi \\ {\scriptstyle D_{10}}\downarrow & & \downarrow \\ \Psi & \longrightarrow & Q_{\rm eq}\Psi \end{array}$$

Φ "elementarv" boson

Ψ "elementarv" fermion

Then the 1-loop determinant reduces to

$$Z^{Q_{\rm eq}\mathcal{V}}_{\text{1-loop}} = \sqrt{\frac{\det K_f}{\det K_b}} = \sqrt{\frac{\det_\Psi H}{\det_\Phi H}}\,. \qquad H \equiv Q_{eq}^2$$

- To see the reduction, let us formally write the QV action as

$$\mathcal{V} \;=\; \int d^4x\,\sqrt{\mathring{g}}\left[(Q_{\rm eq}\Phi\,,\Psi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11}\end{pmatrix}\begin{pmatrix}\Phi \\ Q_{\rm eq}\Psi\end{pmatrix}\right]$$

$$\Rightarrow Q_{\rm eq}\mathcal{V} \;=\; \int d^4x\,\sqrt{\mathring{g}}\left[(\Phi\,,Q_{\rm eq}\Psi)\,K_b\begin{pmatrix}\Phi \\ Q_{\rm eq}\Psi\end{pmatrix} + (Q_{\rm eq}\Phi\,,\Psi)\,K_f\begin{pmatrix}Q_{\rm eq}\Phi \\ \Psi\end{pmatrix}\right],$$

where

$$K_b \;=\; \begin{pmatrix}-H & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11}\end{pmatrix} + \begin{pmatrix} D_{00}^T & D_{10}^T \\ D_{01}^T & D_{11}^T\end{pmatrix}\begin{pmatrix}H & 0 \\ 0 & 1\end{pmatrix},$$

$$K_f \;=\; \begin{pmatrix}1 & 0 \\ 0 & -H\end{pmatrix}\begin{pmatrix} D_{00}^T & D_{10}^T \\ D_{01}^T & D_{11}^T\end{pmatrix} - \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11}\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & H\end{pmatrix}.$$

Then, by the observation

$$\boxed{\begin{pmatrix}1 & 0 \\ 0 & -H\end{pmatrix}K_b \;=\; K_f\begin{pmatrix}H & 0 \\ 0 & 1\end{pmatrix}}$$

$$Z_{\text{1-loop}}^{Q_{\rm eq}\mathcal{V}} \;=\; \sqrt{\frac{\det K_f}{\det K_b}} \;=\; \sqrt{\frac{\det_\Psi H}{\det_\Phi H}}\,.$$

- This determinant can be reproduced by computing the **equivariant index** of D_{10} with respect to U(1) by H.

$$\mathrm{ind}(D_{10})(t) := \mathrm{Tr}_{\mathrm{Ker}D_{10}}\, e^{-iHt} - \mathrm{Tr}_{\mathrm{Coker}D_{10}}\, e^{-iHt}$$

$$= \mathrm{Tr}_{\Phi} e^{-iHt} - \mathrm{Tr}_{\Psi} e^{-iHt}$$

$$\mathrm{ind}(D_{10})(t) \;=\; \sum_n a(n)\, e^{-i\lambda_n t} \;\longrightarrow\; Z_{\text{1-loop}} \;=\; \prod_n \lambda_n^{-a(n)}$$

- We compute the index using two methods:

1. Direct solving the unpaired eigenmodes
   ($\mathrm{Ker}(D_{10})$ and $\mathrm{CoKer}(D_{10})$)
2. Index theorem (Atiyah-Bott fixed point formula)

Evaluation of 1-loop

Method 1: unpaired eigenvalues

- Directly solve the index by identifying the **kernel** and **cockerel** of D_{10} operator and their spectrum with respect to the Q_{eq}^2 respectively on Φ and Ψ space.

- What is the D_{10} ? It is not unique as the choice of the localizing action not unique.

Solve the 1st order differential equation:

$$D_{10}\Phi \sim \begin{pmatrix} 0 & P_+ \\ P_- & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \widetilde{\phi} \end{pmatrix} = 0\,, \qquad D_{10}^\dagger \Psi \sim \begin{pmatrix} 0 & P_- \\ P_+ & 0 \end{pmatrix} \begin{pmatrix} \epsilon\psi \\ \tilde{\epsilon}\widetilde{\psi} \end{pmatrix} = 0\,,$$

where $P_- = i\epsilon\gamma^\mu\epsilon D_\mu \qquad P_+ = i\tilde{\epsilon}\gamma^\mu\tilde{\epsilon}D_\mu$

- Separation of variables + boundary condition $\phi(z+\Delta z, y) = \phi(z, y)$

$$\phi(y,z) = \exp\left(-2\pi \mathrm{i} n \frac{z}{\Delta z}\right) \phi_n(y)\,, \qquad n \in \mathbb{Z}$$

- Eigenvalue n is integer by the periodic condition.
  Each orbifold in general can have twisted boundary conditions by the orbifold action. However the consistent gluing condition with integer gauge transformation only allows the periodic boundary condition.

- By the regularity at the both poles, $|y - y_{1,2}|^{\#} \neq \infty \to \# \geq 0$

$$n \geq \max\left(\frac{\mathfrak{p}_1}{n_1}\,, \frac{\mathfrak{p}_2}{n_2}\right) \qquad \mathfrak{p}_{1,2} = \frac{r}{2} - q_{\mathcal{G}} \mathfrak{m}_{1,2}$$

- 1-loop can be obtained by taking the product over all allowed eigenvalues of Q_{eq}^2

$$Z_{\text{1-loop}}^{Q_{\text{eq}}\mathcal{V}} = \sqrt{\frac{\prod_{p=0}^{\infty} \mathrm{i}\frac{L_0}{L}\left(p + p_{\min} + q_{\mathcal{G}}\frac{L}{L_0}\Lambda^{\mathcal{G}}\right) \prod_{l=0}^{\infty} -\mathrm{i}\frac{L_0}{L}\left(l + l_{\min} + q_{\mathcal{G}}\frac{L}{L_0}\Lambda^{\mathcal{G}}\right)}{\prod_{n=0}^{\infty} -\mathrm{i}\frac{L_0}{L}\left(n + n_{\min} - q_{\mathcal{G}}\frac{L}{L_0}\Lambda^{\mathcal{G}}\right) \prod_{m=0}^{\infty} \mathrm{i}\frac{L_0}{L}\left(m + m_{\min} - q_{\mathcal{G}}\frac{L}{L_0}\Lambda^{\mathcal{G}}\right)}}$$

From the chiral part $(\phi, \epsilon\psi)$ anti-chiral part $(\tilde{\phi}, \tilde{\epsilon}\tilde{\psi})$

- Introduce the ceiling function,

$$n_{\min} = m_{\min} = \max\left(\left\lceil\frac{\mathfrak{p}_1}{n_1}\right\rceil, \left\lceil\frac{\mathfrak{p}_2}{n_2}\right\rceil\right), \; p_{\min} = l_{\min} = \max\left(\left\lceil\frac{1-\mathfrak{p}_1}{n_1}\right\rceil, \left\lceil\frac{1-\mathfrak{p}_2}{n_2}\right\rceil\right)$$

- Zeta function regularization $\prod_{n=0}^{\infty}\left(\frac{n+x}{Y}\right) = \frac{Y^{-\frac{1}{2}+x}}{\Gamma(x)}$

$$Z_{\text{1-loop}}^{Q_{\text{eq}}\mathcal{V}} = \left(\frac{L}{L_0}\right)^{p_{\min} - n_{\min} + 2q_{\mathcal{G}}\frac{L}{L_0}\Lambda^{\mathcal{G}}} \sqrt{\frac{\Gamma\left(n_{\min} - q_{\mathcal{G}}\frac{L}{L_0}\Lambda^{\mathcal{G}}\right)^2}{\Gamma\left(p_{\min} + q_{\mathcal{G}}\frac{L}{L_0}\Lambda^{\mathcal{G}}\right)^2}}$$

the phase from the product of i are canceled from chiral and anti-chiral contributions

- Redefining the variables, we have

$$Z_{\text{1-loop}}^{Q_{eq}\mathcal{V}} = \left(\frac{L}{L_0}\right)^{1-\frac{r}{2}\chi+2q_{\mathcal{G}}\gamma_{\mathcal{G}}-\mathfrak{c}} \frac{\Gamma\left(\frac{\mathfrak{b}+\mathfrak{c}-1}{2}+\frac{r}{4}\chi-q_{\mathcal{G}}\gamma_{\mathcal{G}}\right)}{\Gamma\left(\frac{\mathfrak{b}-\mathfrak{c}+1}{2}-\frac{r}{4}\chi+q_{\mathcal{G}}\gamma_{\mathcal{G}}\right)}$$

- where we have defined

$$\mathfrak{b} = 1 + \left\lfloor -\frac{\mathfrak{p}_2}{n_2} \right\rfloor - \left\lfloor -\frac{\mathfrak{p}_1}{n_1} \right\rfloor, \qquad \mathfrak{c} = \frac{[\![-\mathfrak{p}_2]\!]_{n_2}}{n_2} + \frac{[\![-\mathfrak{p}_1]\!]_{n_1}}{n_1}\,.$$

$$\gamma_{\mathcal{G}} \equiv i\sigma_0 L + \frac{\mathfrak{m}(n_2-n_1)}{6n_1n_2(n_1+n_2)} = \frac{L}{L_0}\Lambda^{\mathcal{G}} + \frac{1}{2}\left(\frac{\mathfrak{m}_1}{n_1}+\frac{\mathfrak{m}_2}{n_2}\right)$$

- The scaling anomaly will renormalize the FI parameter ξ .

Partition function

- Full partition function for abelian theory

$$Z_{\mathbb{X}} = \sum_{\mathfrak{m}\in\mathbb{Z}} e^{-\mathfrak{m}\left(\frac{2(n_2-n_1)\pi\xi}{3n_1n_2(n_1+n_2)}+\frac{\mathrm{i}\theta}{n_1n_2}\right)} \int_{-\infty}^{\infty} \frac{\mathrm{d}(\sigma_0 L)}{2\pi}\, e^{-4\pi\mathrm{i}\xi\sigma_0}\, Z_{\text{1-loop}}^{Q_{eq}\mathcal{V}}$$

$$Z_{\text{1-loop}}^{Q_{eq}\mathcal{V}} = \left(\frac{L}{L_0}\right)^{1-\frac{r}{2}\chi+2q_{\mathcal{G}}\gamma_{\mathcal{G}}-\mathfrak{c}} \frac{\Gamma\left(\frac{\mathfrak{b}+\mathfrak{c}-1}{2}+\frac{r}{4}\chi-q_{\mathcal{G}}\gamma_{\mathcal{G}}\right)}{\Gamma\left(\frac{\mathfrak{b}-\mathfrak{c}+1}{2}-\frac{r}{4}\chi+q_{\mathcal{G}}\gamma_{\mathcal{G}}\right)}$$

FI -parameter gets renormalized by the anomaly $\gamma_{\mathcal{G}} \equiv i\sigma_0 L + \frac{\mathfrak{m}(n_2-n_1)}{6n_1n_2(n_1+n_2)}$

$$\xi_{\text{ren}} = \xi - \frac{q_{\mathcal{G}}}{2\pi}\ln\frac{L}{L_0}$$

- Contour integration over σ_0 → summation of the residue → using Bessel function representation of the first kind J_μ , Poisson resumption for the summation over m, we finally obtain

$$J_\mu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\mu+1)}\left(\frac{x}{2}\right)^{2k+\mu} \overset{\mu\in\mathbb{Z}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}\mu y} e^{\mathrm{i}x\sin y}\, dy \qquad \sum_{\mathfrak{m}} e^{ix\mathfrak{m}} = 2\pi\sum_n \delta(x+2\pi n)$$

$$\boxed{Z_{\Sigma} = \left(\frac{L_0}{L}\right)^{\frac{r}{2}\chi-1} e^{-\frac{r}{2}\left(\frac{\zeta-\mathrm{i}\theta}{n_1}+\frac{\zeta+\mathrm{i}\theta}{n_2}\right)} e^{-2\mathrm{i}e^{-\zeta}\sin\theta} \frac{\left(1-\left(\frac{L_0}{L}\right)e^{-(\zeta-\mathrm{i}\theta)}\right)\left(1-\left(\frac{L_0}{L}\right)e^{-(\zeta+\mathrm{i}\theta)}\right)}{\left(1-\left(\frac{L_0}{L}\right)^{1/n_1}e^{-(\zeta-\mathrm{i}\theta)/n_1}\right)\left(1-\left(\frac{L_0}{L}\right)^{1/n_2}e^{-(\zeta+\mathrm{i}\theta)/n_2}\right)},}$$

where we have set $\zeta \equiv \frac{2\pi\xi_{\mathrm{ren}}}{q_{\mathcal{G}}}$ $\quad q_{\mathcal{G}} = 1 \quad r \in 2\mathbb{Z}$

- As a consistency check, we note that setting $n_1 = n_2 = 1$ reduces the S² partition function.

$$Z_{S^2} = Z_{\Sigma}\big|_{n_1=n_2=1} = \left(\frac{L}{L_0}\right)^{1-\frac{r}{2}\chi} e^{-\zeta r} e^{-2\mathrm{i}e^{-\zeta}\sin\theta}$$

- In the case when action on BPS locus vanishes by setting

$$Z_{\Sigma}|_{\xi_{\mathrm{ren}}=\theta=0} = \left(\frac{L}{L_0}\right)^{1-\frac{r}{2}\chi} \frac{\left(1-\frac{L_0}{L}\right)^2}{\left(1-\left(\frac{L_0}{L}\right)^{1/n_1}\right)\left(1-\left(\frac{L_0}{L}\right)^{1/n_2}\right)}$$

This seems to count some vacuum degeneracies on the spindle.

$$\lim_{L\to L_0} Z_{\Sigma}|_{\xi_{\mathrm{ren}}=\theta=0} = n_1 n_2$$

$$\boxed{Z_{\Sigma} = \left(\frac{L_0}{L}\right)^{\frac{r}{2}\chi-1} e^{-\frac{r}{2}\left(\frac{\zeta-\mathrm{i}\theta}{n_1}+\frac{\zeta+\mathrm{i}\theta}{n_2}\right)} e^{-2\mathrm{i}e^{-\zeta}\sin\theta} \frac{(1-\left(\frac{L_0}{L}\right)e^{-(\zeta-\mathrm{i}\theta)})(1-\left(\frac{L_0}{L}\right)e^{-(\zeta+\mathrm{i}\theta)})}{(1-\left(\frac{L_0}{L}\right)^{1/n_1}e^{-(\zeta-\mathrm{i}\theta)/n_1})(1-\left(\frac{L_0}{L}\right)^{1/n_2}e^{-(\zeta+\mathrm{i}\theta)/n_2})},}$$

- The full partition function nicely separates into the Product of two contributions from north and south poles with anti- and holomorphic parameters.

- Although we have performed only the coulomb branch localization, this factorization is suggestive of the mechanism of vortex and anti-vortex contributions in Higgs branch localization.

- It would also be interesting to see if one can reproduce the result by gluing two partition function on the singular discs along with their boundary effects.

Summary

- We have computed exact partition function of $\mathcal{N}$ = (2, 2) theories on **spindle** $\Sigma = \mathbb{WCP}^1_{[n_1,n_2]}$ in **untwisted** supersymmetry.

- We computed the 1-loop determinant using two methods: **unpaired eigenvalues** and **fixed point formula** and obtained full partition function for abelian theory.

- The result is confirmed by S^2 limit, and matching with the dimensional reduction of the result of [Inglese-Martell-Pitteli '23] for $\mathcal{N}$ =2 on $\Sigma \times S^1$

Future directions

- Spindle with twisted supersymmetry (from the solution of 5D STU supergravity)

- Non-abelian theories with generic gauge group.

- Relation to the world sheet string theory?
  Z_{S^2} provides the instant corrections, and computes the exact Kähler potential on the quantum Kähler moduli space of Calabi-Yau manifolds [Gomis-Lee '12] [Jockers-Jumar-Lapan-Morrison-Romo '12].
  What about Z_Σ ?

$$Z(\tau_a, \bar{\tau}_a) = e^{-\mathcal{K}(\tau_a, \bar{\tau}_a)}$$

- Mirror symmetry via equivalence with Landau-Ginzburg theory with some twisted superpotential? [Hori-Vafa '00]

- Dualities? Seiberg-like duality in 2d ?

Thank you for your attention.