

Extending the finite volume formalism to the $N\pi\pi$ system at maximal isospin



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Based on work with Max Hansen & Fernando Romero-López,
arXiv:2509.24778



Ultimate motivation: Roper

Citation: S. Navas *et al.* (Particle Data Group), Phys. Rev. D **110**, 030001 (2024) and 2025 update

$N(1440) 1/2^+$

$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$ Status: * * * *

Discovered in 1963!

Breit-Wigner parameters: $M = 1410 - 1470$ MeV, $\Gamma = 250 - 450$ MeV

$N(1440)$ DECAY MODES

The following branching fractions are our estimates, not fits or averages.

	Mode	Fraction (Γ_i/Γ)
Γ_1	$N\pi$	55–75 %
Γ_2	$N\eta$	<1 %
Γ_3	$N\pi\pi$	17–50 %

Decays to both 2- and 3-particle channels

A copy of the proton with 50% greater mass:

- Can we understand using LQCD?
- No methodology exists at present

Theoretical challenges

- Decays into three-particle channels ($N\pi\pi$)
 - Work in RFT approach [Hansen & SRS, 2014, ...]
- Dealing with nondegenerate particles with spin
 - Combine nondegenerate formalism of [Blanton & SRS, 2020] with an adaptation of inclusion of spin in three-neutron system [Draper, Hansen, Romero-López, SRS, 2023]
- Incorporating flavor structure for different total isospins
 - $\frac{1}{2} \otimes 1 \otimes 1 = \frac{1}{2} \otimes (0 \oplus 1 \oplus 2) = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \oplus \frac{3}{2} \oplus \frac{5}{2}$
 - In each case can have either a nucleon or pion spectator
- Understanding singularities of 3-particle amplitudes if one pair is below threshold
- Dealing with the presence of both 2- and 3-particle channels ($N\pi + N\pi\pi$)

Status

- Decays into three-particle channels ($N\pi\pi$)
 - Use QC3—work in RFT approach [Hansen & SRS, 2014, ...]
- Dealing with nondegenerate particles with spin
 - Combine nondegenerate formalism of [Blanton & SRS, 2020] with an adaptation of inclusion of spin in three-neutron system [Draper, Hansen, Romero-López, SRS, 2023]



- Incorporating flavor structure for different total isospins



- $\frac{1}{2} \otimes 1 \otimes 1 = \frac{1}{2} \otimes (0 \oplus 1 \oplus 2) = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \oplus \frac{3}{2} \oplus \frac{5}{2}$

- In each case can have either a nucleon or pion spectator



- Understanding singularities of 3-particle amplitudes if one pair is below threshold

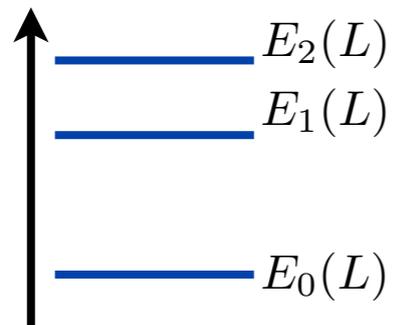


- Dealing with the presence of both 2- and 3-particle channels ($N\pi + N\pi\pi$)

- Formalism only complete for $I = 5/2$, where mixing with $N\pi$ is forbidden
- Have partial formalism for $I = 1/2, 3/2$, ignoring mixing with $N\pi$

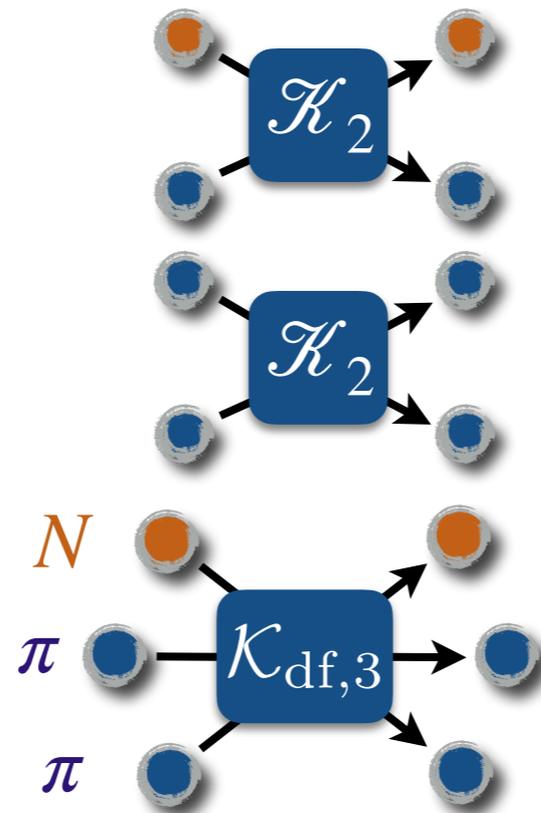
Sketch of result

- Result takes standard form

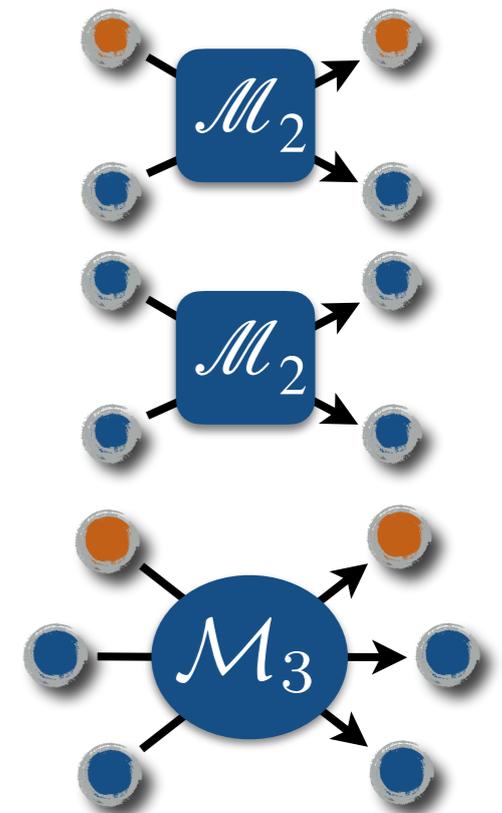


2- and 3-particle spectra
($N\pi$, $\pi\pi$, $N\pi\pi$)

Quantization Conditions
(QC2 & QC3)



Infinite-volume integral eqs.



QC3:

$$\det_{i\mathbf{p}lmm_s} \left(1 + \widehat{\mathcal{K}}_{\text{df},3}(E^*) \widehat{F}_3(E, \mathbf{P}, L) \right) = 0$$

Matrix indices are:
channel, spectator momentum,
pair angular momentum,
lab frame nucleon spin

$$\widehat{F}_3 \equiv \frac{\widehat{F}}{3} - \widehat{F} \frac{1}{1 + \widehat{\mathcal{M}}_{2,L} \widehat{G}} \widehat{\mathcal{M}}_{2,L} \widehat{F}, \quad \widehat{\mathcal{M}}_{2,L} \equiv \frac{1}{\widehat{\mathcal{K}}_{2,L}^{-1} + \widehat{F}}$$

\widehat{F} and \widehat{G} and known kinematic matrices

Dealing with nucleon spin

- Use nucleon spin component in lab frame
 - Complication arises when incorporating \mathcal{K}_2 for $N\pi$ scattering
 - Then want nucleon spin defined in pair CM frame, so it can be combined with ℓ
 - Relating spin components in two frames requires Wigner rotations

$$|\mathbf{a}^*, m_s(\mathbf{a})\rangle \equiv U(L(\boldsymbol{\beta}_{N\pi}(\mathbf{p}))) |\mathbf{a}, m_s(\mathbf{a})\rangle = |\mathbf{a}^*, m'_s(\mathbf{a}^*)\rangle \mathcal{D}_{m'_s m_s}^{(1/2)}(R(\theta(\mathbf{a}, \mathbf{p}), \hat{\mathbf{n}}(\mathbf{a}, \mathbf{p})))$$

Boost
operator

Lab frame spin
component

Pair frame spin
component

Wigner
rotation

- Introduces extra rotation in \mathcal{K}_2 :

$$\mathcal{K}_2^{N\pi, I}(q_\pi^*(\mathbf{p})) = D(\mathbf{p}) \cdot \mathcal{K}_2^{N\pi, *, I}(q_\pi^*(\mathbf{p})) \cdot \bar{D}(\mathbf{p})$$

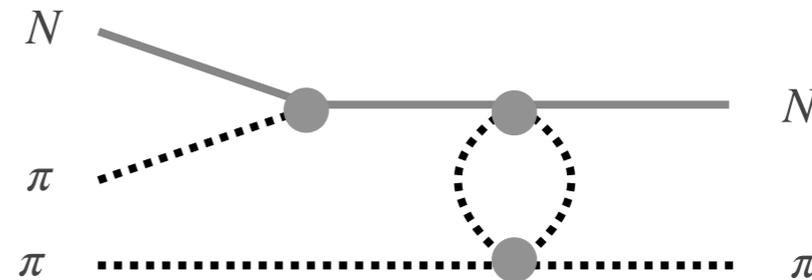
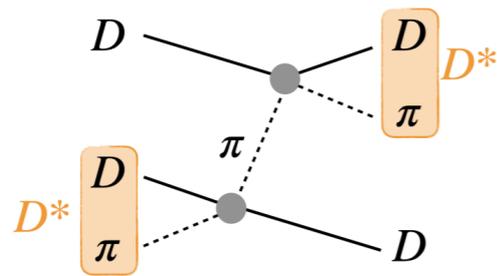
$$[D(\mathbf{p})]_{\ell m m_s; \ell^* m^* m_s^*} = \int d\Omega_{\hat{\mathbf{a}}^*} Y_{\ell m}^*(\hat{\mathbf{a}}^*) [\mathcal{D}^{(1/2)}(\mathbf{a}, \mathbf{p})^{-1}]_{m_s m_s^*} Y_{\ell^* m^*}(\hat{\mathbf{a}}^*),$$

Projection of
Wigner rotation

$$\mathcal{D}_{m_s^* m_s}^{(1/2)}(\mathbf{a}, \mathbf{p}) \equiv \mathcal{D}_{m_s^* m_s}^{(1/2)}(R(\theta(\mathbf{a}, \mathbf{p}), \hat{\mathbf{n}}(\mathbf{a}, \mathbf{p})))$$

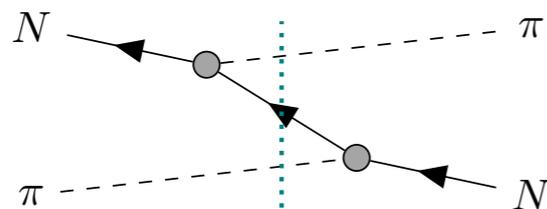
Challenge of $N\pi\pi + N\pi$

- Initial hope: use the “LSZ” method
 - Include $N\pi$ automatically with $N\pi\pi$, by having N as subthreshold pole in p-wave $N\pi$
 - Method has been successful in $DD\pi \leftrightarrow DD^*$, having D^* as pole in p-wave $D\pi$

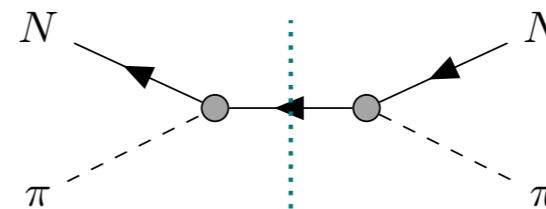


[Hansen, Romero-López, SRS: 2401.06609,
Dawid, Romero-López, SRS: 2409.17059, ...]

- Fails because of u-channel nucleon exchange singularity, which “shields” nucleon pole



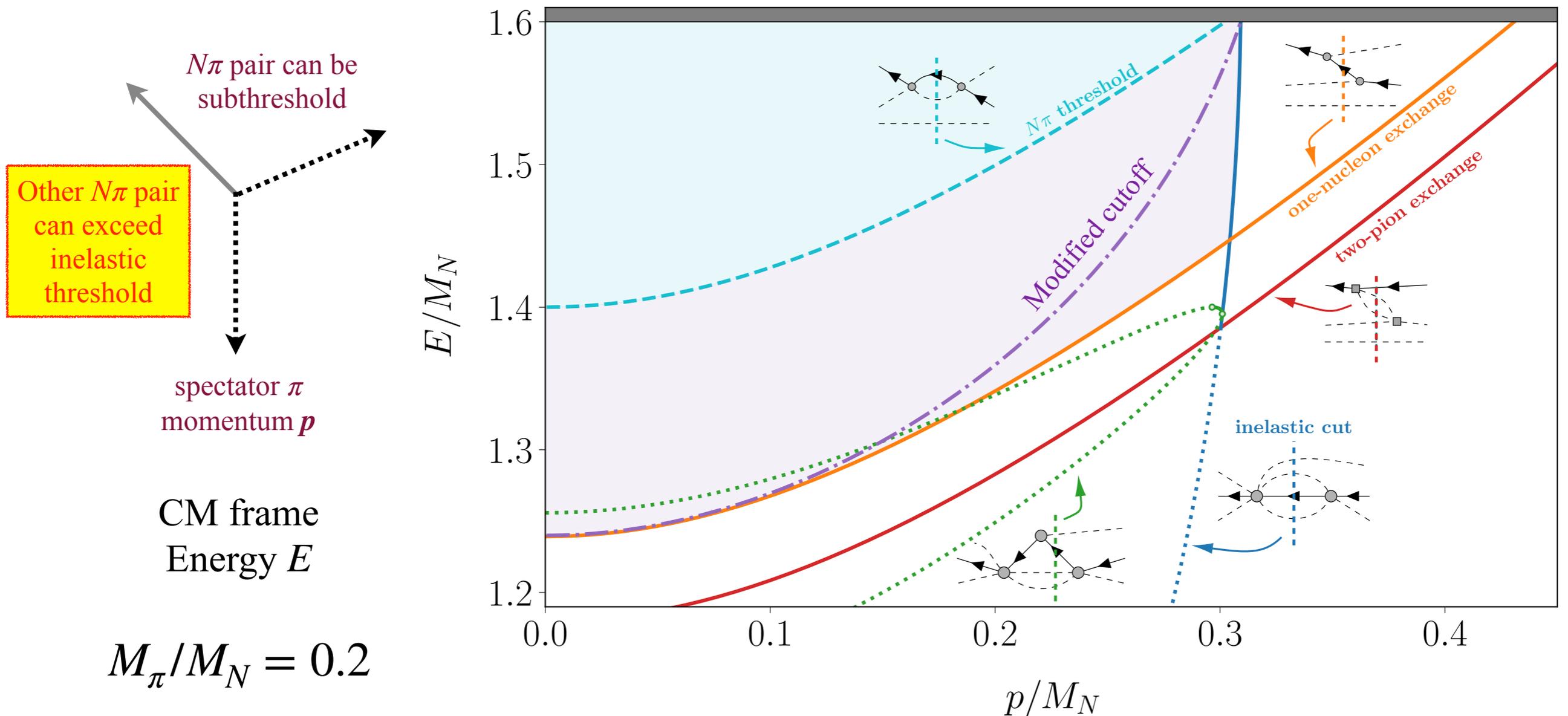
$$s_{\pi N} = M_N^2 + 2M_\pi^2$$



$$s_{\pi N} = M_N^2$$

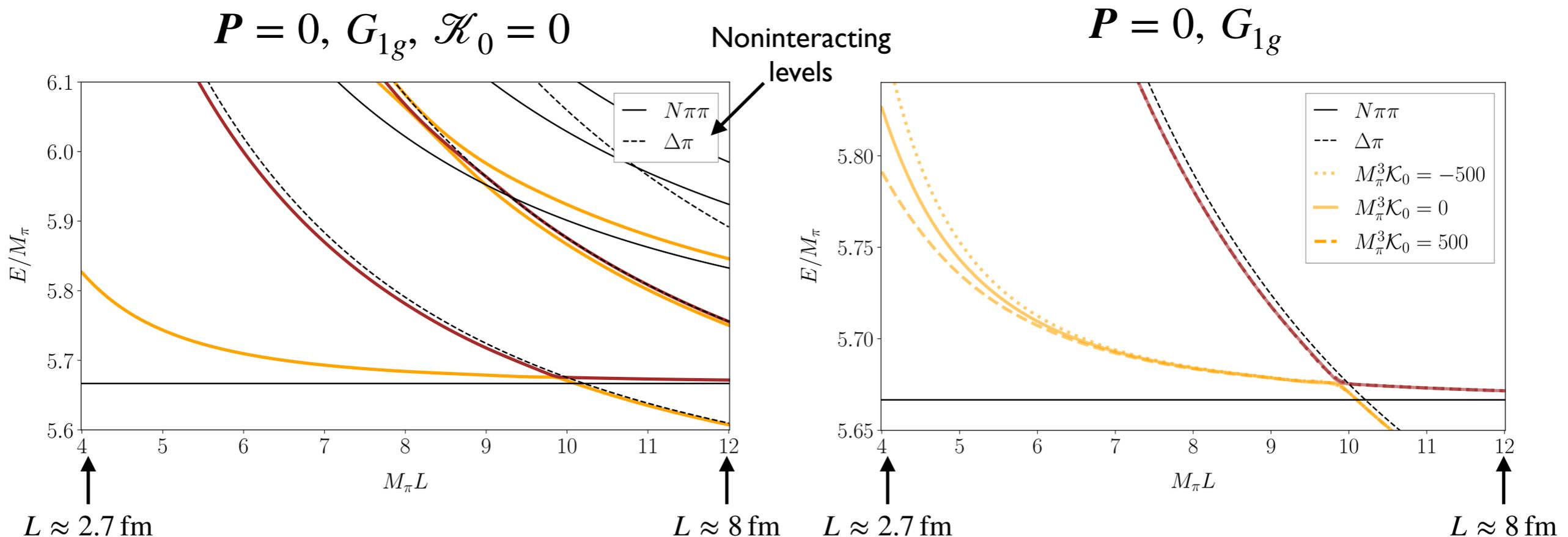
Subthreshold singularities

- 3PI 3-particle Bethe-Salpeter kernel must be singularity-free in our kinematic regime
 - If not, then there are additional, uncontrolled power-law finite-volume effects
 - For nondegenerate systems, we find a new locus of singularities in subthreshold region
 - Requires modifications to the smooth cutoff function $H(\mathbf{p})$



Numerical exploration

- $p\pi^+\pi^+$ with $M_\pi = 0.3$ GeV, $M_p = 1.1$ GeV, $M_\Delta = 1.335$ GeV (bound by 65 MeV)
 - Nontrivial system since $p\pi^+$ has p-wave bound state—included by LSZ method
 - Assume 2-particle interactions from ChPT ($\pi^+\pi^+$ & $p\pi^+$) or LQCD Δ results
 - Use $\mathcal{K}_{\text{df},3} = \mathcal{K}_0 \bar{u}(\mathbf{p}')u(\mathbf{p}_p)$, with $M_\pi^3 \mathcal{K}_0 = \{-500, 0, 500\}$
- Solve QC3 to determine spectrum
 - Observe avoided level crossings and mild \mathcal{K}_0 dependence



Summary & Outlook

- Formalism for $I = 5/2$ $N\pi\pi$ is ready to use
 - Numerical simulations should be possible in next few years
 - Formalism applies also to maximal isospin $\Sigma\pi\pi$ and NKK
- Need further study of forms for $\mathcal{K}_{\text{df},3}$ away from threshold
- Extension to a full Roper ($I = 1/2$) formalism requires explicit inclusion of $N\pi + N\pi\pi$
 - Extension of “2+3” formalism [Briceño, Hansen, SRS: 1701.07465] under study
- Beware of new singularities in subthreshold amplitudes!

Thanks
Any questions?