# Renormalized quark masses using gradient flow

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color meets flavor

Lattice 2025 MUMBAI

Collaborative Research Center TRR 257
PPH
Particle Physics Phenomenology after the Higgs Discovery

in collaboration with

Matthew Black, Robert Harlander, Anna Hasenfratz, Antonio Rago

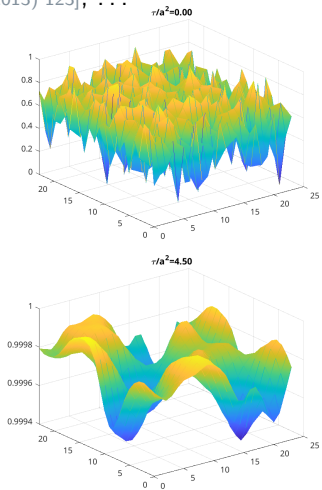
[arXiv:2506.16327]

# Gradient flow (GF)

- Standard tool for calculating scale setting ($\sqrt{8t_0}$), RG $\beta$-function, $\Lambda$ parameter
  [Narayanan, Neuberger JHEP 03 (2006) 064] [Lüscher JHEP 08 (2010) 071][JHEP 04 (2013) 123], ...
- Introduce auxiliary dimension, flow time $\tau$ to regularize UV
  → Well-defined smearing of gauge and fermion fields
  → Smoothening UV fluctuations
- First order differential equation

$$\partial_t B_\mu(\tau, x) = \mathcal{D}_\nu(\tau) G_{\nu\mu}(\tau, x), \quad B_\mu(0, x) = A_\mu(x)$$
$$\partial_t \chi(\tau, x) = \mathcal{D}^2(\tau)\chi(\tau, x), \quad \chi(0, x) = q(x)$$

- Consider GF as an RG transformation
  [Carosso et al. PRL 121 (2018) 201601] [Hasenfratz et al. PoS Lattice 2021 155]
  [Harlander, Lange, Neumann JHEP 08 (2020) 109] ...
- Match to $\overline{\text{MS}}$ scheme using short flow-time expansion (SFTX)
  [Lüscher, Weisz JHEP 02 (2011) 051] [Suzuki PTEP 2013 (2013) 083B03]
  [Lüscher PoS Lattice 2013 016] [Makino, Suzuki PTEP (2014) 063B02] ...



# Short flow-time expansion (SFTX)

- Re-express effective Hamiltonian in terms of 'flowed' operators

$$\mathcal{H}_{\text{eff}} = \sum_n C_n \mathcal{O}_n = \sum_n \widetilde{C}_n(\tau)\widetilde{\mathcal{O}}_n(\tau)$$

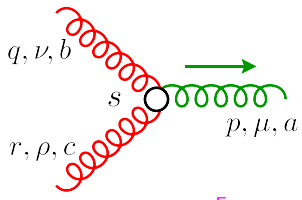
- Relate to regular operators in SFTX

ME of flowed operator (lattice)

$$\widetilde{\mathcal{O}}_n(\tau) = \sum_m \zeta_{nm}(\tau)\mathcal{O}_m + O(\tau)$$

PT calculated matching matrix

$$\sum_n \zeta^{-1}_{nm}(\mu,\tau)\langle\widetilde{\mathcal{O}}_n\rangle(\tau) = \langle\mathcal{O}_m\rangle(\mu)$$




new Feynman diagrams

- Matrix element $\langle\mathcal{O}_m\rangle(\mu)$ in the $\overline{\text{MS}}$ scheme found after taking the $\tau \to 0$
  - → Large systematic effects at very small flow times
  - → Large flow time dominated by operators $\propto O(\tau)$

# Renormalized quark masses [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

- GF renormalized partially conserved axial current (PCAC) relation (Ward-Identity)

$$\left(m_{\mathrm{GF}}^{(r)}(\tau) + m_{\mathrm{GF}}^{(s)}(\tau)\right) = M_{\mathrm{PS}}^{(rs)} \bar{R}_{rs}^{\mathcal{O}}(\tau) \qquad \text{with} \quad \bar{R}_{rs}^{\mathcal{O}}(t;\tau) = \lim_{t\to\infty} -\frac{\langle A_0(t;\tau)\mathcal{O}(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)\mathcal{O}(t=0;\tau=0)\rangle_{rs}}$$

  → Wave-function renormalization $Z_\chi$ cancels in ratio
  → $M_{\mathrm{PS}}$ pseudoscalar meson with flavors $r$ and $s$
  → $m_{\mathrm{GF}}^{(r)}(\tau)$ GF renormalized quark mass for flavor $r$
  → Smearing radius must remain small compared to Euclidean time: $\sqrt{8\tau} \ll t$

- Match to the $\overline{\mathrm{MS}}$ scheme

$$m_{\overline{\mathrm{MS}}}(\mu_{\mathrm{UV}}) = \lim_{\tau\to 0} \zeta_{AP}^{-1}(\mu_{\mathrm{UV}},\tau) m_{\mathrm{GF}}(\tau)$$

  → $\zeta_{AP}^{-1}(\mu_{UV},\tau)$ perturbatively calculated SFTX coefficient

# RBC-UKQCD's 2+1 flavor SDWF+Iwasaki gauge field configurations

| | L | $a^{-1}$(GeV) | $am_l$ | $am_s$ | $M_\pi$ (MeV) | $N_{\text{src}} \times N_{\text{conf}}$ | |
|---|---|---|---|---|---|---|---|
| C1 | 24 | 1.785 | 0.005 | 0.040 | 340 | 32×101 | [PRD 78 (2008) 114509] |
| C2 | 24 | 1.785 | 0.010 | 0.040 | 433 | 32×101 | [PRD 78 (2008) 114509] |
| M1 | 32 | 2.383 | 0.004 | 0.030 | 302 | 32×79 | [PRD 83 (2011) 074508] |
| M2 | 32 | 2.383 | 0.006 | 0.030 | 362 | 32×89 | [PRD 83 (2011) 074508] |
| M3 | 32 | 2.383 | 0.008 | 0.030 | 411 | 32×68 | [PRD 83 (2011) 074508] |
| F1S | 48 | 2.785 | 0.002144 | 0.02144 | 267 | 24×98 | [JHEP 1712 (2017) 008] |

- Lattice spacing determined from combined analysis [Blum et al. PRD 93 (2016) 074505]
- $a$: $\sim 0.11$ fm, $\sim 0.08$ fm, $\sim 0.07$ fm

# Extract ratios $\bar{R}(\tau/a^2)$

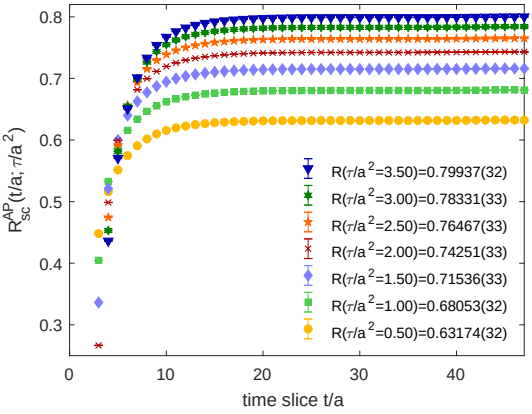
- $\bar{R}^{AP}_{rs}(t;\tau) = \lim_{t\to\infty} \sqrt{\frac{\langle A_0(t;\tau)A_0(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)A_0(t=0;\tau=0)\rangle_{rs}} \frac{\langle A_0(t;\tau)P(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)P(t=0;\tau=0)\rangle_{rs}}}$

- Example F1S:
  $a^{-1} = 2.785$ GeV, $M_\pi = 267$ MeV

- Statistical errors smaller than symbol size

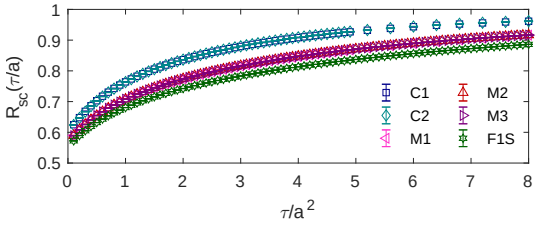
- Correlated fits for $t \in (36, 46)$

- Extract ratios for other gauge field ensembles

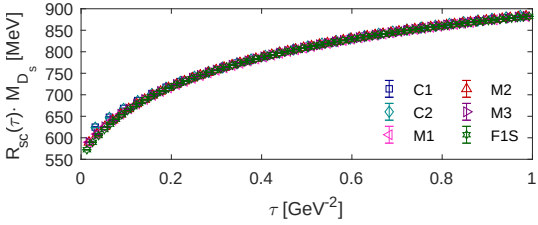


# Obtain $\bar{R}$ as a function of flow time $\tau$

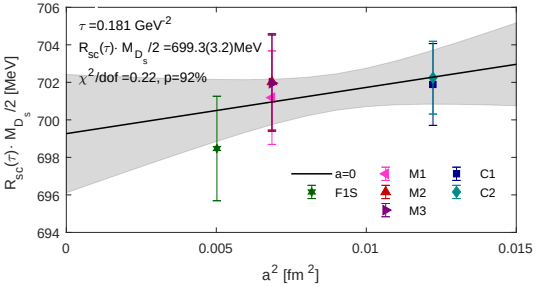
- Results for flow time in lattice units
  → No sea light quark mass dependence

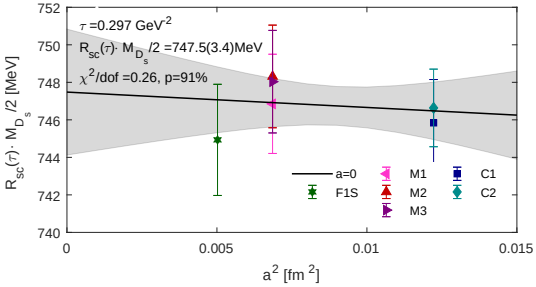



- Convert to "physical" flow time in $\mathrm{GeV}^{-2}$
  → Mild continuum limit

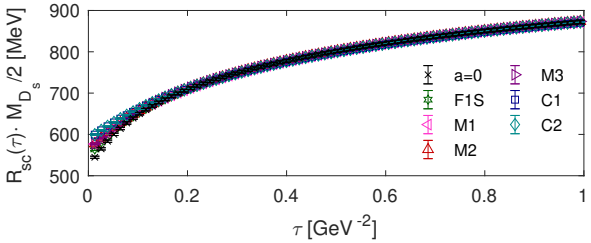


# $a \to 0$ continuum limit




- GF renormalizes our ratios
  - → Take $a \to 0$ continuum limit for $\tau > 0$
- No sea light quark mass dependence
- All actions $O(a)$ improved
  - ⇒ Linear ansatz in $a^2$



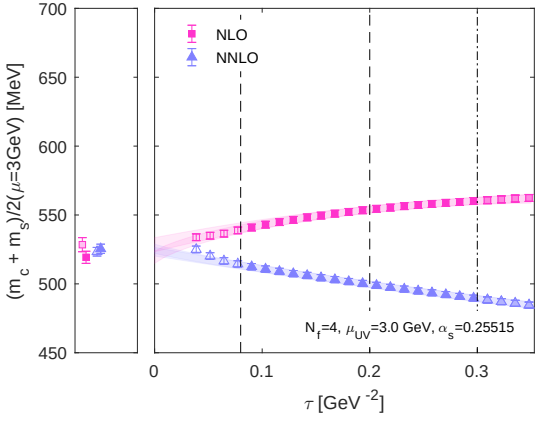
# Multiply by SFTX coefficient and extrapolate $\tau \to 0$

- Multiply $\zeta_{AP}^{-1}$ to match to $\overline{\text{MS}}$ scheme and take $\tau \to 0$ limit
- $\zeta_{AP}^{-1}$ calculated at NLO and NNLO
  [Borgulat et al. JHEP 05 (2024) 179]

- Choose two fit ansätze
  → Lin-log: $f_l(\tau) = \tau(c_l \log(\tau\mu^2) + c_1) + c_0$
  → Quadratic: $f_2(\tau) = \tau^2 c_2 + \tau c_1 + c_0$

- Vary range of $\tau$
  $\tau_{\text{min}} \in (0.08, 0.2)$ GeV$^{-2}$ with $\tau_{\text{max}} = 0.3$ GeV$^{-2}$
  $\tau_{\text{max}} \in (0.25, 0.35)$ GeV$^{-2}$ with $\tau_{\text{min}} = 0.14$ GeV$^{-2}$

- Uncorrelated fit to central values shifted by $\pm 1\sigma$



# Improve by using RG running in GF scheme

- Resum logarithmic terms using RG equation

$$m_{\overline{\text{MS}}}(\mu_{\text{UV}}) = \lim_{\tau\to 0}(\zeta_{AP}^{\text{imp}}(\mu_{\text{UV}},\tau_\mu))^{-1} m_{\text{GF}}(\tau)$$

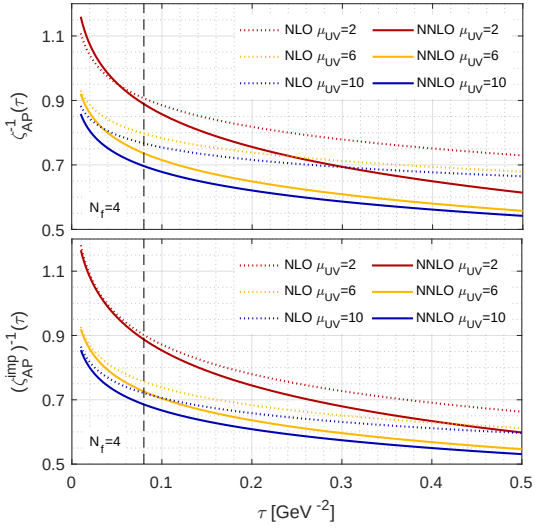
$$(\zeta_{AP}^{\text{imp}}(\mu_{\text{UV}},\tau_\mu))^{-1} =$$

$$\zeta_{AP}^{-1}(\mu_{\text{UV}},\tau_\mu) \times \exp\left(-\int_{\tau_\mu}^{\tau} d\tau' \frac{\gamma_m^{\text{GF}}(\tau')}{\tau'}\right)$$

$\rightarrow \gamma_m^{GF}$ gradient flowed anomalous dimension

$\rightarrow \tau_\mu = e^{-\gamma_E}/(2\mu_{\text{UV}}^2)$

- This work: $\gamma_m^{GF}$ calculated perturbatively
  [Borgulat et al. JHEP 05 (2024) 179]
- Known how to calculate $\gamma_m^{GF}$ on the lattice
  [Hasenfratz et al. PoS Lattice2021 155]
  [Hasenfratz et al. PRD 107(2023)11504][PRD 108(2023) L071503]

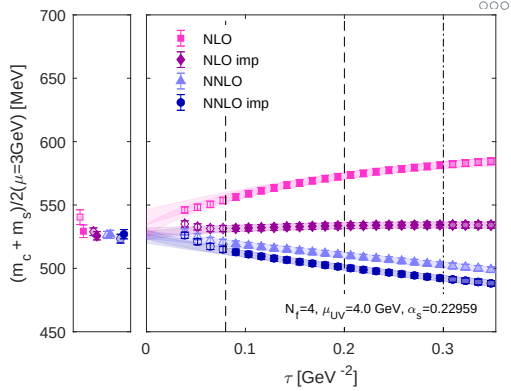


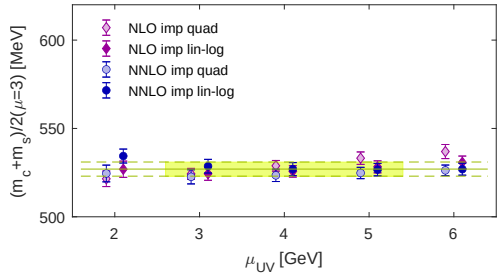
# Final result

- Multiply $\zeta_{AP}^{-1}$ and take $\tau \to 0$ limit to get $\overline{\text{MS}}$ mass
  - → Caclulate $\zeta_{AP}^{-1}$ at NLO and NNLO
  - → Improve using RG running with PT $\gamma_m^{GF}$
- Choose two fit ansätze
  - → Lin-log: $f_l(\tau) = \tau(c_l \log(\tau\mu^2) + c_1) + c_0$
  - → Quadratic: $f_2(\tau) = \tau^2 c_2 + \tau c_1 + c_0$
- Vary range of $\tau$

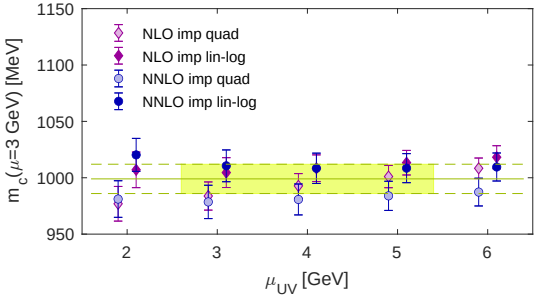
  $\tau_{\text{min}} \in (0.08, 0.2)$ GeV$^{-2}$ with $\tau_{\text{max}} = 0.3$ GeV$^{-2}$

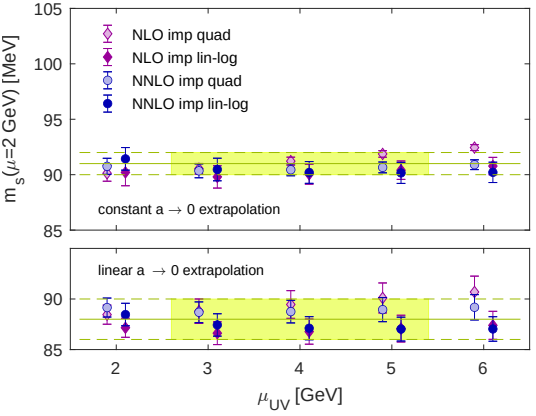
  $\tau_{\text{max}} \in (0.25, 0.35)$ GeV$^{-2}$ with $\tau_{\text{min}} = 0.14$ GeV$^{-2}$
- Uncorrelated fit to central values $\pm 1\sigma$
- Vary $\mu_{UV} \in (2, 3, 4, 5, 6)$ GeV
  - → Run back to $\mu = 3$ GeV using 4-loop $\overline{\text{MS}}$ running
- Calculate correlated average using lin-log and quad for NLO imp, NNLO imp at $\mu_{UV} = (3, 4, 5)$ GeV



# Analyzing $\eta_c$ and $\eta_s$ correlators



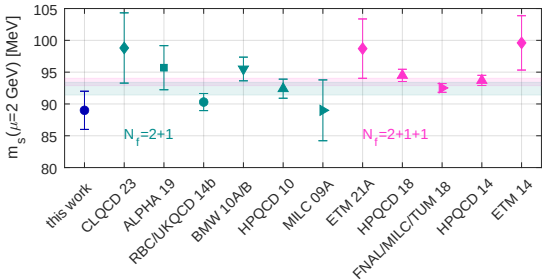



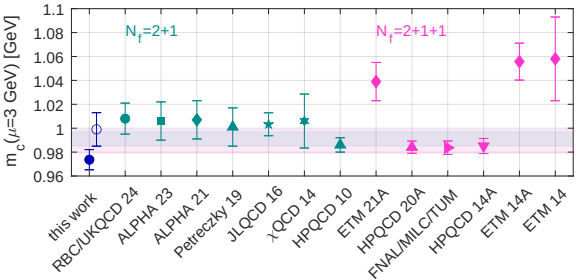

- $m_c^{\eta_c}(\mu = 3\,\text{GeV})$=999(14) MeV
- $m_c(\mu = 3\,\text{GeV})$=974(8) MeV
- $\dfrac{m_c}{m_s} = 12.1(4)$

$\Leftarrow$

- $m_s^{\eta_s}(\mu = 2\,\text{GeV})$=89(3) MeV
- $\left(\dfrac{m_c + m_s}{2}\right)(\mu = 3\,\text{GeV}) = 527(4)$ MeV

# Strange and charm quark masses [M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]




[FLAG 2024 arXiv:2411.04268] [$\chi$QCD PRD92(2015)034517] [ETM PRD104(2021)074515] [FNAL/MILC/TUM PRD98(2018)054517] [HPQCD PRD91(2015)054508] [ETM NPB887(2014)19] [CQCD PRD109(2024)054507] [Alpha EPJC80(2020)169] [RBC-UKQCD PRD93(2016)074505] [HPQCD PRD82(2010)034512] [BMW JHEP08(2011)146] [FNAL/MILC PoS CD09(2009)007] [HPQCD PRD98 (2018) 014513] [Alpha EPJC84(2024)506] [Alpha JHEP05(2021)288] [Petreczky,Weber PRD100(2019)034519] [JLQCD PRD94(2016)054507] [HPQCD PRD102(2020)054511] [HPQCD PRD102(2020)054511] [RBC-UKQCD PRD110(2024)054512]

- $m_s^{\eta_s}(\mu = 2\text{ GeV}) = 89(3)$ MeV
- $m_c^{\eta_c}(\mu = 3\text{ GeV}) = 999(14)$ MeV
- $m_c(\mu = 3\text{ GeV}) = 974(8)$ MeV ($D_s$ and $\eta_s$)
- $\frac{m_c}{m_s} = 12.1(4)$

# Summary and Outlook

- GF+SFTX is a powerful concept and we are just starting to leverage its potential
- Simple prescription to obtain renormalized quark masses
  → Most likely competitive uncertainties compared to other method e.g. RI/(S)MOM
  → No need for gauge-fixing, no "window" problem
- First lattice calculation of heavy meson lifetimes ⇝ Talk by Matthew Black

[Video]

[Website]

[Proceedings arXiv:2510.21494]

# Acknowledgment

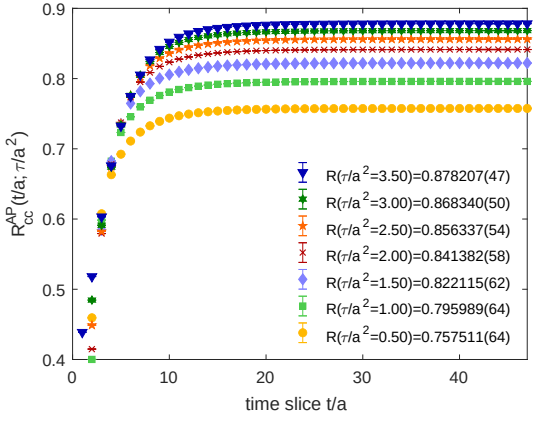

- Grid [Peter Boyle et al.]
- Hadrons [Antonin Portelli et al.]
- Feyngame [Harland er et al.]
- OMNI, Universität Siegen
- HAWK, HLR Stuttgart
- LumiG, DEIC

# Extra

# Analyzing $\eta_c$ correlators

[M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
  - → F1S: $a^{-1} = 2.785$ GeV, $M_\pi = 267$ MeV
  - → Correlated fits $t \in (42, 46)$



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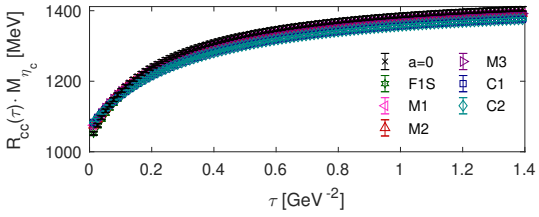


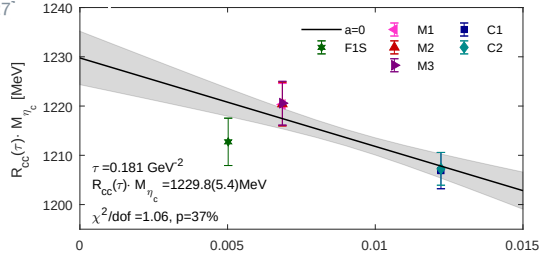


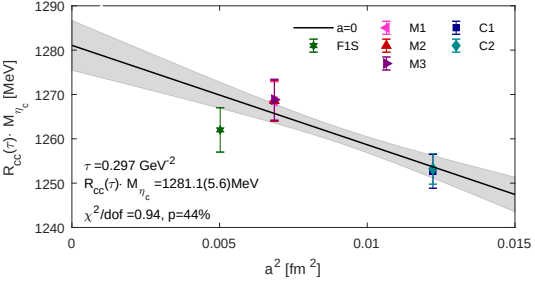

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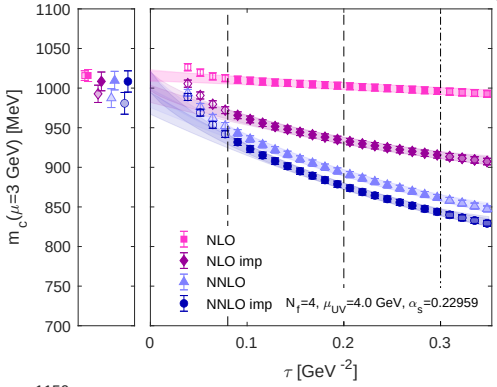
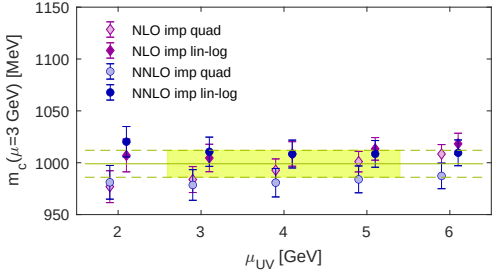
- Multiply $\zeta_{AP}^{-1}$ to match to $\overline{\mathrm{MS}}$ scheme and take $\tau \to 0$ limit
  - → Improve using RG running
  - → Lin-log and quadratic fit ansaetze
  - → Correlated average: (N)LO imp at $\mu = (3, 4, 5)\,\mathrm{GeV}$
- $m_c^{\eta_c}(\mu = 3\,\mathrm{GeV}) = 999(14)\,\mathrm{MeV}$



# Analyzing $\eta_s$ correlators

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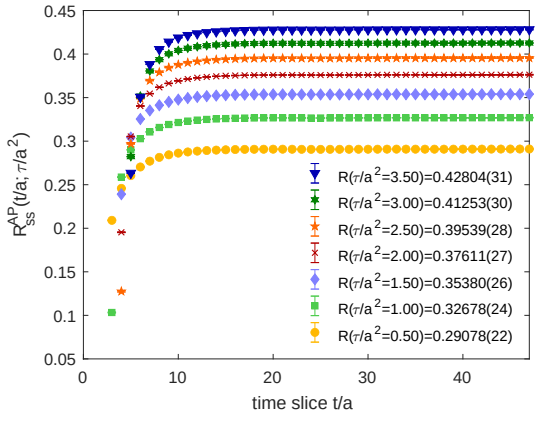
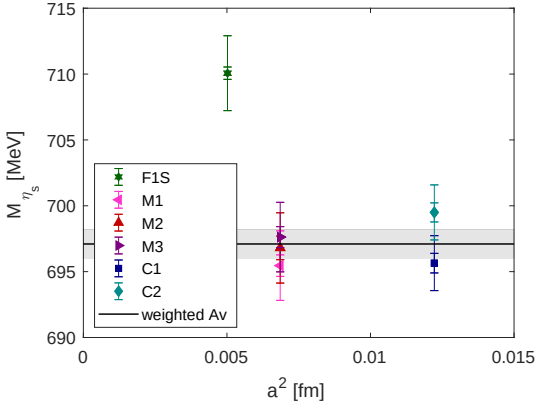


# Analyzing $\eta_s$ correlators

[M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
- Discard F1S: strange quark mistuned



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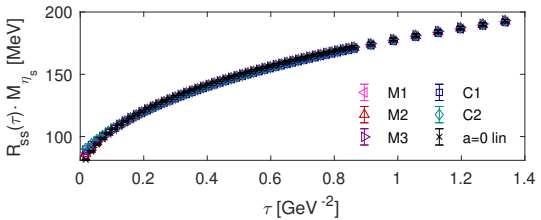
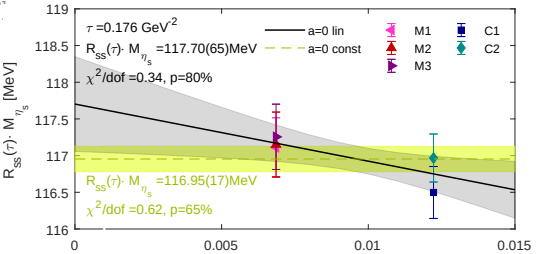
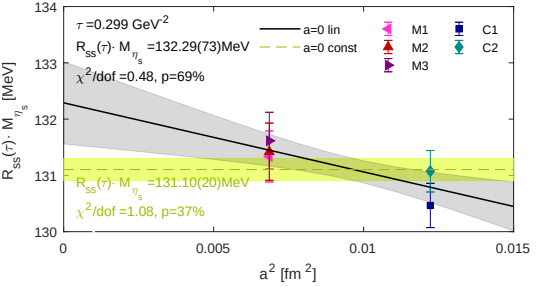
- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
- Convert to "physical" flow time in $\mathrm{GeV}^{-2}$
- Take $a \to 0$ continuum limit for $\tau > 0$
  $\to$ Linear in $a^2$ fit and fit to a constant



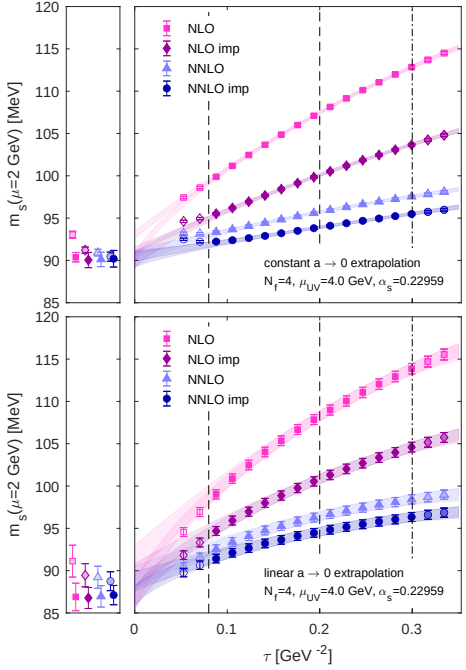



# Analyzing $\eta_s$ correlators

[M. Black, R. Harlander, A. Hasenfratz A. Rago, OW, arXiv:2506.16327]

- Extract $\bar{R}(\tau/a)$ for all gauge field ensembles
- Convert to "physical" flow time in $\mathrm{GeV}^{-2}$
- Take $a \to 0$ continuum limit for $\tau > 0$
- Multiply $\zeta_{AP}^{-1}$ to match to $\overline{\mathrm{MS}}$ scheme and take $\tau \to 0$ limit
  - → Improve using RG running
  - → Lin-log and quadratic fit ansaetze
  - → Correlated average: (N)LO imp at $\mu = (3, 4, 5)\,\mathrm{GeV}$
  - → Repeat for both $a \to 0$ limits



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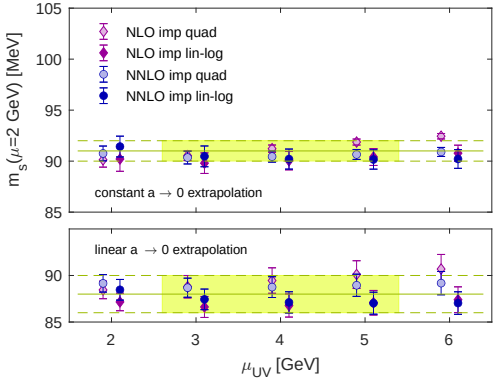


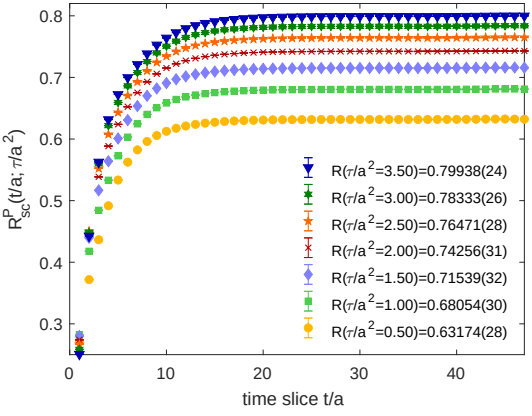

- $m_s(\mu = 2\,\mathrm{GeV}) = 89(3)\,\mathrm{MeV}$

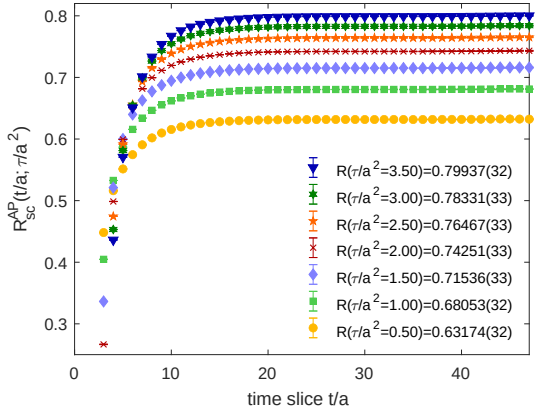
# Ratios $\bar{R}(\tau/a^2)$

- $\bar{R}^{AP}_{rs}(t;\tau) =$

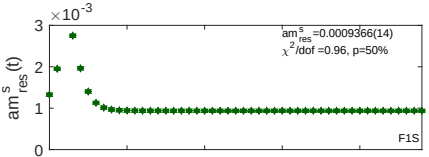
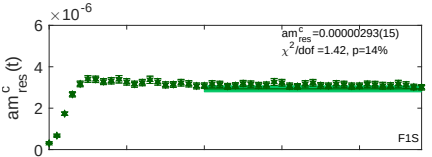
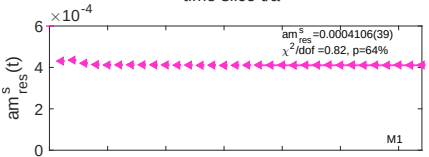
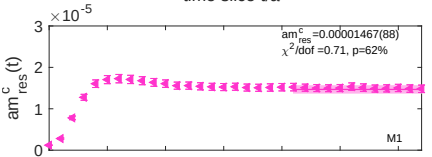
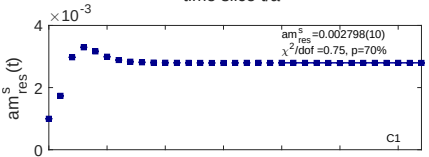
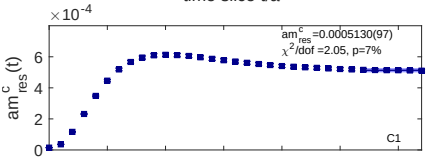
$$\lim_{t\to\infty} \sqrt{\frac{\langle A_0(t;\tau)A_0(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)A_0(t=0;\tau=0)\rangle_{rs}} \frac{\langle A_0(t;\tau)P(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)P(t=0;\tau=0)\rangle_{rs}}}$$

- $\bar{R}^{P}_{rs}(t;\tau) = \lim_{t\to\infty} -\frac{\langle A_0(t;\tau)P(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)P(t=0;\tau=0)\rangle_{rs}}$



# Numerical setup GF calculations

- RBC/UKQCD's 2+1 flavor DWF + Iwasaki gauge action ensembles, 3 lattice spacings
  [PRD 78 (2008) 114509] [PRD 83 (2011) 074508] [PRD 93 (2016) 074505] [JHEP 12 (2017) 008]
- Setup of the lattice calculation follows [Boyle et al. 1812.08791]
  - → Z2 wall sources for all quark propagators [Boyle et al. JHEP 08 (2008) 086]
  - → Gaussian source smearing for strange quarks [Allton et al. PRD 47 (1993) 5128]
  - → Multiple source separations $\Delta T \in \{10, 30\}$
- Fully-relativistic, chiral action for all quarks
  - → Shamir domain-wall fermions for light and strange quarks
    [Kaplan PLB 288 (1992) 342] [Shamir NPB 406 (1993) 90] [Furman, Shamir NPB 439 (1995) 54]
  - → Stout-smeared Möbius domain-wall fermions for heavy quarks
    [Morningstar, Peardon PRD 69 (2004) 054501] [Brower, Neff, Orginos CPC 220 (2017) 1]
- Simulate "neutral" charm-strange mesons
  - → Easy to tune to physical strange and charm quarks
  - → Avoid more expensive chiral or heavy quark extrapolation

# Residual mass