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Thermodynamics of the various Black Holes

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Contents

I. Motivation

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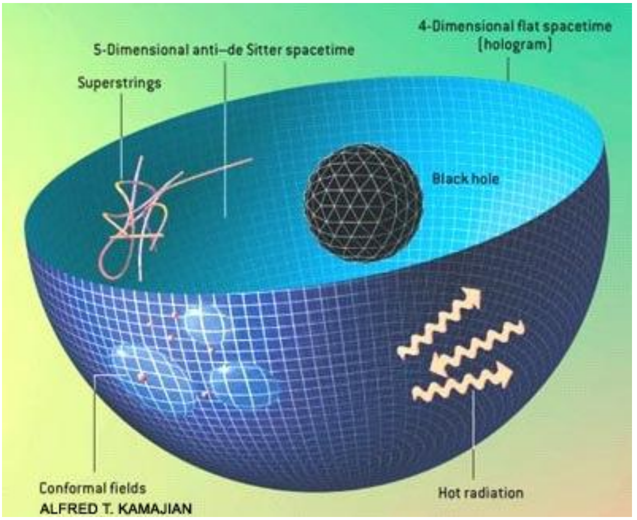
Gravity is the theory with least known quantum properties.

Modified Gravity beyond Einstein - Is it needed?

I-1. Theoretical Aspect: Gravity beyond Einstein?



- GR is an **effective theory :**

  valid below $M_{UV} \sim M_{Pl} \sim 10^{19} GeV$

  Ex) String theory $\xrightarrow[E \lesssim M_{Pl}]{} \sum_{n\geq 0}(\sim R)^n$

  Einstein Grav ($\sim R$): linear in R, the simplest, but **incomplete**

$$Z_{str}[\phi_0(x)] = \int_{\phi_0} \mathcal{D}\phi \exp(-S[\phi, g_{\mu\nu}])$$

$$= Z[\phi_0(x)] = \left\langle \exp \int_{bdru} d^d x\, \phi_0\, \mathcal{O} \right\rangle$$

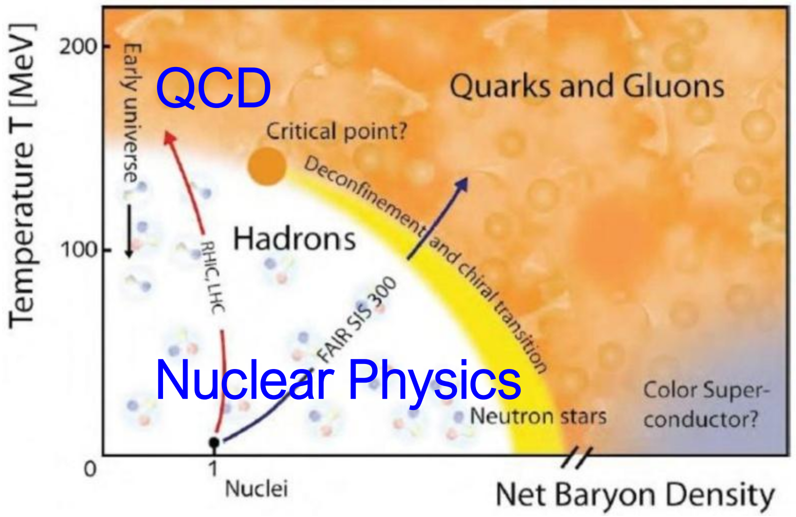
- **Holography**:

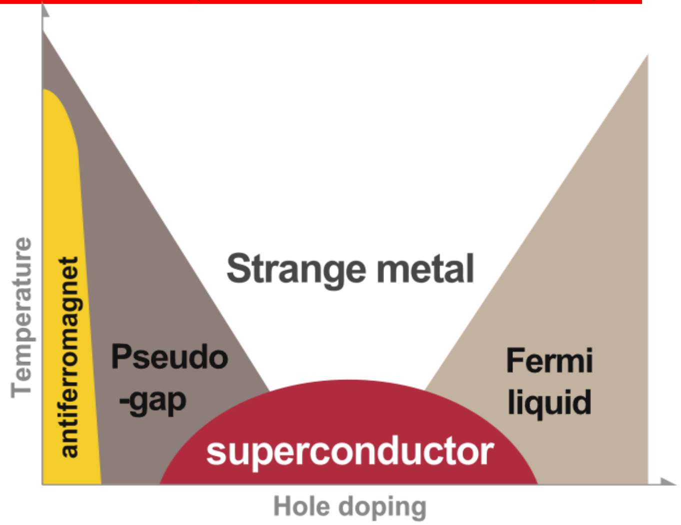
  Needs Anti deSitter (AdS) space $\Lambda < 0$ as the dual (beyond Einstein) geometry

  5dim. classical gravity ⇔ 4dim. strongly interacting theory

  Ex) AdS/QCD & AdS/CMT: Phases, etc?

  Black Holes are thermal system.





I-2. Observational Aspect:

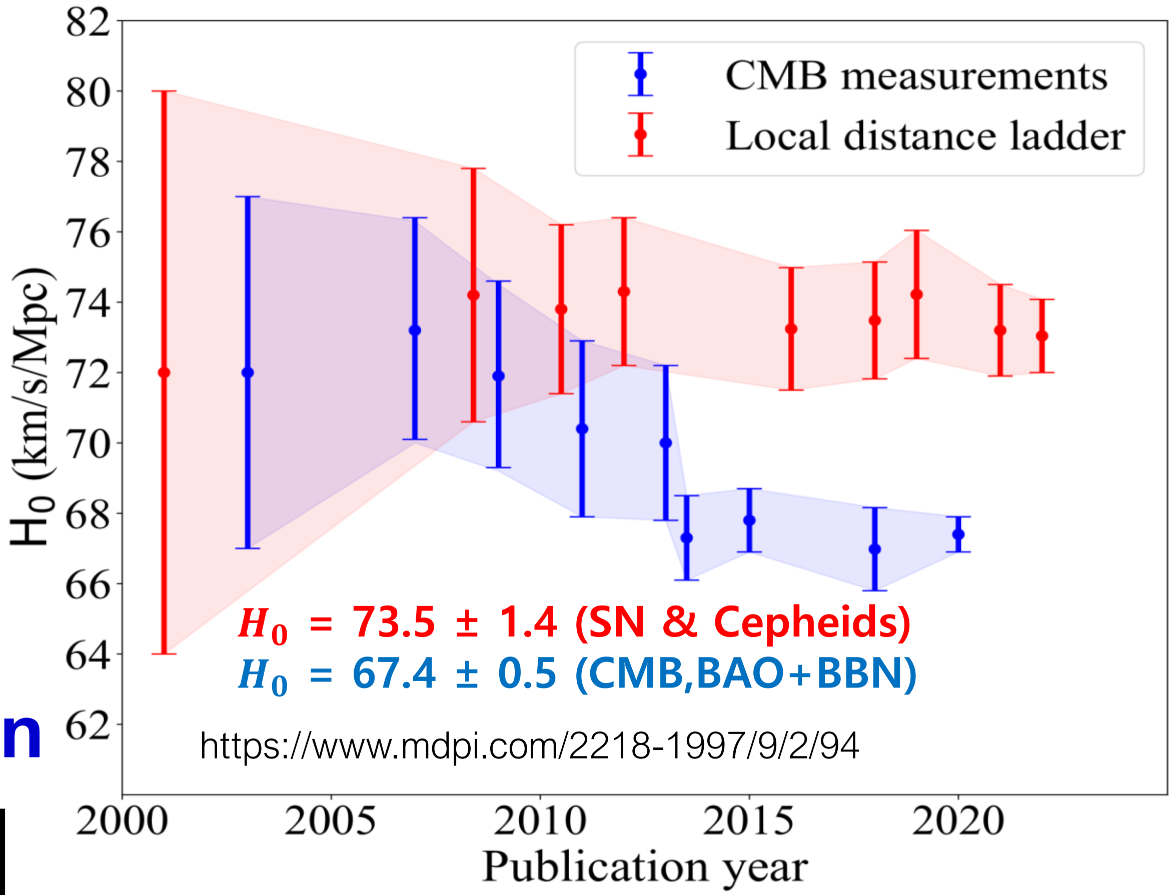
Are alternatives to Λ CDM, the Standard Model for Cosmology, needed?

- **Some challenging observations**

H_0 tension (~5.8σ)
etc.

J. Kochappan, L. Yin, B-HL, T. Ghosh e-Print: 2408.09521

Colgaine, B-HL, W. Lee, Sheikh-Jabbari, Thakur, JCAP 04 (2022)



I-3. Extended Gravity beyond Einstein

Q: Any guide for higher curvature corrections, etc?
A: Yes, **Black Hole** and **Cosmology** !

Black Holes : the simplest objects only with gravity, cf) H-atom: a guide for the discovery for the Quantum Physics, mainly due to its simplicity

Cosmology : Any extra d.o.f. (such as higher curvature, scalar, etc.) will give deviation from ΛCDM

⇒ We investigate

1) the Black Hole properties &

2) the implication to the cosmology

in the Gravity theory beyond Einstein

II. Gravity with Gauss-Bonnet (G-B) term

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II-1. Lovelock theory (dim. $D = 2t + 1$ or $2t$)

Lagrangian with only **1) metric 2)2nd order e.o.m** (for no ghosts and instabilities) will be in the following form

$$\mathcal{L}_D = \sqrt{-g} \sum_{n=0}^{t} \alpha_n L^n$$

L^n : Lovelock term, topological in 2n D

Ex) $L^1 = R$ Einstein-Hilbert term topol in 2 D

$L^2 = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$

$= R_{GB}^2$ Gauss-Bonnet term. topol in 4 D

$$L^m = \frac{1}{2^m} \delta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_m\nu_m}_{a_1b_1a_2b_2\cdots a_mb_m} R_{a_1b_1}{}^{\mu_1\nu_1} R_{a_2b_2}{}^{\mu_2\nu_2} \cdots R_{a_mb_m}{}^{\mu_m\nu_m}$$

Euler characteristic of dim 2m topol in $2m$ D

$$\delta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_m\nu_m}_{a_1b_1a_2b_2\cdots a_mb_m} = (2m)!\, \delta^{\mu_1}_{[a_1} \delta^{\nu_1}_{b_1} \cdots \delta^{\mu_m}_{a_m} \delta^{\nu_m}_{b_m}$$

Ex) D-dim

$\mathcal{L}_2 = L^1 = \sqrt{-g}\ R$ topological

$\mathcal{L}_3 = L^1 = \sqrt{-g}\ R$

$\mathcal{L}_4 = L^1 + L^2 = \sqrt{-g}(R + R_{GB}^2) \approx \sqrt{-g}\ R$

$\mathcal{L}_5 = L^1 + L^2 = \sqrt{-g}(R + R_{GB}^2)$

$\mathcal{L}_6 = L^1 + L^2 + L^3 = \sqrt{-g}(R + R_{GB}^2 + R_{E.C}^3) \approx \sqrt{-g}(R + R_{GB}^2)$

$\mathcal{L}_7 = L^1 + L^2 + L^3 = \sqrt{-g}(R + R_{GB}^2 + R_{E.C}^3)$

Lovelock's theorem (in dim =4 (& 3)

The Einstein eqns (w/ c.c.) are the only possible 2nd-order eqns derived in 4 dim. solely from the metric.

Metric Modification of GR - only by G-B (E.C.)
Further modification needs to add a **new degree of freedom (such as scalars).**

II -2. Einstein Gauss-Bonnet Gravity

higher derivative theories may have ghosts and Ostrogradsky instability :

1) The general theory w/ quadratic curvature terms in $d > 4$

$$S_{quad} = \int d^d x \sqrt{-g} \left[\frac{1}{2\kappa} \left(R - 2\Lambda + aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\rho\sigma}R^{\mu\nu\varrho\sigma} \right) + \mathcal{L}_m^{matt} \right]$$

2nd order e.o.m. only if $b = -4a$ & $c = a$

$R_{GB}^2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ Gauss-Bonnet term

2) the Einstein-Gauss-Bonnet (EGB)- Λ Gravity (GB-AdS) in $d > 4$

$$S_{EGB-\Lambda} = \int d^d x \sqrt{-g} \left[\frac{1}{2\kappa} \left(R - 2\Lambda + \alpha R_{GB}^2 \right) + \mathcal{L}_m^{matt} \right]$$

Note : $\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}$

$\kappa = 8\pi G$, $g = \det g_{\mu\nu}$

$[\alpha]$= [length]^2

3) in $d \geq 7$

$$S_{d=7} = \int d^7 x \sqrt{-g} \left[\frac{1}{2\kappa} \left(R - 2\Lambda + \alpha R_{GB}^2 + aR_{E.C}^3 \right) + \mathcal{L}_m^{matt} \right]$$

4) the Dilaton-Einstein-Gauss-Bonnet (DEGB) Gravity in $d = 4$

$$S_{dEGB} = \int d^4 x \sqrt{-g} \left[\frac{1}{2\kappa} \left(R - 2\Lambda e^{\lambda\phi(r)} + f(\phi) R_{GB}^2 \right) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_m^{matt} \right]$$

$f(\phi) = \alpha e^{\gamma\phi}$

polynomial

etc.

Horndeski Theory - the most general scalar-tensor theory w/ 2nd-order field eqn in 4D

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\nu\mu}]$$

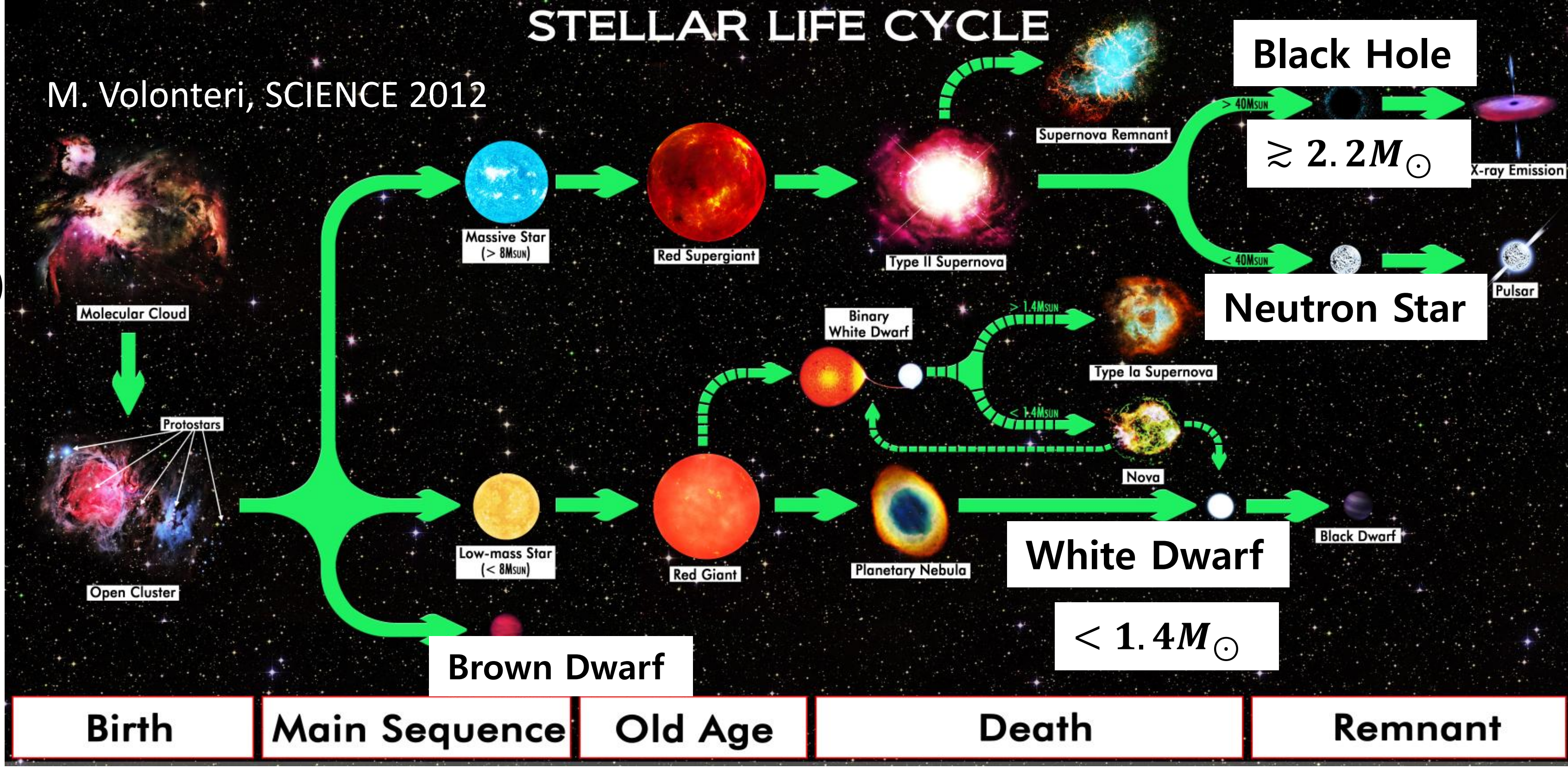
$$+ G_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} - \frac{G_{5X}}{6}[(\Box\phi)^3 - 3\Box\phi\phi_{\mu\nu}\phi^{\nu\mu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi_\lambda{}^\mu]$$

Dilaton-Einstein-Gauss-Bonnet (DEGB) Gravity belongs to Horndeski Theory

III. Black Holes

-Formation & Discovery

III. Black Hole
- Formation & Discovery



A. Formation

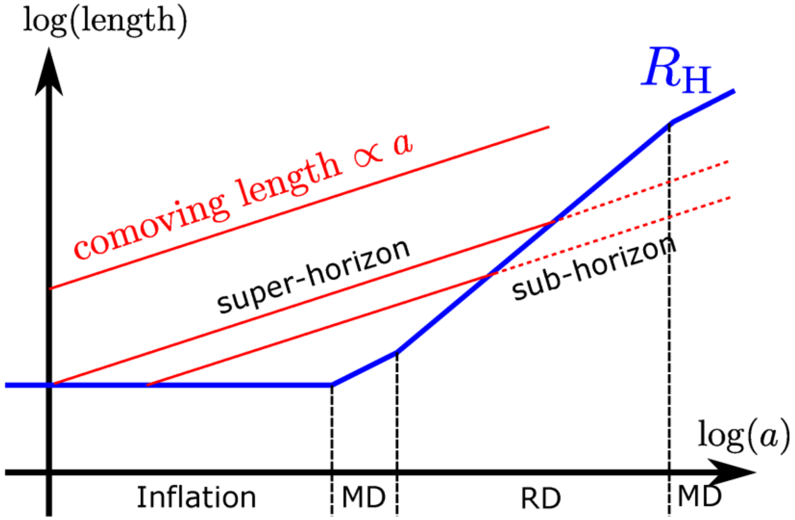
- **By Stellar Evolution** $(2 - 150M_\odot)$
- **Primordial BH (no mass limit)**

A-1) By Stellar Evolution

'Death Body' of heavy stars

An upper bound to the mass of WD and NS

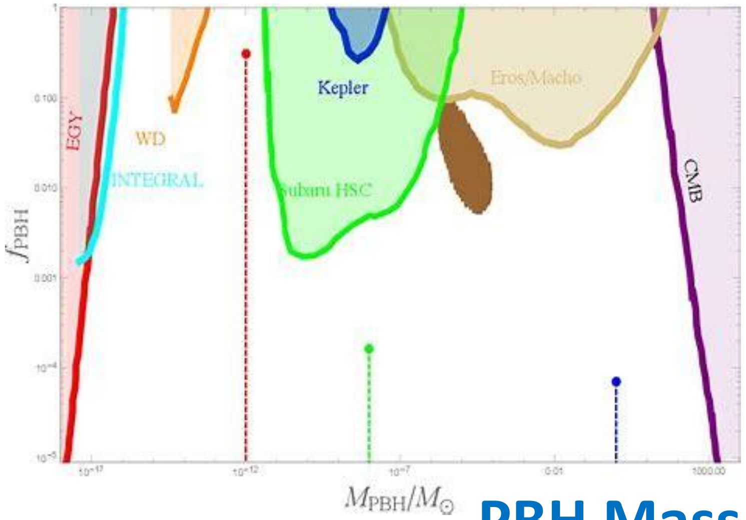
- **The Chandrasekhar limit**
  $\approx 1.44\, M_\odot$ (the max M of a white dwarf star (**WD**)
- **The Tolman–Oppenheimer–Volkoff (TOV) limit:** (for M of NSs)
  $1.5 - 3M_\odot$; (original mass of $15 - 20M_\odot$)
  Cf) GW170817 : $2.01 - 2.17M_\odot$ the merging NSs (into a BH)



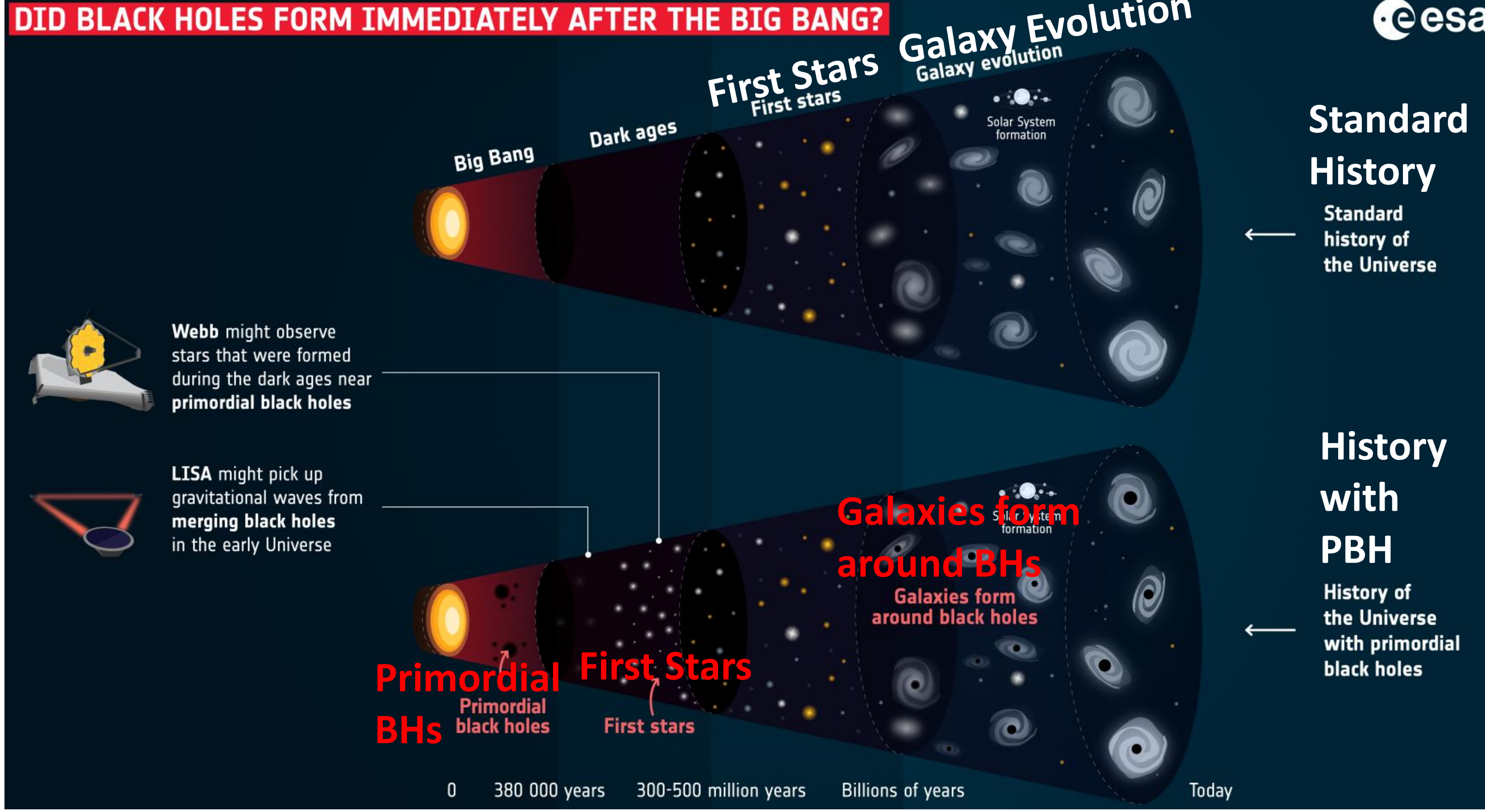
A-2) Primordial BHs (PBHs)

BHs with M below the TOV limit cannot be by the stellar evolution

⇝ Primordial BH



PBH Mass



BH Formation

- By Stellar Evolution $(2-150M_\odot)$
- Primordial BH (no mass limit)

Masses of BHs

1. Stellar BH $(2-150M_\odot)$
2. Intermediate mass BHs (IMBHs) $(10^2-10^5M_\odot)$
3. Supermassive BHs (SMBHs) $(10^6-10^9M_\odot)$: Inside Galaxies

Black Holes are affecting the evolution of the Universe.

Question : How much do they depend on the gravity theories?

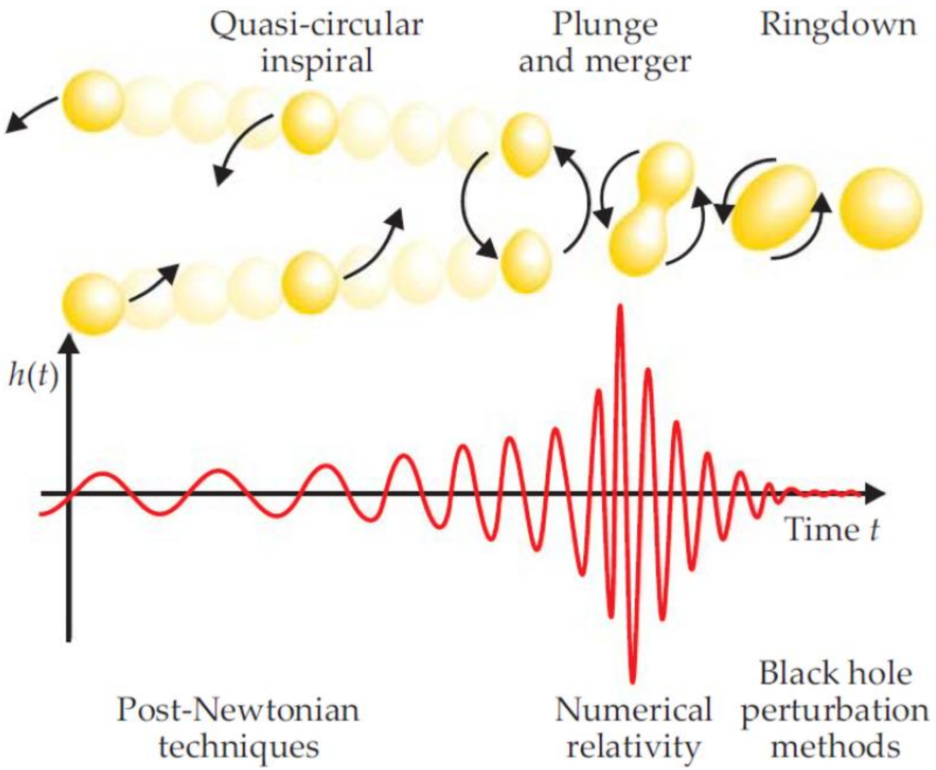
B. Observational Evidences

B-1. Stellar-Mass Black Holes $(2-150M_{\odot})$

- 100 Millions of stellar mass BHs in each galaxy.
- Observation as

1) Black hole X-ray binaries (BHXRBs)

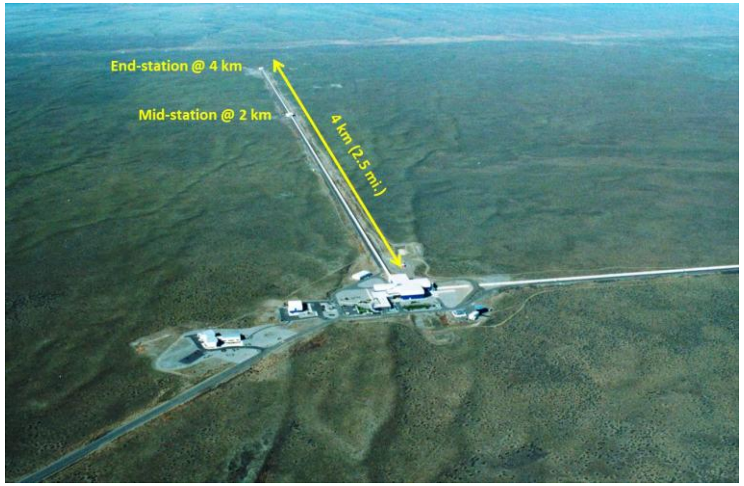
- The 1st stellar-mass BH was Cyg X-1
- Total # observed BHXRBs$\approx$ **20**

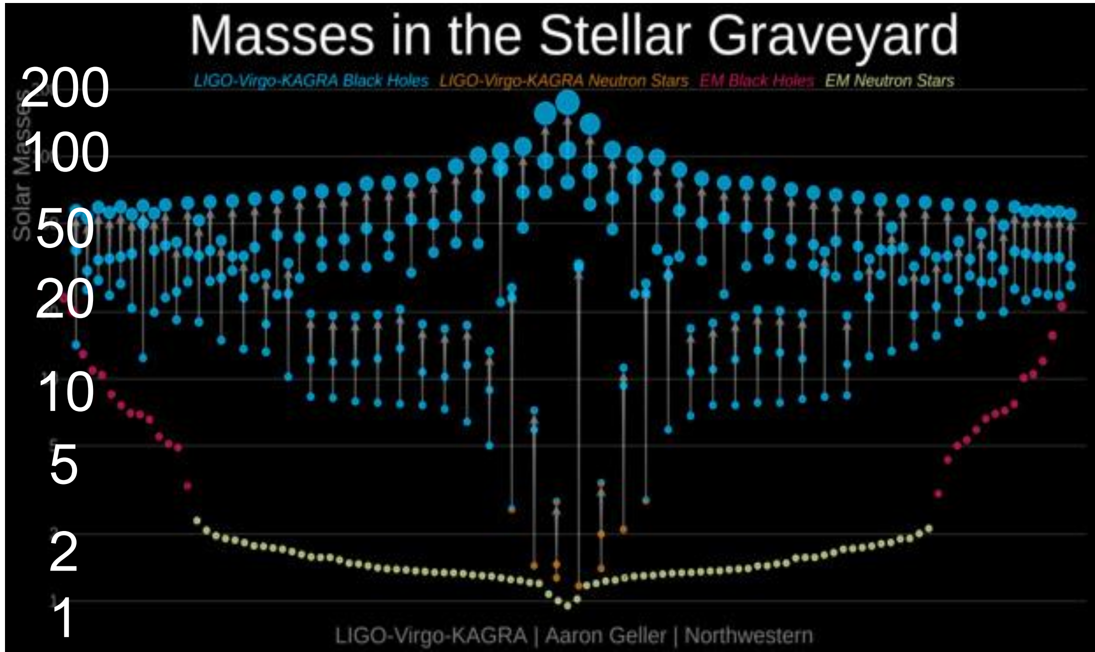


2)GW observed by LIGO, VIRGO, KAGRA

Sources:
BBH & NSBH binaries

Ex) GW150914

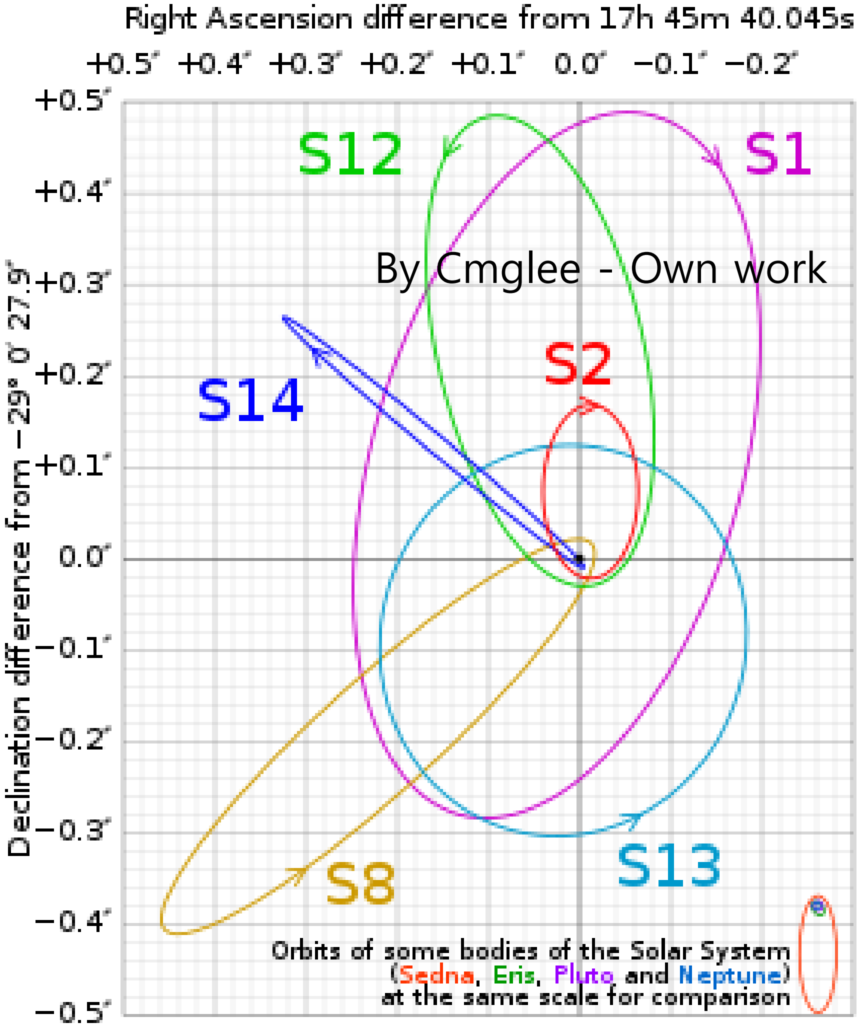




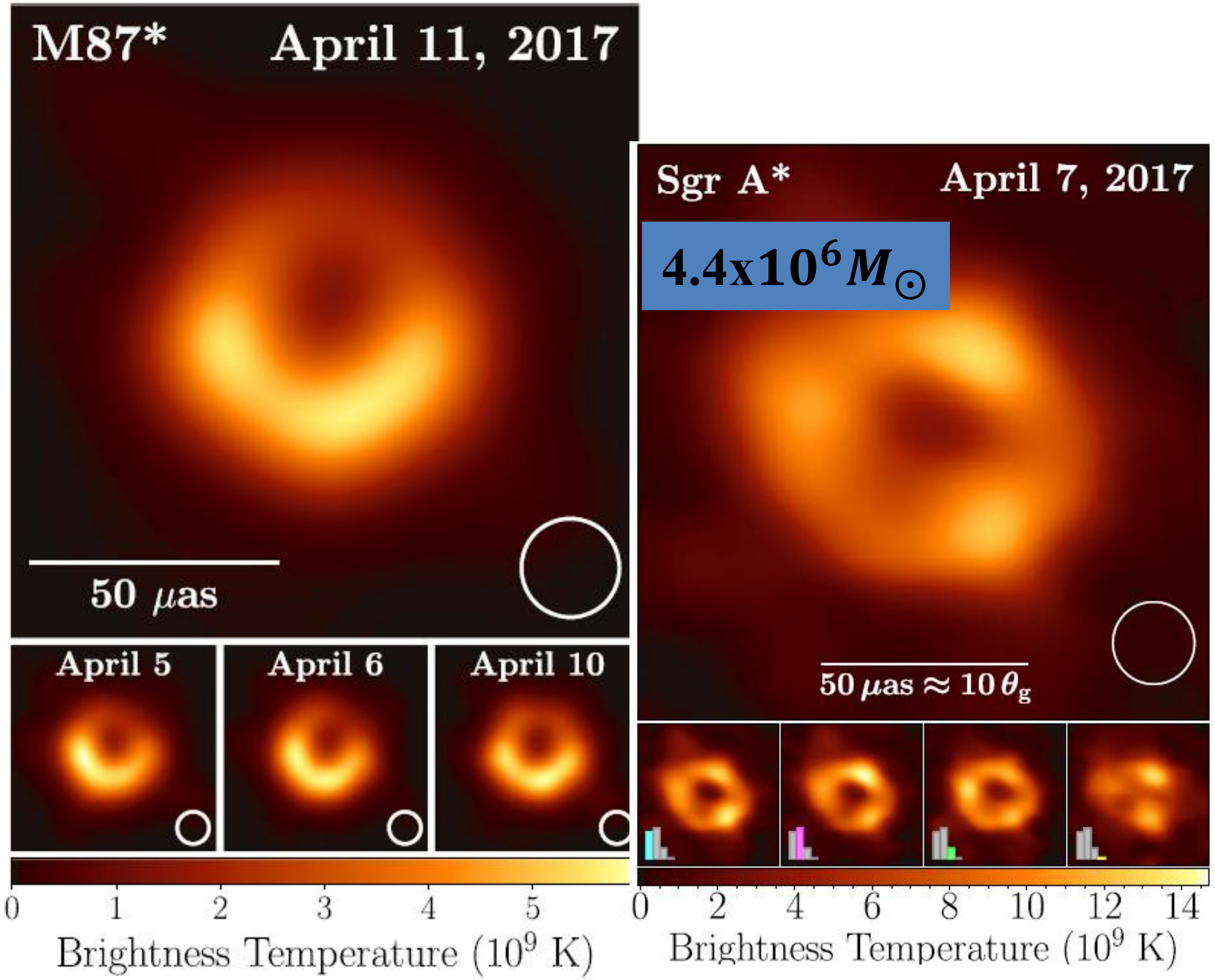
B-2. Supermassive Black Holes (SMBHs) $(10^6 - 10^9 M_\odot)$

Q : What if the theory is beyond Einstein Gravity?

1) Tracing the ellipses



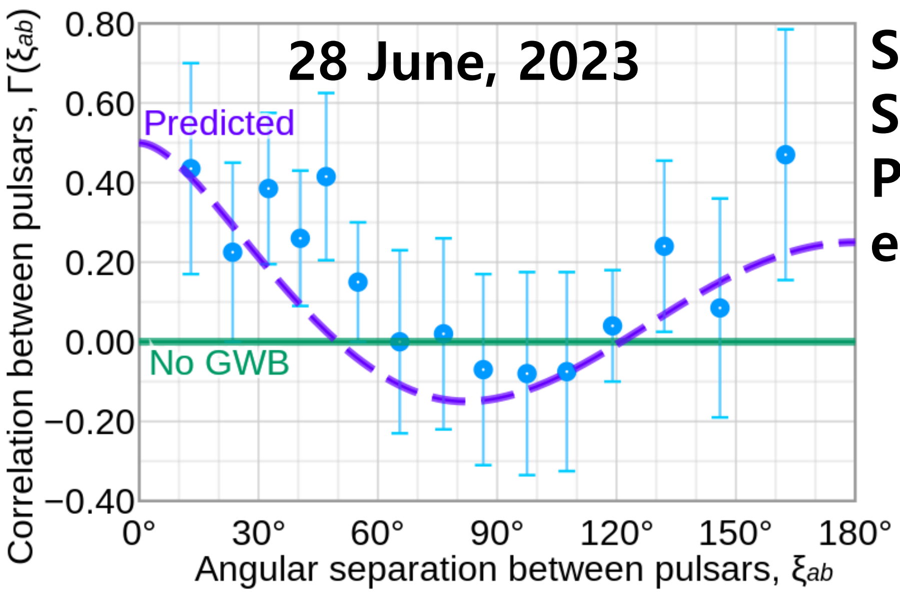
2) Event Horizon Telescope Collab



APJL, 2019 April 10

APJL, 2022 May 10

3) the Stochastic GW Backgrounds



Sources :
SMBHs,
PBHs,
etc.

NanoGrav; EPTA; CPTA

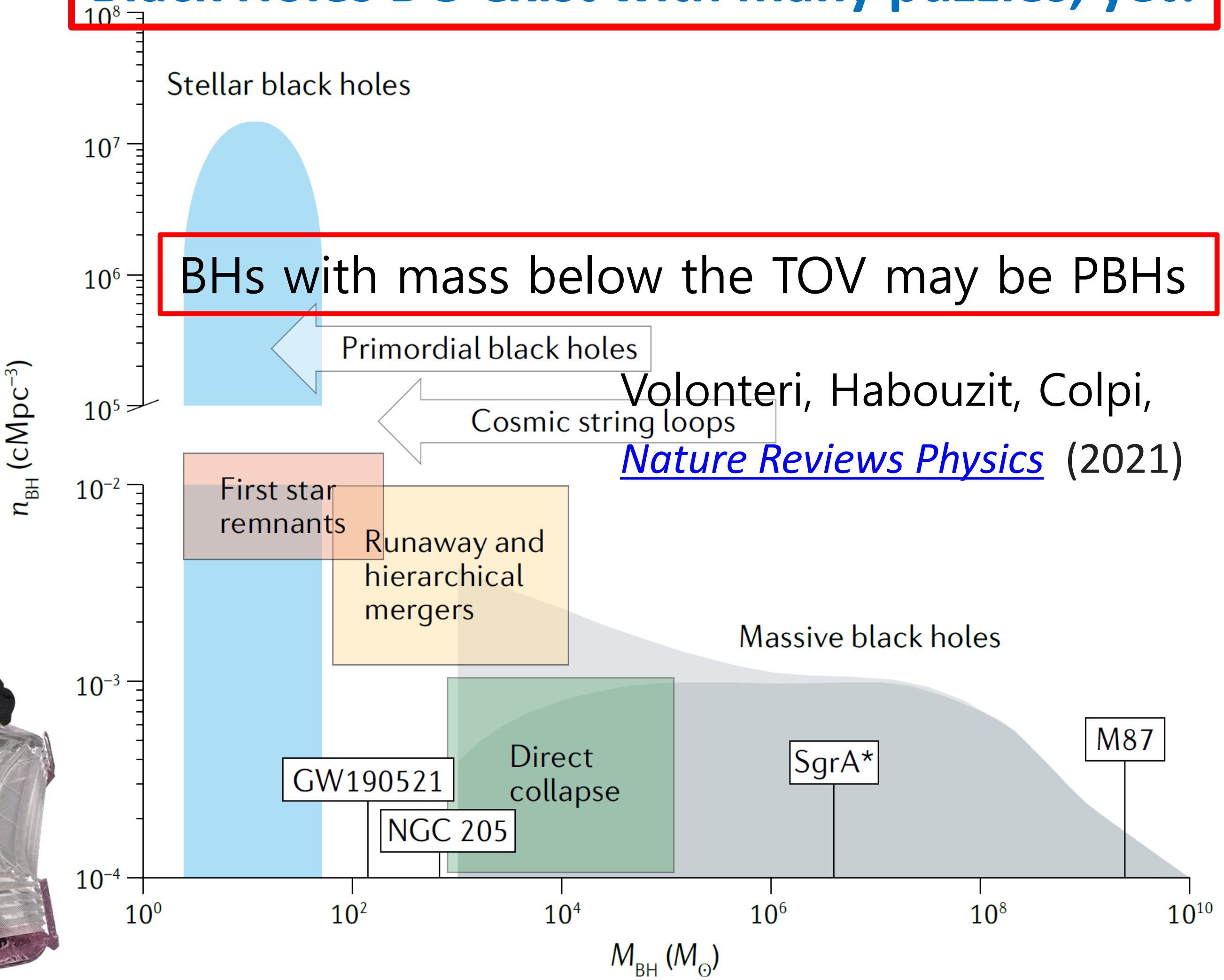
Parkes Observatory;

4) Observation by the James Webb Space Telescope (at $z \approx 10.3$ behind the cluster lens Abell 2744)

In summary,

1. Stellar BH $(2 - 150 M_\odot)$ **Yes, Observed!**
2. IMBHs $(10^2 - 10^5 M_\odot)$ **No, Not Yet!**
3. SMBHs $(10^6 - 10^9 M_\odot)$ **Yes, Observed!**

Black Holes DO exist with many puzzles, yet!



BHs with mass below the TOV may be PBHs

Volonteri, Habouzit, Colpi, *Nature Reviews Physics* (2021)

IV. Black Holes Thermodynamics

- Schwarzschild BH, AdS BH, RN AdS BH,
- RN Gauss-Bonnet AdS BH
- dEGB BH

Review: Dimensional Analysis

0) Classical, relativistic, w/o gravity effect

- Time is “identified” with Length - (units with c):

  $[c]$[Time] = [Length] or [Time] $\Leftrightarrow$ [Length]

1) Classical (w/o $\hbar$) General Relativity - (units with c & G_N)

- Mass is “identified” with Length (as well as Time),

  [Length] $\Leftrightarrow$ [Mass] [Time] $\Leftrightarrow$ [Length]

  ex) $r_H = 2\frac{G_N}{c^2}M_\odot = 2.953km$ Schwarzschild radius of $\odot$

cf) quantum, relativistic, w/o gravity effect

$[c = [L]/[T]$ or [Time] $\Leftrightarrow$ [Length]

$[\hbar c] =$ [E][L] $\quad \left[\frac{\hbar}{c}\right] =$ [M] [L]

2) General Relativity w/ Quantum Effect - (units with c, G_N & $\hbar$):

- “Characteristic” [M] [L] [T]

Planck Mass $m_P = \sqrt{\frac{\hbar c}{G_N}} = 1.221 \times \frac{10^{19}GeV}{c^2} = 2.176 \times 10^{-8}kg$

Planck length $\ell_P = \sqrt{\frac{\hbar G_N}{c^3}} = 1.616 \times 10^{-35}m;$

Planck time $t_P = \sqrt{\frac{\hbar G_N}{c^5}} = 5.391 \times 10^{-44}s;$

Dimensions :[M], [L], [T]

Physical constants:

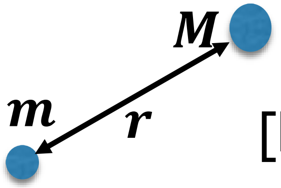
c: relativistic (nonrel if $\frac{v}{c} \to 0$)

$\hbar$: quantum (classical if $\hbar \to 0$)

G_N: Gravity (no grav if $G_N \to 0$)

Gravitational Constant G_N

$$= 6.674 \times 10^{-11} m^3 kg^{-1} s^{-2}$$



$$U = -G_N \frac{mM}{r}$$

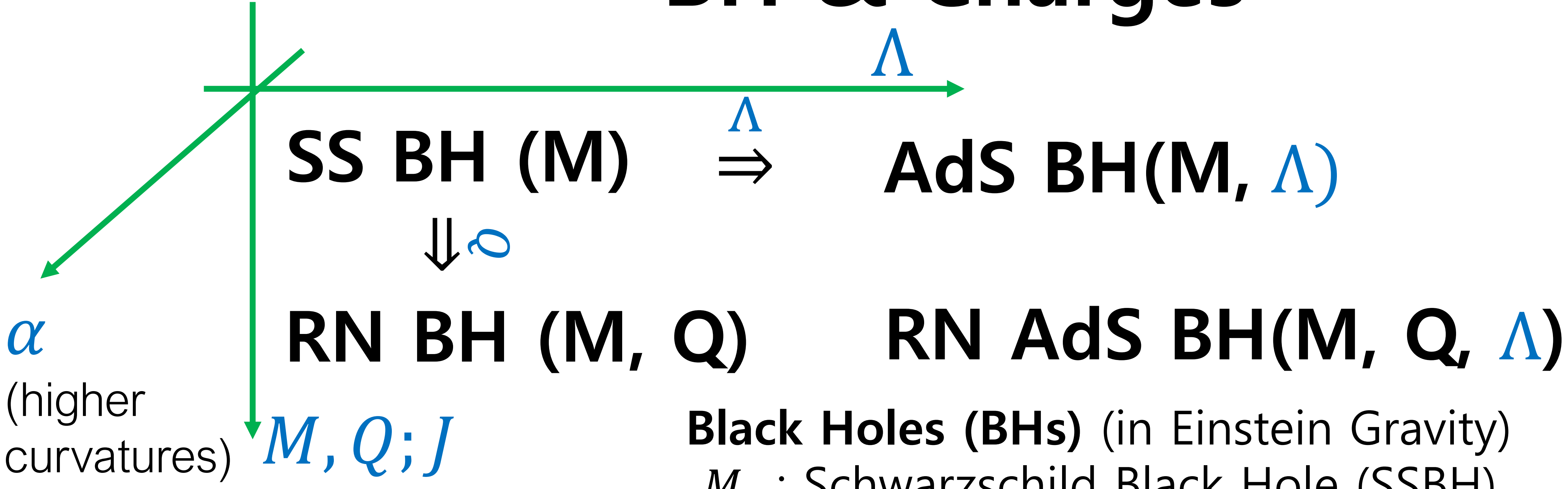
$$[M][c^2] = [G_N][M]^2[L]^{-1}$$

$$\Rightarrow [G_N]/c^2 = \frac{[\text{Length}]}{[\text{Mass}]} \quad \text{cf)} \ L = \frac{G_N}{c^2}M$$

Note: $\frac{G_N}{c^2} = \frac{\ell_P}{m_P}\left(= \frac{2.953\ km}{2M_\odot}\right)$

$(m_P c)\ell_P = \hbar$ or $\ell_P = \frac{\hbar}{m_P c}$

BH & Charges



Black Holes (BHs) (in Einstein Gravity)

M : Schwarzschild Black Hole (SSBH)

M & J : Kerr Black Hole (KBH)

M & Q : Reissner-Nordström Black Hole (RN BH)

M & J & Q : Kerr-Newman Black Hole (KN BH)

Question:

What are the BH thermodynamic phase diagrams?

Schwarzschild(SS) BH

III. Black Holes (BHs) (in d-dim)

1) Einstein theory – Schwarzschild(SS) BH

$r_H = 2M$

Action (d-dim)

$$S = \frac{1}{2\kappa}\int d^d x\,\sqrt{-g}\,R$$

$\kappa = 8\pi G$, $g = \det g_{\mu\nu}$

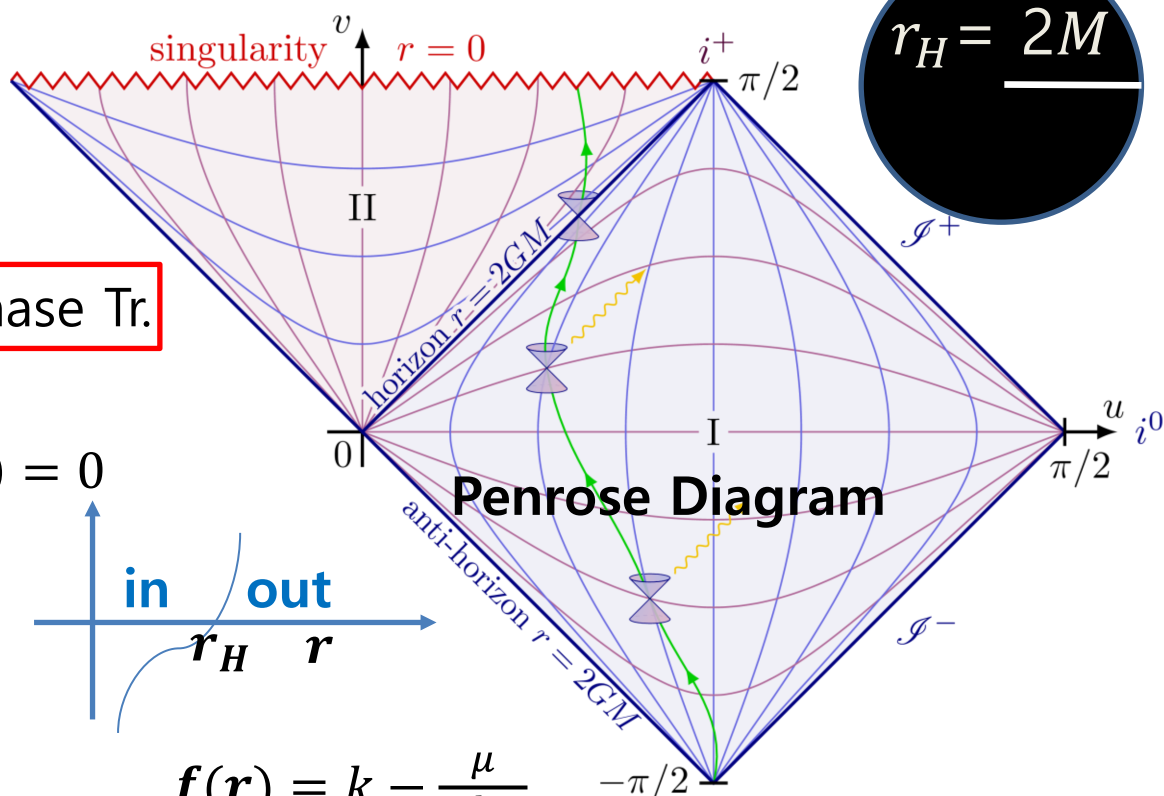
Note: Scale inv → no phase Tr.

Einstein Eqns (vacuum Eq.)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} = 0$$

becomes

$$\Rightarrow r f' - (d-3)(k-f) = 0$$



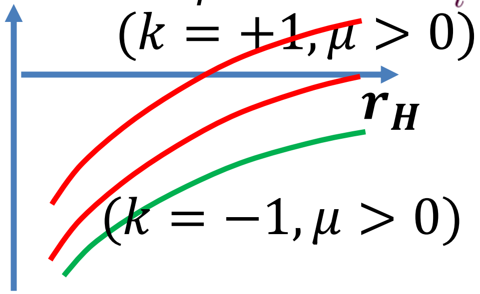
Black Hole solution Ansatz

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + +r^2 d\Sigma_k^{d-2}$$

Black Hole solution : **Metric Function $f(r)$**

$$f(r) = k - \frac{\mu}{r^{d-3}} \xrightarrow{d=4;k=1} 1 - \frac{\mu}{r} = 1 - \frac{2GM}{r}\ (\mu > 0),$$

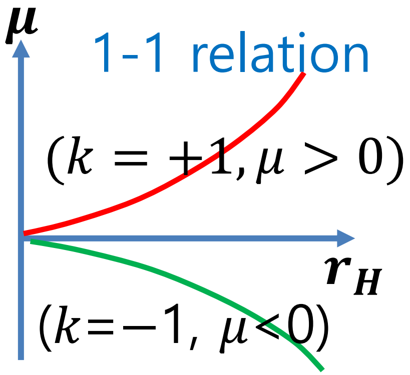
$f(r) = k - \frac{\mu}{r^{d-3}}$



BH Horizon r_H: $f(r_H) = 0$ ($r = r_H$ null hypersurf) & $\frac{df}{dr}|_{r=r_H} > 0$

$$f(r_H) = 0 \to \frac{\mu}{r_H{}^{d-3}} = k \xrightarrow{d=4;k=1} \frac{\mu}{r_H} = 1\ (\mu > 0);\ r_H \equiv 2GM$$

$$\frac{df}{dr}|_{r=r_H} = (d-3)\frac{\mu}{r^{d-2}} > 0 \text{ only for } \mu > 0$$



Note: BH exists only for $k = +1$ & $\mu = k r_H^{d-3} > 0$ (No minimum mass)

Note: 1) One-parameter (M) BH solution.

2) If $M = 0$, then Minkowski. space-time.

Schwarzschild(SS) BH in d-dim.

$$f(r) = k - \frac{\mu}{r^{d-3}}$$

Horizon r_H: $(f(r_H) = 0)$ & $\frac{df}{dr}|_{r=r_H} > 0$

$$f(r_H) = 0 \to \frac{\mu}{r_H{}^{d-3}} = k$$

$$\xrightarrow{d=4;k=1} \frac{\mu}{r_H} = 1\ (\mu > 0);\ r_H \equiv 2GM$$

$$\frac{df}{dr}|_{r=r_H} = (d-3)\frac{\mu}{r^{d-2}} > 0 \text{ only for } \mu > 0$$

$$f'(r_H) = \frac{(d-3)}{r_H}\frac{\mu}{r_H{}^{d-3}} = \frac{1}{r_H}k(d-3)$$

Note: 1) far region $r \gg r_H \to$

$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ Minkowski spacetime

2) Near horizon $r = r_H + \delta r \to$ AdS2 x S^2

Singularity at $r = r_H$: coordinate singularity

Singularity at $r = 0$ physical

The Kretschmann invariant **(spacelike singularity)**

$$I \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \mathcal{O}\left(\frac{\mu^2}{r^{2(d-1)}}\right)$$

the tortoise coord $r^* \equiv \int^r dr\, f^{-1}$ finite

Classical rel Grav - (units c & G_N)

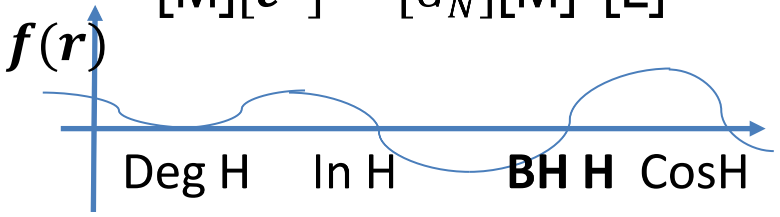
[L] ⇔ [M]; [T] ⇔ [L]

ex) $M_\odot = 2.953km$ SSBH rad of ⊙

$$r_H = \frac{2GM}{c^2}$$

In d-dim. (c=1), $U = -G_N \frac{mM}{r^{d-3}}$ $\quad [GM] = L^{d-3} = [\mu]$;

$[\mathrm{M}][c^2] = [G_N][\mathrm{M}]^2[\mathrm{L}]^{-(d-3)}$ $\quad [S] = ML$;



Σ_k^{d-2}: Einstein mfld curvature = k

$(R_{ij} \propto h_{ij})$, codim.2

$$d\Sigma_k^{d-2} = h_{ij}(x)\, dx^i dx^j$$

Ex)Σ_k^2: $(d = 4)$

$$\Sigma_1^{d-2} = \int d^{d-2}x\sqrt{|h_{ij}|}$$

$$\Sigma_1^2 = S^2;\ \Sigma_0^2 = T^2;\ \Sigma_{-1}^2 = H^2$$

$$\Sigma_1^1 = 2\pi;\ \Sigma_1^2 = 4\pi;\ \Sigma_1^3 = 2\pi^2$$

$$d\Sigma_k^2 = \begin{cases} d\Omega_{d-2}^2 & \text{for } k = +1 \\ \Sigma\, dx_i^2 & \text{for } k = 0 \\ dH_{d-2}^2 & \text{for } k = -1 \end{cases}$$

ADM mass M:

$$\mu = \frac{16\pi}{(d-2)\Sigma_1^{d-2}}GM;\quad \Sigma_1^{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma[\frac{d-1}{2}]}$$

Ex)

$\mu = 2GM\ (d = 4)$;

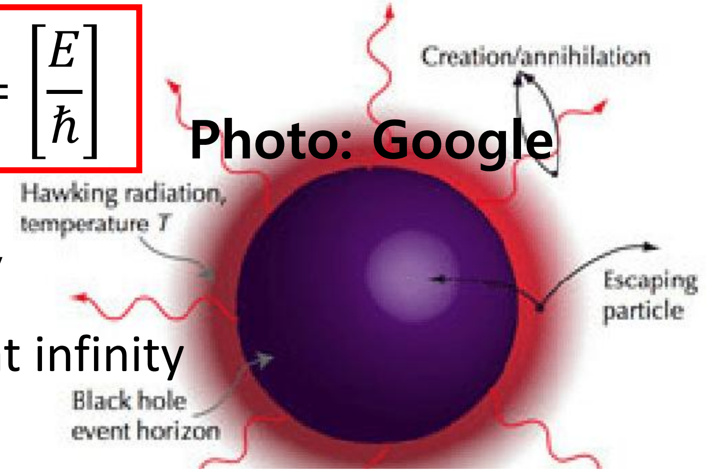
$\mu = \frac{8}{3\pi}GM\ (d = 5)$

Surface Gravity κ_{SG}

$$\kappa_{SG} = \frac{f'(r_H)}{2} = \frac{d-3}{2}\frac{1}{r_H}$$

BHs are Thermodynamical Systems

$$\left[\frac{k_B T}{\hbar}\right] = L^{-1} = \left[\frac{E}{\hbar}\right]$$



Hawking Temperature T_H

$$f'(r_H) = \frac{1}{r_H}$$

$$\kappa_{SG} = \frac{f'(r_H)}{2}$$ **Surface Gravity**

$$\frac{k_B T_H}{\hbar} = \frac{\kappa_{SG}}{2\pi} = \frac{1}{4\pi} f'(r_H) = \frac{1}{4\pi}\frac{1}{r_H}\frac{(d-3)\mu}{{r_H}^{d-3}}$$

$$= \frac{1}{4\pi}\frac{1}{r_H}(d-3)k$$

$$\xrightarrow{d=4;k=1} \frac{1}{8\pi GM} \quad \left(\text{or } \frac{c^3}{8\pi GM}\right) = \frac{k_B}{\hbar}\frac{6\times10^{-8}}{M_\odot}$$

Hawking radiated Particle ($m = 0$) at infinity

Wavelength~ r_H;

energy~1/ r_H

\# evaporated γs~ $M/T_H \sim GM^2$

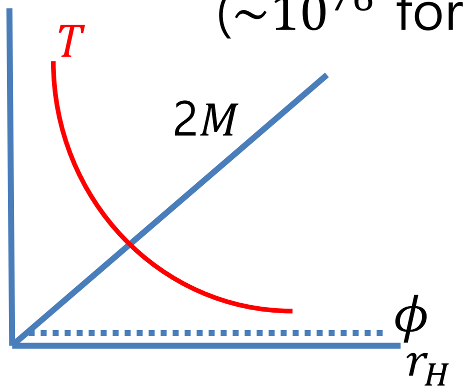
($\sim 10^{76}$ for $M_\odot$)

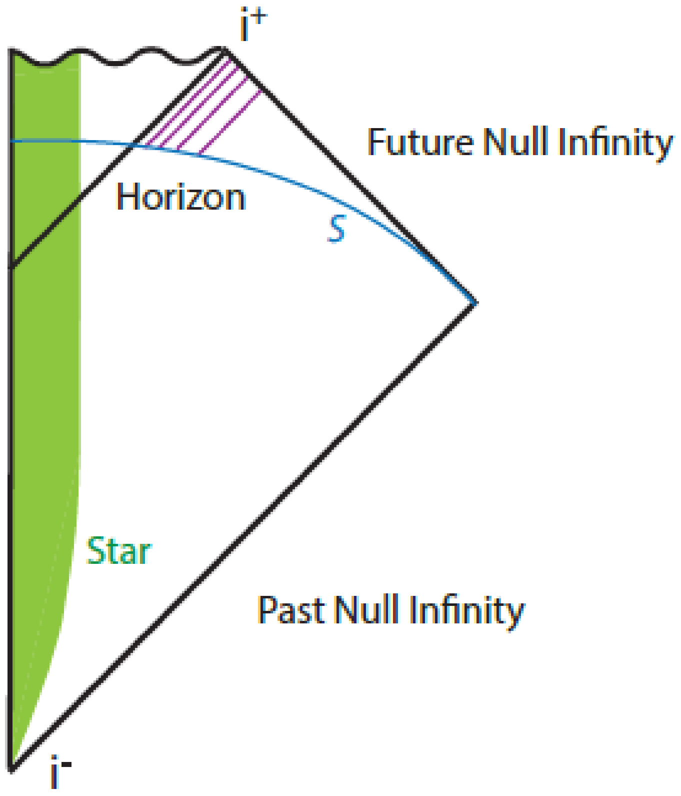
Hawking Radiation

Radiation Rate $\frac{dM}{dt} \propto AT_H^4 \sim \frac{1}{M^2}$

life time $\tau = \int dt \sim \int M^2 dM \sim M^3$;

$$\frac{\tau}{10^{10} yr} = \left(\frac{M}{10^{12} kg}\right)^3; \ \tau_\odot \sim 10^{67} yr$$





Bekenstein Entropy

$$S/k_B = \frac{\text{Area}}{4{\ell_P}^2} = \frac{Ac^3}{4\hbar G} \equiv \frac{\text{A}}{4G} = \frac{4\pi{r_H}^2c^3}{4\hbar G}$$

$$= \frac{4\pi G^2M^2}{\hbar G} = \frac{4\pi GM^2}{\hbar} = \frac{1}{16\pi\hbar G}\frac{1}{T^2}$$

$$e^S \sim \epsilon^{E^2} \quad (r_H = 2GM)$$

Ex) $S_{BH}(M_\odot) = 10^{77} = 10^{18} S(M_\odot)$

$$s = \int T^{-1} d\mu = \int_0^{r_H} T^{-1}\left(\frac{\partial\mu}{\partial r_H}\right) dr_H$$

$$= \int_0^{r_H} \frac{4\pi r_H}{(d-3)k}\frac{(d-3)}{r_H} k r_H^{d-3} dr_H$$

$$= 4\pi\int_0^{r_H} r_H^{d-3} dr_H = \frac{4\pi}{(d-2)} r_H^{d-2}$$

$$S = \frac{(d-2)\Sigma_k^{d-2}}{16\pi G} s = \frac{(d-2)\Sigma_k^{d-2}}{16\pi G}\frac{4\pi}{(d-2)} r_H^{d-2} = \frac{\Sigma_k^{d-2} r_H^{d-2}}{4G} = \frac{Area}{4G}$$

Note

- No length scales in the Einstein Gravity (except the overall G_N) "scale-invariance"
- As a result, there exists only "single" phase
- No minimum mass of BH : there is no lower bound on $r_H = 2GM$

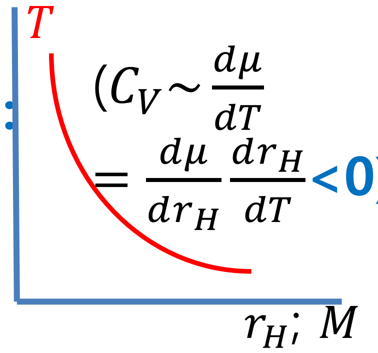
$$F(T) = M - TS = \frac{r_H}{4G} = \frac{1}{16\pi G}\frac{1}{T};$$

$dF = -SdT$, [Chase, CMR 19, 276 (1970)]

Heat Capacity

$$C_V = \frac{dM}{dT} = -\frac{1}{8\pi G}\frac{1}{T^2} < 0 \;;$$

Unstable (thermal)



$$\left(C_V \sim \frac{d\mu}{dT} = \frac{d\mu}{dr_H}\frac{dr_H}{dT} < 0\right)$$

Note: For a nonextremal stationary BH, with the metric

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2 d\Sigma_k^{d-2}$$

Near the horizon, (Hor $r = r_H$ collapses to a pt $\rho = 0$)

$$f(r) = f'(r_H)(r - r_H) = 2\kappa_{SG}(r - r_H)$$

Hence, get the Euclidean BH metric after "Wick rotation"

$$ds^2 = 2\kappa_{SG}(r - r_H)d\tau^2 + \frac{1}{2\kappa_{SG}(r - r_H)}dr^2 + r^2 d\Sigma_k^{d-2}$$

$$= d\rho^2 + \kappa_{SG}^2\rho^2 d\tau^2 + r^2 d\Sigma_k^{d-2}; \quad \rho = \frac{1}{\kappa_{SG}}\sqrt{2\kappa_{SG}(r - r_H)}$$

For no conical singularity at the origin,

$$\tau\text{-period} = \frac{2\pi}{\kappa_{SG}} = \frac{4\pi}{f'(r_H)} \equiv \beta = \frac{1}{T_H}$$

A **BH** in **asymptotically flat space** is **thermodynamically unstable (**Hawking Radiation)..

Question: How to make the BH thermally stable?

1) Place the BH inside **a finite spherical cavity**. T: fixed at the surface of the cavity, a heat hath. **Or**
2) Put the BH in AdS space ($\Lambda < 0$), which stabilizes BH by acting as a reflecting box.

Black Hole Thermodynamics

1st law

$dM = T_H dS + \Omega dJ + \Phi dQ\ (+Vd\Lambda)$

(dM=TdS+ ωdJ+ΦdQ (+VdΛ))

Temperature

$$kT_H = \frac{\hbar\kappa_{SG}}{2\pi} = \frac{\hbar}{4\pi} f'(r_H) = \frac{\hbar}{4\pi}\frac{(d-3)}{r_H} = \frac{\hbar}{8\pi GM}$$

$r_H = 2GM$ Horizon

2nd Law :

$\Delta(\text{Area}) \geq 0$

Entropy ($r_H = 2GM$)

$$S = \frac{\text{Area}}{4{\ell_P}^2} = \frac{Ac^3}{4\hbar G} = \frac{4\pi {r_H}^2 c^3}{4\hbar G} = \frac{4\pi G^2 M^2}{\hbar G} = \frac{4\pi G M^2}{\hbar} = \frac{1}{16\pi\hbar G}\frac{1}{T^2}$$

$e^S \sim \epsilon^{E^2}$

(Generalized 2nd) Law

$\frac{dS_{gen}}{dt} \geq 0$ $\quad S_{gen} = S_{BH} + S_{out}$

Mass ⇔ Energy

Area ⇔ Entropy

Surf. Grav ⇔ Temp

Thermal Variables

Potential	↔Charge
μ	N
Φ	Q
ω	L

cf) BH charges (M,J,Q)

dM=TdS + ωdJ+ΦdQ(+VdΛ)

Thermodynamics

(1st) Law

Microcanonical Ensemble

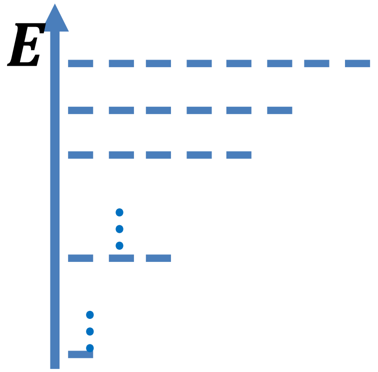
$dE = TdS - pdV + \mu dN$

Canonical Ensemble

$F(T) = M - TS = \frac{r_H}{4G} = \frac{1}{16\pi G}\frac{1}{T'}$; $dF = -SdT$

Temperature

$\frac{1}{T} = \frac{dS}{dE}$



2nd Law :

$\Delta S \geq 0$

Entropy

e^S= # of states

$S \sim \ln$(# of states)

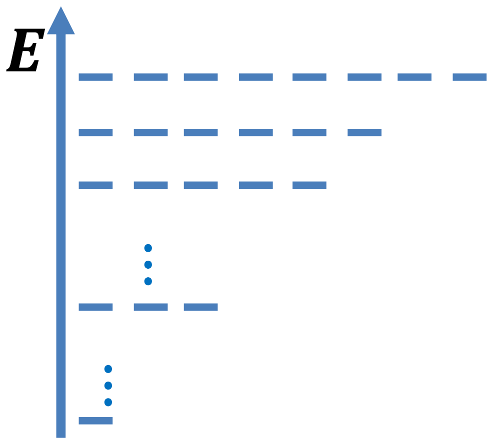
Thermodynamic Stability $\frac{\partial^2 \mathcal{F}}{\partial x \partial y} \geq 0$

Ex) Heat Capacity $C = \frac{dE}{dT}$

$S \sim E^n \rightarrow T \sim E^{1-n}$,

$C = \frac{dE}{dT} \sim \frac{1}{1-n} E^n$

Unstable if $n > 1$ $(C < 0)$



$$dM = T_H dS = \frac{dS}{8\pi GM}; \; kT_H = \frac{\hbar}{8\pi GM}$$

$$dS = 8\pi GM dM = d(4\pi GM^2)$$

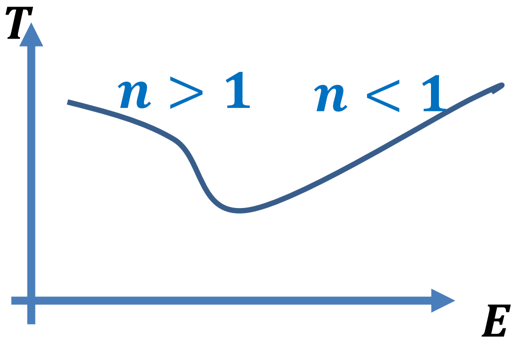
$$S = 4\pi GM^2 = \frac{\pi {r_H}^2}{G} = \frac{A}{4G}$$

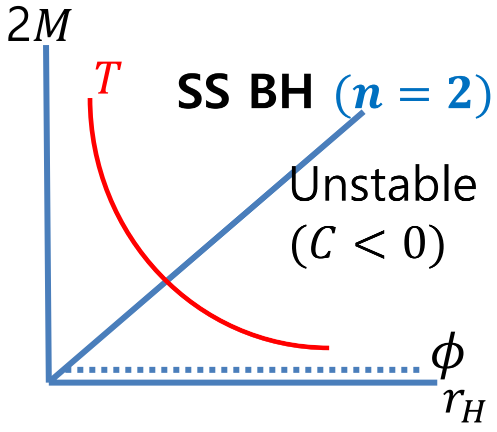
Note:

$e^S \sim E^{\epsilon}$, then $S \sim \ln E \ll E^2$ $(n = 0_+)$

$e^S \sim \epsilon^E$, then, $S \sim E \ll E^2$ $(n = 1)$

$e^S \sim E! \sim E^E \rightarrow S \sim E \ln E \ll E^2 (n = 1_+)$





Questions

What are the states for the BH entropy? etc.

The no-hair theorem (a hypothesis) "black holes have no hair"

All stationary BH solutions of the Einstein–Maxwell eqns of general relativity with electromagnetism can be completely characterized by only three independent observables :

Mass M; **Angular momentum** J and **Charge** Q. "Macroscopic thermal equilibrium states"

Other characteristics are uniquely determined by these three parameters, and all other information ("hair") about the matter that formed a BH "disappears" behind the BH event horizon and is therefore permanently inaccessible to external observers after the BH "settles down".

AdS BH

2. Schwarz **AdS Black Holes**

Action Birmingham 1999 CQG

$$S = \int d^d x \sqrt{-g}\left[\frac{1}{2\kappa}\left(R + \frac{(d-1)(d-2)}{\ell^2}\right)\right] + S_{matter}$$

$\kappa = 8\pi G,\ g = \det g_{\mu\nu};$

Note:

The only length scale is ℓ^2, which governs the phase structure.

Note: Dimension (c=1)

$[S] = ML\,;[GM] = L^{d-3} = [\mu];$

$[\ell^2] = L^2$

$\mu = \frac{16\pi G}{(d-2)\Sigma_k^{d-2}} M;\ M$:ADM mass

$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}$

$\Lambda = -\frac{3}{\ell^2}\ (d = 4);$

$\Lambda = -\frac{6}{\ell^2}\ (d = 5)$

Note: BH(Horizon geo) exists for

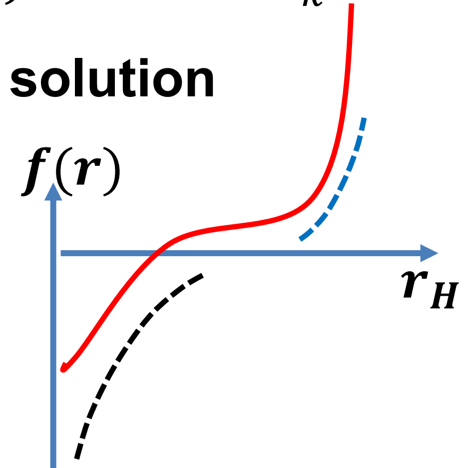
$k = +1$(sphere), 0(plane), -1 (hyperbolic)

Black Hole (BH) Ansatz

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2 d\Sigma_k^{d-2}$$

Metric Function $f(r)$ - BH solution

$$f(r) = k - \frac{\mu}{r^{d-3}} + \frac{r^2}{l^2}$$



Horizon: $f(r_H) = 0$

$$\rightarrow \frac{\mu}{r_H^{d-3}} = \left(k + \frac{r_H^2}{\ell^2}\right)$$

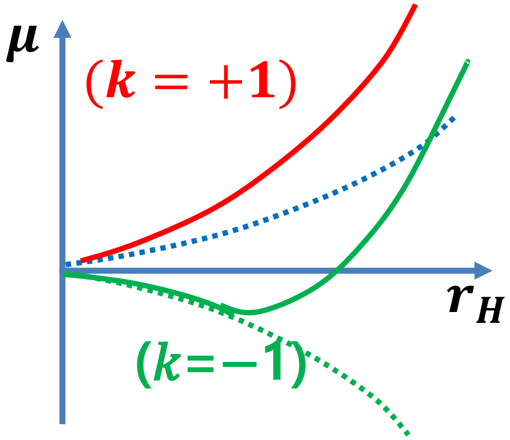
$$f'(r_H) \geq 0 \rightarrow f'(r_H) = \frac{1}{r_H}\left[(d-3)\frac{\mu}{r_H{}^{d-3}} + \frac{2r_H^2}{l^2}\right]$$

$$= \frac{1}{r_H}\left[(d-3)k + (d-1)\frac{r_H^2}{\ell^2}\right] > 0$$

Singularity (spacelike) at r = 0

The Kretschmann invariant $I \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \mathcal{O}\left(\frac{\mu^2}{r^{2(d-1)}}\right)$

the tortoise coordinate $r^* \equiv \int^r dr\, f^{-1}$ finite



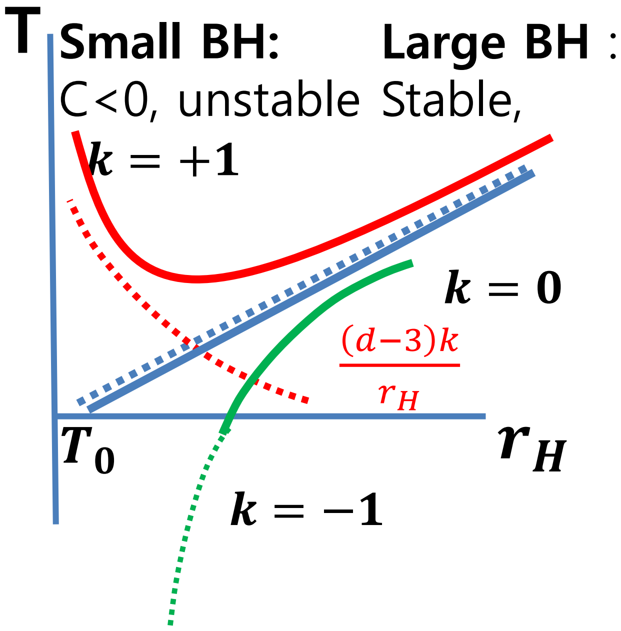
Note: Horizon exists for all k $(k = +1, 0, -1)$

Schwarz **AdS Black Holes**

Hawking Temperature

$$T_H = \frac{1}{4\pi} f'(r_H) = \frac{1}{4\pi}\left((d-3)\frac{\mu}{r_H^{d-2}} + 2\frac{r_H}{\ell^2}\right)$$
$$= \frac{1}{4\pi}\left(\frac{k(d-3)}{r_H} + (d-1)\frac{r_H}{\ell^2}\right)$$

Note: Natural reference scale :
ℓ (in addition to μ).



Note :

1) Two branches:
   Small BH ($r_H \ll \ell$): unstable
   Large BH ($r_H \gg \ell$) : stable.

2) Horizon geometry can be
   sphere ($k = +1$),
   plane ($k = 01$),
   or hyperbolic ($k = -1$).

3) For $k = +1$, (Schw. AdS BH)
   - $T \geq T_0 = \frac{\sqrt{2}}{\pi\ell}$,
   - Hawking-Page Tr.

Canonical Ensemble

Energy

$$E = -\frac{\partial}{\partial\beta}\ln Z = \frac{\partial}{\partial\beta}\beta F = \frac{\partial I_{Euc}}{\partial\beta}$$
$$= \frac{(d-2)\Sigma_{d-2}^1(\ell^{-2} r_H^{d-1} + r_H^{d-3})}{16\pi G} = M$$

Entropy

$$S = \beta E - I = \frac{\Sigma_{d-2}^1 r_H^{d-2}}{4G} = \frac{A}{4G}$$

Note :ADM mass M

$$\mu = \frac{16\pi G}{(d-2)\Sigma_{d-2}^1} M;\ \Sigma_{d-2}^1 = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma[\frac{d-1}{2}]}$$

Remark Gravitational **Partition function**) (the Euclidean path integral) :

Canonical Ensemble

(1-parameter)

$$Z[\beta] = \int [dg][d\Phi_{matter}]\, e^{-I_{Euc}} = e^{-\beta F} \quad -\ln Z = I_{Euc} = \beta F$$

Witten (9803131)

Hawking-page'83

Hawking-page Transition

Action ($k = +1$) (for X_2 = **AdS SS BH** wrt X_1 = **AdS$_d$**/Z)

$$2\Lambda = -\frac{(d-1)(d-2)}{\ell^2}$$

$$V = \int d^d x \sqrt{-g} \text{ Volume}$$

$$I_{Euc} = -\frac{1}{16\pi G}\int d^d x \sqrt{-g}\,[R - 2\Lambda] = -\frac{1}{16\pi G}\int d^d x \sqrt{-g}\left[\frac{2}{d-2}2\Lambda\right]$$
$$= -\frac{1}{16\pi G}\int d^d x \sqrt{-g}\left[\frac{2}{d-2}(-1)\frac{(d-1)(d-2)}{\ell^2}\right] = \frac{(d-1)}{8\pi G\ell^2}\int d^d x \sqrt{-g} = \frac{(d-1)}{8\pi G\ell^2}V$$

For boundary manifold $\partial \mathbf{X}^d = M^{d-1} = \mathbf{S}^{d-2}\times \mathbf{S}^1$, two solutions

X_1^d = **AdS$_d$**/Z (thermal AdS) and X_2^d = AdS BH

1) $\boldsymbol{X_1}$ = **AdS$_d$**/Z (**Thermal AdS$_d$**)

$$ds^2 = \left(1+\frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{l^2}} + r^2 d\Sigma_{k=1}^{d-2}$$

Here, t is a periodic variable of arbitrary period.

Topology : $\mathbf{B}^{d-1}\times\mathbf{S}^1$ (or $\mathbf{R}^{d-1}\times\mathbf{S}^1$)

2) X_2 = **Schwarzschild AdS$_d$ BH**

$$ds^2 = \left(1-\frac{w_d M}{r^{d-3}}+\frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1-\frac{w_d M}{r^{d-3}}+\frac{r^2}{l^2}} + r^2 d\Sigma_{k=1}^{d-2}$$

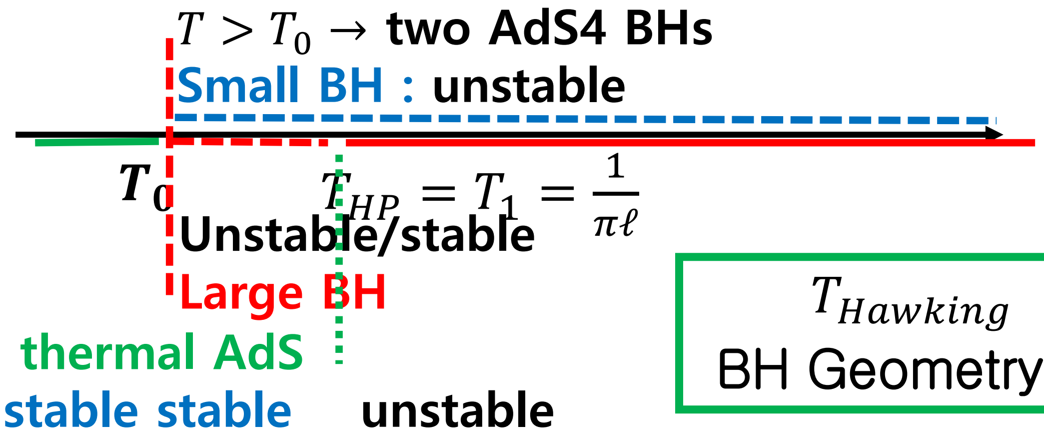
$$f(r) = 1-\frac{\mu}{r^{d-3}}+\frac{r^2}{l^2} = 1-\frac{w_d M}{r^{d-3}}+\frac{r^2}{l^2}$$

$$\mu = \frac{16\pi G}{(d-2)\Sigma_1^{d-2}}M = w_d M, \qquad w_d = \frac{16\pi G}{(d-2)\Sigma_1^{d-2}}$$

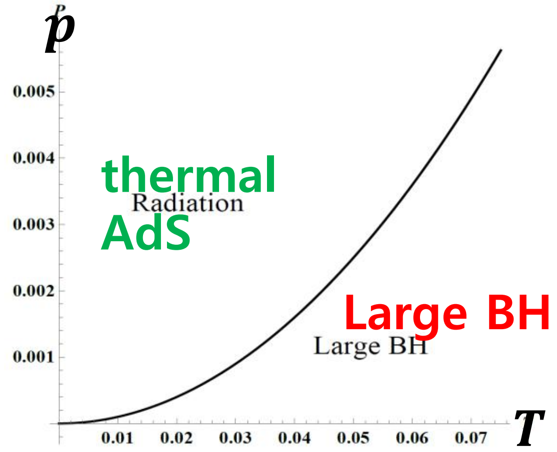
$$I_{Euc} = I_{BH} - I_{AdS} = -\frac{1}{16\pi G}\int d^d x\sqrt{-g}\,[R-2\Lambda] = \frac{(d-1)}{8\pi G\ell^2}\big(V_2(R)-V_1(R)\big)$$
$$= \frac{1}{8\pi G\ell^2}\Sigma_1^{d-2}\left(\beta_0\left(R^{d-1}-r_H^{d-2}\right)-\beta'\left(R^{d-1}\right)\right) = \frac{\Sigma_1^{d-2}}{4G}\,\frac{\ell^2 r_H^{d-2}-r_H^d}{(d-1)r_H^2+(d-3)\ell^2}$$

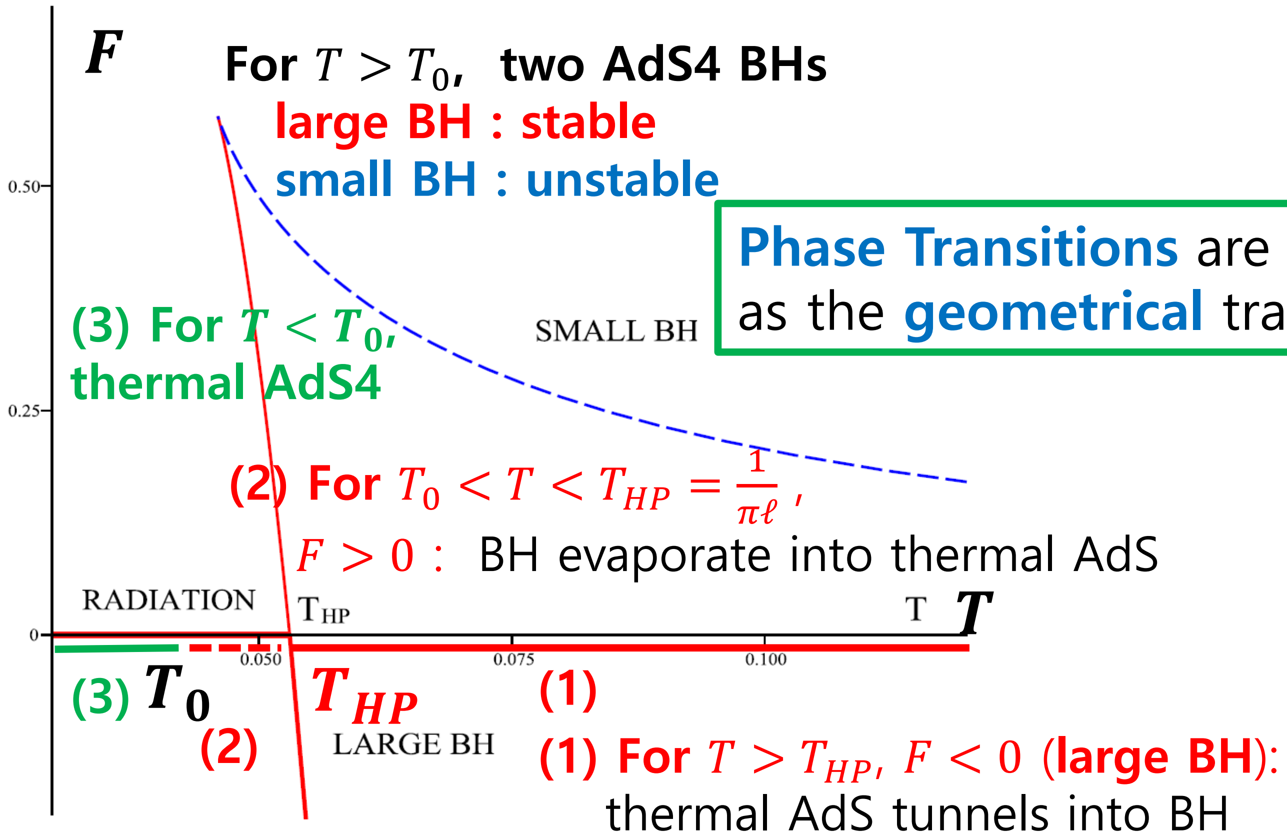
Hawking-page Transition Hawking-page'83

The Hawking-Page phase trans btwn stable large BHs and thermal gas in AdS space, explain the confinement/deconfinement phase trans of gauge field

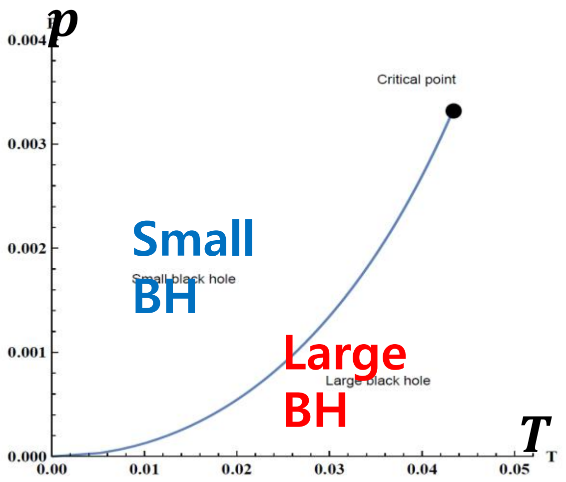


$T_{Hawking} = T$:
BH Geometry $\Leftrightarrow$ Finite T system





Phase Transitions are realized as the **geometrical** transition.



RN AdS BH

3. RNAdS Black Holes

$2\Lambda = -\frac{(d-1)(d-2)}{\ell^2}$

$2\Lambda \xrightarrow{d=4} -\frac{6}{\ell^2} \xrightarrow{d=5} -\frac{12}{\ell^2}$

Action

$$S = \int d^d x \sqrt{-g} \left[\frac{1}{2\kappa}(R - 2\Lambda)\right] + S_{matter}$$

$$S_{matter} = -\frac{1}{4\pi g^2}\int d^d x \sqrt{-g}\, F_{\mu\nu}F^{\mu\nu}$$

$\kappa = 8\pi G,$
$g = \det g_{\mu\nu};$

Note:

The length scale is ℓ^2 and $[g^2]$ (or q^2) which governs the phase structure.

Note: Dimension (c=1)

$[S] = ML\,;[G] = \frac{L^{d-3}}{M}\,;[\mu] = L^{d-3};[\ell^2] = L^2;\ [q^2] = L^{2(d-3)}$

Note: mass M & charge Q

$$\mu = \frac{16\pi G}{(d-2)\Sigma_k^{d-2}} M \quad q^2 = \frac{8\pi G}{2(d-2)\pi g^2} Q^2$$

$\frac{[Q^2]}{[g^2]} = ML^{d-3};$

$\frac{[GQ^2]}{[g^2]} = [q^2]$

Ex) $q = GQ(d=4),$
$= \frac{2G}{\sqrt{3}\pi} Q(d=5)$

Metric Ansatz

$$ds^2 = -f(r)\, dt^2 + f^{-1}(r)\, dr^2 + r^2 d\Sigma_k^{d-2}$$

Eqns of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda\, g_{\mu\nu} = \kappa T_{\mu\nu};\quad T_{\mu\nu} = \frac{1}{4\pi g^2}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}\right)$$ Einstein Eq.

$$\nabla_\alpha F^{\alpha\mu} = 0$$ Maxwell Eq.

Black Hole solution

$$f(r) = k - \frac{\mu}{r^{d-3}} + \frac{d-3}{2}\frac{q^2}{r^{2(d-3)}} + \frac{r^2}{l^2}$$

$$A = \left(-\frac{1}{c}\frac{q}{r^{d-3}} + \Phi\right) dt$$ (gauge choice) $A(r_H) = 0$

$$c = \sqrt{\frac{2(d-3)}{d-2}}$$ and $\Phi = \frac{1}{c}\frac{q}{r_H^{d-3}}$

Singularity (timelike) at r = 0

The Kretschmann invariant

$$I \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \mathcal{O}\left(\frac{q^2}{r^{4(d-2)}}\right)$$

$f(r) > 0\ (r \sim 0)$ coord $r^* \equiv \int^r dr\, f^{-1} < \infty$

(Electric field) $E(r) = \frac{Q}{r^{(d-2)}}$

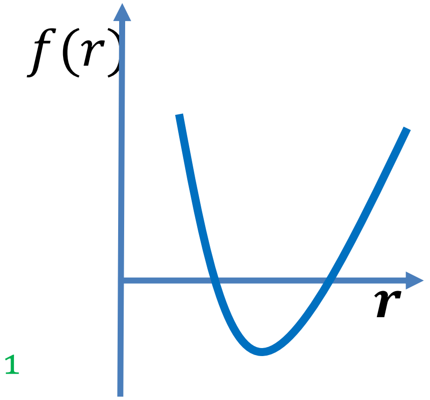
RNAdS Black Holes

$$2\Lambda = -\frac{(d-1)(d-2)}{\ell^2}$$

$$2\Lambda \xrightarrow{d=4} -\frac{6}{\ell^2} \xrightarrow{d=5} -\frac{12}{\ell^2}$$

Metric Function $f(r)$ BH solution

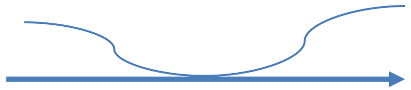
$$f(r) = k - \frac{\mu}{r^{d-3}} + \frac{d-3}{2}\frac{q^2}{r^{2(d-3)}} + \frac{r^2}{l^2}$$



Horizon $f(r_H) = 0$

$$\mu = r_H^{d-3}\left(k + \frac{d-3}{2}\frac{q^2}{r_H^{2(d-3)}} + \frac{r_H^2}{\ell^2}\right) = k r_H^{d-3} + \frac{d-3}{2}\frac{q^2}{r_H^{(d-3)}} + \frac{r_H^{d-1}}{\ell^2}$$

Extremal solution $f(r_{ex}) = \left.\frac{df}{dr}\right|_{r=r_{ex}} = 0$



In H= **BH H**

$$\boldsymbol{f(r_H) = 0} \Rightarrow \frac{\mu}{r_H^{d-3}} = k + \frac{d-3}{2}\frac{q^2}{r_H^{2(d-3)}} \quad (1)$$

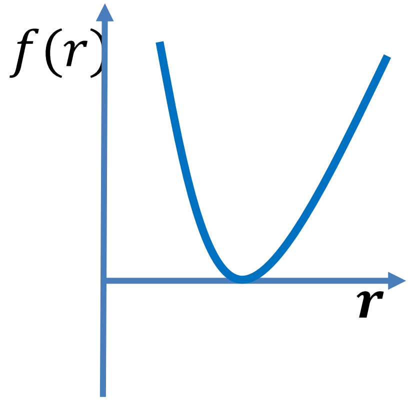
$$\boldsymbol{f'(r_H) = 0} \Rightarrow \frac{\mu}{r_H^{d-3}} = 2\frac{d-3}{2}\frac{q^2}{r_H^{2(d-3)}} \quad (2)$$

$$\mu_{ex} = 2r_{ex}^{d-3}\left(k + \frac{(d-2)r_{ex}^2}{(d-3)\ell^2}\right)$$

$$q_{ex}^2 = \frac{2r_{ex}^{2(d-3)}}{(d-3)^2}\left(\frac{(d-1)r_{ex}^2}{\ell^2} + (d-3)k\right)$$

Note $\mu \geq \mu_{ex} \quad (k = +1)$

there are extreme solns only if Λ < 0 and/or k = 1.



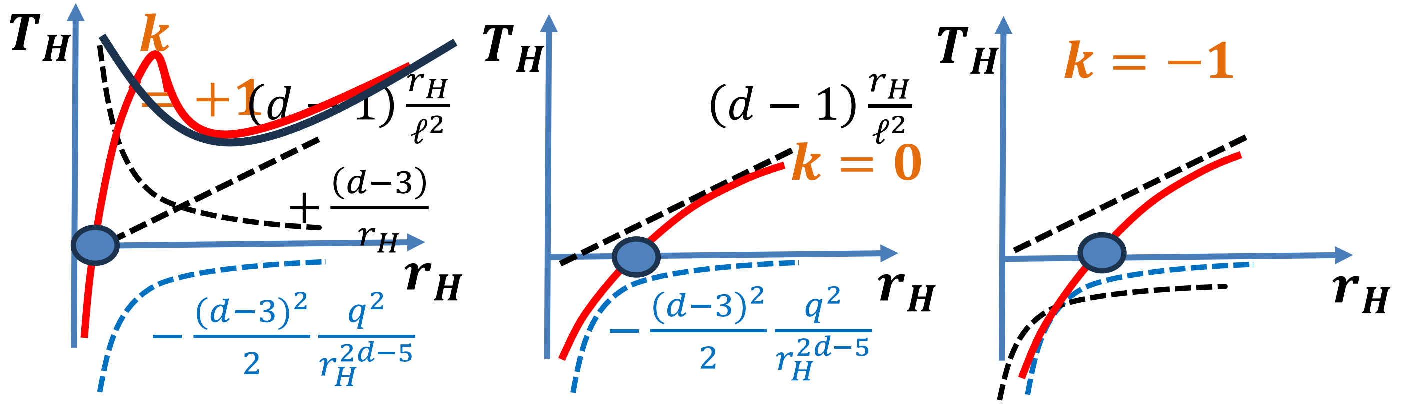
Extremal black hole is non-supersymmetric

$$\mu \geq \mu_e(q,\ell) > \mu_{SUSY}(q,\ell) \equiv 2\sqrt{\frac{d-3}{2}}\,q$$

RN AdS BH

$$f(r) = k - \frac{\mu}{r^{d-3}} + \frac{d-3}{2}\frac{q^2}{r^{2(d-3)}} + \frac{r^2}{l^2}$$

$$f'(r) = (d-3)\frac{\mu}{r^{d-2}} - (d-3)^2\frac{q^2}{r^{2d-5}} + 2\frac{r}{\ell^2}$$

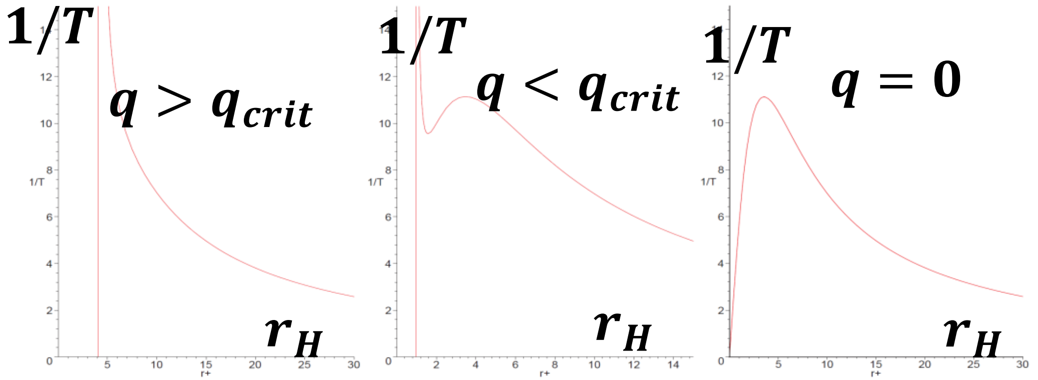


Temperature

$$T_H = \frac{1}{4\pi}f'(r_H) = \frac{1}{4\pi}\left((d-3)\frac{\mu}{r_H^{d-2}} - (d-3)^2\frac{q^2}{r_H^{2d-5}} + 2\frac{r_H}{\ell^2}\right)$$

$$= \frac{1}{4\pi}\left(\frac{(d-3)k}{r_H} - \frac{(d-3)^2}{2}\frac{q^2}{r_H^{2d-5}} + (d-1)\frac{r_H}{\ell^2}\right)$$

$$= \frac{1}{4\pi}\frac{1}{r_H}\left((d-3)k - \frac{(d-3)^2}{2}\frac{q^2}{r_H^{2(d-3)}} + (d-1)\frac{r_H^2}{\ell^2}\right)$$



Note

U(1) charge allows the extremal BH.

Free Energy (fixed potential)

Euclidean Action

$$I_E = \frac{1}{16\pi G}\int_{\mathcal{M}} d^d x\,\sqrt{-g}\left[R + \frac{(d-1)(d-2)}{\ell^2}\right] + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\,\sqrt{-h}\mathcal{K} - I_{\text{subtr}}$$

$$g = \mu - Ts - \phi q = f - \phi q$$

$$= \frac{r_H^{d-3}}{d-2}\left(k + \frac{(d-3)(2d-5)}{2}\frac{q^2}{r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}\right) - (d-3)\frac{q^2}{r_H^{d-3}}$$

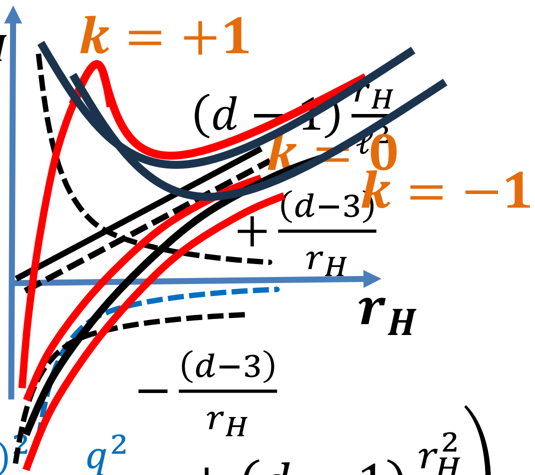
$$= \frac{r_H^{d-3}}{d-2}\left(k - \frac{(d-3)}{2}\frac{q^2}{r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}\right)$$

$$T_H = \frac{1}{4\pi}\frac{1}{r_H}\left((d-3)k - \frac{(d-3)^2}{2}\frac{q^2}{r_H^{2(d-3)}} + (d-1)\frac{r_H^2}{\ell^2}\right)$$

$$= \frac{1}{4\pi}\frac{1}{r_H}\left((d-3)k - \frac{1}{2}\phi^2 + (d-1)\frac{r_H^2}{\ell^2}\right)$$

$$\beta = \frac{4\pi r_H}{k(d-3) - \frac{1}{2}\phi^2 + (d-1)\frac{r_H^2}{\ell^2}}$$

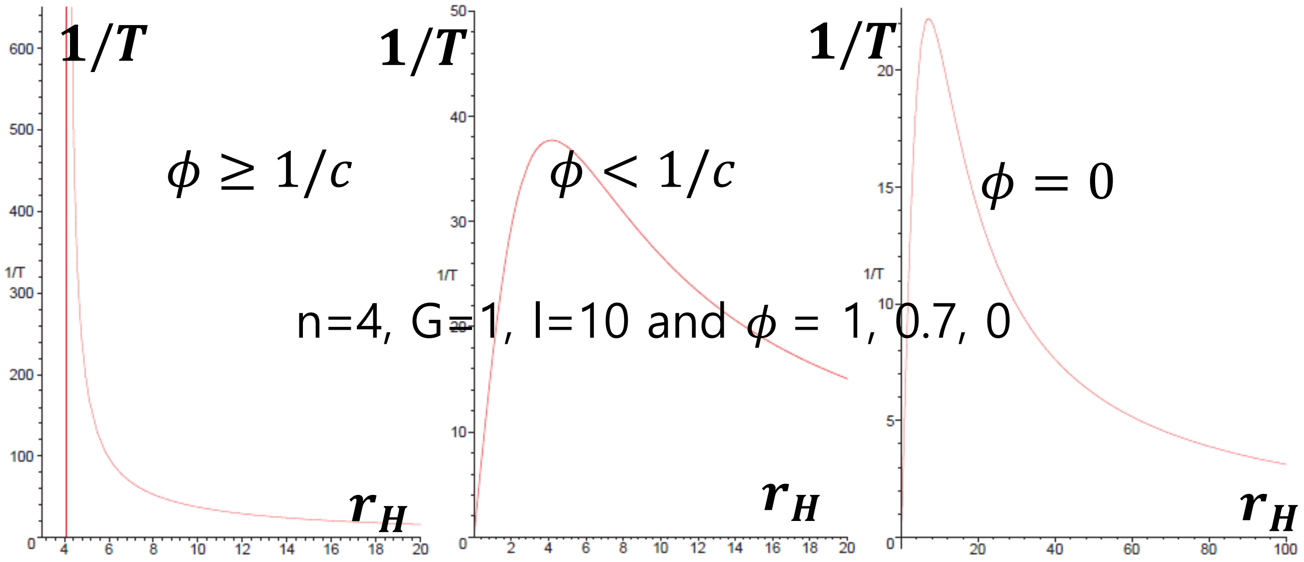
Chamblin Emparan Johnson Myers 1999 PRD



$$T_H = \frac{1}{4\pi}\frac{1}{r_H}\left((d-3)k - \frac{(d-3)^2}{2}\frac{q^2}{r_H^{2(d-3)}} + (d-1)\frac{r_H^2}{\ell^2}\right)$$

$$\mu = r_H^{d-3}\left(k + \frac{d-3}{2}\frac{q^2}{r_H^{2(d-3)}} + \frac{r_H^2}{\ell^2}\right);\quad s = \frac{4\pi}{(d-2)}r_H^{d-2}$$

$$\phi = (d-3)\frac{q}{r_H^{d-3}}$$



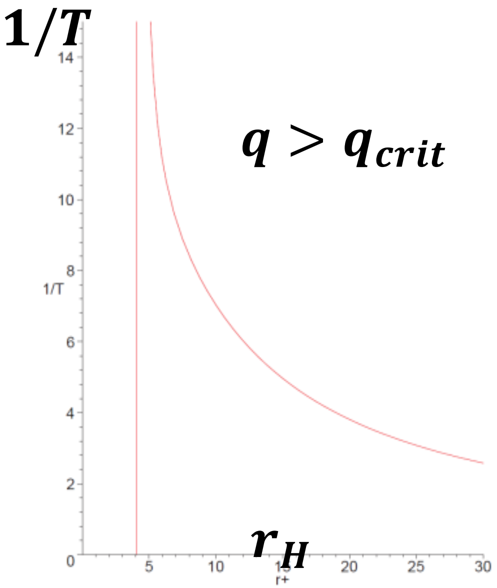
$1/T$ vs. r_H, at fixed potential

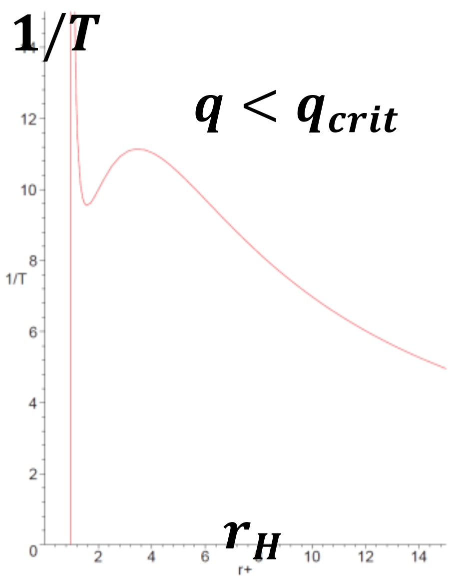
For $\phi \geq 1/c$, BH is extremal at div ($T_H = 0$ re=4.08)

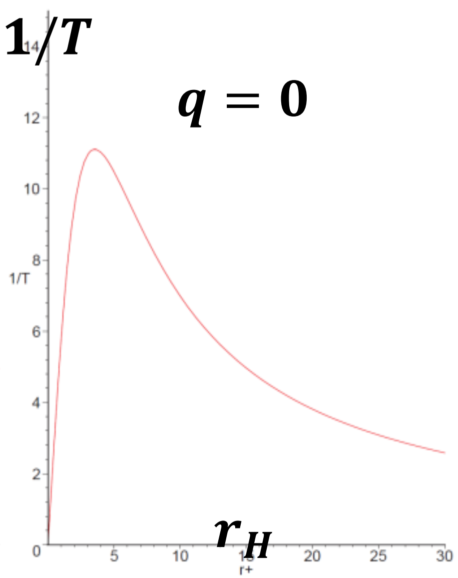
For $\phi < 1/c$, divergence goes away, in general,

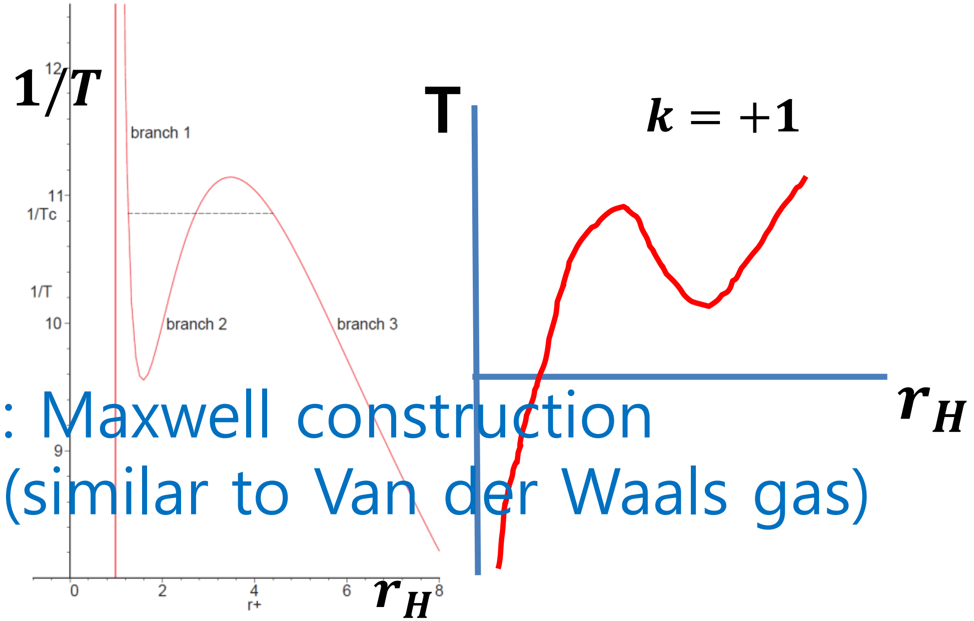
For $\phi = 0$, similar to that of AdS BH.

RN**AdS** : **Thermodynamics** $dM = TdS + \Phi dQ$

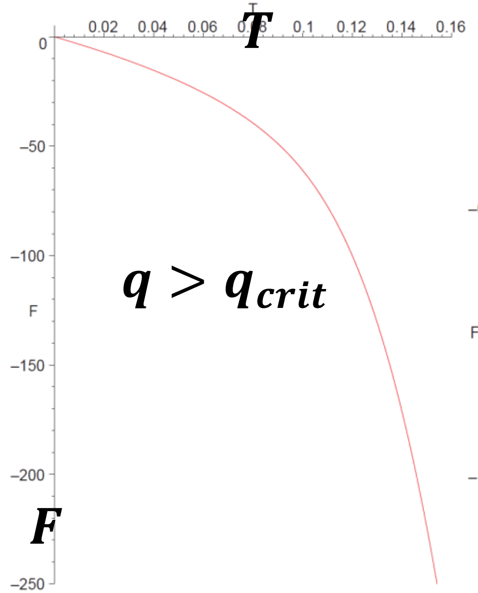


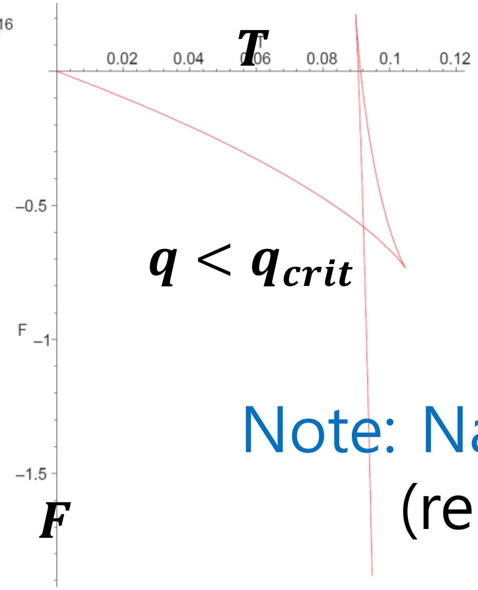


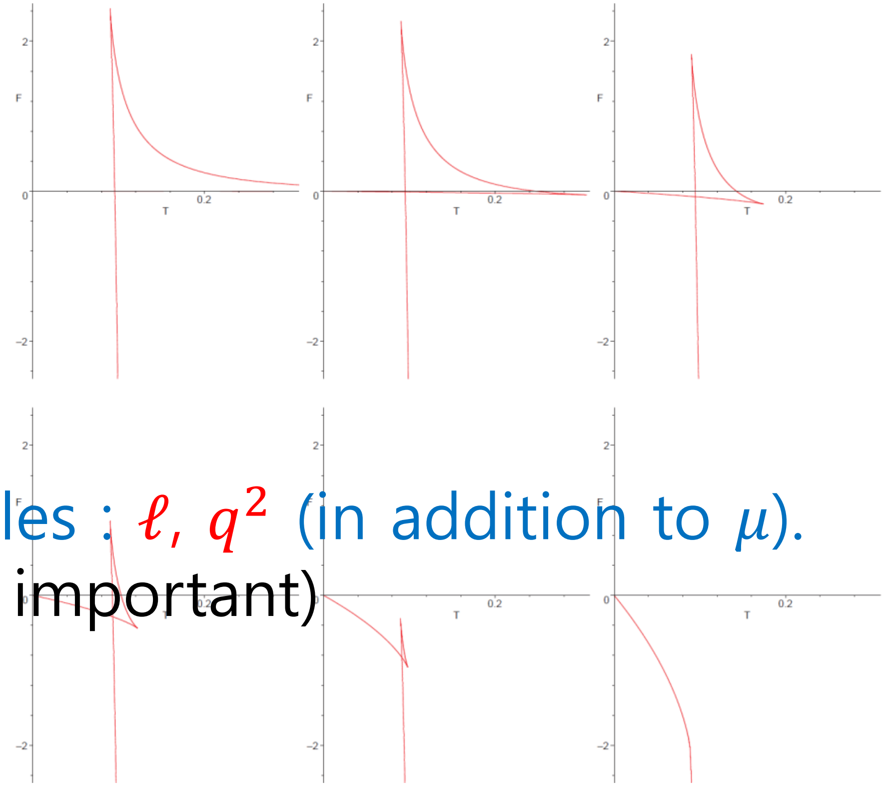




: Maxwell construction
(similar to Van der Waals gas)







Note: Natural reference scales : ℓ, q^2 (in addition to μ).
(relative magnitude is important)

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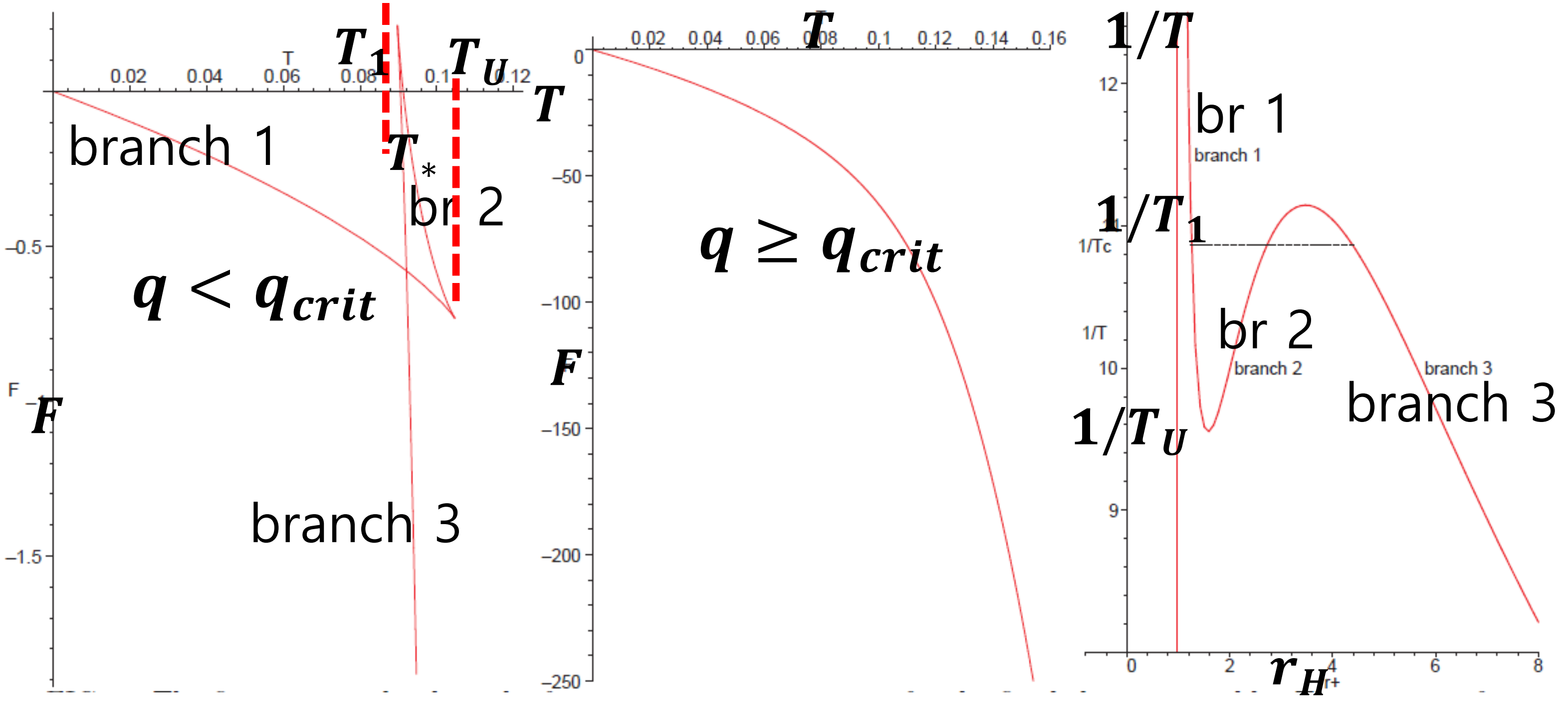
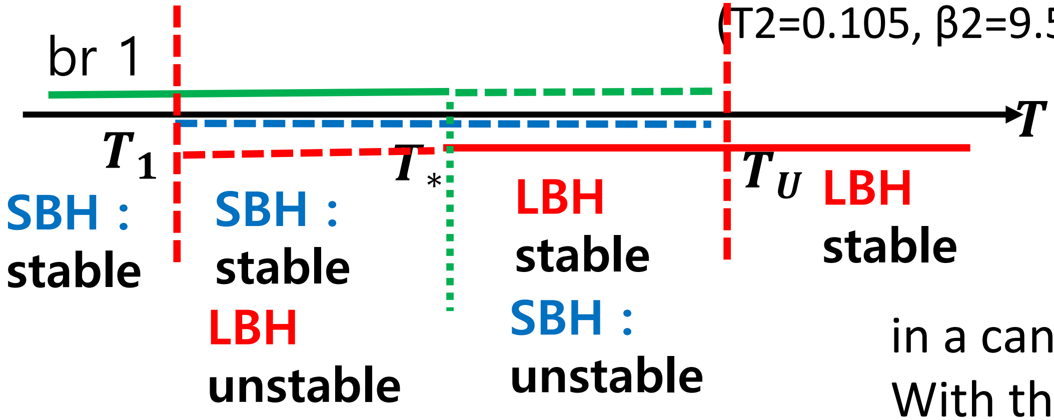
the fixed charge ensemble

$d = 5$, G=1, $\ell = 5, q = 1, 25$



(T2=0.105, β2=9.55 in the plot)



E measures the energy above the ground state, which is the extremal black hole.

three branches

For low T can only be one solution ("branch 1") for the BH radius.

At some T1=1/β1, the origin of two new branches ("branches 2 and 3") of solutions appears.

Above this T (below β1), there are therefore three distinct branches of solution until at T2=1/β2, two of the branches (1 and 2) coalesce and disappear, leaving again only a single branch (3), which persists for all higher temperatures.

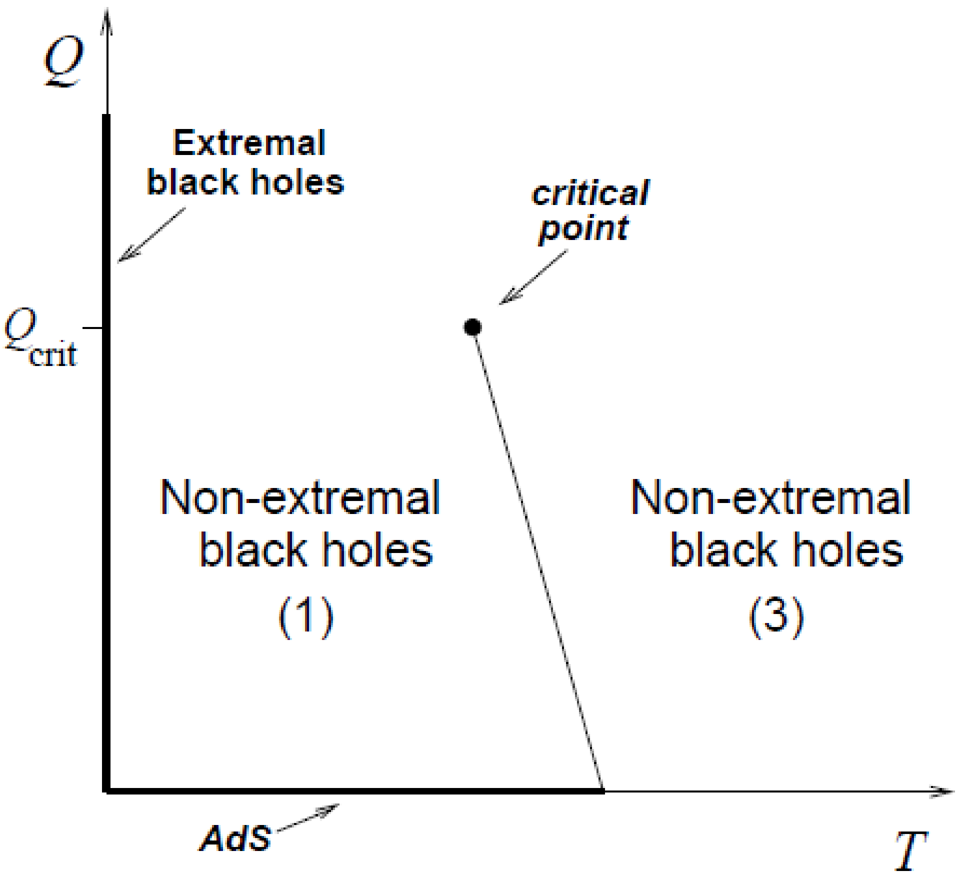
in a canonical ensemble the BH system has a **SBH/LBH ph tr.** With the increasing of the charge, such phase transition terminates at a critical point.

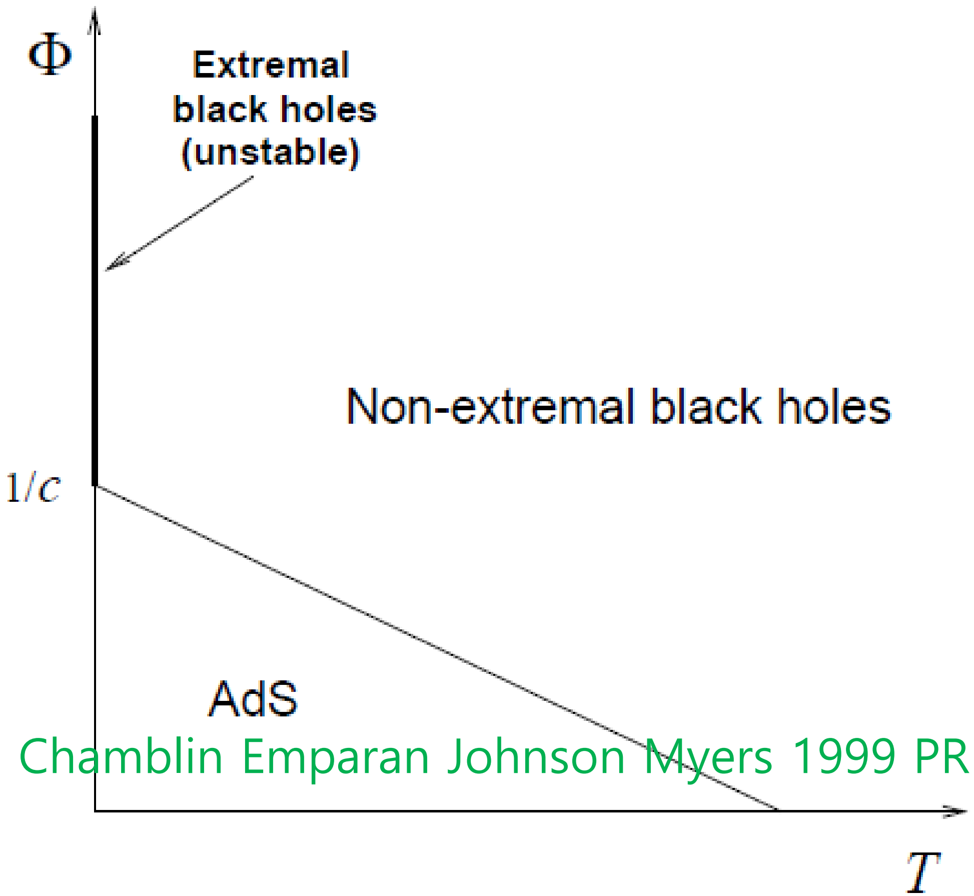
Thermodynamics RN AdS BH

Evaluate other thermodynamic quantities, such as energy, entropy, etc.

5D gauge symmetry & **Charge** ↔
(global symmetry **Charge e.g., Baryon Number**),
conjugate variable is the chemical potential

Thermodynamics are **ensemble dependant**.





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BH in G-B Gravity

1) G-B AdS BH

AdS Gauss-Bonnet BH

R. -G. Cai, PRD (2002); T. Torii, H.Maeda (2005)
S.W. Wei Y.-X. Liu, PRD (2013)

Gauss-Bonnet term
$R_{GB}^2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$

Action

$$S_{GBAdS} = \int d^d x \sqrt{-g}\left[\frac{1}{2\kappa}\left(R + \frac{(d-1)(d-2)}{\ell^2} + \alpha_{GB} R_{GB}^2\right)\right] + S_m + S_b$$

$$S_m = -\frac{1}{4\pi g^2}\int d^d x \sqrt{-g}\, F_{\mu\nu}F^{\mu\nu} \qquad \kappa = 8\pi G, \quad g = \det g_{\mu\nu};$$

$$2\Lambda = -\frac{(d-1)(d-2)}{\ell^2} \xrightarrow{d=4} -\frac{6}{\ell^2}; \xrightarrow{d=5} -\frac{12}{\ell^2}$$

Note: Dimension (c=1)

$[S] = ML\,; [G] = \frac{L^{d-3}}{M}; \quad [\ell^2] = L^2 = [\alpha_{GB}];$

$[\mu] = L^{d-3}; \ [q^2] = L^{2(d-3)};$

Ex) $d = 4 \to \Sigma_k^2 = \frac{\tilde{\Sigma}_k^2}{\Gamma}; \tilde{\Sigma}_k^2 = S^2, T^2, H^2$

Ex) $d = 5 \to$ similar to $d = 4$

Ex) $d > 5 \to \Sigma_k^{d-2}$:rich structure e.g., nonconst curv. etc.

Eqns of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda\, g_{\mu\nu} + \alpha_{GB} H_{\mu\nu} = \kappa T_{\mu\nu} = 0 \quad \text{(Einstein Eq.)}$$

$$H_{\mu\nu} = 2\left(RR_{\mu\nu} - 2R_{\mu\alpha}R^{\alpha}{}_{\nu} - 2R^{\alpha\beta}R_{\mu\nu\alpha\beta} + R_{\mu}{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}\right) - \frac{1}{2}g_{\mu\nu}R_{GB}^2$$

$$\nabla_\alpha F^{\alpha\mu} = 0 \quad \text{(Maxwell Eq.)}$$

Ansatz for Vac ($T_{\mu\nu} = 0$) solutions

$$ds^2 = -f(r)\, dt^2 + f^{-1}(r)\, dr^2 + r^2 d\Sigma_k^{d-2}$$

$d\Sigma_k^{d-2} = h_{ij}\,(x)\, dx^i dx^j$ Metic

Σ_k^{d-2}: Einstein mfld (codim 2) ($R_{ij} = (d-3)k h_{ij}$)

$\Sigma_k^{d-2} = \int d^{d-2}x\sqrt{|h_{ij}|}$; Volume $R = (d-2)(d-3)k$ const curv

$$d\Sigma_k^{d-2} = \begin{cases} d\Omega_{d-2}^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\Omega_{d-4}^2 & k = +1 \text{ sph} \\ \sum_{i=1}^{d-2} d\,x_i^2 = d\theta^2 + d\phi^2 + dx_{d-4}^2 & k = 0 \text{ plane} \\ dH_{d-2}^2 = d\theta^2 + \sinh^2\theta\, d\phi^2 + \cosh^2\theta\, d\Omega_{d-4}^2 & -1 \text{ hyperbol} \end{cases}$$

Equation for the metric function $f(r)$

$$\tilde{\alpha} \equiv \alpha = (d-3)(d-4)\alpha_{GB}$$

$$rf' - (d-3)(k-f) - \frac{d-1}{\ell^2}r^2 + \frac{\tilde{\alpha}}{r^2}(k-f)\{2rf' - (d-5)(k-f)\} = -\frac{\kappa q^2}{4(d-2)\pi r^{2d-6}}$$

RN GB AdS BH solution

Horizon $f(r_H)=0$ & (μ–r_H) relation

At Horizon r_H, $f(r_H)=0$

$$\pm\left(1+\frac{2\tilde{\alpha}k}{r_H^2}\right)=\sqrt{1+\frac{4\tilde{\alpha}}{r_H^2}\left[\frac{\mu}{r_H^{d-3}}-\frac{(d-3)}{2}\frac{q^2}{2r_H^{2(d-3)}}-\frac{r_H^2}{\ell^2}\right]}$$

$$k+\frac{\tilde{\alpha}k^2}{r_H^2}=\left(\frac{\mu}{r_H^{d-3}}-\frac{(d-3)q^2}{2r_H^{2(d-3)}}-\frac{r_H^2}{\ell^2}\right)$$

Mass

$$\mu=r_H^{d-3}\left\{\frac{(d-3)}{2}\frac{q^2}{r_H^{2(d-3)}}+k\left(1+\frac{\tilde{\alpha}k}{r_H^2}\right)+\frac{r_H^2}{\ell^2}\right\}$$

$$=r_H^{d-3}\left\{\frac{(d-3)}{2}\frac{q^2}{r_H^{2(d-3)}}+\frac{\tilde{\alpha}k^2}{r_H^2}+k+\frac{r_H^2}{\ell^2}\right\}$$

$$=\frac{(d-3)}{2}\frac{q^2}{r_H^{(d-3)}}+r_H^{d-3}k\left(1+\frac{\tilde{\alpha}k}{r_H^2}\right)+\frac{r_H^{d-1}}{\ell^2}$$

$$=\frac{(d-3)}{2}\frac{q^2}{r_H^{(d-3)}}+\tilde{\alpha}k^2r_H^{d-5}+kr_H^{d-3}+\frac{r_H^{d-1}}{\ell^2}$$

$$\tilde{\alpha}=\alpha=(d-3)(d-4)\alpha_{GB}$$

$$f(r)=k+\frac{r^2}{2\tilde{\alpha}}\left(1\mp\sqrt{1-\frac{4\tilde{\alpha}}{\ell^2}}\sqrt{1+\frac{\bar{\mu}}{r^{d-1}}-\frac{\bar{q}^2}{r^{2(d-2)}}}\right)$$

$$\bar{\mu}=\frac{4\tilde{\alpha}}{1-\frac{4\tilde{\alpha}}{\ell^2}}\mu\qquad \bar{q}^2=\frac{2\tilde{\alpha}(d-3)}{1-\frac{4\tilde{\alpha}}{\ell^2}}q^2$$

$$\pm\left(1+\frac{2\tilde{\alpha}k}{r_H^2}\right)=\sqrt{1-\frac{4\tilde{\alpha}}{\ell^2}}\sqrt{1+\frac{\bar{\mu}}{r^{d-1}}-\frac{\bar{q}^2}{r^{2(d-2)}}}$$

$$\left(1+\frac{4\tilde{\alpha}k}{r_H^2}+\frac{4\tilde{\alpha}^2k^2}{r_H^4}\right)=\left(1-\frac{4\tilde{\alpha}}{\ell^2}\right)\left(1+\frac{\bar{\mu}}{r^{d-1}}-\frac{\bar{q}^2}{r^{2(d-2)}}\right)$$

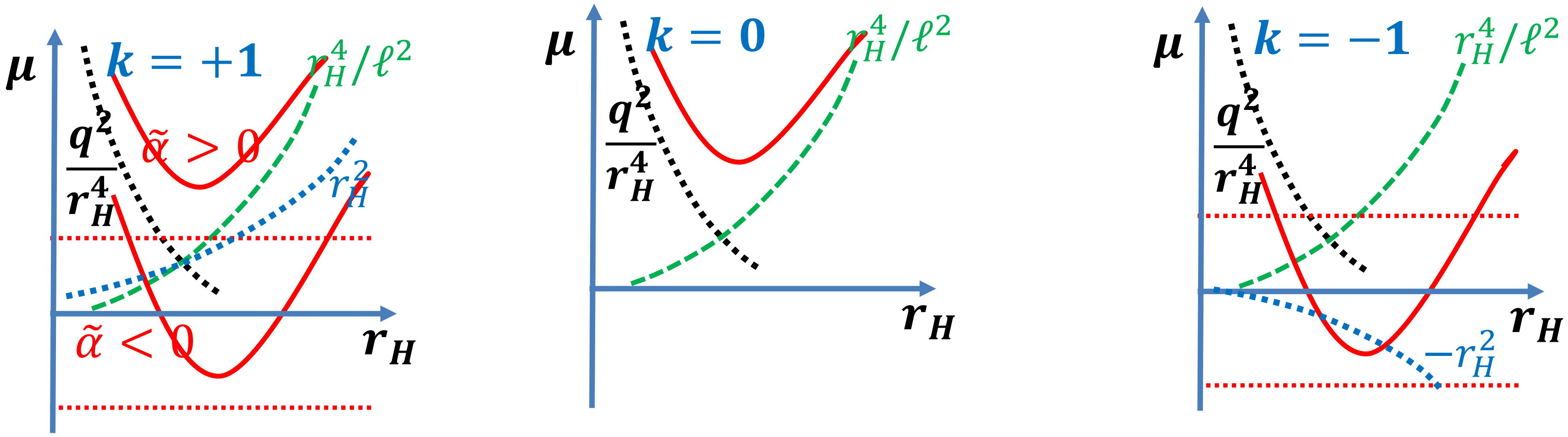
$$\left(1-\frac{4\tilde{\alpha}}{\ell^2}\right)\frac{\bar{\mu}}{r^{d-1}}=\frac{4\tilde{\alpha}}{r_H^2}k\left(1+\frac{\tilde{\alpha}k}{r_H^2}\right)+\left(1-\frac{4\tilde{\alpha}}{\ell^2}\right)\frac{\bar{q}^2}{r^{2(d-2)}}+\frac{4\tilde{\alpha}}{\ell^2}$$

$$\frac{\bar{\mu}}{4\tilde{\alpha}}=r_H^{d-3}\left\{k\frac{\left(1+\frac{\tilde{\alpha}k}{r_H^2}\right)}{1-\frac{4\tilde{\alpha}}{\ell^2}}+\frac{1}{4\tilde{\alpha}}\frac{\bar{q}^2}{r_H^{2(d-3)}}+\frac{r_H^2}{\ell^2}\right\}$$

$\frac{2\tilde{\alpha}k}{r_H^2}>-1;\ r_H^2<-2\tilde{\alpha}k$ (Upper sign + branch)

$\frac{2\tilde{\alpha}k}{r_H^2}<-1;\ r_H^2>-2\tilde{\alpha}k$ (Lower sign – branch)

$$\mu = \frac{q^2}{r_H^4} + \tilde{\alpha} k^2 + k r_H^2 + \frac{r_H^4}{\ell^2} \quad (d = 5)$$



At horizon,

$$f(r_H) = 0 = k + \frac{r_H^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1 + \frac{4\tilde{\alpha}}{r_H^2}\left(\frac{\mu}{r_H^{d-3}} - \frac{(d-3)q^2}{2r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}\right)}\right)$$

$$\pm\left(1 + \frac{2\tilde{\alpha}k}{r_H^2}\right) = \sqrt{1 + \frac{4\tilde{\alpha}}{r_H^2}\left(\frac{\mu}{r_H^{d-3}} - \frac{(d-3)q^2}{2r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}\right)}$$

$$1 + \frac{4\tilde{\alpha}k}{r_H^2} + \frac{4\tilde{\alpha}^2k^2}{r_H^4} = 1 + \frac{4\tilde{\alpha}}{r_H^2}\left(\frac{\mu}{r_H^{d-3}} - \frac{(d-3)q^2}{2r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}\right)$$

If $k = 0$,

$$f(r_H) = 0 = \frac{r_H^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1 + \frac{4\tilde{\alpha}}{r_H^2}\left(\frac{\mu}{r_H^{d-3}} - \frac{(d-3)q^2}{2r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}\right)}\right)$$

$$\pm 1 = \sqrt{1 + \frac{4\tilde{\alpha}}{r_H^2}\left(\frac{\mu}{r_H^{d-3}} - \frac{(d-3)q^2}{2r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}\right)}$$

Lower $(-)$ branch doesn't exist.

$$0 = \frac{\mu}{r_H^{d-3}} - \frac{(d-3)q^2}{2r_H^{2(d-3)}} - \frac{r_H^2}{\ell^2}$$

Hawking Temperature T_H

R. -G. Cai, PRD (2002).

$$f(r) = k + \frac{r^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1 + \frac{4\tilde{\alpha}}{r^2}\left(\frac{\mu}{r^{d-3}} - \frac{(d-3)q^2}{2\,r^{2(d-3)}} - \frac{r^2}{\ell^2}\right)}\right)$$

Horizon $\mu = r_H^{d-3}\left\{k + \frac{\tilde{\alpha}k^2}{r_H^2} + \frac{(d-3)}{2}\frac{q^2}{r_H^{2(d-3)}} + \frac{r_H^2}{\ell^2}\right\}$

$$f'(r_H) = -\frac{2k}{r_H} + \frac{\left\{+\frac{\mu(d-1)}{r_H{}^{d-3}} - (d-3)(d-2)\frac{q^2}{r_H{}^{2(d-3)}}\right\}}{r_H\left(1+\frac{2\tilde{\alpha}k}{r_H^2}\right)}$$

$$= \frac{1}{r_H\left(1+\frac{2\tilde{\alpha}k}{r_H^2}\right)}\left\{-2k - \frac{4\tilde{\alpha}k^2}{r_H^2} + (d-1)\frac{\mu}{r_H^{d-3}} - (d-3)(d-2)\frac{q^2}{r_H^{2(d-3)}}\right\}$$

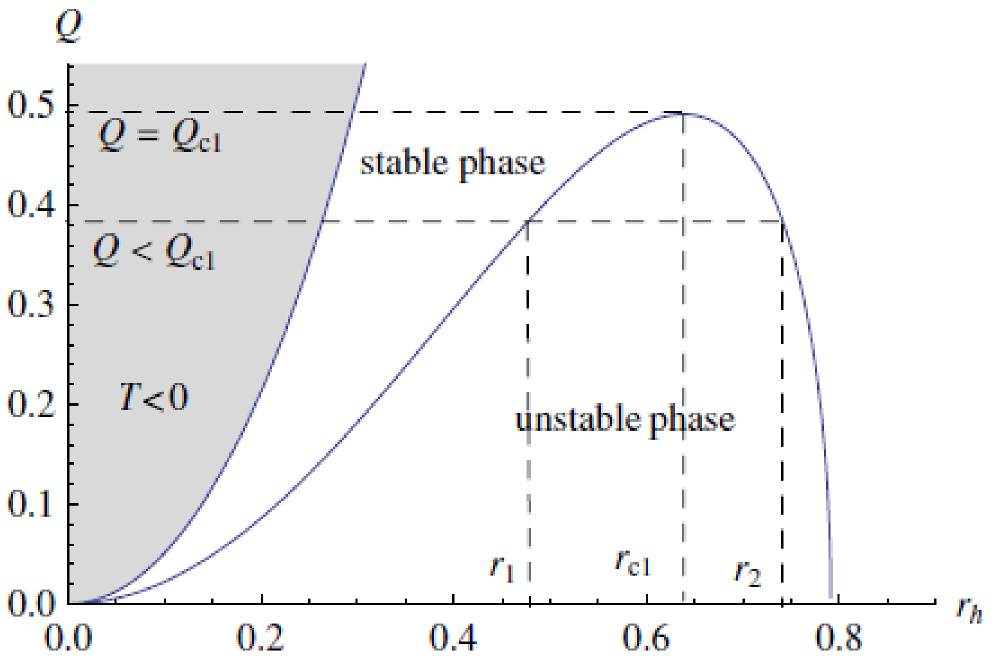
$$= \frac{1}{r_H\left(1+\frac{2\tilde{\alpha}k}{r_H^2}\right)}\left\{(d-3)k + (d-5)\frac{\tilde{\alpha}k^2}{r_H^2} - \frac{(d-3)^2}{2}\frac{q^2}{r_H^{2(d-3)}} + (d-1)\frac{r_H^2}{\ell^2}\right\}$$

$$T_H = \frac{1}{4\pi}f'(r_H)$$

$$= \frac{1}{4\pi}\frac{1}{r_H\left(1+\frac{2\tilde{\alpha}k}{r_H^2}\right)}\left\{-\frac{(d-3)^2}{2}\frac{q^2}{r_H^{2(d-3)}} + (d-5)\frac{\tilde{\alpha}k^2}{r_H^2} + (d-3)k + (d-1)\frac{r_H^2}{\ell^2}\right\}$$

$$= \frac{1}{4\pi}\frac{1}{\left(1+\frac{2\tilde{\alpha}k}{r_H^2}\right)}\left\{-\frac{(d-3)^2}{2}\frac{q^2}{r_H^{2(d-3)+1}} + (d-5)\frac{\tilde{\alpha}k^2}{r_H^3} + \frac{(d-3)k}{r_H} + (d-1)\frac{r_H}{\ell^2}\right\}$$

3-2) RNAdS in Einstein-Gauss-Bonnet : Phases



$$f(r) = k + \frac{r^2}{2\tilde{\alpha}}\left(1 \mp \sqrt{1-\frac{4\alpha}{\ell^2}}\sqrt{1+\frac{\mu}{r^{d-1}}-\frac{q^2}{r^{2(d-2)}}}\right)$$

Note : $Q^2 = \frac{\pi(d-2)(d-3)\left(1-\frac{4\alpha}{\ell^2}\right)}{2G\alpha}q^2$

Wei & Liu, PRD (2013) mass

$$M = \frac{(d-1)Q^2 r_H^8 + 2\pi r_H^{2d}(d-3)\left((d^2-3d+2)(kr_H^2+k^2\alpha)-2\Lambda r_H^4\right)}{8\pi^2(d^2-4d+3)r_H^{d+5}}\Sigma_{\mathrm{d}-2}^{k}$$

Hawking Temperature

$$T_H = \frac{1}{4\pi}f'(r_H) = \frac{-Q^2 r_H^8 + 2\pi r_H^{2d}\left((d-2)k\left((d-3)r_H^2+(d-5)k\alpha\right)-2\Lambda r_H^4\right)}{32\pi^2(d-2)r_H^{2d+1}(2k\alpha+r_H^2)}$$

Near Extremal behavior etc.
I. Jeon, BHL, W. Lee, M. Mishra, (2025)

BH in G-B Gravity

2) dEGB BH

4. dEGB theory - Black Holes

Guo,Ohta & Torii, Prog.Theor.Phys. (2008); (2009); (2010);
Maeda,Ohta Sasagawa, PRD(2009);(2011) Ohta Torii, PRD (2013).

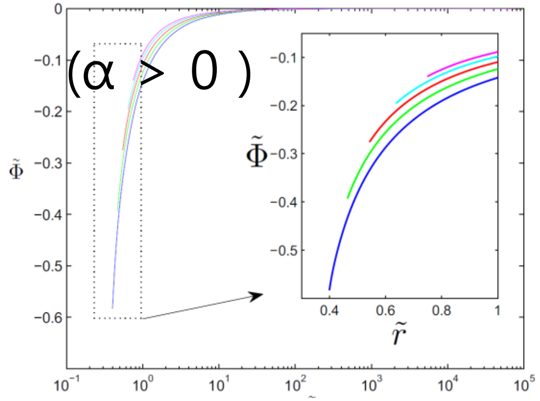
BHL, W. Lee, D. Rho, PRD(2019)

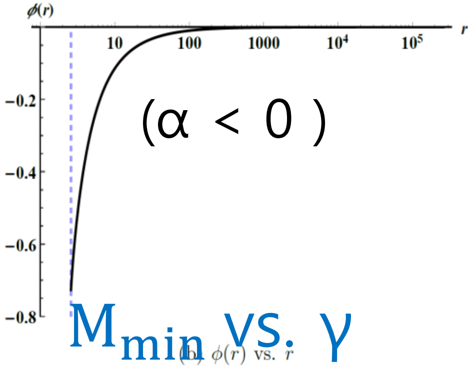
$$S_{dEGB} = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa}\left(R + \alpha e^{-\gamma\phi(r)} R^2_{GB}\right) + \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}^{matt}_m\right]$$

New Properties of BHs

1) Scalar Hair - BH hair ↘ as M ↗

- All DEGB BHs have hairs.
- If Φ = 0, e.o.m. impose R^2_{GB} = 0.
- (consistent with the no hair theorem).
- Hair Charge is dependent : 2nd ary charge





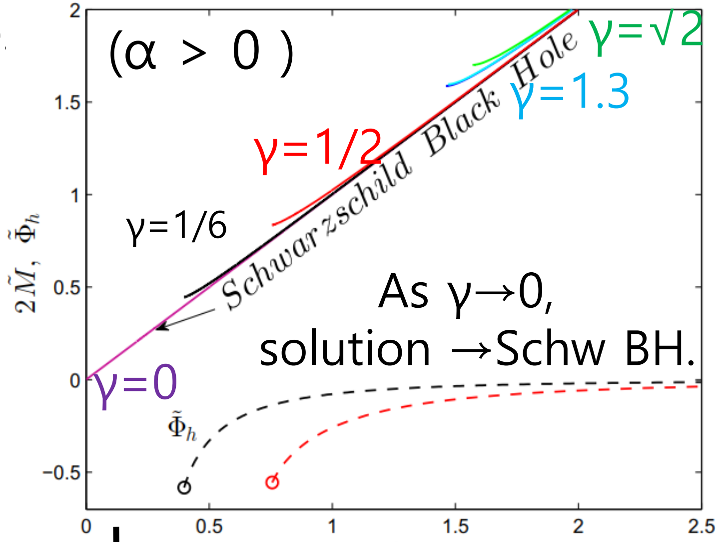
2) Minimum Mass

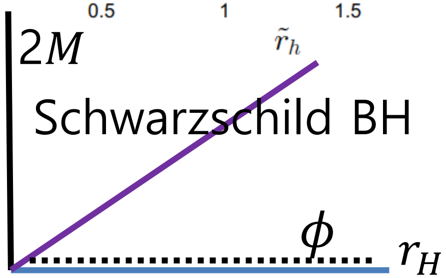
BH mass $M \geq M_{min}$ New Phase?

The BH properties strongly depends on the sign of the G-B term (as well as Λ).

Soliton Star? Black Holes

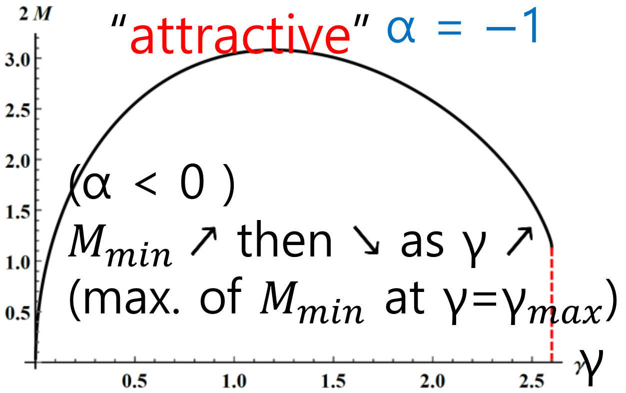
M=0 M_{min} → ∞

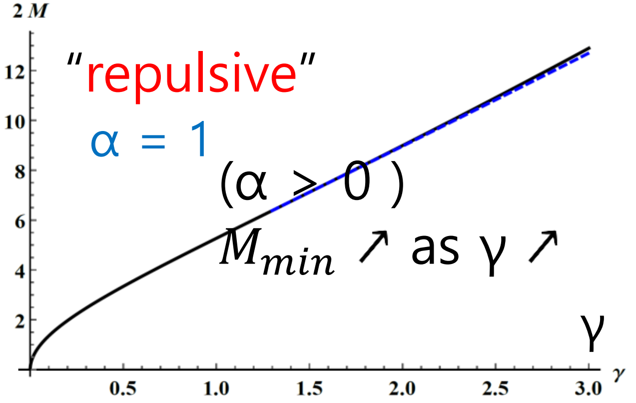




GB term → makes gravity "less attractive"
(for α >0) (making the black hole "smaller") !!!

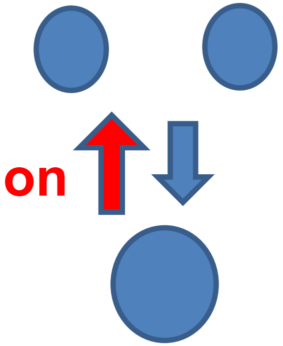
M_{min} vs. γ





3)(In)stability of the DEGB Blackholes under fragmentation

B. Gwak & BHL, PRD (2015).
B.Gwak, BHL, D. Rho, PL.B (2016)



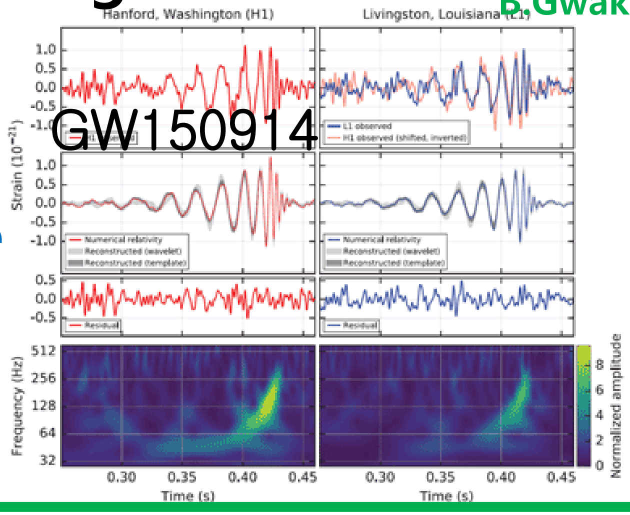
B) fragmentation

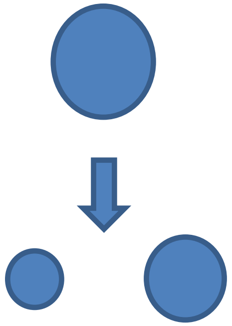
?

A) Merging

+ Gravitational Wave

Observed!





δ 1-δ

Fragmentation Process : one BHs → two BH ?

Schwarzschild BHs are marginally stable under the fragmentation of shooting off the infinitesimal mass BH .

$$\frac{S_f}{S_i} = \frac{M_1^2+M_2^2}{(M_1+M_2)^2} = \frac{(\delta r_h)^2+((1-\delta)r_h)^2}{r_h^2} = \delta^2 + (1-\delta)^2 \leq 1$$

(equality only if $\delta \to 0$))

These phenomena could happen in the theory with the higher order of curvature term with appropriate parameters.

DEGB Cosmology

- New Phases exists at high enough temperature

NEW PHASEs⟶ | ⟵Rad Dom⟶ | ⟵ Matt ⟶ | ⟵ Λ(DE) ⟶

- Effects at each stage of ΛCDM

A. Biswas, A. Kar, **BHL**, H. Lee, W. Lee, **S. Scopel**, Velasco-Sevilla, L. Yin **JCAP08 (2023) 023**

Biswas, Kar, **BHL**,H.Lee, W.Lee, **Scopel**, Velasco-Sevilla, L. Yin **JCAP (2024)**

V. Summary

V.Summary

Ex) The String theory at low Energy

→ Einstein Grav + higher curvature terms

Modified Gravity beyond Einstein needed?

Theoretical Aspect

- an **effective theory** below UV cut-off, $M_{Pl} \sim 10^{19} GeV \rightarrow$ Einstein Grav + higher curvature terms
- Standard Cosmology (**ΛCDM**) : extremely fine-tuned ($\Lambda = 2{,}888 \text{x} 10^{-122} \ell_P^{-2}$)
- **Holography**

Observational Aspect - H_0 tension, Cosmological Birefringence etc.

Modification of GR - needs to introduce additional d.o.f.

- Ex) higher derivatives : Generically, ghosts & Ostrogradsky instability :

In dim=4 the Dilaton-Einstein-Gauss-Bonnet (dEGB) Gravity (belongs to Horndeski theory)

$$S_{dEGB} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R + f(\phi) R_{GB}^2 - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_m \right]$$

$$f(\phi) = \alpha e^{\gamma\phi}$$

- **hairy Black Hole : minimum mass.**

In dim>4, GB term allows 2nd order e.o.m.

consider the Einstein-Gauss-Bonnet (EGB)- Λ Gravity (GB-AdS)

$$S_{EGB-\Lambda} = \int d^dx \sqrt{-g} \left[\frac{1}{2\kappa} (R - 2\Lambda + \alpha R_{GB}^2) + \mathcal{L}_m^{matt} \right]$$

$$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}$$

$$\kappa = 8\pi G, \quad g = \det g_{\mu\nu}$$

IV.Summary (continued)

We briefly introduced the black hole solutions, thermodynamics, and phases:

- Schwarzschild BH
- AdS Schwarzchild BH,
- RN AdS BH,
- AdS GB Black Holes
- charged GB AdS BH.
- dEGB BH, , etc

Q: What is the phase diagrams of “realistic” gravity systems,
and the properties of the corresponding quantum systems?

Q: How to deal with the system of BH + DM/DE ?
Parts of this directional research has been done and in progress

Thank you!