Electromagnetic pion mass splitting using PV-regulated photon propagator

Alessandro De Santis, on the behalf of the Mainz group

Introduction

First-principles Lattice QCD simulations have entered the era of sub-percent precision which makes the inclusion of QED and isospin-breaking effects mandatory

- a_μ^{HVP} — Parrino's talk 03/11, Sitison's talk 03/11, Hoferichter's talk 04/11
- Inclusive τ decays — Di Carlo's talk 05/11
- New strategies — Albandea's talk 06/11, Erb's talk 03/11
- Methods comparison — Marinkovic's talk 04/11
- Baryon masses — Rosso's talk 06/11
- Decay constants — Cotellucci's talk 07/11

Introduction

- Two viable approaches
  - `RM123` **expand the isospin symmetric action** in powers of $(m_d - m_u)$ and $\alpha_{\rm em}$ de Divitiis et al. (2013)
  - **simulate directly QCD+QED** with $C^\star$ boundary conditions RC* (2022)

- Several regularizations/actions for the photon
  - QED$_{\rm L}$ Hayakawa, Uno (2008)
  - QED$_{\rm TL}$ Borsanyi et al. (2014)
  - QED$_\infty$ Green et al. (2016)
  - QED$_C$ Lucini et al. (2015)
  - QED$_m$ Endres et al. (2015)
  - QED$_r$ Di Carlo et al. (2025)

- Most common problems
  - sizeable (IR) Finite Volume Effects (FVEs)
  - treatment of UV-divergences is scheme dependent
  - validation and cross-checks to phenomenology or between collaborations is challenging

The formalism

Key idea proposed in Biloshytskyi et al. (2023): introduce a Pauli-Villars-regulated photon propagator and split the calculation in two parts

- short distance : in finite volume and UV-regulated by lattice spacing
- long distance : in infinite-volume and UV-regulated by a PV mass Λ

$$\frac{1}{k^2} \mapsto \left(\frac{1}{k^2} - \frac{1}{k^2 + \Lambda^2} \right) + \frac{1}{k^2 + \Lambda^2}$$

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Advantages

- Infinite-volume and analytical photon propagator UV- and IR-regulated
- Suppression of finite volume effects
- Allow for easier comparisons to phenomenology

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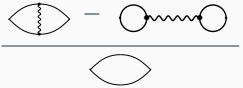
Applied in Erb et al. (2025) in conjunction with the Covariant Coordinate-space (CCS) method Meyer (2017) to compute the isospin-violating part $a_\mu^{\mathrm{HVP},38}$ at **unphysical point** + Erb's talk 03/11

The pion mass splitting

As a **proof-of-concept**, we want to **validate the method** for the $\mathcal{O}(\alpha)$ electromagnetic correction to the positive/neutral pion mass difference for which $\Delta M_\pi^{\mathrm{PDG}} = 4.5936(5)$ PDG 2025

$$\Delta M_\pi \equiv M_{\pi^+} - M_{\pi^0} = \frac{e^2}{2}(q_u - q_d)^2 \partial_t$$



- **connected diagram**: preliminary results in this talk
- **disconnected diagram**: work in progress

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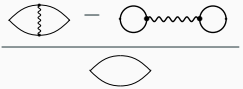

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ΔM_π already successfully computed on the lattice in other regularizations (ΔM_π is UV-finite)

- $\Delta M_\pi^{\rm latt} = 4.622(95)$ ETMC (2022)
- $\Delta M_\pi^{\rm latt} = 4.534(60)$ Feng et al. (2022)

de Divitiis et al. (2013)

Pauli-Villars-regulated photon propagator

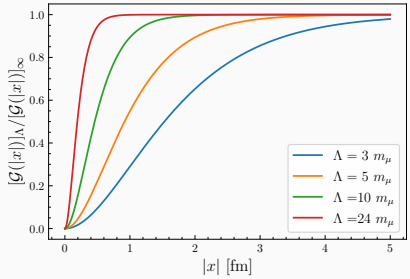
Doubly Pauli-Villars-regulated photon propagator in coordinate space

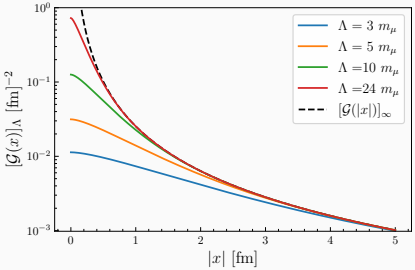
$$\mathcal{G}_\Lambda(|x|)\delta^{\mu\nu} = \left[\frac{1}{4\pi^2|x|^2} - 2G_{\frac{\Lambda}{\sqrt{2}}}(|x|) + G_\Lambda(|x|)\right]\delta^{\mu\nu}$$

- massive scalar propagator

$$G_m(|x|) = m\frac{K_1(m|x|)}{4\pi^2|x|}$$

- $\Delta M_\pi = \lim_{\Lambda\mapsto\infty} \Delta M_\pi(\Lambda)$





Setup proposed in Erb et al. (2025)

MID-POINT evaluation of the electromagnetic correction at $\mathcal{O}(\alpha)$

$$\Delta M_\pi(\Lambda) = \frac{e^2}{2}(q_u - q_d)^2 \lim_{t_{\text{sep}} \mapsto \infty} \int_{-\infty}^{+\infty} \mathrm{d}z_0 \, f_\Lambda^{\text{latt}}(z_0)$$

$$f_\Lambda^{\text{latt}}(z_0) = \frac{\sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \mathcal{G}_\Lambda(\, z_0 \,, \boldsymbol{z}) \delta^{\mu\nu} \, C_{\mu\nu}^{\text{4pt}}(t_{\text{sep}}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}{\sum_{\boldsymbol{x},\boldsymbol{y}} C^{\text{2pt}}(t_{\text{sep}}, \boldsymbol{x}, \boldsymbol{y})}$$

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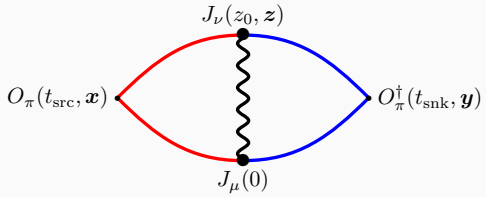
- Conserved point-split currents
- Suppression of excited states
  - $t_{\text{src}} = -t_{\text{sep}}/2$
  - $t_{\text{snk}} = +t_{\text{sep}}/2$
- sequential propagators
- z_0 temporal separation between currents



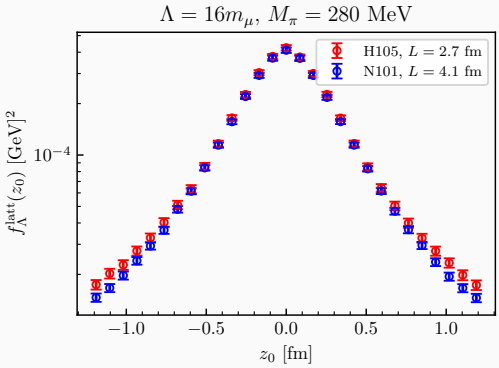
Lattice gauge ensembles

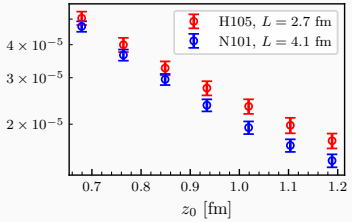
ID	$L^3 \times T$	a [fm]	β	bc	L [fm]	M_π [MeV]	$M_\pi L$
H105	$32^3 \times 96$	0.085	3.40	o	2.7	283	3.9
N101	$48^3 \times 128$	0.085	3.40	o	4.1	282	5.8
C101	$48^3 \times 96$	0.085	3.40	o	4.1	222	4.6
D450	$64^3 \times 128$	0.075	3.46	p	4.8	219	5.3
D452	$64^3 \times 128$	0.075	3.46	p	4.8	156	3.8
D251	$64^3 \times 128$	0.064	3.55	p	4.1	286	5.9
D200	$64^3 \times 128$	0.064	3.55	o	4.1	202	4.2
J303	$64^3 \times 192$	0.049	3.70	o	3.1	260	4.1

- $N_f = 2 + 1$ CLS ensembles with $\mathcal{O}(a)$-improved Wilson fermions
- Tree-level improved Lüscher-Weisz gauge action
- Check FVEs

Bruno et al. (2014),Bali et al. (2016)

Analytical description of the long-distance contribution





- FVEs show up at large time separations

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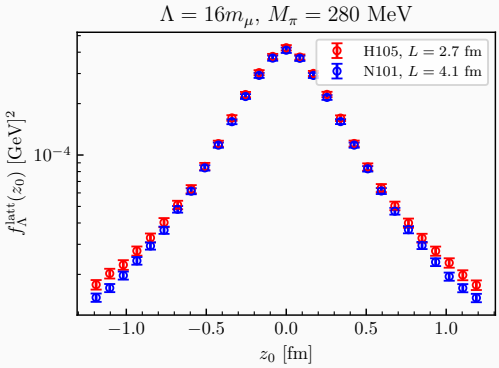
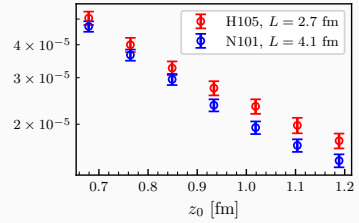


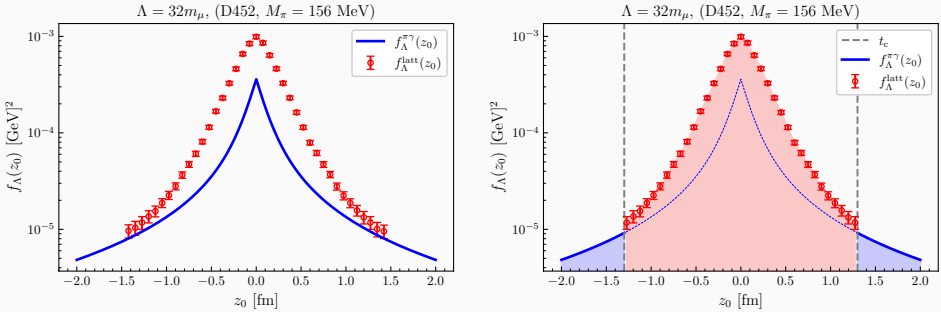


- FVEs show up at large time separations

$$f(z_0) = \frac{e^2}{2} \int \mathrm{d}^3 z\, \delta^{\mu\nu} \mathcal{G}_\Lambda(z)\, \langle \pi, \mathbf{0} | \hat{T} \left[J_\mu^{\mathrm{em}}(z) J_\nu^{\mathrm{em}}(0) \right] | \pi, \mathbf{0} \rangle$$

- At large z_0 the intermediate states between the currents are **dominated by single pion states** $\mapsto f^{\pi\gamma}(z_0)$
- analytical expression for $f_\Lambda^{\pi\gamma}(z_0)$ in the **continuum** and **finite/infinite volume**
- More details in backup slides

Separation of short- and long-distance contributions

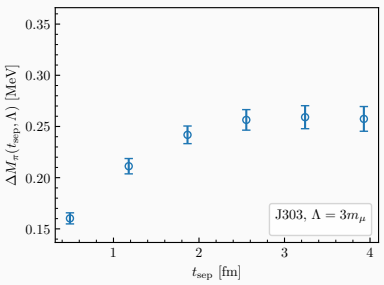


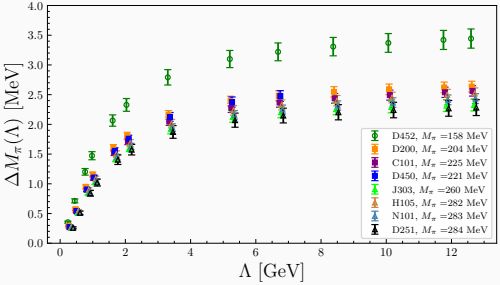
$$\Delta M_\pi(\Lambda) = \frac{e^2}{2}(q_u - q_d)^2 \lim_{t_{\text{sep}} \mapsto \infty} \int_{-t_c}^{+t_c} \mathrm{d}z_0 \; f_\Lambda^{\text{latt}}(z_0) + 2\int_{t_c}^{\infty} \mathrm{d}z_0 \; f_\Lambda^{\pi\gamma}(z_0)$$

- short-distance contribution with exponentially suppressed FVEs on the lattice
- long-distance contribution in continuum and infinite volume

Inspired by infinite-volume reconstruction Feng et al. (2018)

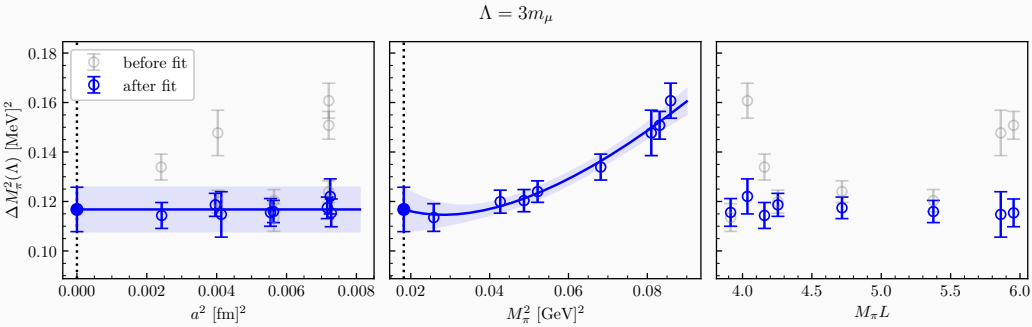
- Several value of $t_{\rm sep}$, saturation around 3 fm
- Several value of Λ from 0.3 GeV to 13 GeV





- At $\mathcal{O}(\alpha)$ we can work with $\Delta M_\pi^2 \equiv M_{\pi^+}^2 - M_{\pi^0}^2 = 2M_\pi \Delta M_\pi$

Chiral-continuum extrapolation at small Λ

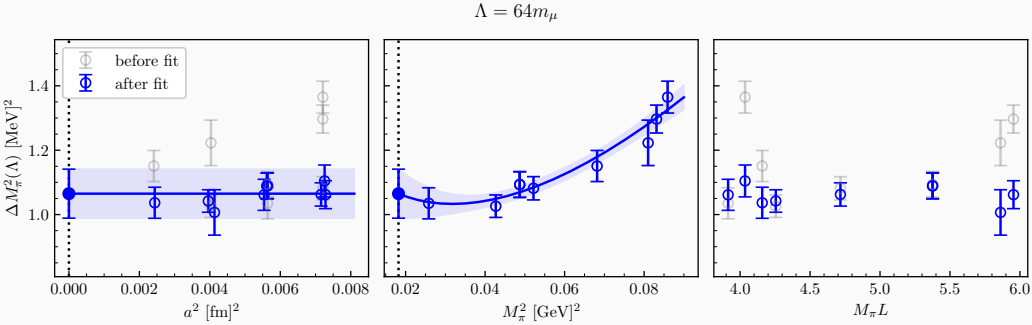


- Naive Chiral-continuum extrapolation Λ

$$\Delta M_\pi^2(a^2, M_\pi) = e^2 \left[c_0 + c_\pi^{(1)} M_\pi^2 \log \frac{M_\pi^2}{\mu^2} + c_\pi^{(2)} M_\pi^2 \right]$$

- ChPT-based work in progress
- Good description of the data

Chiral-continuum extrapolation at large Λ



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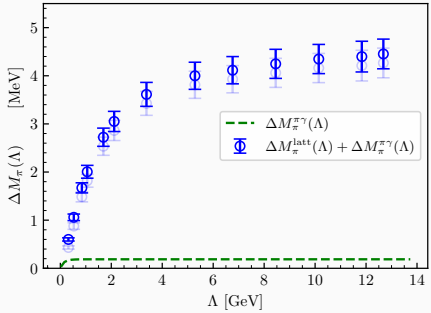
$\Lambda \mapsto \infty$ and addition of intermediate states

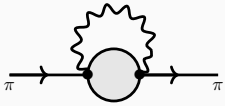
- These are preliminary results
- Addition of single pion intermediate states already in continuum and infinite volume at given Λ
- residual FVEs on $\Delta M_\pi^{\text{latt}}$ to be checked
- How to address the Λ dependence?



Elastic contribution from phenomenology

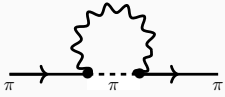
- Cottingham formula to evaluate electromagnetic correction in terms of **forward Compton tensor**

$$M_\pi^2 = \frac{ie^2}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{T_\mu^\mu}{k^2 + i\varepsilon}$$



- One can derive an analytical expression for the **elastic contribution** with the PV-regulated photon propagator

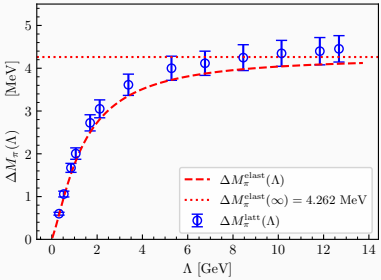
$$(M_\pi^2)^{\text{elast}}(\Lambda) = \frac{\alpha}{8\pi} \int_0^\infty \mathrm{d}s \left[F_\pi(-s)\right]^2 I(s, \Lambda)$$

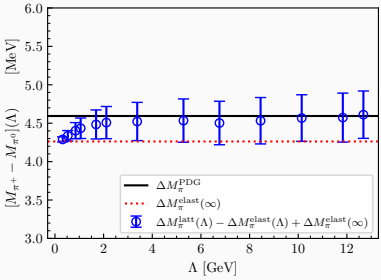


- Vector Meson Dominance parametrization for the pion form factor

$$F_\pi(-s) = \frac{1}{1 + s/M_\rho^2}$$

- The elastic contribution saturates almost completely the pion mass splitting
- After subtraction of the elastic contribution there is no residual dependence on Λ allowing for controlled extrapolation
- Results in perfect agreement with experiments





Conclusions and outlooks

- We show preliminary results for the charged/neutral pion mass difference at $\mathcal{O}(\alpha)$ using a Pauli-Villars-regulated photon propagator as **test-case to validate the method**
- This strategy leads to **suppression of finite volume effects** and the dependence on Λ can be controlled with the help of phenomenological inputs
- The preliminary result is in perfect agreement with experiments
- This strategy opens the way to a more **controlled determination of electromagnetic corrections** to hadronic observables computed on the lattice such as a_μ^{HVP} (Erb's talk 03/11) or proton/nucleon mass splitting

Thank you for the attention!

Backup slides

Analytical description of the long-distance contribution

Starting from

$$f(z_0) = \frac{e^2}{2} \int \mathrm{d}^3 z \, \delta^{\mu\nu} \mathcal{G}_\Lambda(z) \, \langle \pi, \mathbf{0} | \, \hat{T} \big[J^{\mathrm{em}}_\mu(z) J^{\mathrm{em}}_\nu(0) \big] \, | \pi, \mathbf{0} \rangle$$

We introduce the 1-particle completeness of states $\sum_{\boldsymbol{p}} \frac{1}{4E_{\boldsymbol{p}} M_\pi} \, |\pi, \boldsymbol{p}\rangle \, \langle \pi, \boldsymbol{p}|$ and introduce the electromagnetic pion form factor

$$\langle \pi, \boldsymbol{p}| \, J^{\mathrm{em}}_\mu(0) \, |\pi, \boldsymbol{k}\rangle = -i(p_\mu + k_\mu) F_\pi \big[-(p-k)^2 \big]$$

In infinite-volume limit and rest frame of the pion

$$f^{\pi\gamma}(z_0) = \frac{e^2}{8\pi^2} \int_0^\infty \mathrm{d}p \, p^2 e^{(M_\pi - E_{\boldsymbol{p}})|z_0|} \frac{E_{\boldsymbol{p}} + M_\pi}{E_{\boldsymbol{p}}} F_\pi^2 \big[-(p-k) \big] \times$$
$$\times \left[\frac{e^{-|\boldsymbol{p}||z_0|}}{2|\boldsymbol{p}|} - \frac{e^{-\sqrt{|\boldsymbol{p}|^2 + \Lambda^2/2}|z_0|}}{\sqrt{|\boldsymbol{p}|^2 + \Lambda^2/2}} + \frac{e^{-\sqrt{|\boldsymbol{p}|^2 + \Lambda^2}|z_0|}}{2\sqrt{|\boldsymbol{p}|^2 + \Lambda^2}} \right]$$

Finite-volume version also possible

More details in Erb et al. (2025)

Elastic contribution

Elastic contribution to mass shift

$$(M_\pi^2)^{\text{elast}}(\Lambda) = \frac{\alpha}{8\pi}\int_0^\infty \mathrm{d}s\,\big[F_\pi(-s)\big]^2 I(s,\Lambda)$$

with

$$I(s,\Lambda) = s\bigg[4w(s) + \frac{s}{M_\pi^2}\big(w(s)-1\big)\bigg]\bigg[\Delta_0(s) - 2\Delta_{\frac{\Lambda}{\sqrt{2}}}(s) + \Delta_\Lambda(s)\bigg]$$

and

$$w(s) = \sqrt{1+\frac{4M_\pi^2}{s}}, \qquad \Delta_m(s) = \frac{1}{s+m^2}$$

Stamen et al. (2022)