Preliminary study of the H dibaryon in $N_f = 2 + 1$ lattice QCD

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BaSc collaboration

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Setting the stage

- Six-quark state with $J^P = 0^+$, $I = 0$ and $S = -2$ (flavor content $uuddss$) and $m_H \approx 2100\,\mathrm{MeV}$ was predicted in 1977 with the MIT bag model [R. L. Jaffe (1977)]

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- **Nagara event** sets an upper bound to its binding energy to $B_H < 7\,\mathrm{MeV}$ from decay of the ${}^{\Lambda\Lambda}{}^{6}_{}\mathrm{He}$ hypernucleus [H. Takahashi et al (2001), B. Kim et al. (Belle, 2013)]

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✘ No evidence of its existence.

Motivations

- Most of the lattice calculations are in the $SU(3)$ limit with heavier than physical pion mass where dibaryon states can be described as [T. Inoue et al (2012), S.R. Beane et al (2012)]

$$\mathbf{8} \otimes \mathbf{8} = \underbrace{\mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8_A}}_{\text{anti-symmetric}} \oplus \underbrace{\mathbf{8_S} \oplus \mathbf{27} \oplus \mathbf{1}}_{\text{symmetric}}$$

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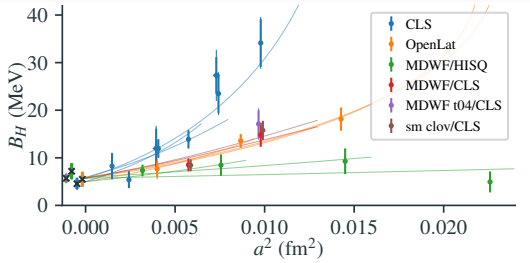
✘ No conclusive evidence at physical pion masses.

Lessons from the $SU(3)$-symmetric point - I

Mainz '21 paper and the (ongoing) universality study show that the binding energy has large cut-off effects

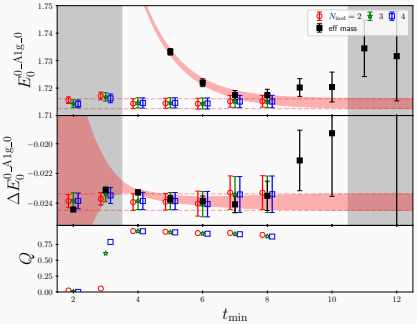
[J.R. Green et al (2021), Jeremy Green's talk on Monday]



- B_H decreases as $a \to 0$: cut-off effects are relevant and discretization-dependent
- At $m_\pi \approx 420\text{MeV}$ the H dibaryon is weakly bound ($B_H \approx 5\text{MeV}$).

Lessons from the $SU(3)$-symmetric point - II

Conspiracy-type models can exploit correlations and excited state cancellations between single and dibaryon correlators [J. Bulava et al (2025), Amy Nicholson's talk on Monday]



- The only underlying assumption is that baryons are weakly interacting
- Useful when plateaux are short or excited state contamination is large (as $m_\pi \to m_\pi^{\text{phys}}$)
- Exploit cancellations of the excited states without the problem of false plateaux in ratio fits

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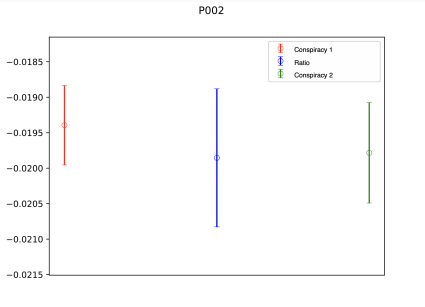

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How the conspiracy works

Global fit of single and dibaryon correlators

$$C_{B_i}(t) = \sum_{n=0}^{N_i-1} A_n^i e^{-E_n^i t}$$

$$C_{B_1B_2}(t) = \sum_{n,m} (A_n^1 A_m^2 + \epsilon_{nm}) e^{-(E_n^1+E_m^2+\delta_{nm})t}\,, \qquad E_n^i = E_0^i + \Delta_{n0}^i$$

Number of exponentials M for the dibaryon correlator is fixed by number of exponentials in single baryon correlators

- For non-identical scatterers $M = N_1 N_2$
- For identical scatterers $M = N_1(N_1-1)/2$

For weakly interacting scatterers cancellations occur between single and dibaryon correlators. Conspiracy fit allows to exploit these cancellations and preserves monotonicity of the correlator.

Methodology

- Correlation matrices of bilocal two-baryon operators (no hexa-quark operators) computed with distillation

[A. Francis et al (2019), M. Peardon et al (2009)]

- Perambulator computation performed with the `QUDA_laph` library

[https://github.com/cosmon-collaboration/quda_laph]

- Generalized eigenvalue problem to obtain energy spectrum

[M. Lüscher et al (1990)]

- Solve the two-body quantization condition to constrain K-matrix parameterization

[M. Lüscher (1991)]

$$\det\left[K^{-1}(E_{\mathrm{CM}}) - F(E_{\mathrm{CM}}, L)\right] = 0$$

Away from the SU(3) limit

Preliminary study on the D251 CLS ensemble:

$m_\pi\,(\mathrm{MeV})$	$m_K\,(\mathrm{MeV})$	$m_\pi L$	$a\,(\mathrm{fm})$	N_{lapH}
286	465	5.91	0.064	160

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  - In moving frames $N\Xi$ couples ${}^1S_0$ with ${}^3P_1$ (see subduction of the irreps of O_h^D).
  - Scattering amplitude couples ${}^1P_1$ with ${}^3P_1$

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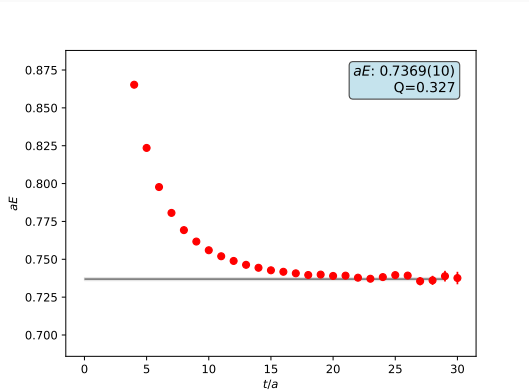
⚠ Multiple left-hand cuts: their position might be troublesome

[A. B. Raposo et al (2023, 2025), L. Meng et al (2024)]

⚠ Second left-hand cut due to two pseudoscalar mesons exchange might be even more problematic

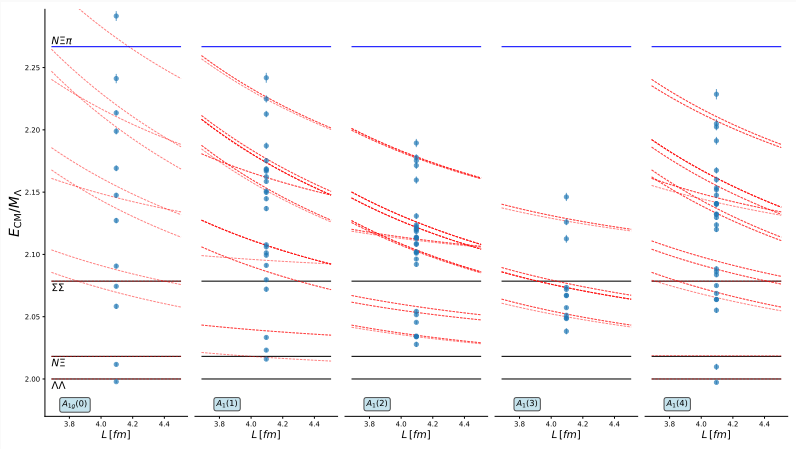
Spectrum extraction



- Same procedure for single baryon and dibaryon correlators
- Direct fit to correlation function (no conspiracy fit at this stage)
- $(t_{\min}, t_{\max})$ and model chosen with an information theory based model (and dataset) selection procedure.

Energy spectrum

Irreps $\Lambda(d^2)$ where $\vec{P} = (2\pi/L)\vec{d}$ and $d = |\vec{d}|$



- - = Non-interacting spectrum

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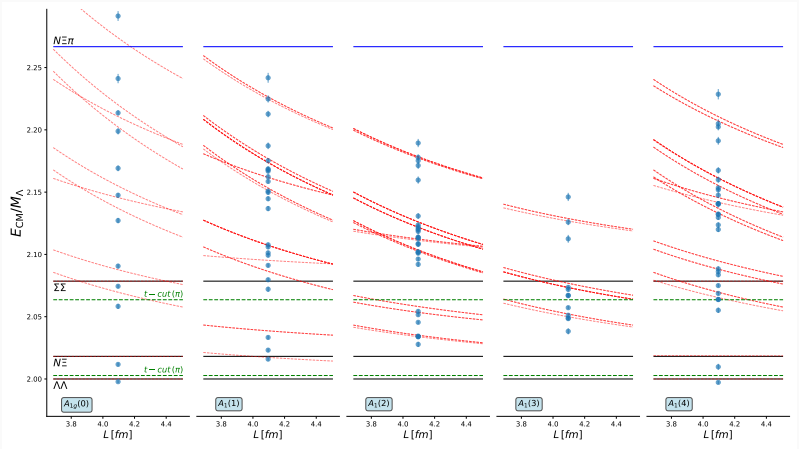
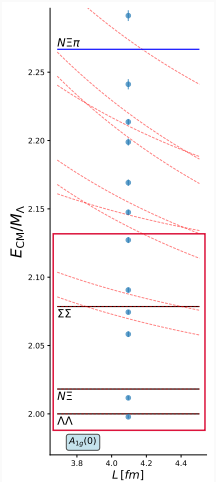


- - = Left-hand cuts due to one-pion exchange

Energy spectrum

- We restrict the preliminary analysis to the rest frame
- Select only flavor-symmetric states which couple to ${}^1S_0$: no partial wave mixing
- Select only the 6 lowest energy levels
- Ignore the presence of left-hand cuts



$SU(3)$-symmetric inspired K-matrix

Inspired by ChEFT, the inverse K-matrix can be parameterized as

$$K^{-1}(E_{\mathrm{CM}}) = U \begin{pmatrix} K^{-1}_{SU(3)} & & \\ & & \\ & & \end{pmatrix} U^{-1}$$

with

$$K^{-1}_{SU(3)}(E_{\mathrm{CM}}) = \underbrace{\begin{pmatrix} c_{\mathbf{1}} & 0 & 0 \\ 0 & c_{\mathbf{8_s}} & 0 \\ 0 & 0 & c_{\mathbf{27}} \end{pmatrix}}_{SU(3)\text{ symmetric}}$$

and U contains $SU(3)$ Clebsch-Gordan coefficients [J. Haidenbauer et al (2011, 2015), T. Inoue et al (2010)]

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$$K^{-1}(E_{\mathrm{CM}}) = U\left(K^{-1}_{SU(3)} + K^{-1}_{\text{breaking}}\right)U^{-1}$$

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K-matrix parameterization

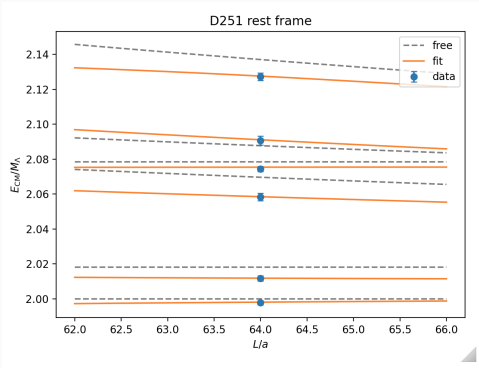
Minimal K-matrix parameterization with

- Constant $c_{\mathbf{27}}$ and $c_{\mathbf{8_s}}$
- $c_{\mathbf{1}} = c_{10} + \frac{c_{11}}{2}\left(E_{\mathrm{CM}} - 2M_{\Lambda}\right)$
  - Both ChEFT and lattice results suggest the $\mathbf{1}$ irrep is responsible for attractive interaction [J. Haidenbauer et al (2011, 2015), T. Inoue et al (2010)]
- Constant $c_{\mathbf{1}\leftrightarrow\mathbf{27}}$
- $c_{\mathbf{1}\leftrightarrow\mathbf{8_s}} = c_{\mathbf{8_s}\leftrightarrow\mathbf{27}} = 0$
  - Direct inspection of the $SU(3)$-breaking terms shows that $c_{\mathbf{1}\leftrightarrow\mathbf{27}}$ is the most relevant

Results and K-matrix parameters

- $\chi^2/\mathrm{dof} \approx 0.7$ with parameters

c_{10}	$-0.158(13)$
c_{11}	$3.68(28)$
$c_{\mathbf{8_S}}$	$0.28(7)$
$c_{\mathbf{27}}$	$-0.23(7)$
$c_{\mathbf{1}\leftrightarrow\mathbf{27}}$	$2.5(7)$



- $SU(3)$-breaking term is crucial to have a good quality-of-fit
- With more energy levels the model is not general enough: left-hand cut effects? Higher-partial waves?

Summary and outlook

Preliminary analysis on the $I = 0$, $S = -2$ H dibaryon on a single CLS ensemble away from the $SU(3)$ symmetric point with $m_\pi \approx 280\,\text{MeV}$.

- ✔ Interacting spectrum determined in multiple momentum frames for all the three relevant channels $\Lambda\Lambda$, $\Sigma\Sigma$ and $N\Xi$.
- ✔ Very simple model for K-matrix is able to describe the lowest lying energy levels in the rest frame

Many things to do now

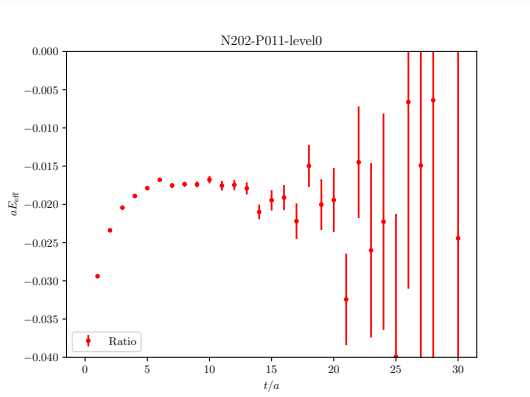
- ✎ Employ conspiracy-type models and model averaging for spectrum extraction
- ✎ Include more energy levels in the Lüscher analysis, e.g. anti-symmetric states
- ✎ Explore more advanced K-matrix parameterizations.
- ✎ Rôle of left-hand cuts: can we treat them properly?

Questions?

Ratio fits

Ratio fits are

- ✔ Great: they exploit correlations between single and two-hadron correlators
- ✘ Bad: they don't have a monotonically decreasing excited state contamination, leading to false plateaux identification

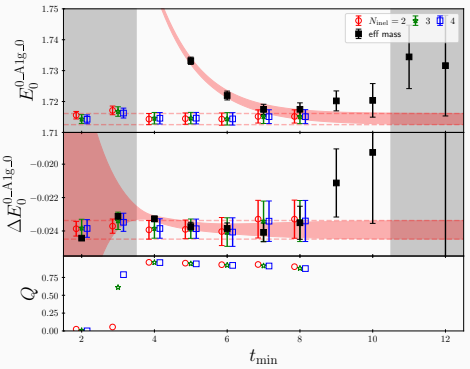


The conspiracy model: priors

We choose physics-based priors, e.g.

$$E_0 \sim m_{eff}(t_{ref}) \times \mathcal{N}(1, 0.1)$$
$$\Delta_{n,n-1} \sim \text{LogNormal}(2m_\pi, m_\pi)$$
$$\delta_{00} = m^R_{eff}(t_{ref}) \times \mathcal{N}(1, 1)$$
$$\dots$$

Be careful: choice of priors hyperparameters might induce biases in the results!



[J. Bulava et al, 2025]

Model averaging

Many arbitrary choices $\rightarrow$ many sources of systematic error ($t_{\mathrm{min}}, t_{\mathrm{max}}$, GEVP parameters, models, priors hyperparameters and many more)
Perform a model average on the fit parameters with a goodness-of-fit based weight and propagate the error with a **bootstrap** procedure.

- ✔ Most of the variability in the fit due to arbitrary choices is taken into account
- ✔ Correlations among energy levels are preserved even after systematic error is included.

Akaike information criterion

- It was introduced in 1973 as a tool for model selection [Akaike (1973)]
- It has a direct relation with the KL divergence D_{KL}

Given a vector of parameters θ and a set of likelihoods associated with a candidate model

$$\mathcal{F} = \{\mathcal{L}(y|\theta), \theta \in \Theta\}$$

If the true distribution $P(y) = P(y|\theta_0) \in \mathcal{F}$ for some $\theta_0 \in \Theta$, then the KL divergence reads

$$2D_{KL}(P|\mathcal{L}) = \underbrace{\mathbb{E}_0\left[-2\log \mathcal{L}(y|\theta)\right]}_{\equiv d(\theta|\theta_0)} - \underbrace{\mathbb{E}_0\left[-2\log P(y|\theta_0)\right]}_{\text{unknown}}$$

If $D_{KL} = 0$: perfect match between true and candidate model!

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Minimizing $d(\theta|\theta_0)$ wrt θ is equivalent to minimize D_{KL} and one can prove that un unbiased estimator for $d(\theta|\theta_0)$ is given by

$$\mathrm{AIC} = -2\log\mathcal{L}(y|\hat{\theta}) + 2n_p = \chi^2 + 2n_p\,, \qquad \begin{cases}\hat{\theta} = \mathrm{argmax}_\theta\left\{\mathcal{L}(y|\theta)\right\} \\ n_p = \#\,\text{of parameters}\end{cases}$$

Under reasonable assumptions it can be extended to dataset selection

$$\mathrm{AIC} = \chi^2 + 2n_p + 2n_c\,, \qquad n_c = \#\text{of points excluded from the fit}$$

- ✔ Trade-off between goodness-of-fit (χ^2), complexity of the model (n_p) and size of the dataset (n_c)
- ✔ Small values of AIC imply small values of D_{KL} and a better agreement between candidate and (unknown) true model

Interactions

At the $SU(3)$ symmetric point, different $SU(3)$ irreps do not interact with each other.
At leading order in ChEFT (baryon-baryon contact interactions)

$$V_{SU(3)} = \begin{pmatrix} c_{\mathbf{1}} & 0 & 0 \\ 0 & c_{\mathbf{8_S}} & 0 \\ 0 & 0 & c_{\mathbf{27}} \end{pmatrix}$$

Different flavor irreps are decoupled from each other.
Away from the $SU(3)$ symmetric point different $SU(3)$ irreps can interact, i.e.

$$V = \begin{pmatrix} c_{\mathbf{1}} & c_{\mathbf{1}\leftrightarrow\mathbf{8_S}} & c_{\mathbf{1}\leftrightarrow\mathbf{27}} \\ c_{\mathbf{1}\leftrightarrow\mathbf{8_S}} & c_{\mathbf{8_S}} & c_{\mathbf{8_S}\leftrightarrow\mathbf{27}} \\ c_{\mathbf{1}\leftrightarrow\mathbf{27}} & c_{\mathbf{8_S}\leftrightarrow\mathbf{27}} & c_{\mathbf{27}} \end{pmatrix}$$

Flavor basis and two-baryon states

Flavor-symmetric physical states can be written as

$$\begin{pmatrix} |\Lambda\Lambda\rangle \\ |\Sigma\Sigma\rangle \\ |N\Xi\rangle_s \end{pmatrix} = \underbrace{\frac{1}{\sqrt{40}} \begin{pmatrix} \sqrt{12} & \sqrt{8} & \sqrt{20} \\ -\sqrt{1} & -\sqrt{24} & \sqrt{15} \\ \sqrt{27} & -\sqrt{8} & -\sqrt{5} \end{pmatrix}}_{SU(3)\text{ Clebsch Gordan coeffs}} \begin{pmatrix} |\mathbf{1}\rangle \\ |\mathbf{8_S}\rangle \\ |\mathbf{27}\rangle \end{pmatrix} \equiv U \begin{pmatrix} |\mathbf{1}\rangle \\ |\mathbf{8_S}\rangle \\ |\mathbf{27}\rangle \end{pmatrix}$$

Equivalently, the $SU(3)$-symmetric states can be written in terms of physical states as

$$|\mathbf{1}\rangle = -\sqrt{\frac{1}{8}}\,|\Lambda\Lambda\rangle + \sqrt{\frac{3}{8}}\,|\Sigma\Sigma\rangle + \sqrt{\frac{4}{8}}\,|N\Xi\rangle_s$$

$$|\mathbf{8_S}\rangle = -\sqrt{\frac{1}{5}}\,|\Lambda\Lambda\rangle - \sqrt{\frac{3}{5}}\,|\Sigma\Sigma\rangle + \sqrt{\frac{1}{5}}\,|N\Xi\rangle_s$$

$$|\mathbf{27}\rangle = +\sqrt{\frac{27}{40}}\,|\Lambda\Lambda\rangle - \sqrt{\frac{1}{40}}\,|\Sigma\Sigma\rangle + \sqrt{\frac{12}{40}}\,|N\Xi\rangle_s$$

Final parameterization

The final parameterization reads

$$K^{-1}_{SU(3)} = \begin{pmatrix} \frac{1}{8}c_{\mathbf{1}} + \frac{27}{40}c_{\mathbf{27}} + \frac{1}{5}c_{\mathbf{8_S}} & -\frac{1}{4}c_{\mathbf{1}} + \frac{9}{20}c_{\mathbf{27}} - \frac{1}{5}c_{\mathbf{8_S}} & -\frac{1\sqrt{3}}{8}c_{\mathbf{1}} - \frac{3\sqrt{3}}{40}c_{\mathbf{27}} + \frac{\sqrt{3}}{5}c_{\mathbf{8_S}} \\ -\frac{1}{4}c_{\mathbf{1}} + \frac{9}{20}c_{\mathbf{27}} - \frac{1}{5}c_{\mathbf{8_S}} & \frac{1}{2}c_{\mathbf{1}} + \frac{3}{10}c_{\mathbf{27}} - \frac{1}{5}c_{\mathbf{8_S}} & -\frac{\sqrt{3}}{4}c_{\mathbf{1}} - \frac{\sqrt{3}}{20}c_{\mathbf{27}} + \frac{\sqrt{3}}{5}c_{\mathbf{8_S}} \\ -\frac{1\sqrt{3}}{8}c_{\mathbf{1}} - \frac{3\sqrt{3}}{40}c_{\mathbf{27}} + \frac{\sqrt{3}}{5}c_{\mathbf{8_S}} & -\frac{\sqrt{3}}{4}c_{\mathbf{1}} - \frac{\sqrt{3}}{20}c_{\mathbf{27}} + \frac{\sqrt{3}}{5}c_{\mathbf{8_S}} & \frac{3}{8}c_{\mathbf{1}} + \frac{1}{40}c_{\mathbf{27}} + \frac{3}{5}c_{\mathbf{8_S}} \end{pmatrix}$$

$$K^{-1}_{\text{breaking}} = c_{\mathbf{1}\leftrightarrow\mathbf{27}}\left(v_{\mathbf{1}}v^T_{\mathbf{27}} + v_{\mathbf{27}}v^T_{\mathbf{1}}\right)$$

where $v_{\mathbf{1}} = (-\sqrt{1/8}, \sqrt{4/8}, \sqrt{3/8})^T$, $v_{\mathbf{27}} = (\sqrt{27/40}, \sqrt{12/40}, -\sqrt{1/40})^T$