Nonlinear Theory for Plasma Wakefield Produced by Elliptical Beams via Data-Driven Physical Modeling

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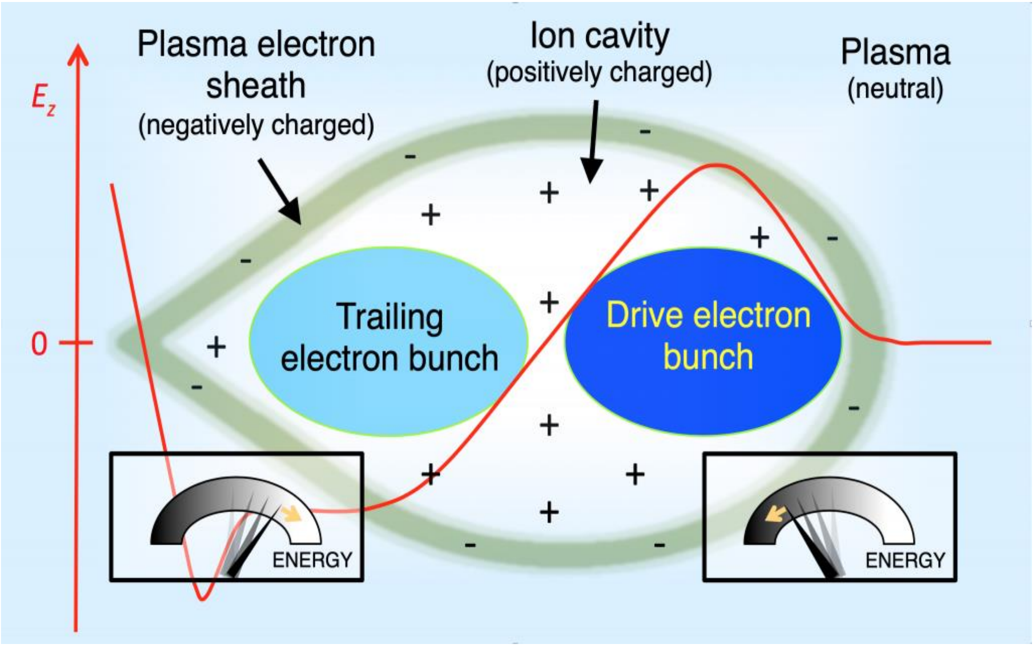
The Blowout Regime and Plasma Wakefields

- Plasma wakefield acceleration (PWFA) is a promising candidate for future linear colliders
- Electron beam traverses underdense plasma, expelling plasma electrons
- When these electrons are pulled back, a wake is formed
- A trailing beam accelerates from this wake to reach relativistic speeds

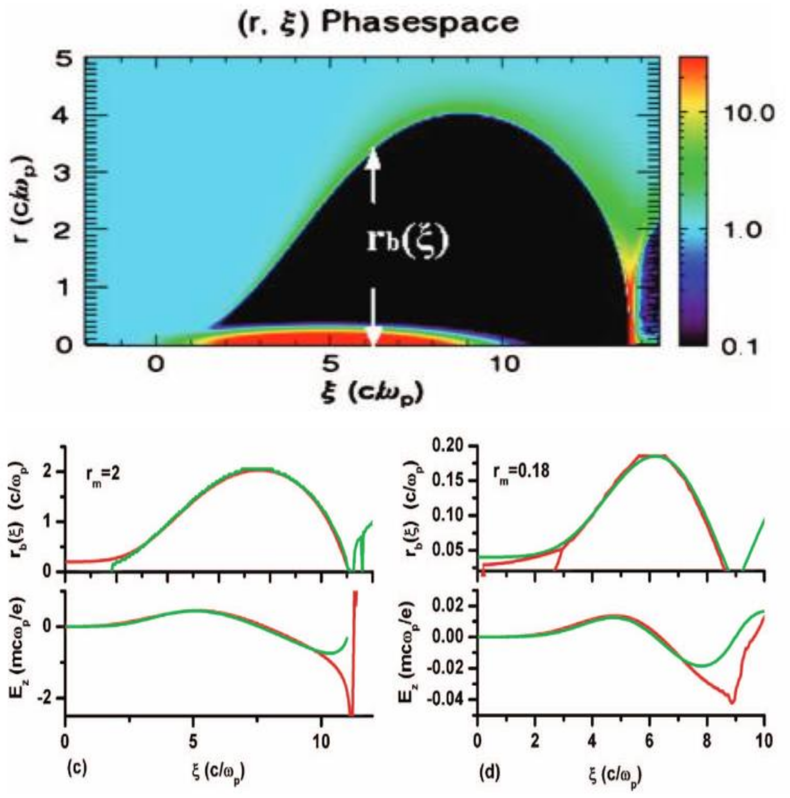


Characterizing this wakefield is of the utmost importance!

Chen, et al, Phys. Rev. Lett. 55, 1537, 1985
Rosenzweig, et al, Phys. Rev. A 44, R6189(R), 1991
Lindstrøm, et al, arXiv:2504.05558, 2025

Visualization from Laboratoire d'Optique Appliquée

Nonlinear Theory of Symmetric Driver Beams



Wakefield Equation:

$$E_z = \frac{\partial}{\partial \xi}\psi(r_\perp, \xi) \qquad (1)$$

Wake Potential Equation:

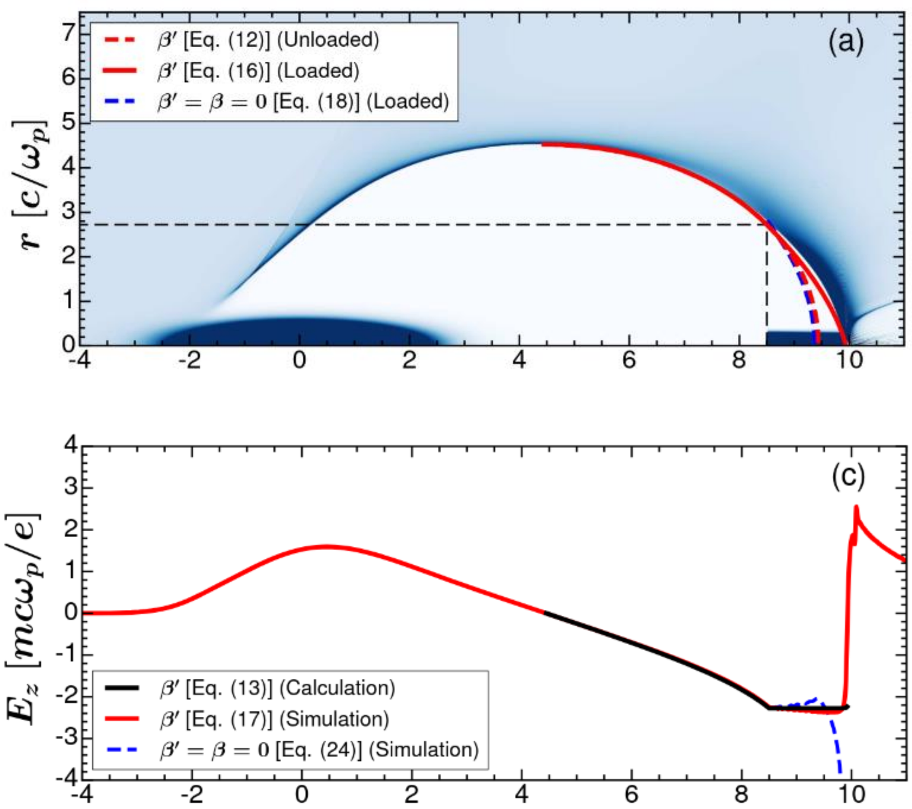
$$\psi = \frac{r_b^2(\xi)}{4}(1+\beta(\xi)) - \frac{r^2}{4} \qquad (2)$$

Blowout Trajectory Equation:

$$\frac{d}{d\xi}\left[(1+\psi)\frac{d}{d\xi}r_b\right] = r_b\left\{-\frac{1}{4}\left[1+\frac{1}{(1+\psi)^2}+\left(\frac{dr_b}{d\xi}\right)^2\right] + \left(\frac{d\sigma}{d\xi}\right)+\frac{\lambda(\xi)}{r_b^2}\right\} \qquad (3)$$

Lu, et al, Physics of Plasmas 13, 056709, 2006
Golovanov et al. Physical review letters 130, 105001 2023

Multi-Sheath Model of Symmetric Driver Beams



Updated Wake Potential:

$$\psi(r,\xi) = \frac{r_b^2(\xi)}{4}(1+\beta') - \frac{r^2}{4} \tag{4}$$

Refined β' Equation:

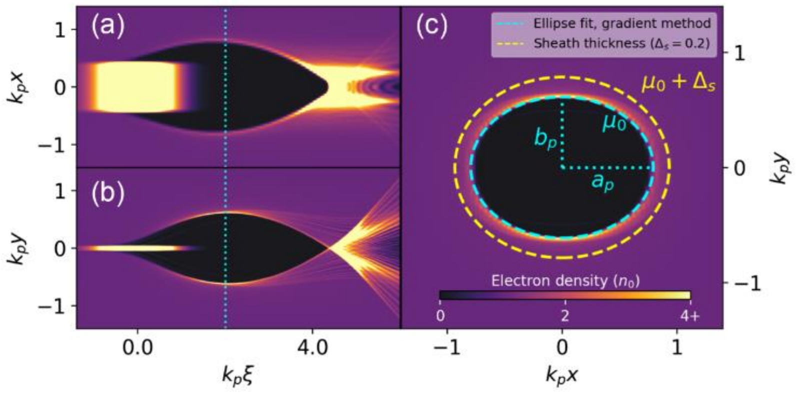
$$\begin{aligned}\beta' =& 2(1+n_1)\ln(1+\alpha_1) - 1 \\ &+ 2n_2(1+\alpha_1+\alpha_2)^2 \ln\left(1+\frac{\alpha_2}{1+\alpha_1}\right)\end{aligned} \tag{5}$$

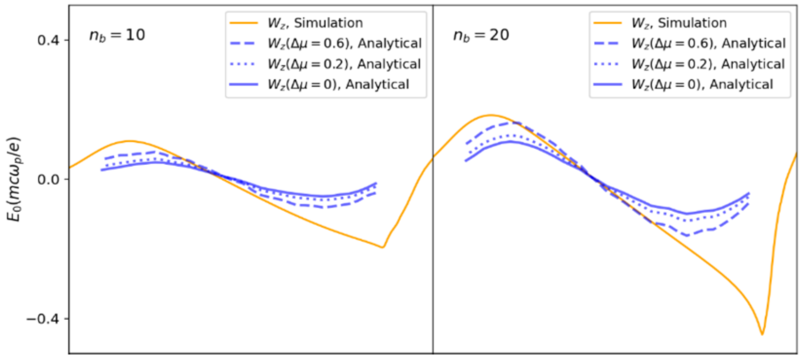
Updated Blowout Trajectory:

$$A'(r_b)\frac{d^2r_b}{d\xi^2} + B'(r_b)r_b\left(\frac{dr_b}{d\xi}\right)^2 + C'(r_b)r_b = \frac{\lambda(\xi)}{r_b} \tag{6}$$

Dalichaouch, et al, Physics of Plasmas 28, 063103, 2021

Elliptical Driver Beam Theory





Wake Potential:

$$\psi(x, y) = -\frac{x^2 b_p^2 + y^2 a_p^2 - p_p^4}{2d_p^2} \qquad (7)$$

Accelerating Wakefield:

$$W_z = E_z = \frac{\partial \psi}{\partial \xi} \qquad (8)$$

Focusing Wakefield:

$$W_y = E_y + B_x = -\frac{\partial \psi}{\partial y} \qquad (9)$$

$$W_x = E_x - B_y = -\frac{\partial \psi}{\partial x} \qquad (10)$$

Manwani et al, Phys. Rev. Lett. 135, 095001, 2025

Equation of Blowout Trajectory

Lu, et al Model

$$\frac{d}{d\xi}[(1+\psi)\frac{d}{d\xi}r_b] = r_b\left\{-\frac{1}{4}\left[1+\frac{1}{(1+\psi)^2}+(\frac{dr_b}{d\xi})^2\right] + \frac{d\sigma}{d\xi}+\frac{\lambda(\xi)}{r_b^2}\right\}$$

Our Model

$$\frac{d}{d\xi}(1+\psi(r_{bx},0,\xi))\frac{d}{d\xi}r_{bx} = -\frac{1}{2}\left[1+\frac{1}{(1+\psi(r_{bx},0,\xi))^2}+\left(\frac{dr_{bx}}{d\xi}\right)^2+\left(\frac{dr_{by}}{d\xi}\right)^2\right]\frac{r_{bx}r_{by}}{(r_{bx}+r_{by})} + \frac{\partial A_x}{\partial\xi}\bigg|_{r_{bx}} + \frac{\partial A_z}{\partial x}\bigg|_{r_{bx}}$$

$$\psi(x,y,\xi) = \psi_0(\xi) - \frac{r_{by}}{2(r_{bx}+r_{by})}x^2 - \frac{r_{bx}}{2(r_{bx}+r_{by})}y^2$$

Now, we require an analytical form for both ψ_0 **and** $\frac{\partial A_z}{\partial x}$

Introduction of QuickPIC

Quasi-Static Approximation

Cartesian coordinates $(x, y, z; t)$

Co-moving coordinates

$(x, y, \xi = ct - z; s = z)$

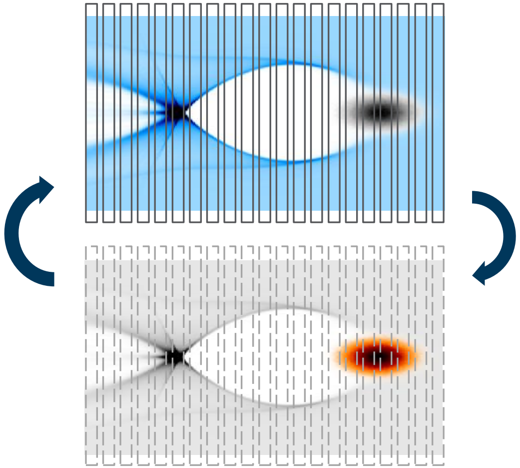
$\partial_s \ll \partial_\xi$

Plasma: $(x, y; \xi)$

Beam: $(x, y, \xi; s)$

Quasi-Static Loop

Beam frozen, plasma & field evolves



3D Loop

Plasma & field frozen, beam evolves

Huang, et al, Journal of Computational Physics 217 658–679, 2006
Weiming, et al, Journal of Computational Physics 250 165-177, 2013
Su, et al, Journal of Computational Physics 539 114225, 2025

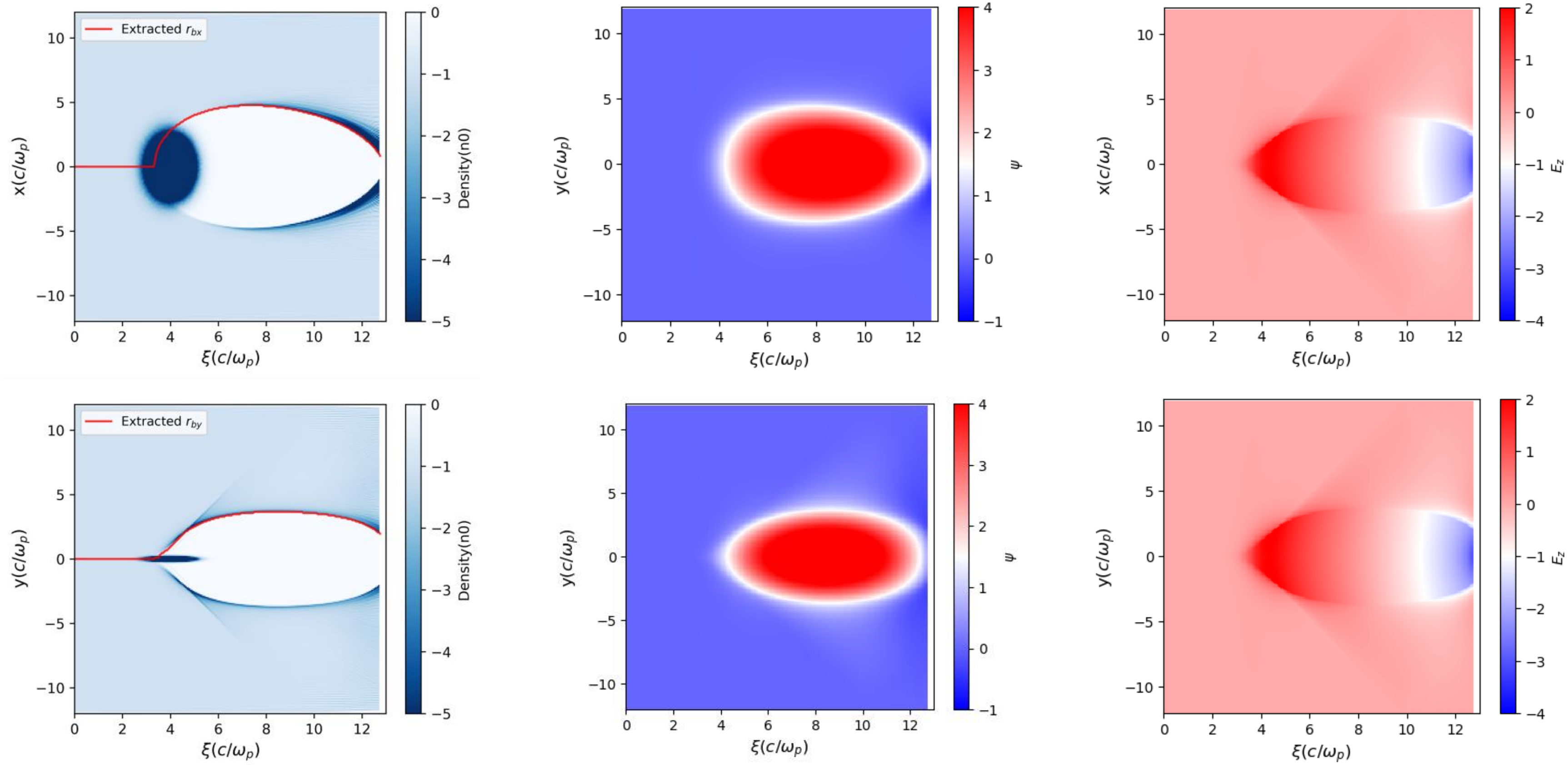
QuickPIC Dataset for Asymmetric Blowout Modeling

- 100 one-step 3D simulations from QuickPIC, the quasi-static, Particle-in-Cell simulation code
- 256x256x256 grid space spanning 3 dimensions
- Randomized asymmetric drive beam values selected from a Gaussian distribution
- Resulting dataset offers a range of blowout shapes and wakefield responses to act as training data

Beam Parameters in Dataset		
Variable	Range	Description
λ $(n_p c^2/\omega_p^2)$	[4.0, 8.0]	Normalized beam current
σ_x (c/ω_p)	[0.1, 2.0]	Beam size in the x-axis
σ_y (c/ω_p)	[0.1, 2.0]	Beam size in the y-axis
σ_z (c/ω_p)	[0.4, 0.8]	Beam size in the z-axis

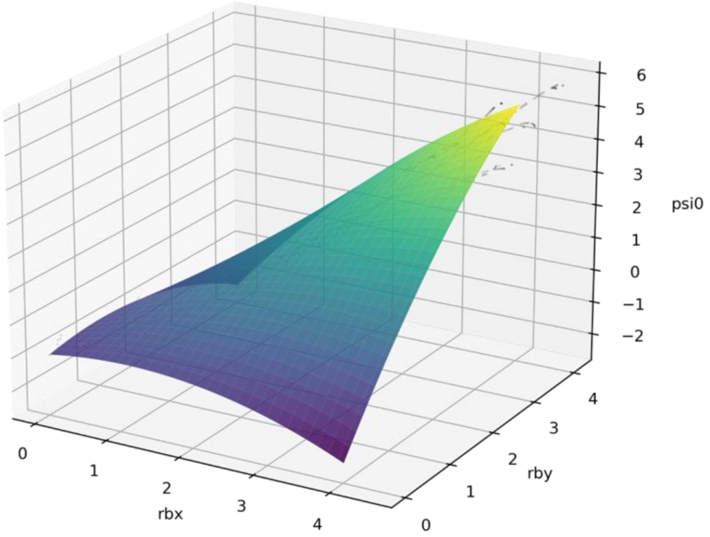
Huang, et al, Journal of Computational Physics 217 658–679, 2006
Weiming, et al, Journal of Computational Physics 250 165-177, 2013
Su, et al, Journal of Computational Physics 539 114225, 2025

Targeted Quantities from QuickPIC



ψ Term Modeling

- Utilized 10 simulations from QuickPIC augmented with multi-sheath theory to perform polynomial regression for ψ_0
- With this ψ_0 form, we can add it to established asymmetric theory to determine a form of ψ
- Later work could open up other beam parameters

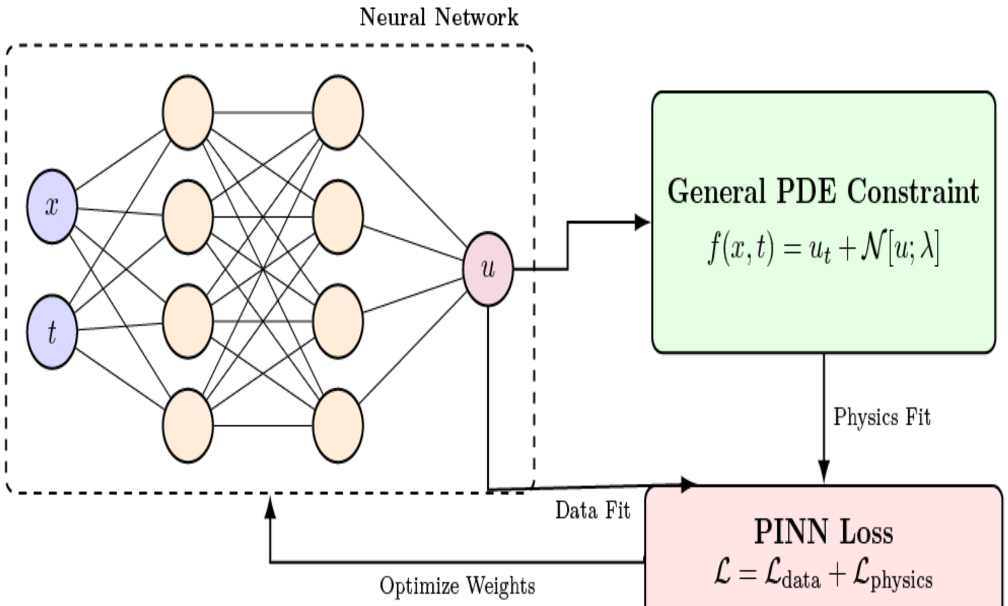


$$\psi_0(\xi) = -0.979 + 0.379(r_{bx} + r_{by}) + 0.420(r_{bx} * r_{by}) - 0.118(r_{bx}^2 + r_{by}^2) + (3.4e-5)(r_{bx}^2 * r_{by}^2)$$

$$\psi(x, y, \xi) = \psi_0(\xi) - \frac{r_{by}}{2(r_{bx} + r_{by})}x^2 - \frac{r_{bx}}{2(r_{bx} + r_{by})}y^2$$

Machine Learning and Physics Informed Neural Networks

- Neural network training is a **global optimization problem**
- Physics-Informed Neural Networks (PINNs) provide means to encode physical laws into neural networks
- This approach allows for more accurate surrogate models with a substantially lower dataset requirement



Raissi et al, Elsevier Volume 378 686-707, 2019

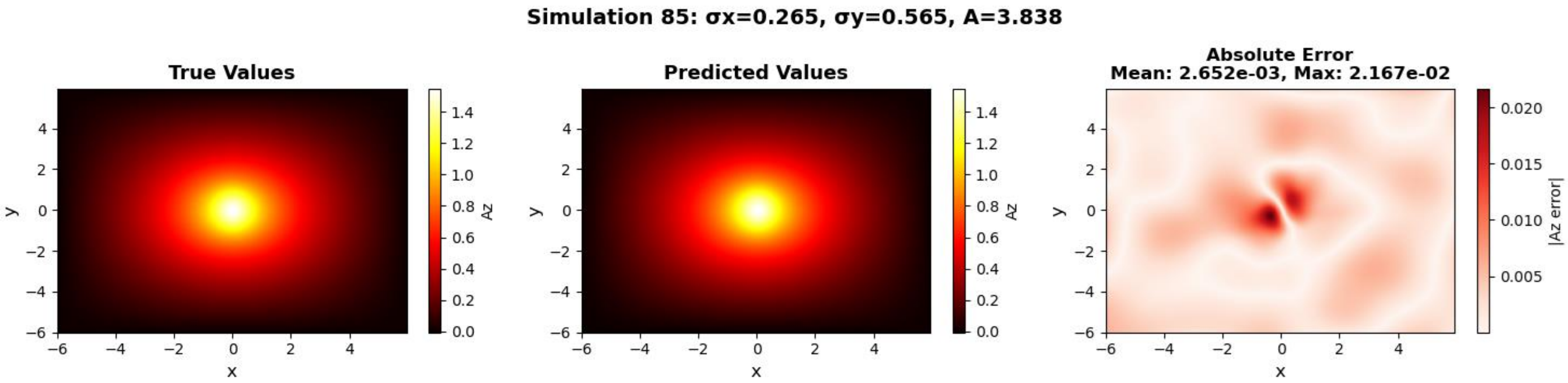
A_z Modeling - Physics Informed Neural Network

For the analytic function, A_z, we solve the 2D Poisson equation:

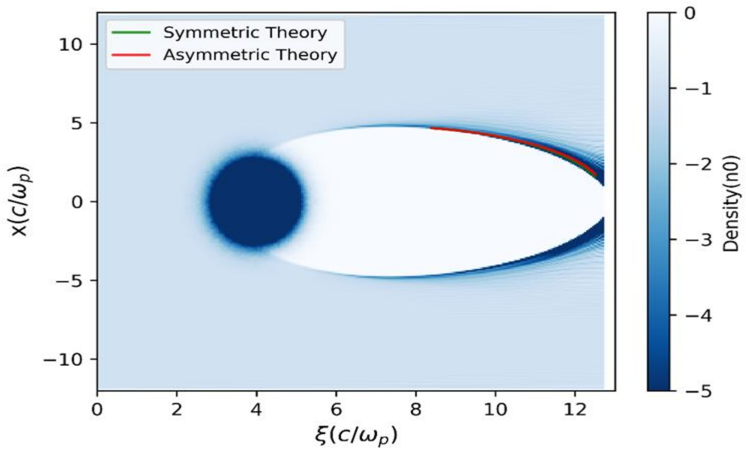
$-\nabla_\perp^2 A_z = J_z$ wherein the beam current takes the form of:

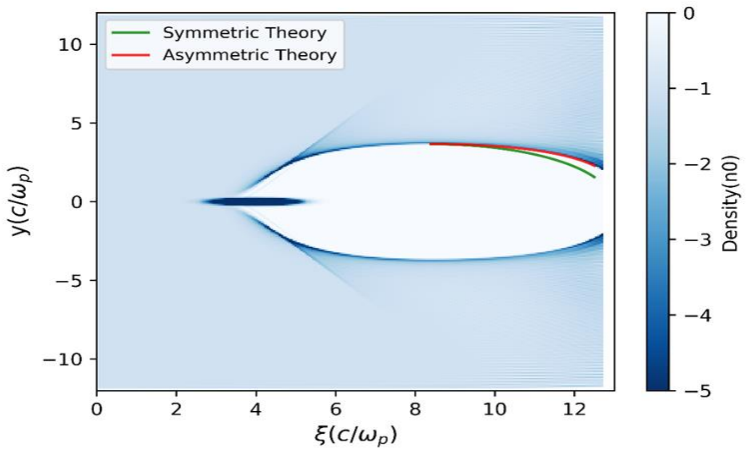
$J_z = A \cdot \exp(\frac{-x^2}{2\sigma_x^2} - \frac{-y^2}{2\sigma_y^2})$ and transverse laplacian operator: $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$

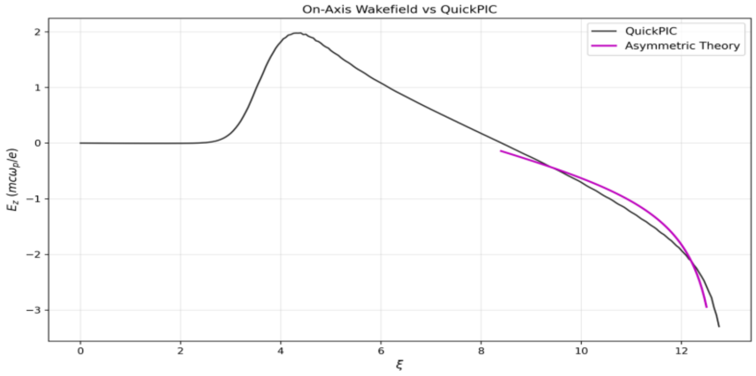
We model this function using a Physics-Informed Neural Network



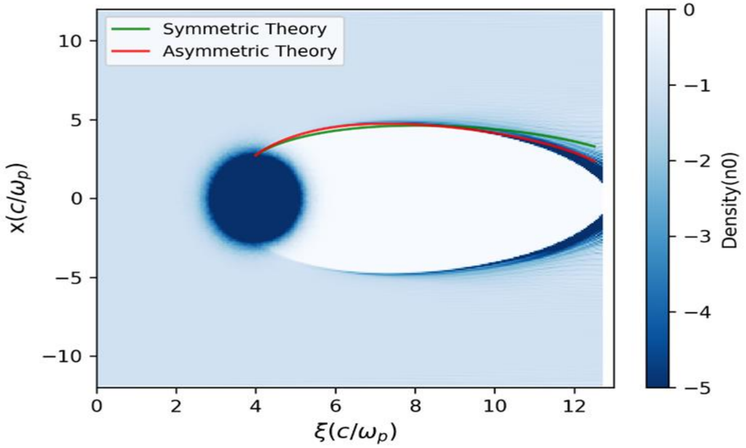
Blowout Trajectory - Center of the Blowout Predictions

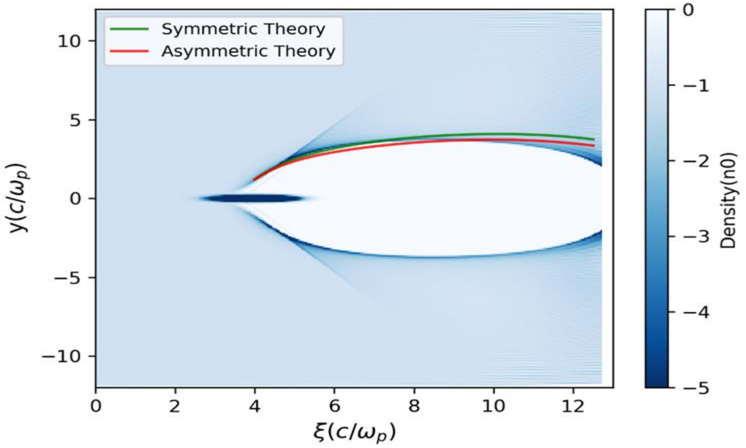


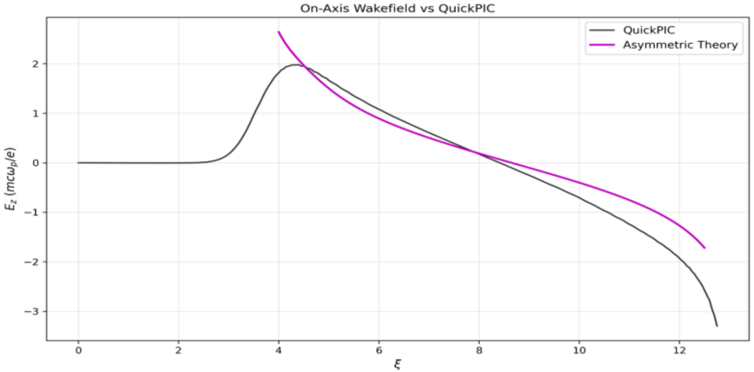




Blowout Trajectory - Start of the Blowout Results







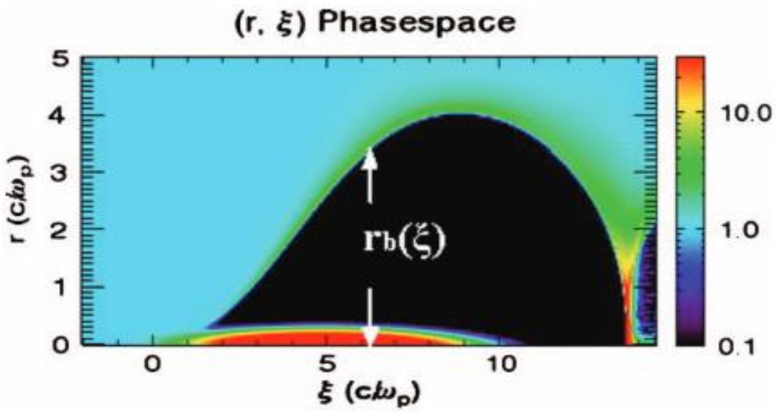
Conclusion & Future Work

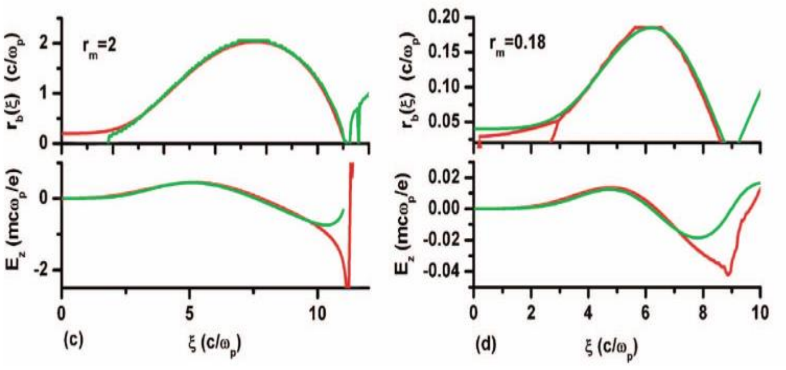
- We leveraged data from QuickPIC simulations to model components of the blowout trajectory
- There are good preliminary results on modeling the blowout trajectory with machine learning and neural network components
- Now, we move to modeling the full blowout trajectory and creating a standalone machine learning model for this trajectory based on input beam parameters
- This will unlock the ability to model the wakefield and develop a complete blowout theory

Thank you for your attention! Questions?

Backup Slides

Nonlinear Theory of Symmetric Driver Beams





In the 3D blowout regime, we can predict the wakefield by modeling the blowout trajectory!

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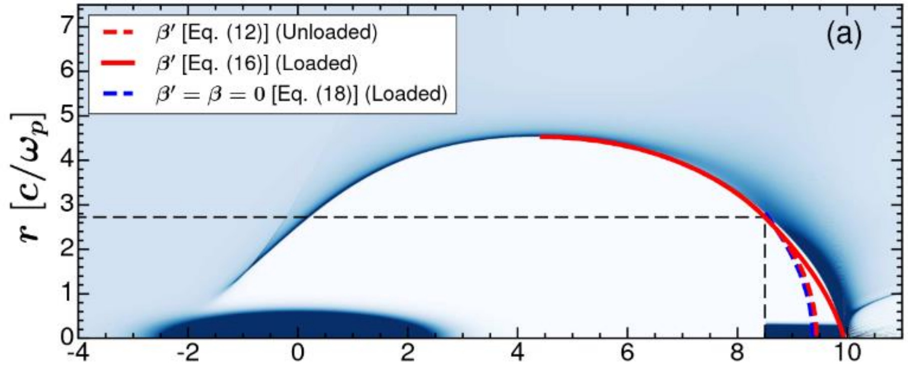
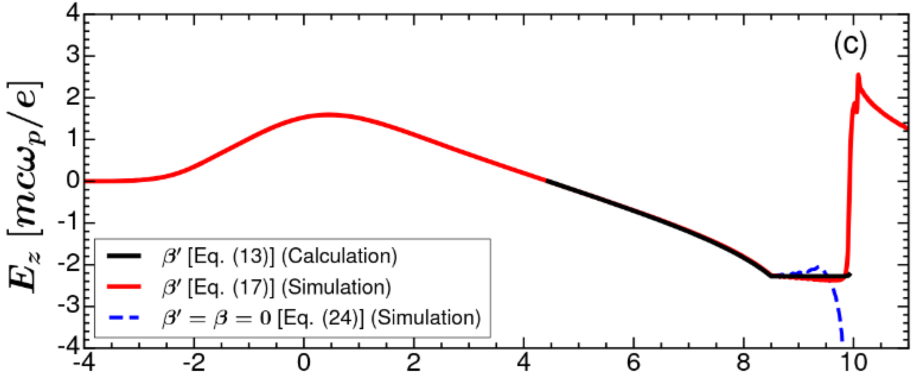
Lu et al, Physics of Plasmas 13, 056709, 2006

Lu et al Equation (Other Form From Letter, eq9)

$$\frac{d}{d\xi}[(1+\psi)\frac{d}{d\xi}r_b] = r_b \left\{ \frac{-1}{4}\left[1 + \frac{1}{(1+\psi)^2} + (\frac{dr_b}{d\xi})^2\right] - \frac{1}{2}\frac{d^2\psi_0}{d\xi^2} + \frac{\lambda(\xi)}{r_b^2} \right\}$$

Multi-Sheath Model of Symmetric Driver Beams





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Updated Wake Potential:

$$\psi(r,\xi) = \frac{r_b^2(\xi)}{4}(1+\beta') - \frac{r^2}{4} \qquad (5)$$

Dalichaouch et al, arXiv:2103.12986, 2021

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Su, et al, Journal of Computational Physics 539 114225, 2025

Rear of Beam Wakefields