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Towards four-pion effects in multi-hadron decays

Rajnandini Mukherjee
based on work with Maxwell T. Hansen

THE UNIVERSITY *of* EDINBURGH

Motivation

Three and four-pion states in many multi-hadron processes:

- weak decays: $K \to 3\pi,\ D \to 2\pi, K\overline{K}, 4\pi,\ \eta \to 3\pi,\ \eta' \to 4\pi$
- precision observables: $e^+e^- \to 2\pi, 3\pi, 4\pi$ for a_μ^{HVP}
- scattering, resonances: $3\pi \to 3\pi,\ 2\pi \to 4\pi,\ 2\rho \to 4\pi$

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key challenge: complicated interpretation of finite volume states

$$\langle n, L | H_W | D, L\rangle \to C_{2\pi}\langle 2\pi, \mathrm{out} | \mathscr{H}_W | D\rangle + C_{4\pi}\langle 4\pi, \mathrm{out} | \mathscr{H}_W | D\rangle$$

[Lellouch & Lüscher (2000)], Briceño, Davoudi, Hansen, Rusetsky, Sharpe, Walker-Loud, …

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$\Longrightarrow$ need N-particle finite volume (FV) formalisms

- 3-particles: major theoretical progress **[J Green (Monday), S Sharpe (Saturday)]**
- 4-particles: currently unexplored

Towards a four-pion formalism

Hierarchy of complexity		
2 particles	**3 particles**	**4 particles**
single phase shift, well understood *[Lüscher (1986)]*	coupled integral equations, topology of diagrams important *Hansen, Sharpe, Döring, Mai, Hammer, Pang, Rusetsky*	all 3 particle issues, both 2 and 3-particle subprocesses, many diagrams *[Huang & Yang (1958), Beane, Detmold & Savage (2007)]*

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Goal: **physical intuition** for future non-perturbative framework

As a *first* step, use **perturbation theory** (PT)

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Goal: **physical intuition** for future non-perturbative framework

As a *first* step, use **perturbation theory** (PT) to study

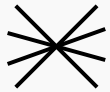
- size of 4π contributions to the FV spectra
- level repulsion and state mixing near thresholds
- leading order (LO) shifts of FV energies and relation to LO amplitudes

PT setup

Following the approach in *Hansen & Sharpe (2015)*

$$S = \int_{-\infty}^{\infty} d\tau \int_{L^3} d^3x \left[\frac{1}{2}\partial_\mu \pi \partial_\mu \pi + \frac{m_\pi^2}{2}\pi^2 + \; cts + \frac{g_{2\to 2}}{4!}\pi^4 + \frac{g_{2\leftrightarrow 4}}{6!}\pi^6 + \frac{g_{4\to 4}}{8!}\pi^8 \right]$$

vertices in the theory:

$g_{2\to 2}$: $g_{4\to 2}$: $g_{2\to 4}$: $g_{4\to 4}$: 

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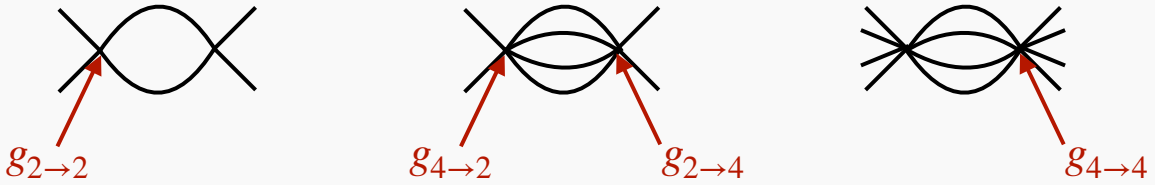

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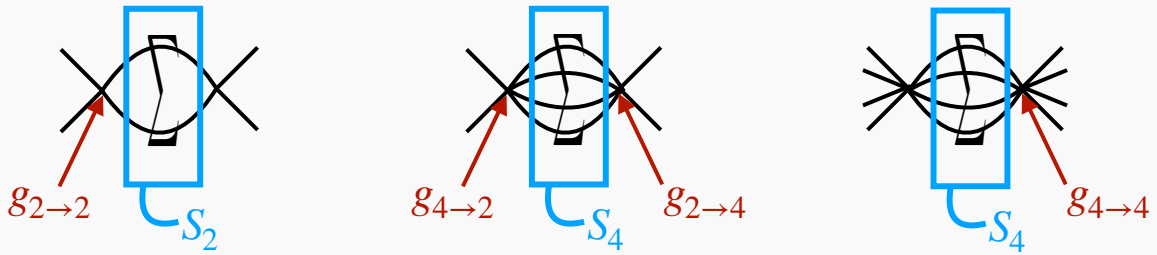
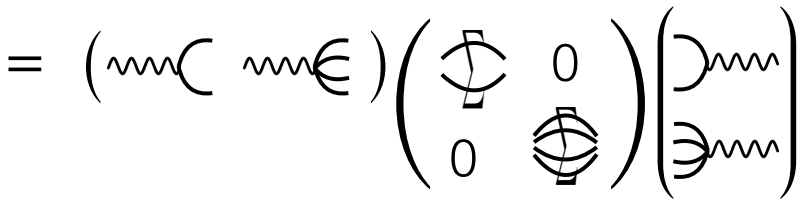


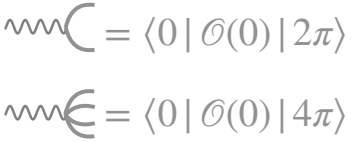
PT setup

Two-point function using coupled channel expansion

[Hansen & Sharpe (2012), Kim, Sachrajda & Sharpe (2005)]

$$C_L(\tau) = \left\langle 0 \left| \mathcal{O}(\tau)\mathcal{O}^\dagger(0) \right| 0 \right\rangle_L$$





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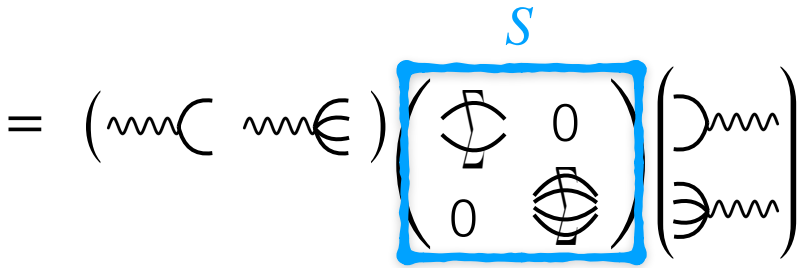

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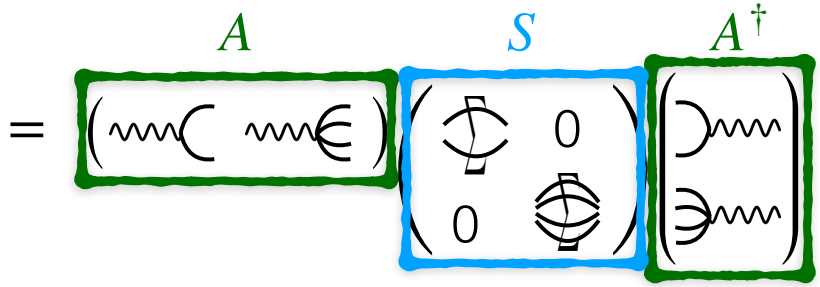

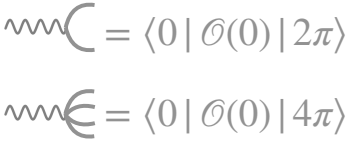

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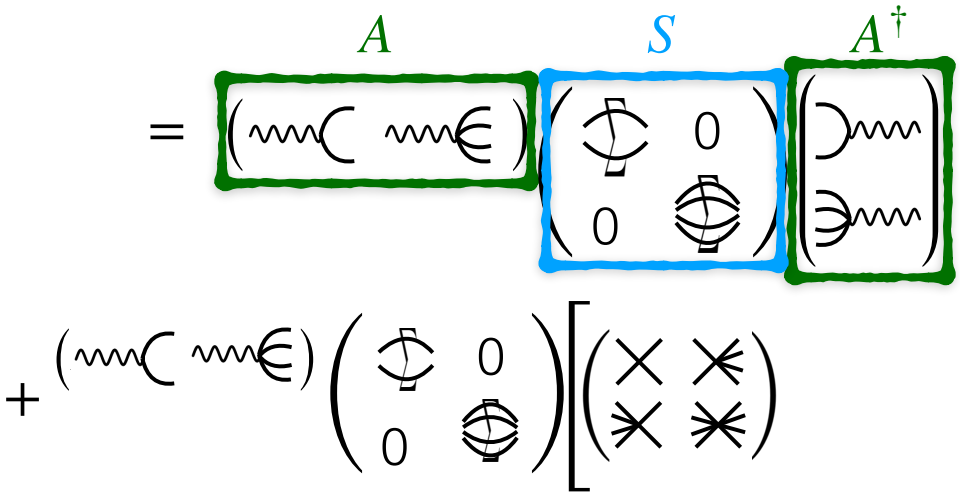

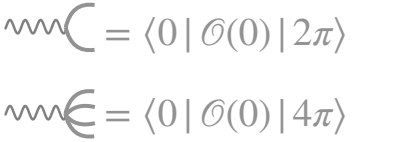

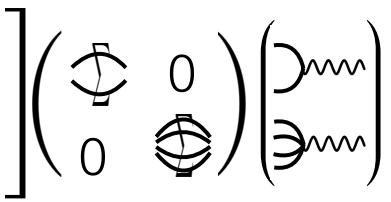

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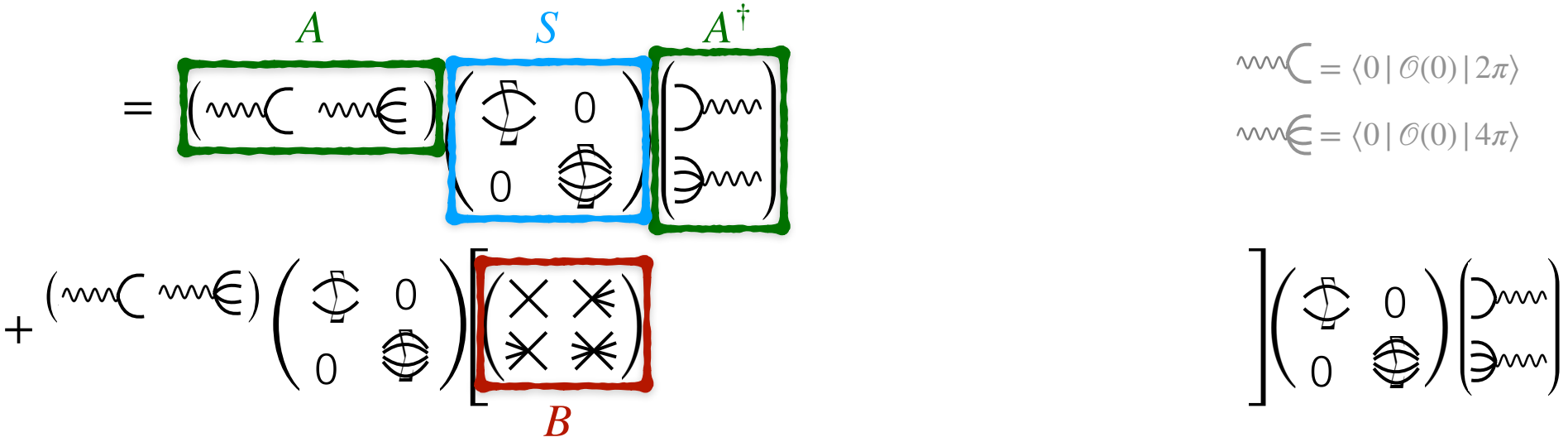

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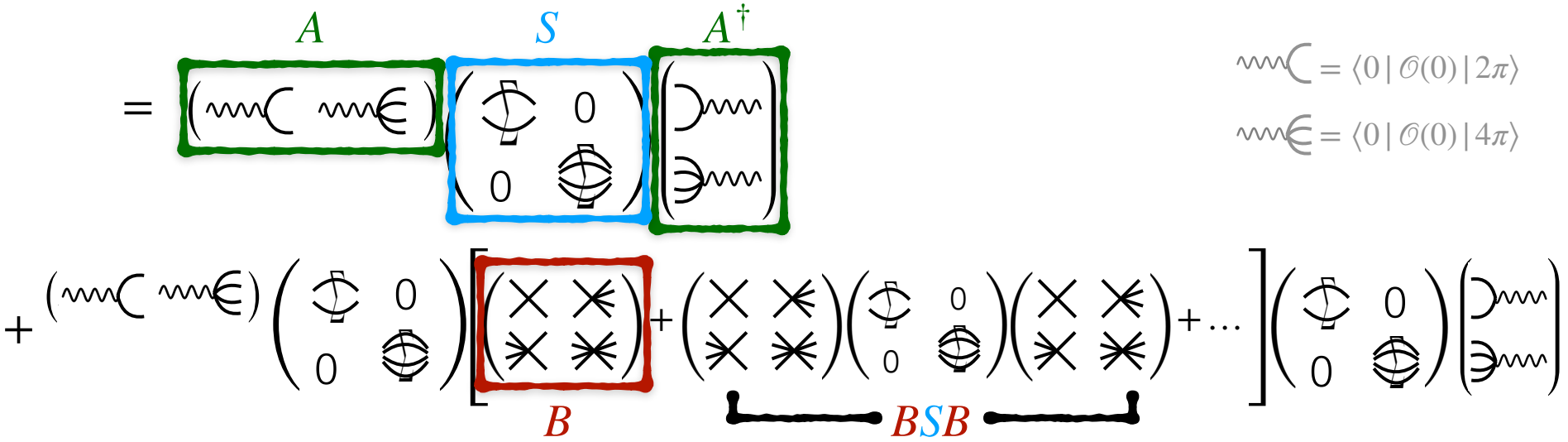

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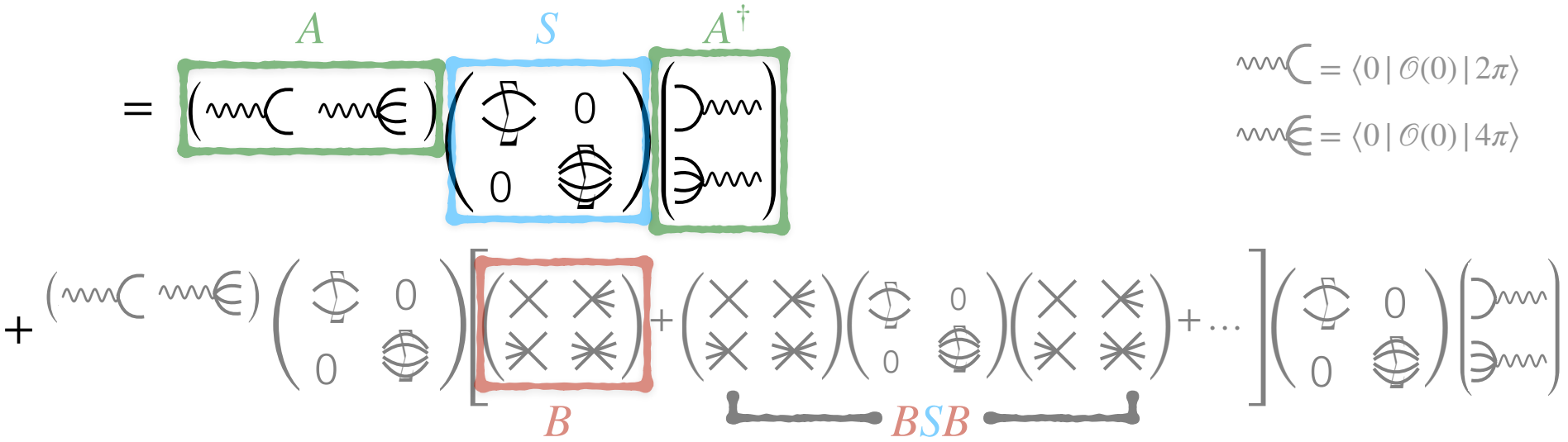

$$= ASA^\dagger + AS\left(\frac{1}{B^{-1} - S}\right)SA^\dagger$$

FV energies are solutions to $\det[B^{-1} - S] = 0$

Building blocks

$$\det[B^{-1} - S] = 0$$

$$B = \begin{pmatrix} \times & \times \\ \times & \times \end{pmatrix} = \begin{pmatrix} g_{2\to 2} & g_{4\to 2} \\ g_{2\to 4} & g_{4\to 4} \end{pmatrix}, \quad S = \begin{pmatrix} \text{bubble} & 0 \\ 0 & \text{bubble} \end{pmatrix} = \begin{pmatrix} S_2 & 0 \\ 0 & S_4 \end{pmatrix}$$

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use time-ordered perturbation theory (TOPT) *[Sterman (1993), Blanton & Sharpe (2020)]*

$$S_k(E, L) = \sum_{\mathbf{p}_{1,\ldots,k}} \frac{\delta_{\sum \mathbf{p},\mathbf{0}}}{\prod_{i=1}^{k} 2\omega_{\mathbf{p}_i}} \left(\frac{1}{E - \sum_{i=1}^{k} \omega_{\mathbf{p}_i} + i\varepsilon} \right) + O(e^{-m_\pi L}),$$

where $\mathbf{p} = \frac{2\pi}{L}\mathbf{n}$, with $\mathbf{n} \in \mathbb{Z}^3$.

A_1^+

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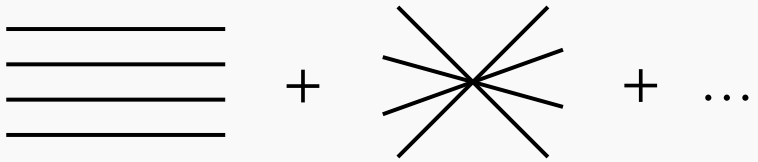
A_1^+

in practice, regulate integral by truncating sum to $\Lambda : \mathbf{n} \in [-N, N]^3$

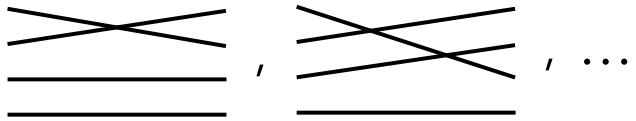
Additional ingredient

$$\det[B^{-1} - S] = 0$$

So far, the $4 \to 4$ interactions accounted for are



does not include diagrams with
internal $2 \to 2$, $3 \to 3$ scattering



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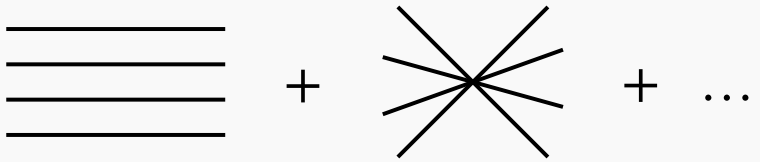

contributions to $E_{4,n}$: $O(1)$, $O\left(\frac{g_{2\to2}}{L^3}\right)$, $O\left(\frac{g_{4\to4}}{L^9}\right)$

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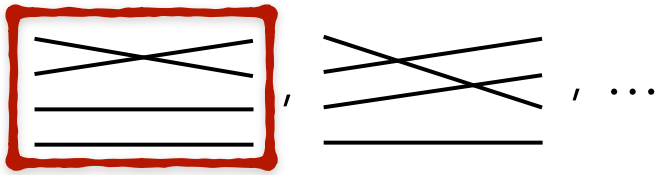

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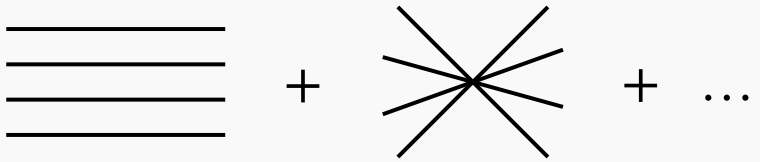

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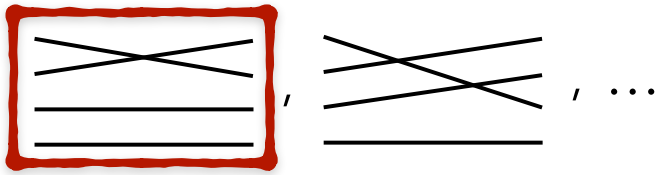


- We compute in TOPT for the first several energy levels and add it to $S_4(E, L)$ by hand

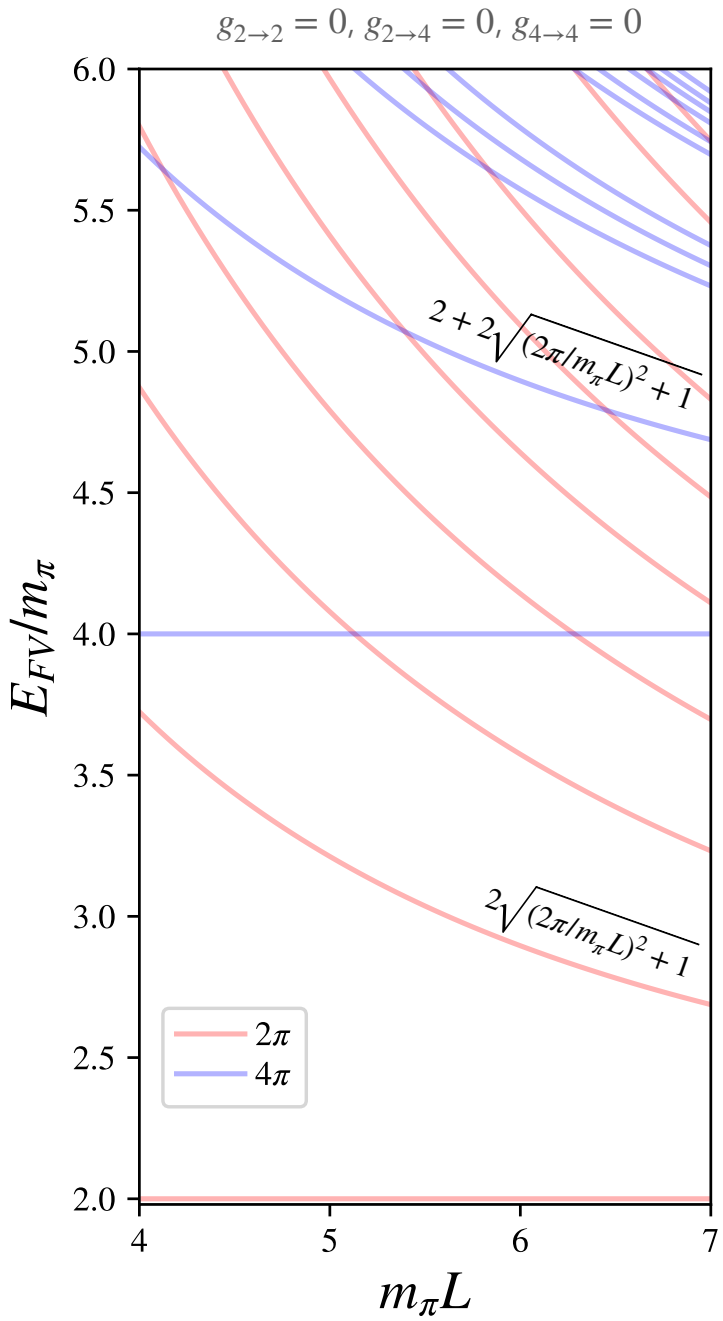
$$S = \begin{pmatrix} S_2(E,L) & 0 \\ 0 & S_4'(E,L,g_{2\to2}) \end{pmatrix}$$

Non-interacting levels

(centre of mass frame)

all couplings set to zero

$\Longrightarrow$ levels are poles of S_2 and S_4



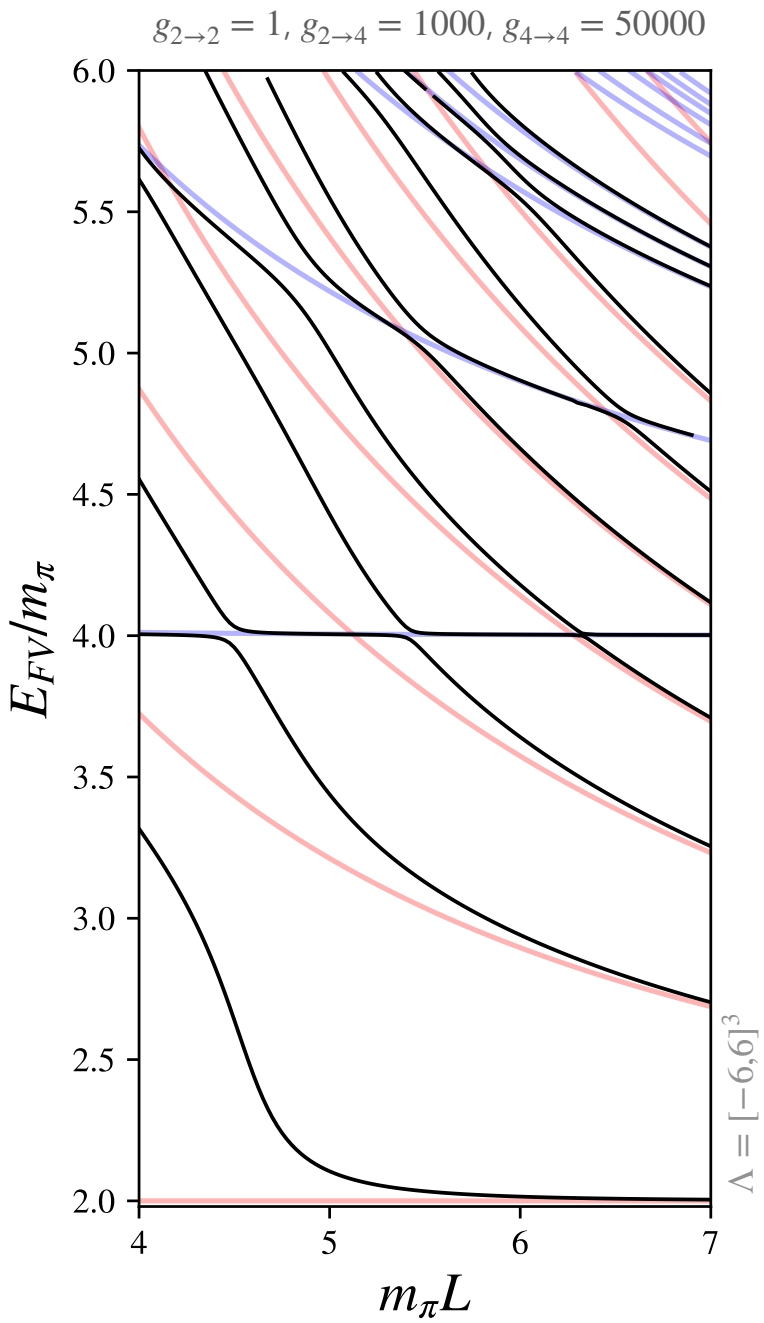
Interacting levels

(centre of mass frame)

$$B = \begin{pmatrix} g_{2\to2} & g_{4\to2} \\ g_{2\to4} & g_{4\to4} \end{pmatrix},\ S = \begin{pmatrix} S_2 & 0 \\ 0 & S_4' \end{pmatrix},\ E_n(L):\ \det[B^{-1} - S] = 0$$

interactions couple 2 and 4 pion sectors

avoided level crossings: signs of inelastic thresholds and multi-particle mixing



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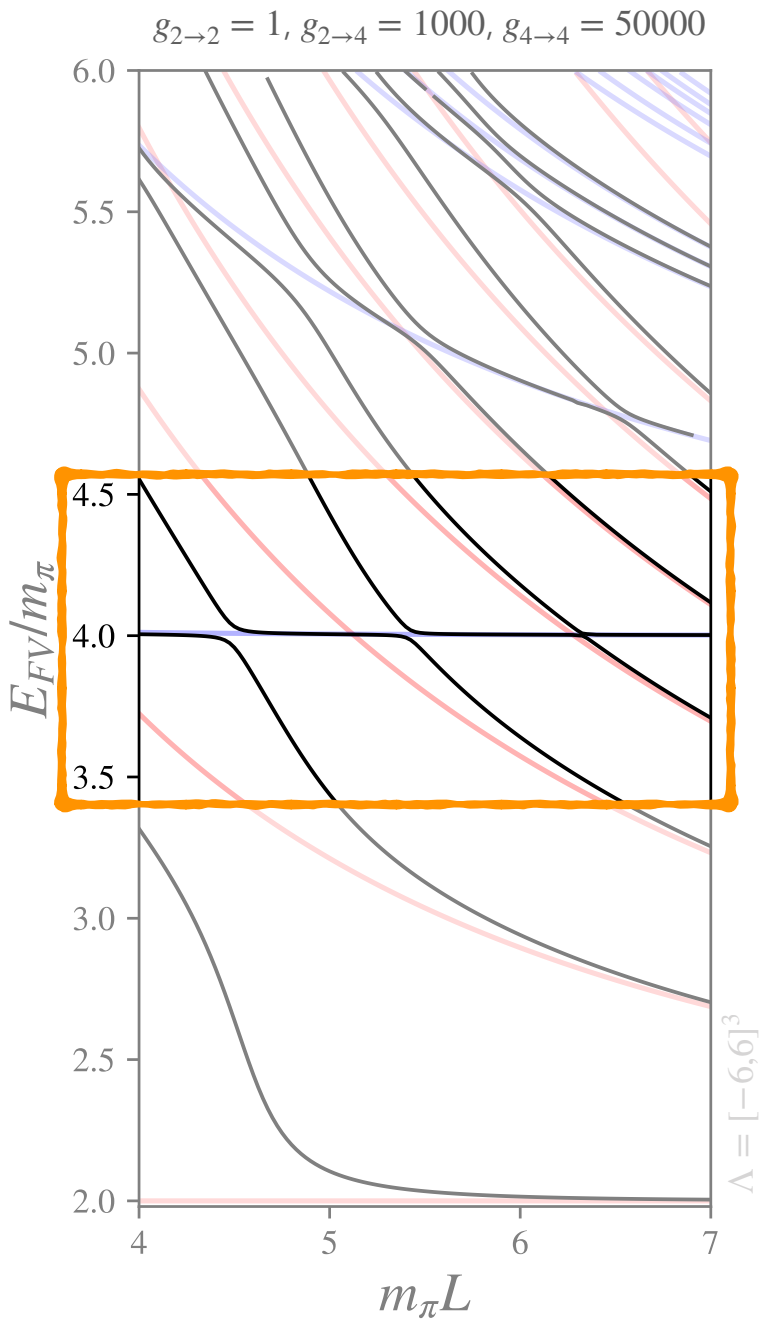
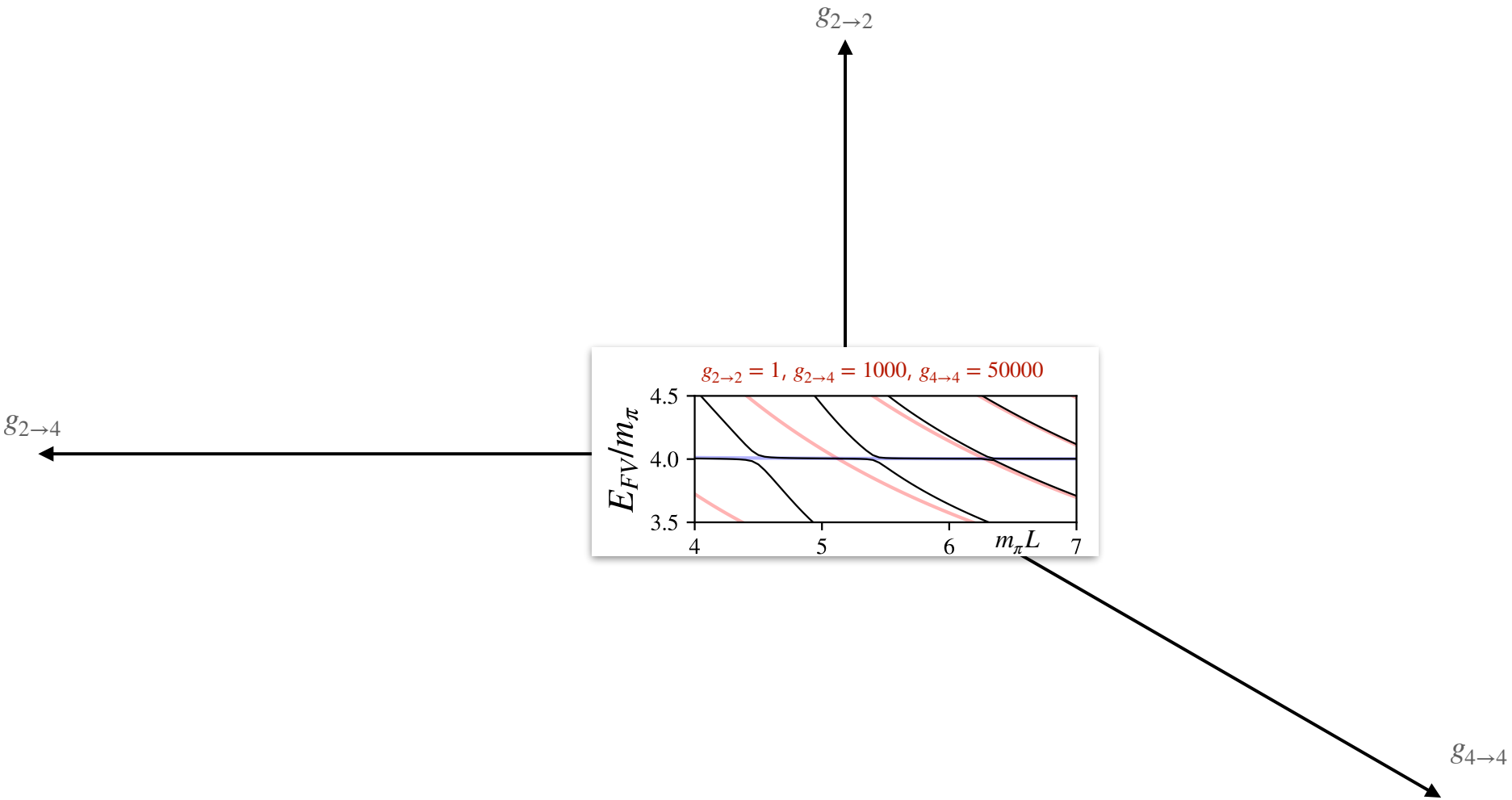
many toy results for different choices of

$$g_{2\to2},\ g_{2\to4}{}^{(=g_{4\to2})},\ g_{4\to4}$$



Varying couplings



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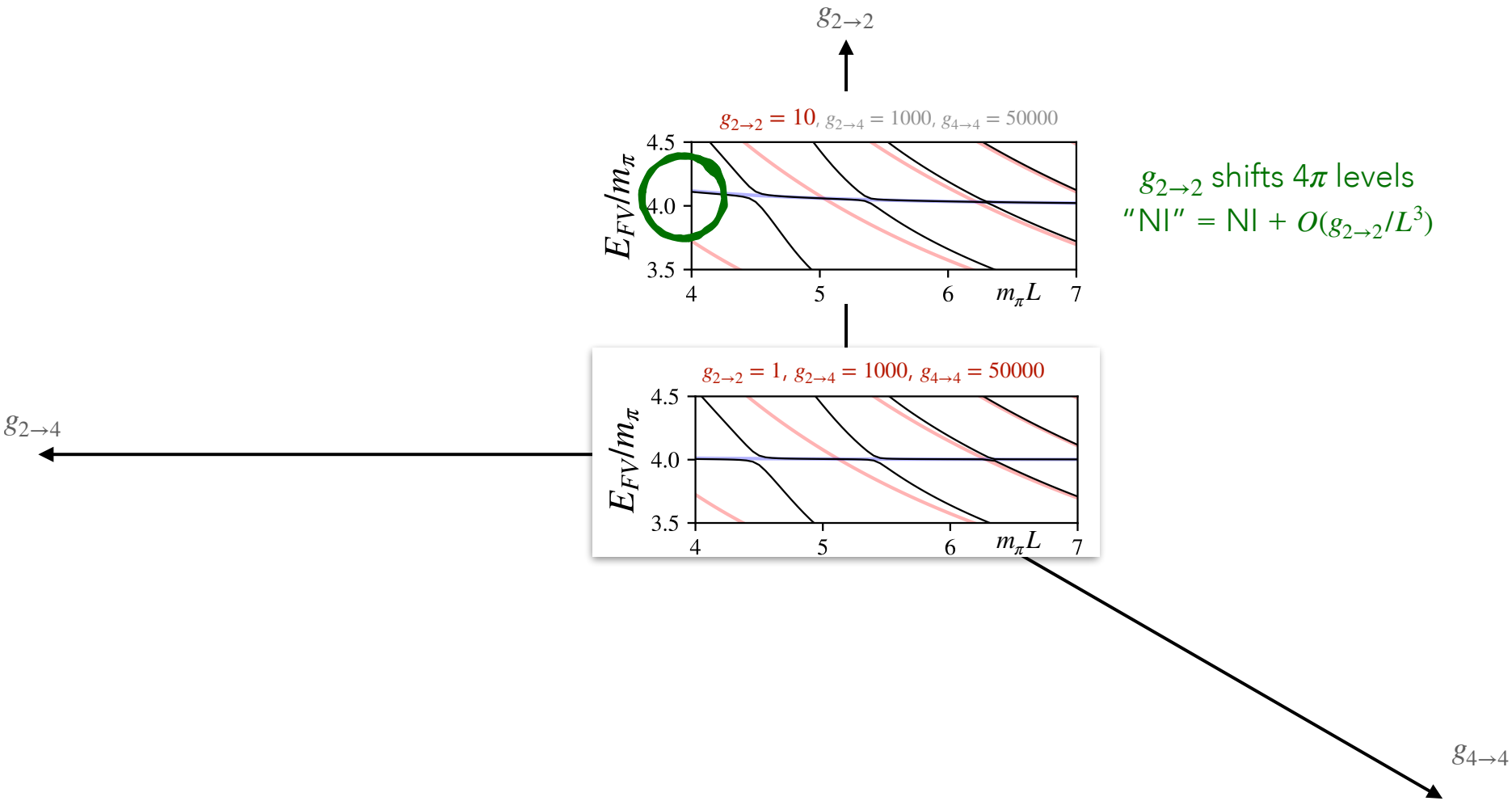
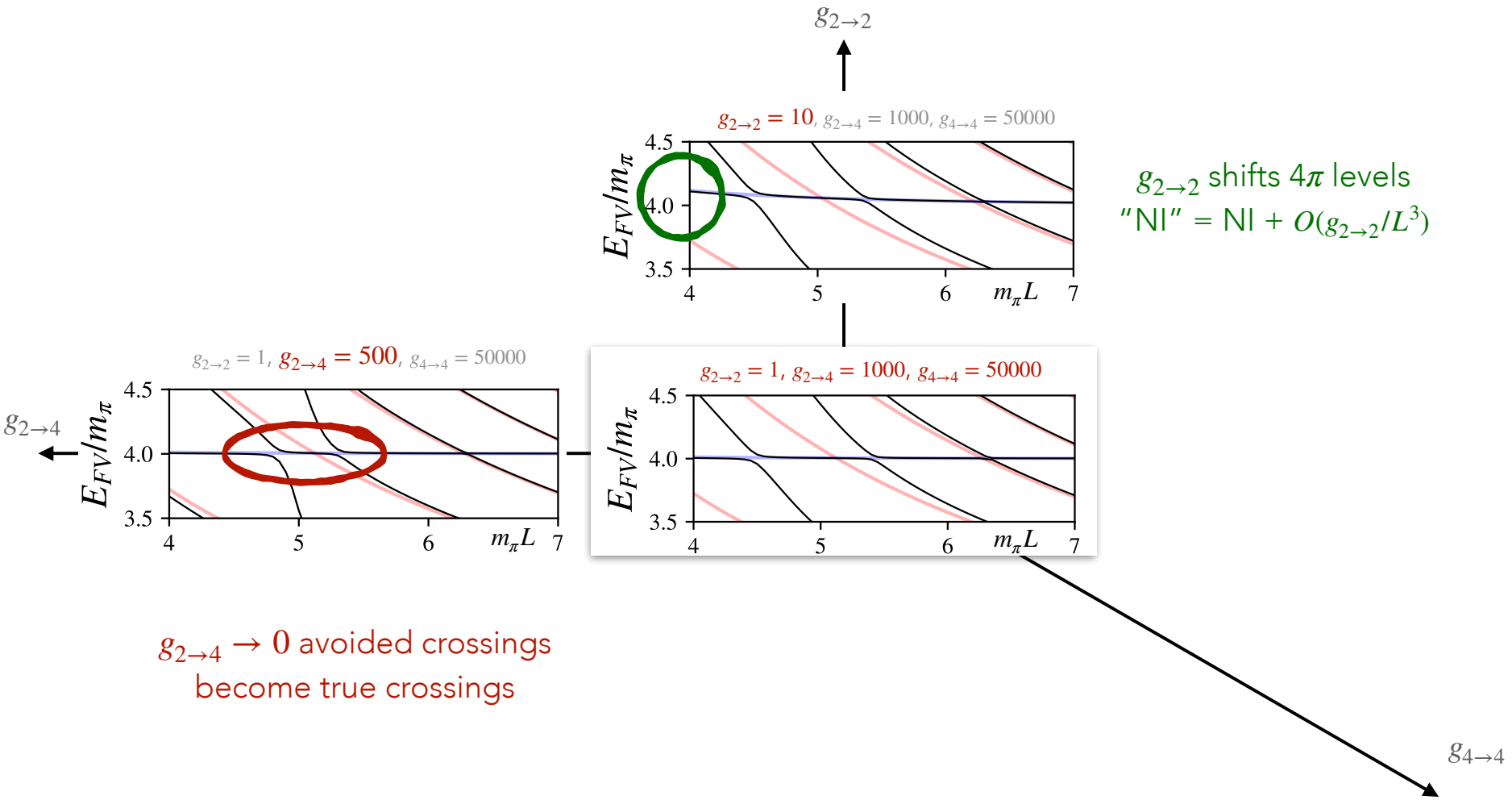


$g_{2\to 2}$ shifts 4π levels

"NI" = NI + $O(g_{2\to 2}/L^3)$

Varying couplings



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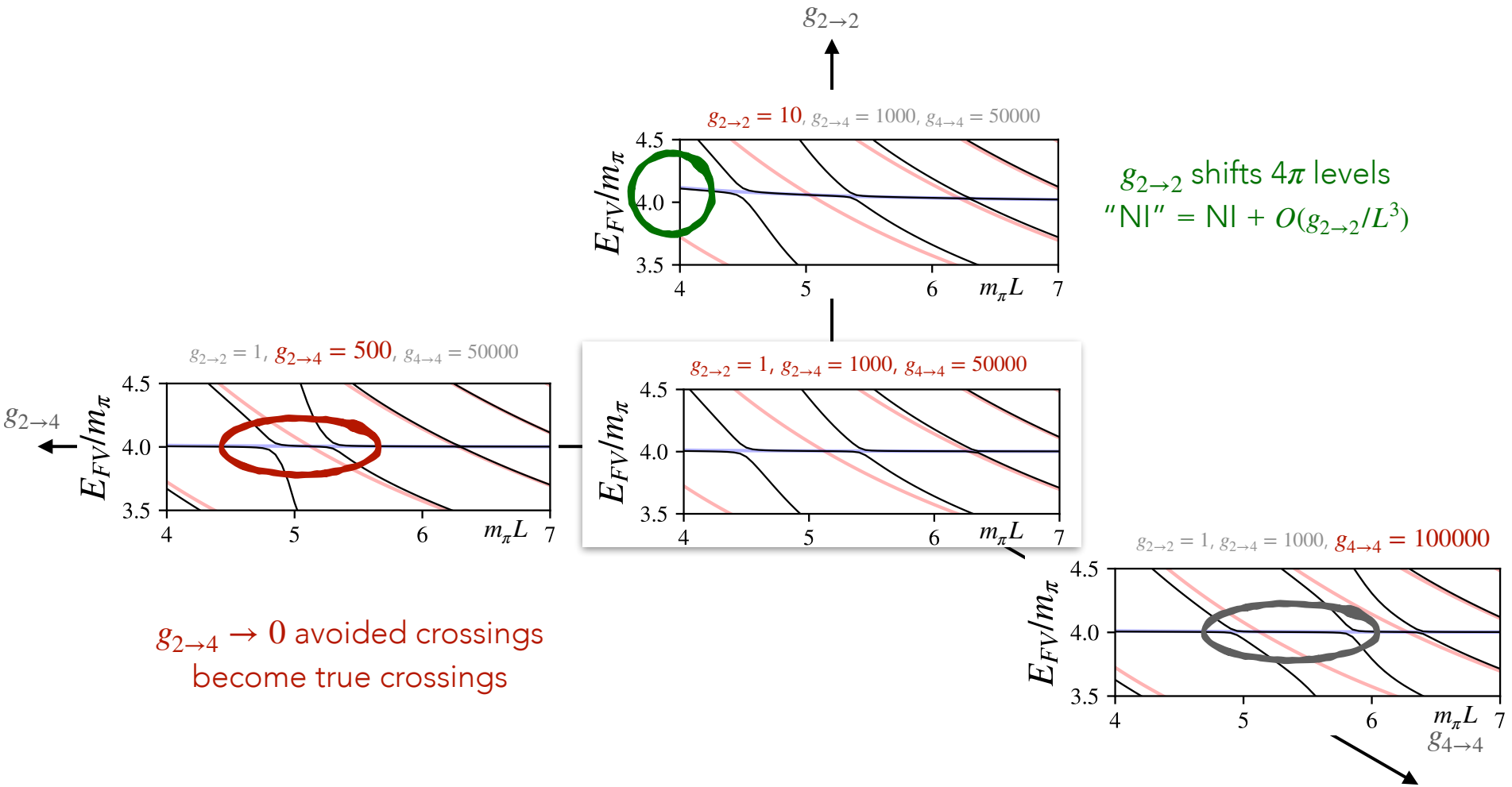

Estimating four-pion effects

key challenge: complicated interpretation of finite volume states

$$\langle n, L | H_W | D, L \rangle = C_{2\pi} \langle 2\pi, \text{out} | \mathscr{H}_W | D \rangle_{\text{KLP}} + C_{4\pi} \langle 4\pi, \text{out} | \mathscr{H}_W | D \rangle_{\text{KLP*}}$$

*K-matrix like piece

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$$\left[\begin{pmatrix} g_{2\to 2} & g_{4\to 2} \\ g_{2\to 4} & g_{4\to 4} \end{pmatrix}^{-1} - \begin{pmatrix} S_2 & 0 \\ 0 & S_4' \end{pmatrix} \right] = (\mathbf{v}_1 \quad \mathbf{v}_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{pmatrix}$$

singular at $E_n(L)$ ~ 0

Relative sizes of 2 and 4π-like contributions given by the zero eigenvector:

[Hansen & Sharpe (2012), Briceño, Hansen & Walker-Loud (2015), Hansen, Romero-Lopez & Sharpe (2021)]

$$\mathbf{v}^{(n)}_{\lambda=0}(L) \propto \begin{pmatrix} C_{2\pi} \\ C_{4\pi} \end{pmatrix}$$

*K-matrix like piece

Estimating four-pion effects

key challenge: complicated interpretation of finite volume states

$$\langle n, L | H_W | D, L \rangle = C_{2\pi} \langle 2\pi, \text{out} | \mathscr{H}_W | D \rangle_{\text{KLP}} + C_{4\pi} \langle 4\pi, \text{out} | \mathscr{H}_W | D \rangle_{\text{KLP*}}$$

Toy example:

hadronic D decay, $m_\pi L = 6$, and $n = 1$:

$g_{2\to 2} = 1$, $g_{2\to 4} = 1000$, $g_{4\to 4} = 50000$

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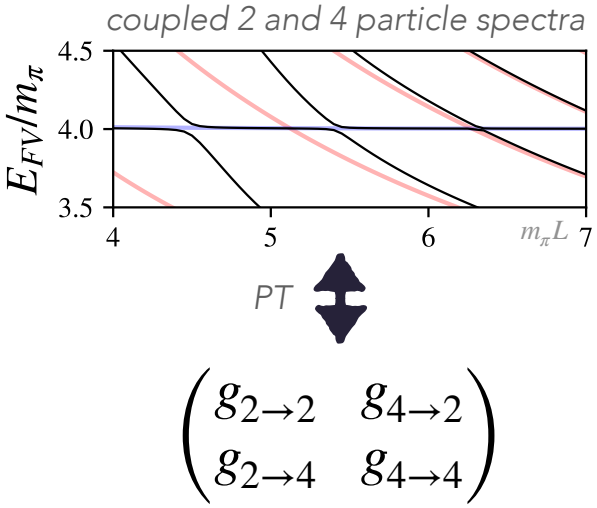
$$\frac{E_{\text{FV}}^{n=1}}{m_\pi}(m_\pi L = 6) = 2.94101938,$$

$$\mathbf{v}_{\lambda=0} = \begin{pmatrix} -0.99995000 \\ -0.00999942 \end{pmatrix} \implies \left.\frac{C_{4\pi}}{C_{2\pi}}\right|_{m_\pi L=6}^{n=1} \sim 1\,\%$$

*K-matrix like piece

Summary

- PT method relating bare couplings to finite-volume energies
- LO effects of $2 \to 4$ and $4 \to 4$ interactions
- estimates for relative size of 4π effects

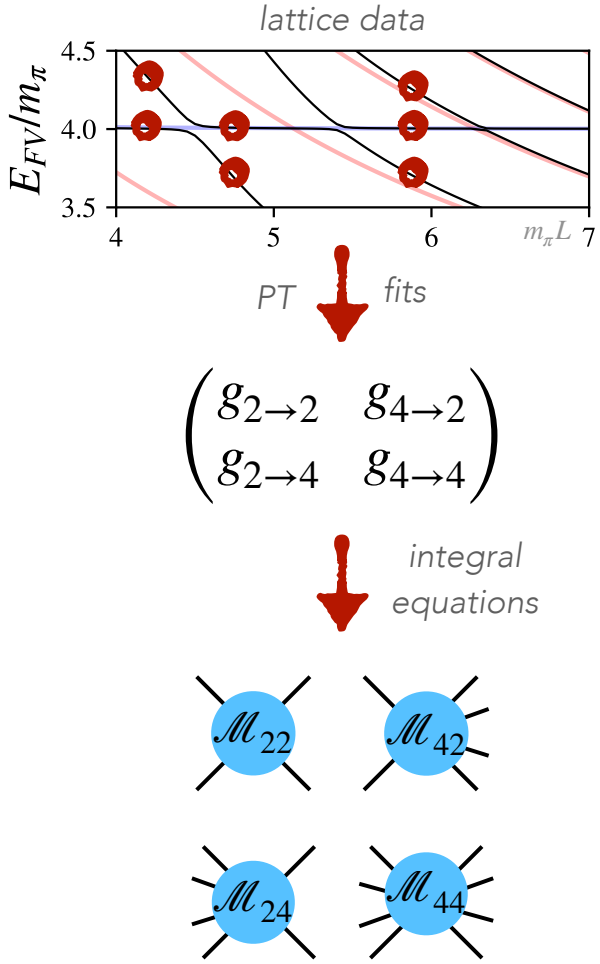


Summary

- PT method relating bare couplings to finite-volume energies
- LO effects of $2 \to 4$ and $4 \to 4$ interactions
- estimates for relative size of 4π effects

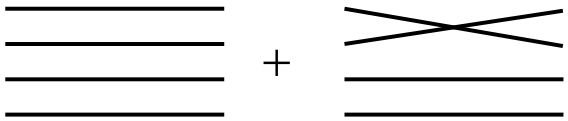
Outlook

- Short term:
  relation to *infinite-volume* amplitudes
  different irreps, non-zero total momentum
- Long term:
  a K-matrix style fully developed 4π formalism



Backups

$O(g_{2\to 2}/L^3)$ contribution to $E_{4\pi}$



$$E_{4\pi}^{n=0}(L) = 4m_\pi + \binom{4}{2}\frac{g_{2\to 2}}{8m_\pi^2 L^3}$$

$$E_{4\pi}^{n=1}(L) = 2m_\pi + 2\omega_{(1)} + \frac{g_{2\to 2}}{8m_\pi L^3}\left[\frac{1}{m_\pi} + \frac{2m_\pi}{\omega_{(1)}^2} + \frac{8}{\omega_{(1)}}\right]$$

$$E_{4\pi}^{n=2}(L) = 2m_\pi + 2\omega_{(2)} + \frac{g_{2\to 2}}{8m_\pi L^3}\left[\frac{1}{m_\pi} + \frac{2m_\pi}{\omega_{(2)}^2} + \frac{8}{\omega_{(2)}}\right]$$

$$E_{4\pi}^{n=3}(L) = 2m_\pi + 2\omega_{(3)} + \frac{g_{2\to 2}}{8m_\pi L^3}\left[\frac{1}{m_\pi} + \frac{2m_\pi}{\omega_{(3)}^2} + \frac{8}{\omega_{(3)}}\right]$$

$\cdots$

$$\frac{g_{2\to 2}}{8m_\pi^2 L^3} = \frac{4\pi a_0}{m_\pi L^3}$$

$$\omega_{(n)} = \sqrt{n\left(\frac{2\pi}{L}\right)^2 + m_\pi^2}$$