Near-extremal dynamics away from the horizon

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25 August 2025

2nd APCTP-INPP Demokritos meeting, APCTP, Korea

Based on arXiv:2507.01126 with Alejandra Castro and Robinson Mancilla

Motivation & Overview

The enhancement of symmetries arising in the near-horizon region of extremal black holes has proved an invaluable tool for decoding both classical and quantum properties of gravity.

The leading temperature effects that break these enhanced symmetries are controlled by Jackiw-Teitelboim (JT) gravity, both at the classical and quantum levels, resulting in the resolution of long standing puzzles about extremal black holes.

The precise relation between extremal or near-extremal black holes in $d+1$ dimensions and JT gravity follows from a dimensional reduction to two dimensions, keeping the relevant near-horizon modes.

The dimensional reduction of the $d+1$ dimensional theory is available only in highly symmetric cases. Typically, a dimensional reduction is possible only after focusing on the near-horizon region, which obscures how the JT sector interacts with the full higher dimensional theory.

An alternative way to identify the JT sector within higher dimensional gravity is the evaluation of the Euclidean path integral around the finite temperature geometry, tracking the quantum corrections in the zero temperature limit. This was done initially for the near-horizon region in [Iliesiu, Murthy, Turiaci '22], and was recently extended to the entire geometry in [Kapec, Law, Toldo '24; Kolanowski, Marolf, Rakic, Rangamani, Turiaci '24; Arnaudo, Bonelli, Tanzini '24].

In [Castro, Mancilla, IP '25] we propose yet another approach to the identification of the JT sector without zooming in the near horizon region, applicable directly to Lorentzian AdS$_{d+1}$ black holes near extremality.

The construction relies on the identification of a suitable reduced phase space of AdS_{d+1} gravity that generalizes the black hole solution and is valid in the entire geometry. This forces us to focus on AdS_3 gravity and the algebraically special sector of AdS_4 gravity.

Utilizing the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ dictionary, the reduced phase space is parameterized in terms of local sources and operators in the dual CFT_d. It is off-shell in the sense that the CFT_d Ward identities are not imposed.

We identify the JT sector within this phase space by introducing a suitable *off-shell* decoupling limit.

Importantly, this decoupling limit can be imposed to any CFT_d, even non-holographic CFTs!

Outline

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JT gravity

JT gravity describes the dynamics of a two-dimensional metric g_{ab} and a dilaton ϕ through the action

$$S_{\mathrm{JT}} = \frac{1}{2\kappa_2^2}\int_{\mathcal{M}_2} \mathrm{d}^2x\sqrt{-g}\,\phi\left(\mathcal{R}+\frac{2}{\ell_2^2}\right) + \frac{1}{\kappa_2^2}\int_{\partial\mathcal{M}_2}\mathrm{d}x\sqrt{-h}\,\phi\left(K-\frac{1}{\ell_2}\right)$$

The equations of motion take the form

$$\mathcal{R}+\frac{2}{\ell_2^2}=0\,, \qquad (\nabla_a\nabla_b - g_{ab}\Box)\phi + \frac{g_{ab}}{\ell_2^2}\phi = 0$$

and imply that there is no propagating degree of freedom. In particular, the first equation fixes the two-dimensional metric to be locally AdS_2.

Bondi gauge phase space

The JT equations of motion can be solved exactly in a number of gauges. The phase space parameterization that facilitates the simplest decoupling limit is the Bondi gauge, in which the 2D metric takes the form

$$ds^2 = -f(u,r)\mathrm{d}u^2 - 2\alpha(u)\mathrm{d}u\mathrm{d}r$$

Solving the radial equation of motion determines that the function $f(u,r)$ and the dilaton $\phi(u,r)$ are given respectively by

$$f(u,r) = \alpha(u)^2 \frac{r^2}{\ell_2^2} + f_1(u)\frac{r}{\ell_2} + f_0(u)$$

and

$$\phi(u,r) = \nu(u)\frac{r}{\ell_2} + \phi_0(u)$$

where $\alpha(u)$, $\nu(u)$, $f_1(u)$, $f_0(u)$ and $\phi_0(u)$ are arbitrary functions.

Local symmetries

There are three distinct local symmetries that preserve the Bondi gauge:

- Time reparameterizations: $u \to \bar{u}(u)$
- Local rescalings of the radial coordinate: $r \to e^{\sigma(u)} r$
- Local shifts of the radial coordinate: $r \to r + \zeta(u)$

The functions $\alpha(u)$, $\nu(u)$, $f_1(u)$, $f_0(u)$ and $\phi_0(u)$ transform nontrivially under these transformations.

In particular, under the local shift symmetry $r \to r + \zeta(u)$, $\alpha(u)$ and $\nu(u)$ are invariant, while $\phi_0(u)$ transforms additively: $\phi_0 \to \phi_0 + \nu\zeta/\ell_2$.

This leads to shift-invariant versions of the functions $f_1(u)$, $f_0(u)$:

$$\widetilde{f}_1 = f_1 - 2\alpha^2 \frac{\phi_0}{\nu}\,, \qquad \widetilde{f}_0 = f_0 + \alpha^2 \frac{\phi_0^2}{\nu^2} - \frac{\phi_0}{\nu} f_1 - 2\ell_2 \alpha \partial_u \Big(\frac{\phi_0}{\nu}\Big)$$

Off-shell reduced phase space

The reduced phase space obtained from the quotient by the local shift symmetry is parameterized by the four invariant functions $\alpha(u)$, $\nu(u)$, $\widetilde{f}_1(u)$ and $\widetilde{f}_0(u)$ that form a symplectic space.

These can be identified with sources and local operators in the dual quantum mechanics. This identification is guided by the (remaining) local symmetries, the symplectic form, and the Ward identities, but contains some inherent ambiguity.

The functions $\alpha(u)$, $\nu(u)$ are unambiguously identified with the *sources* of the stress tensor, $\mathcal{O}_u$, and the scalar operator dual to the dilaton, $\mathcal{O}_\phi$. It follows that the remaining functions $\widetilde{f}_1(u)$ and $\widetilde{f}_0(u)$ are related with the *one-point functions* of these operators.

One way to determine the relation between the variables $\widetilde{f}_1$, $\widetilde{f}_0$ and the dual operators $\mathcal{O}_u$, $\mathcal{O}_\phi$ is to use the remaining (transverse) equations of motion, which take the form

$$\widetilde{f}_1 = 2\ell_2\alpha\frac{\partial_u\nu}{\nu}\,, \qquad \partial_u\left(\frac{\nu^2}{\alpha^2}\widetilde{f}_0\right) = 0$$

Parameterizing $\widetilde{f}_1$, $\widetilde{f}_0$ in terms of the operators $\mathcal{O}_u$, $\mathcal{O}_\phi$ as

$$\begin{aligned}\widetilde{f}_1(u) &= 2\kappa_2^2\gamma^2\frac{\alpha^2}{\nu}\int \mathrm{d}u\,\alpha(\mathcal{O}_u+\mathcal{O}_\phi) + 2(1-\omega^2\gamma^2)\ell_2\alpha\frac{\partial_u\nu}{\nu}\,,\\ \widetilde{f}_0(u) &= 2\kappa_2^2\frac{\alpha^2}{\nu}\ell_2\mathcal{O}_u - \omega^2\ell_2^2\frac{(\partial_u\nu)^2}{\nu^2}\end{aligned}$$

where γ and ω are constants, these equations become the *Ward identities* of the dual conformal quantum mechanics, namely [Cvetič, IP '16]

$$\boxed{\mathcal{O}_u + \mathcal{O}_\phi = \frac{\omega^2\ell_2}{\kappa_2^2\alpha}\partial_u\left(\frac{\partial_u\nu}{\alpha}\right)\,, \qquad \partial_u\mathcal{O}_u - \mathcal{O}_\phi\frac{\partial_u\nu}{\nu} = 0}$$

The local term on the RHS of the trace Ward identity corresponds to a trivial (type C) conformal anomaly, i.e. its coefficient can be modified at will by a local counterterm. The constant ω reflects this ambiguity.

Nevertheless, the coefficient of the 1D conformal anomaly multiplies the Schwarzian effective action of JT gravity, and hence the $T \log T$ correction to the free energy [Charles, Larsen '19; Iliesiu, Turiaci '20].

As an effective low energy theory of gravity, JT gravity does not determine the value of ω. Fixing ω requires input from its "UV completion", i.e. the higher dimensional theory. In particular, JT gravity without any further local boundary counterterm corresponds to $\omega = 1$.

A reduced phase space parameterized by the variables α, ν, $\mathcal{O}_u$, $\mathcal{O}_\phi$ that satisfy the 1D Ward identities can be used to identify a JT sector within AdS$_{d+1}$ gravity, or CFT$_d$. This is the goal of the off-shell decoupling limit that we identify.

Outline

CFT$_2$ on a curved background

Let us consider a 2D CFT on a background of the form

$$ds^2_{\mathrm{2D}} = -\alpha^2 \mathrm{d}u^2 + \nu^2(\mathrm{d}\theta + \mu \mathrm{d}u)^2$$

where $\alpha(u)$, $\nu(u)$, $\mu(u)$ are arbitrary functions corresponding to specific components of the 2D metric g^{2D}_{ab}.

Accordingly, the CFT stress tensor can be decomposed as

$$\begin{aligned}
T_{\theta\theta} &= \nu\, \mathcal{O}^{\mathrm{2D}}_{\theta} \\
T_{u\theta} &= \alpha\, \mathcal{O}^{\mathrm{2D}}_{j} + \mu\, T_{\theta\theta} \\
T_{uu} &= \mu^2 \nu\, \mathcal{O}^{\mathrm{2D}}_{\theta} + 2\mu\alpha\, \mathcal{O}^{\mathrm{2D}}_{j} - \alpha^2 \nu^{-1} \mathcal{O}^{\mathrm{2D}}_{u}
\end{aligned}$$

so that the variables $\mathcal{O}^{\mathrm{2D}}_{u}$, $\mathcal{O}^{\mathrm{2D}}_{\theta}$, $\mathcal{O}^{\mathrm{2D}}_{j}$ are sourced respectively by α, ν, μ:

$$-\frac{1}{2}\int d^2x \sqrt{-g_{\mathrm{2D}}}\, T_{ab}\, \delta g^{ab}_{\mathrm{2D}} = \int d^2x \big(-\mathcal{O}^{\mathrm{2D}}_{j} \nu \delta\mu + \mathcal{O}^{\mathrm{2D}}_{\theta} \alpha \nu^{-1} \delta\nu + \mathcal{O}^{\mathrm{2D}}_{u} \delta\alpha \big)$$

CFT$_2$ Ward identities

The 2D stress tensor satisfies the Ward identities

$$\nabla_a T^{ab} = 0 \ , \quad T^a_{\ a} = \frac{c}{24\pi} R^{(2)}$$

where c the central charge and the Ricci scalar is given by

$$R^{(2)} = \frac{2}{\alpha\nu}\partial_u\Big(\frac{\partial_u \nu}{\alpha}\Big)$$

Decomposing the 2D metric and stress tensor as above, these Ward identities take the form

$$(\partial_u - \mu\partial_\theta)\mathcal{O}_j^{\rm 2D} + \mathcal{O}_j^{\rm 2D}\frac{\partial_u\nu}{\nu} + \frac{\alpha}{\nu}\partial_\theta\mathcal{O}_u^{\rm 2D} = 0$$
$$(\partial_u - \mu\partial_\theta)\mathcal{O}_u^{\rm 2D} - \mathcal{O}_\theta^{\rm 2D}\frac{\partial_u\nu}{\nu} + \frac{\alpha}{\nu}\partial_\theta\mathcal{O}_j^{\rm 2D} = 0$$

$$\mathcal{O}_\theta^{\rm 2D} + \mathcal{O}_u^{\rm 2D} = \frac{c}{24\pi}\frac{2}{\alpha}\partial_u\Big(\frac{\partial_u\nu}{\alpha}\Big)$$

AdS$_3$ gravity reduced phase space

Standard holography allows us to embed the CFT$_2$ phase space and Ward identities within AdS$_3$ gravity in Fefferman-Graham gauge.

In order to identify a phase space that can accommodate the decoupling limit in a clean way, it is necessary to use instead the Bondi gauge metric

$$ds^2_{3\mathrm{D}} = -F(r,u,\theta)\mathrm{d}u^2 - 2\alpha(u)\mathrm{d}u\mathrm{d}r + \Phi^2(u,r)\big(\mathrm{d}\theta + A(r,u,\theta)\mathrm{d}u\big)^2$$

Solving the radial equations of motion determines that functions parameterizing the metric take the form

$$F(r,u,\theta) = \frac{\alpha^2}{\nu^2}\,\Phi^2 + F_1(u,\theta)\,\Phi + F_0(u,\theta) + \frac{F_{-2}(u,\theta)^2}{\Phi^2}$$
$$A(r,u,\theta) = \mu(u) + \frac{F_{-2}(u,\theta)}{\Phi^2}\,, \quad \Phi(u,r) = \nu(u)\frac{r}{\ell_3} + \Phi_0$$

where $F_1(u,\theta)$, $F_0(u,\theta)$ and $F_{-2}(u,\theta)$ are arbitrary functions of the transverse coordinates and Φ_0 a constant.

The remaining equations of motion are differential constraints on the functions $F_1(u,\theta)$, $F_0(u,\theta)$ and $F_{-2}(u,\theta)$. These constraints coincide with the 2D CFT Ward identities provided we parameterize these functions in terms of the variables $\mathcal{O}_\theta^{\rm 2D}(u,\theta)$, $\mathcal{O}_u^{\rm 2D}(u,\theta)$, $\mathcal{O}_j^{\rm 2D}(u,\theta)$ as

$$\begin{aligned} F_1(u,\theta) &= 2\kappa_3^2 \frac{\alpha^2}{\nu^2} \int \mathrm{d}u\, \alpha \left(\mathcal{O}_\theta^{\rm 2D} + \mathcal{O}_u^{\rm 2D}\right) \\ F_0(u,\theta) &= 2\ell_3\kappa_3^2 \frac{\alpha^2}{\nu} \mathcal{O}_u^{\rm 2D} - \ell_3^2 \left(\frac{\partial_u \nu}{\nu}\right)^2 - \Phi_0 F_1(u,\theta) \\ F_{-2}(u,\theta) &= \ell_3 \kappa_3^2\, \alpha\, \mathcal{O}_j^{\rm 2D} \end{aligned}$$

In particular, the central charge c in the trace Ward identity is given by the Brown-Henneaux central charge

$$c = \frac{3\ell_3}{2G_3}$$

BTZ black hole

In order to motivate the decoupling limit it is instructive to see how the BTZ black hole is embedded in the AdS_3 reduced phase space.

The BTZ black hole metric can be written in the form

$$ds^2_{\mathrm{BTZ}} = -\left(\frac{r^2}{\ell_3^2} - m + \frac{j^2}{4r^2}\right)\mathsf{d}u^2 - 2\mathsf{d}u\mathsf{d}r + r^2\left(\mathsf{d}\theta + \frac{j}{2r^2}\mathsf{d}u\right)^2$$

where m and j are constants related to the outer and inner horizons, located respectively at $r_\pm$, via $m = (r_+^2 + r_-^2)/\ell_3^2$ and $j = 2r_+r_-/\ell_3$

In particular, they correspond respectively to the energy and angular momentum of the black hole: $M_{\mathrm{BTZ}} = \frac{m}{8G_3}$ and $J_{\mathrm{BTZ}} = \frac{j}{8G_3}$

The temperature, angular momentum and entropy are also expressed in terms of $r_\pm$ as

$$T = \frac{r_+^2 - r_-^2}{2\pi\ell_3^2 r_+}\ , \quad \Omega = \frac{r_-}{\ell_3 r_+}\ , \quad S = \frac{\pi r_+}{2G_3}$$

Comparing the BTZ metric with the general Bondi gauge metric we determine that values of the sources corresponding to the BTZ are

$$\alpha = 1\,, \quad \nu = \ell_3\,, \quad \mu = 0\,, \quad \Phi_0 = 0$$

while the values of the corresponding one-point functions are

$$\mathcal{O}_u^{\rm 2D} = -\mathcal{O}_\theta^{\rm 2D} = -\frac{1}{2\pi} M_{\rm BTZ}\ , \qquad \mathcal{O}_j^{\rm 2D} = \frac{1}{2\pi\ell_3} J_{\rm BTZ}$$

Outline

Near-extremal limit of BTZ

At extremality, the temperature of the BTZ black hole vanishes, while the energy is determined by the angular momentum through $\ell_3 M_{\mathrm{BTZ}} = J_{\mathrm{BTZ}}$.

These conditions are equivalent to the coincidence of the inner and outer horizons, namely $r_+ = r_- \equiv r_0$.

The low-temperature, near-extremal, regime can be studied by expanding $r_\pm = r_0 \pm \epsilon + \mathcal{O}(\epsilon^2)$ such that J_{BTZ} is kept fixed.

To leading order in ϵ the temperature is then given by

$$T = \frac{2\epsilon}{\pi \ell_3^2}$$

while the deviations of the energy and the entropy away from extremality are given respectively by

$$\Delta M_{\mathrm{BTZ}} = \pi^2 \frac{c}{12} \ell_3 T^2 \ , \qquad \Delta S = \pi^2 \frac{c}{6} \ell_3 T$$

In terms of the operators $\mathcal{O}_\theta^{\rm 2D}$, $\mathcal{O}_u^{\rm 2D}$, $\mathcal{O}_j^{\rm 2D}$ the extremality condition can be expressed as

$$\mathcal{O}_\theta^{\rm 2D,*} + \mathcal{O}_u^{\rm 2D,*} = 0 \ , \quad \mathcal{O}_u^{\rm 2D,*} + \mathcal{O}_j^{\rm 2D,*} = 0$$

Moreover, the angular momentum is related with the radius r_0 of the coincident horizons as

$$\mathcal{O}_j^{\rm 2D,*} = \frac{r_0^2}{\kappa_3^2 \ell_3^2}$$

These relations provide some guidance for identifying the general off-shell decoupling limit.

The off-shell decoupling limit

Recall that as radial coordinate in the Bondi gauge parameterization of AdS_3 gravity solutions we used the function

$$\Phi(u,r) = \nu \frac{r}{\ell_3} + \Phi_0$$

For standard BTZ black hole we have $\nu = \ell_3$ and $\Phi_0 = 0$ so that $\Phi = r$. The extremal limit then zooms in the region $r = r_0$, i.e. $\Phi = r_0$.

However, our goal is to formulate the decoupling limit without reference to the radial coordinate. This motivates the limit $\nu \to 0$, keeping $\Phi_0 \neq 0$.

In order to keep track of the behavior of all variables we implement this limit by defining

$$\nu(u) \equiv \lambda \tilde{\nu}(u) \;, \qquad \lambda \to 0$$

An important ingredient of the extremal limit is keeping the angular momentum fixed. In the presence of a non-trivial $\nu(u)$, the angular momentum is proportional to $\nu\mathcal{O}_j^{\rm 2D}$, which implies that

$$\mathcal{O}_j^{\rm 2D} = \frac{\mathcal{O}_j^{(-1)}}{\lambda}$$

with $\mathcal{O}_j^{(-1)}$ fixed in the decoupling limit.

Demanding that the Bondi gauge metric remains well defined as $\lambda \to 0$ requires that $\mathcal{O}_i^{\rm 2D} = \{\mathcal{O}_u^{\rm 2D}, \mathcal{O}_\theta^{\rm 2D}\}$ admit expansions of the form

$$\mathcal{O}_i^{\rm 2D} = \frac{\mathcal{O}_i^{(-1)}}{\lambda} + \mathcal{O}_i^{(0)} + \mathcal{O}_i^{(1)}\lambda + \cdots$$

where the expansion coefficients satisfy

$$\mathcal{O}_j^{(-1)} = \mathcal{O}_\theta^{(-1)} = -\mathcal{O}_u^{(-1)} = \frac{c}{12\pi\ell_3^2}\frac{\Phi_0^2}{\tilde{\nu}}\,, \qquad \mathcal{O}_\theta^{(0)} = -\mathcal{O}_u^{(0)} = 0$$

Moreover, the function $\mu(u)$ must diverge $\lambda \to 0$ according to

$$\mu = \tilde{\mu} - \frac{\alpha}{\lambda \tilde{\nu}}$$

with $\tilde{\mu}(u)$ and $\alpha(u)$ fixed.

With these conditions, the Bondi gauge metric admits a $\lambda \to 0$ limit:

$$ds^2_{\mathrm{3D}} \underset{\lambda \to 0}{=} -F_{\mathrm{IR}}(r,u,\theta)\mathsf{d}u^2 - 2\alpha(u)\mathsf{d}u\mathsf{d}r + \Phi_0^2\left(\mathsf{d}\theta - \frac{2\alpha(u)}{\ell_3 \kappa_3^2 \Phi_0} r\mathsf{d}u\right)^2 + O(\lambda)$$

where

$$\begin{aligned} F_{\mathrm{IR}}(r,u,\theta) \equiv & \lim_{\lambda \to 0} F(r,u,\theta) \\ = & \frac{\alpha^2 r^2}{(\ell_3/2)^2} + 2r\frac{\alpha^2}{\tilde{\nu}}\frac{\kappa_3^2}{\ell_3}\int \mathsf{d}u\, \alpha(\mathcal{O}_\theta^{(1)} + \mathcal{O}_u^{(1)}) \\ & + 2\frac{\alpha^2}{\tilde{\nu}}\ell_3\kappa_3^2\mathcal{O}_u^{(1)} - \ell_3^2\left(\frac{\partial_u \tilde{\nu}}{\tilde{\nu}}\right)^2 \end{aligned}$$

Decoupling limit of CFT_2 Ward identities

The λ-expansions of the phase space variables can be inserted also in the CFT_2 Ward identities, leading to the conditions

$$\partial_\theta \mathcal{O}_i^{(-1)} = \partial_u(\tilde{\nu}\mathcal{O}_i^{(-1)}) = 0\ , \quad i \in \{u, \theta, j\}$$

$$\partial_\theta \mathcal{O}_i^{(1)} = 0\ , \quad i \in \{u, \theta\}$$

The constraints in the first line are compatible with the conditions imposed by the regularity of the Bondi gauge metric. The second line implies that the variables $\mathcal{O}_i^{(1)}$, $i \in \{u, \theta\}$, are only functions of u.

Moreover, to order λ, the CFT_2 Ward identities imply that

$$\mathcal{O}_\theta^{(1)} + \mathcal{O}_u^{(1)} = \frac{c}{24\pi}\frac{2}{\alpha}\partial_u\left(\frac{\partial_u\tilde{\nu}}{\alpha}\right)\ , \qquad \partial_u\mathcal{O}_u^{(1)} - \mathcal{O}_\theta^{(1)}\frac{\partial_u\tilde{\nu}}{\tilde{\nu}} = 0$$

JT sector within AdS_3 phase space

Having established the existence of a well defined off-shell decoupling limit we can now proceed to identify the JT sector within the AdS_3 and CFT_2 phase spaces.

A direct comparison of the Bondi gauge line elements and of the Ward identities leads to the identifications

$$\ell_2 = \frac{\ell_3}{2} \ , \quad \alpha_{\mathrm{2D}} = \alpha_{\mathrm{3D}}$$

Moreover, the JT dilaton corresponds to

$$\Phi - \Phi_0 = \lambda\tilde{\nu}\frac{r}{\ell_3} \equiv L\phi \quad \leftrightarrow \quad \phi = \nu_{\mathrm{2D}}\frac{r}{\ell_2} \, , \quad \nu_{\mathrm{2D}} = \frac{\lambda}{2L}\tilde{\nu}$$

where the scale L is defined through $2\pi L\kappa_2^2 = \kappa_3^2$.

Finally, the JT operators are identified as

$$L\kappa_2^2\mathcal{O}_u = \lambda\kappa_3^2\mathcal{O}_u^{(1)}\ , \quad L\kappa_2^2\mathcal{O}_\phi = \lambda\kappa_3^2\mathcal{O}_\theta^{(1)}$$

while, in this case, the parameters γ and ω of the JT gravity are fixed by the higher dimensional theory to the values

$$\omega^2 = \gamma^{-2} = 4$$

Outline

Summary & Outlook

- We have identified an off-shell decoupling limit of pure AdS_3 gravity, Topologically Massive Gravity, and the algebraically special sector of AdS_4 Einstein-Maxwell theory that leads to the identification of a JT sector without zooming into the near horizon region.
- This off-shell decoupling limit can be applied directly to 2D and 3D CFTs that do not necessarily admit a holographic dual.
- This limit can be easily extended to rotating algebraically special AdS_4 solutions that include the Kerr-AdS_4 black hole, but its generalization beyond the algebraically special sector seems hard.
- It would be interesting to extend these results to incorporate quantum corrections.

Thank you!