Bayesian Analysis and Analytic Continuation of Scattering Amplitudes from Lattice QCD

Miguel Salg

In collaboration with Fernando Romero-López and William Jay

Based on arXiv: 2506.16161 [hep-lat] **(accepted for publication in PRD)**

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UNIVERSITÄT BERN

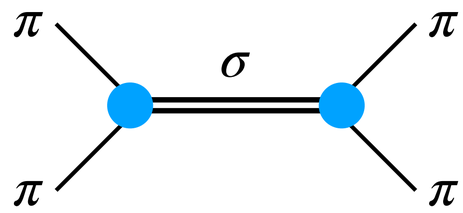
Background

Hadron spectroscopy on the lattice

- Goal: obtain poles of scattering amplitude $\mathscr{M}$ in complex plane from QCD —> masses and decay widths of hadronic resonances, bound states, and virtual states

$$\mathscr{M} \sim -\frac{g^2}{E^2 - E_{\text{pole}}^2}, \qquad E_{\text{pole}} = M_R - i\Gamma/2$$

Mass — Decay width



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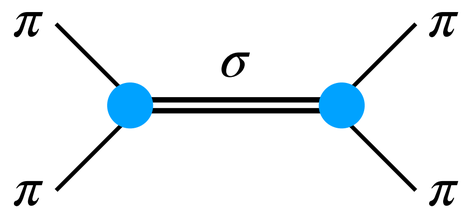

- Finite-volume formalism provides connection between finite-volume spectrum and infinite-volume scattering quantities: $\{E_n \in \mathbb{R}\} \leftrightarrow \mathscr{K}^{-1} \xrightarrow[\text{continuation}]{\text{analytic}} E_{\text{pole}} \in \mathbb{C}$

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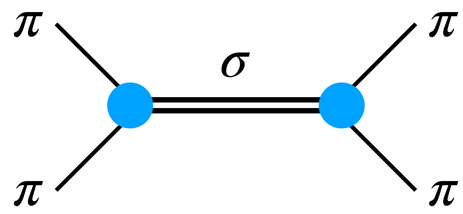

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- Reconstruct function $\mathscr{K}^{-1}(E)$ for $E \in \mathbb{R}$ and analytically continue to $E \in \mathbb{C}$ from finite set of points —> (mild) inverse problem

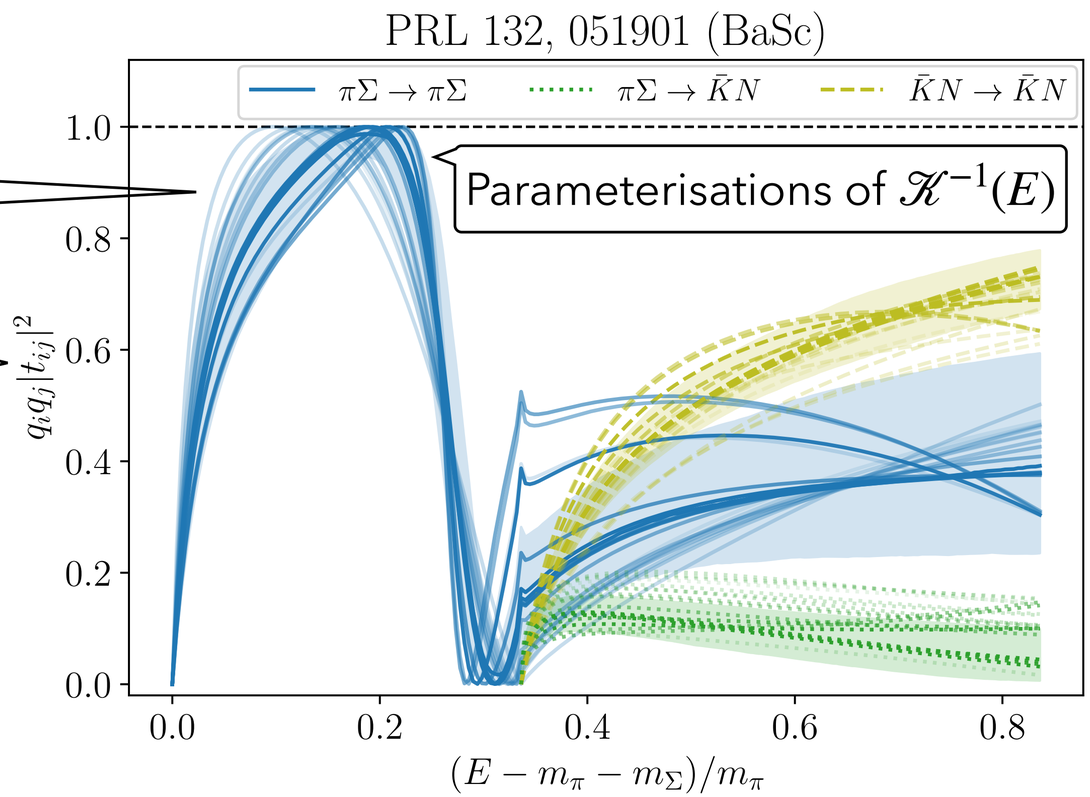
Motivation

Bulava et al. (BaSc) 2024 [PRL 132, 051901; PRD 109, 014511]

- Conventional approach: spectrum fits

Model dependence

$$t^{-1} \propto \mathscr{M}^{-1},$$
$$0 \leq q_i q_j |t_{ij}|^2 \leq 1$$



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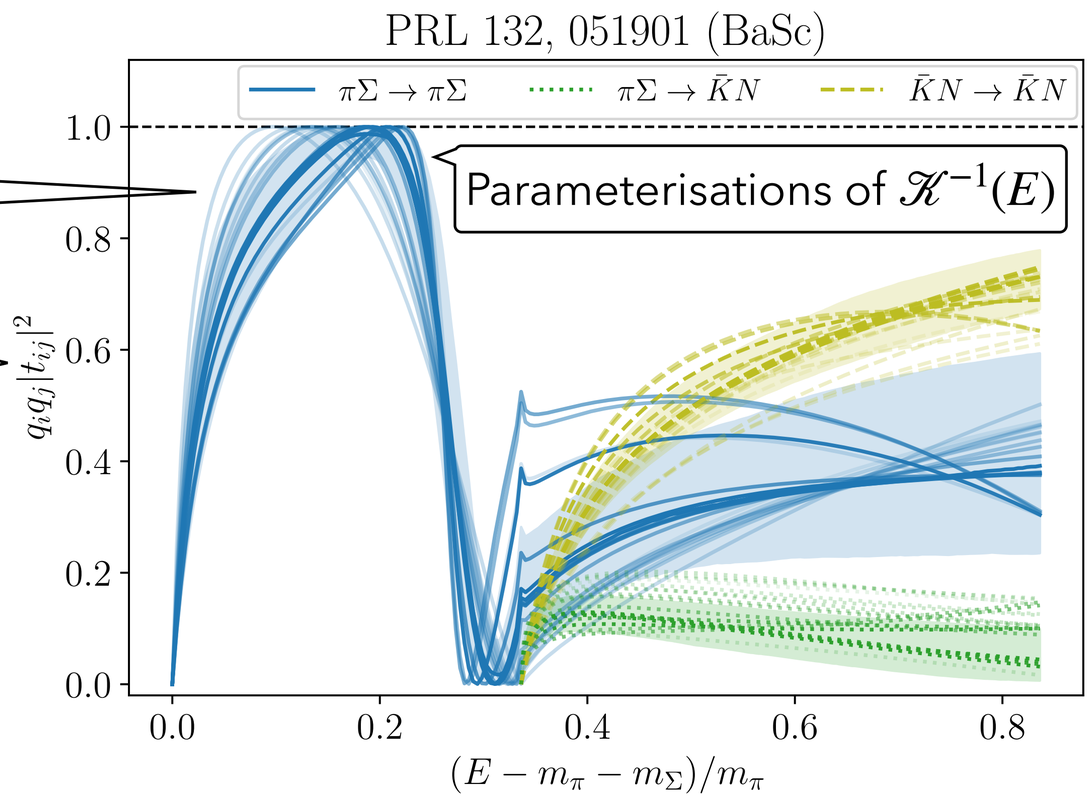

- Our approach: avoid explicit model dependence
  - Bayesian analysis of FV spectrum to constrain K matrix for real energies (inspired by Gaussian processes) Del Debbio, Giani, and Wilson 2022 [EPJC 82, 330]
  - Nevanlinna-Pick interpolation for analytic continuation to complex energies with strict uncertainty quantification —> Wertevorrat

Outline of our method

Bayesian analysis

General framework

Evaluate $\mathscr{K}^{-1}$ at n_p discrete "nodes" $\mathbf{E}$ —> values $\mathbf{u} = \mathscr{K}^{-1}(\mathbf{E})$

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$$p(\mathbf{u}\,|\,\mathbf{u}_0) = \frac{1}{\sqrt{(2\pi)^{n_p}\det\Sigma_p}}\exp\left[-\frac{1}{2}(\mathbf{u}-\mathbf{u}_0)^T\Sigma_p^{-1}(\mathbf{u}-\mathbf{u}_0)\right]$$

Prior central values

Prior covariance matrix,

$$\Sigma_p(E,E') = \sigma(E,E')^2\exp\left[-\frac{(E-E')^2}{2\ell_c^2}\right]$$

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LQCD data

Covariance of $\mathbf{E}_\mathrm{d}$

Solution of QC (nonlinear!)

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Posterior (Bayes' theorem)

$$p(\mathbf{u}\,|\,\mathbf{E}_\mathrm{d},\mathbf{u}_0) = p(\mathbf{E}_\mathrm{d}\,|\,\mathbf{u})\,p(\mathbf{u}\,|\,\mathbf{u}_0)$$

Bayesian analysis

Computing expectation values and errors

- Expectation values given by Bayesian integrals, e.g. $\langle \mathbf{u} \rangle_{\text{post}} = \frac{1}{Z} \int d^{n_p} u \, \mathbf{u} \, p(\mathbf{u} \,|\, \mathbf{E}_{\text{d}}, \mathbf{u}_0)$
- Uncertainties given by width of posterior distribution, e.g. $(\delta \mathbf{u})^2 = \langle \mathbf{u}^2 \rangle_{\text{post}} - \langle \mathbf{u} \rangle^2_{\text{post}}$

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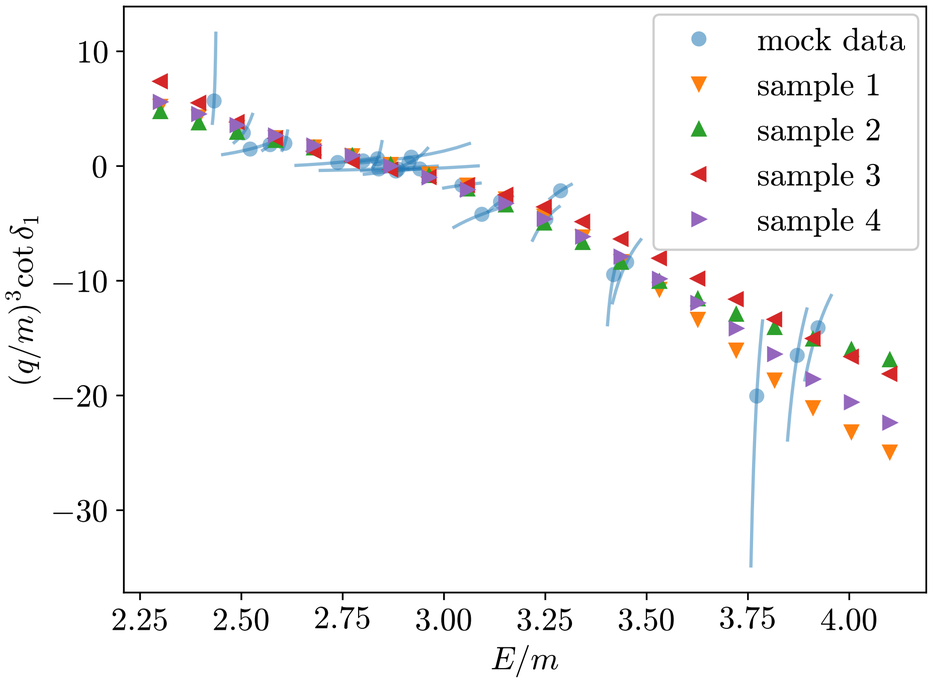
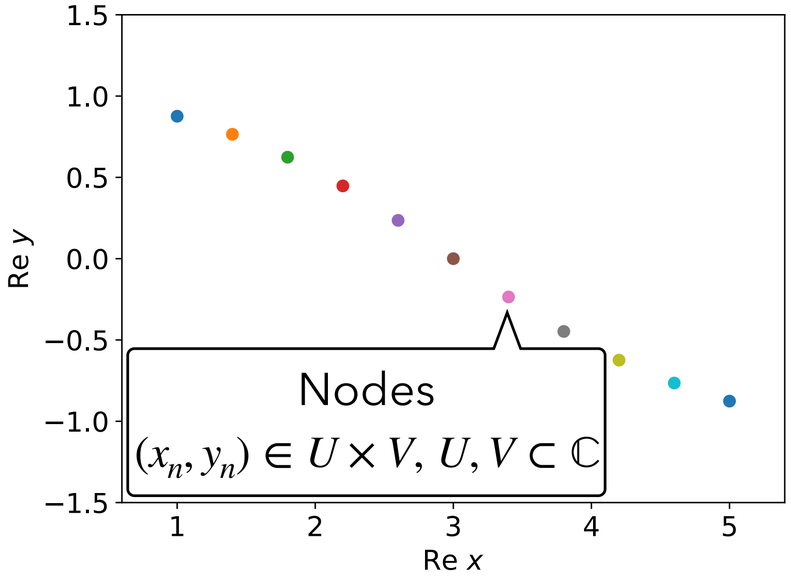
- Monte-Carlo sampling of posterior distribution: Gaussian approximation + reweighting or HMC
- Independent of the approach followed, the result are "functional" samples of $\mathbf{u} = \mathscr{K}^{-1}(\mathbf{E})$
  —> explore broad range of functional behaviours



Nevanlinna-Pick interpolation

General idea



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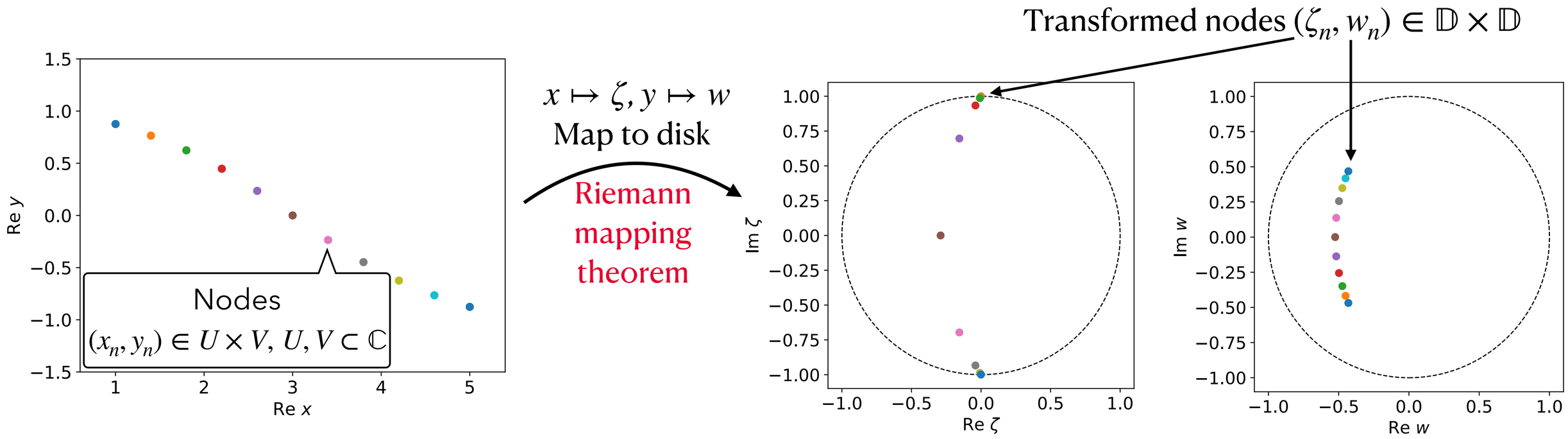

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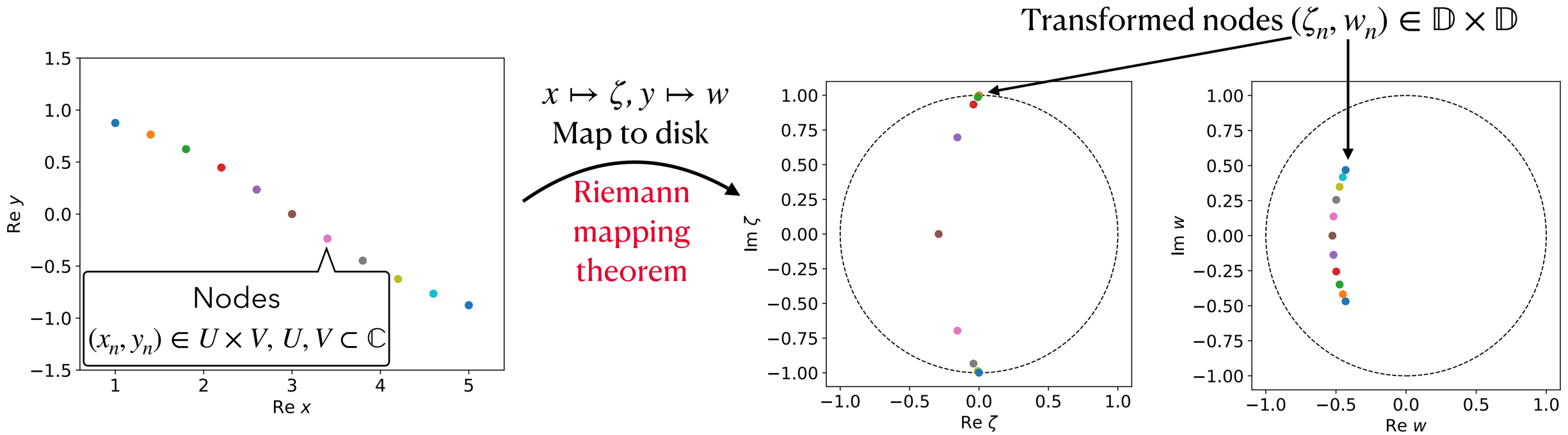

- Any solution can be written as $f(\zeta) = \dfrac{P(\zeta)h(\zeta) + Q(\zeta)}{R(\zeta)h(\zeta) + S(\zeta)}$ with $h : \mathbb{D} \to \mathbb{D}$ analytic, but arb.

  Recursively calculable coefficients

- For fixed ζ, all possible solutions fall within a disk $\Delta(\zeta) \subset \mathbb{D}$ —> Wertevorrat (WV)

Pick 1915 [Math. Ann. 77, 7]; Nevanlinna 1919 [Ann. Acad. Sci. Fenn. Ser. A 13]; Nevanlinna 1929 [Ann. Acad. Sci. Fenn. Ser. A 32]
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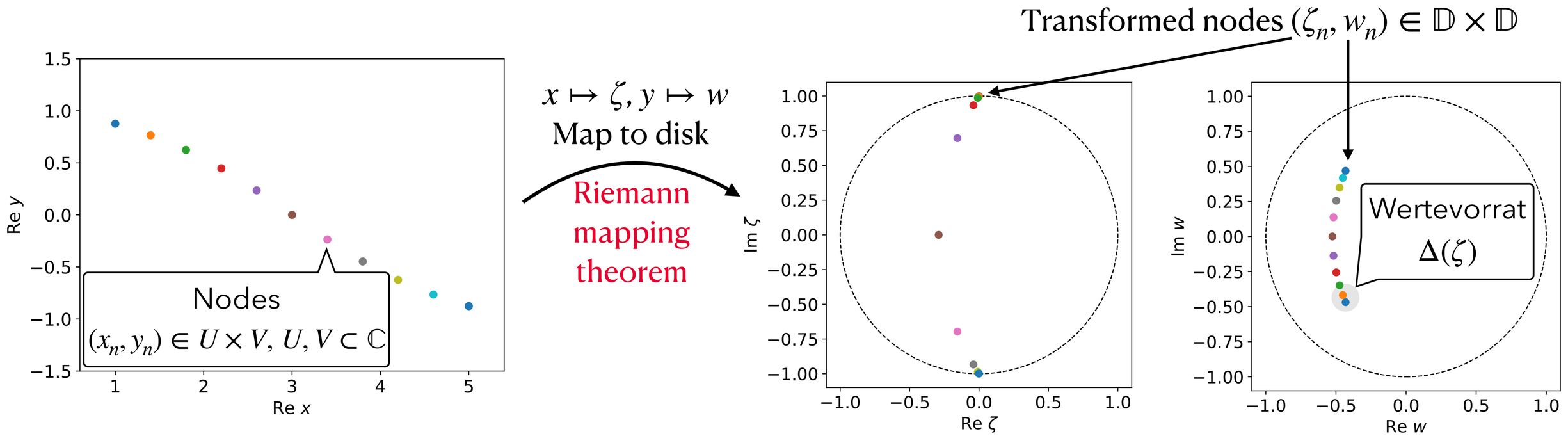

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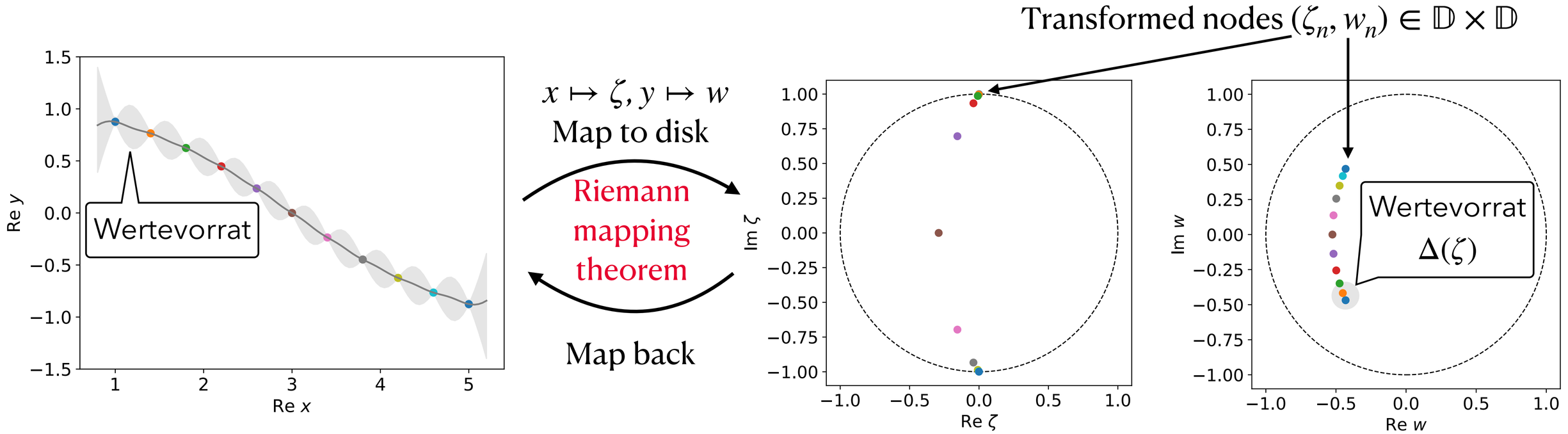

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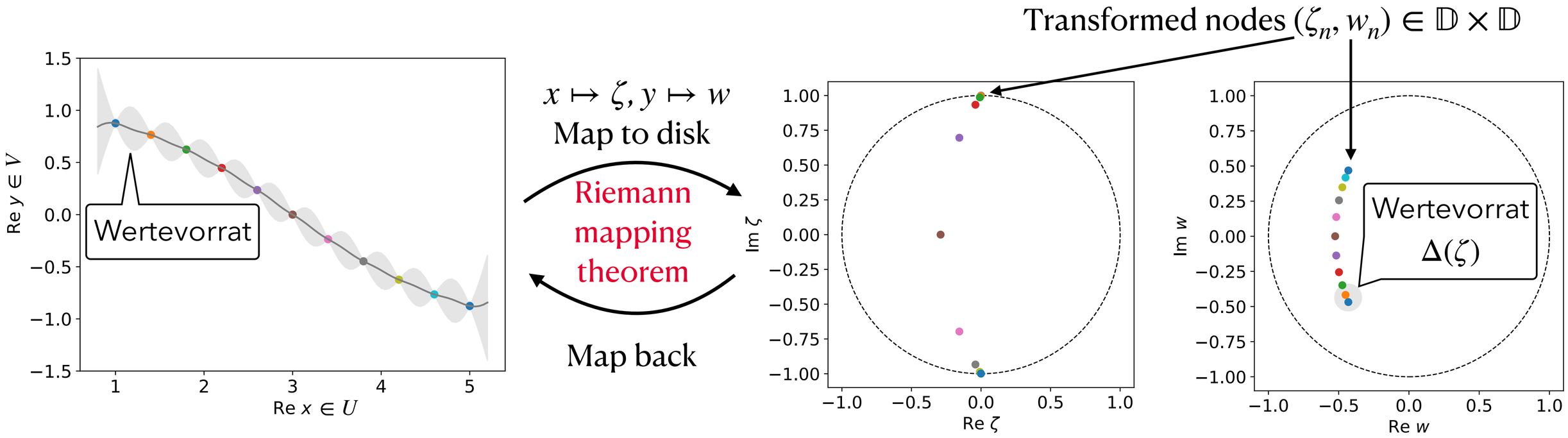

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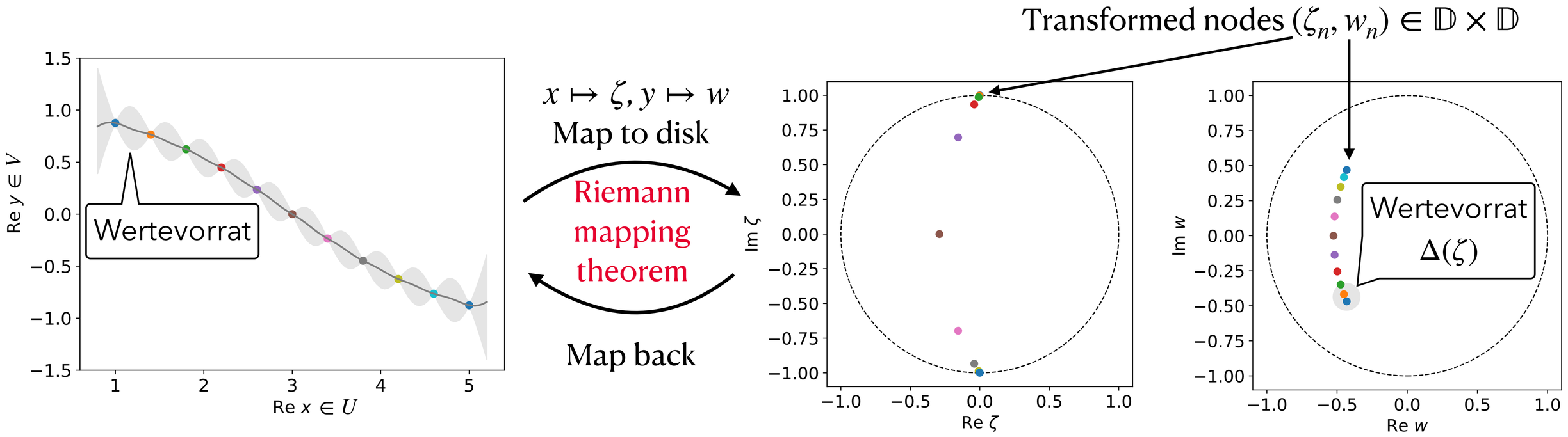

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- Use physics knowledge (locations of poles and zeros...) to guide choice of U and V

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Nevanlinna-Pick interpolation

Application to analytic continuation of scattering amplitudes

Goal: analytic continuation from discrete set of points given by functional samples of $\mathscr{K}^{-1}(\mathbf{E})$

Nevanlinna-Pick interpolation

Application to analytic continuation of scattering amplitudes

For each sample from the posterior:

Nevanlinna-Pick interpolation

Application to analytic continuation of scattering amplitudes

For each sample from the posterior:

Transform to unit disk

e.g. Schwarz-Christoffel mapping

Liu, Chen, and Karageorghis 2017 [J. Sci. Comput. 71, 1035]

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Analytic continuation of $\mathscr{K}^{-1}(E)$

Evaluation of interpolating function

+ uncertainty (WV) for $E \in \mathbb{C}$

due to having only a discrete set of points

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Resonance poles

Solve for zeros of $\mathscr{M}^{-1}(E)$ for $E \in \mathbb{C}$, combining WV uncertainty with statistical error by sampling from WV

Applications to lattice QCD analyses

Isospin-0 NN scattering on D251

$M_\pi = 280$ MeV

Deuteron towards the physical pion mass

- Large pion masses: $I = 0$ NN interaction attractive,
  but not strong enough to form bound state
  —> deuteron is virtual state Amy Nicholson's plenary
- Goal: find value of M_π where deuteron unbinds

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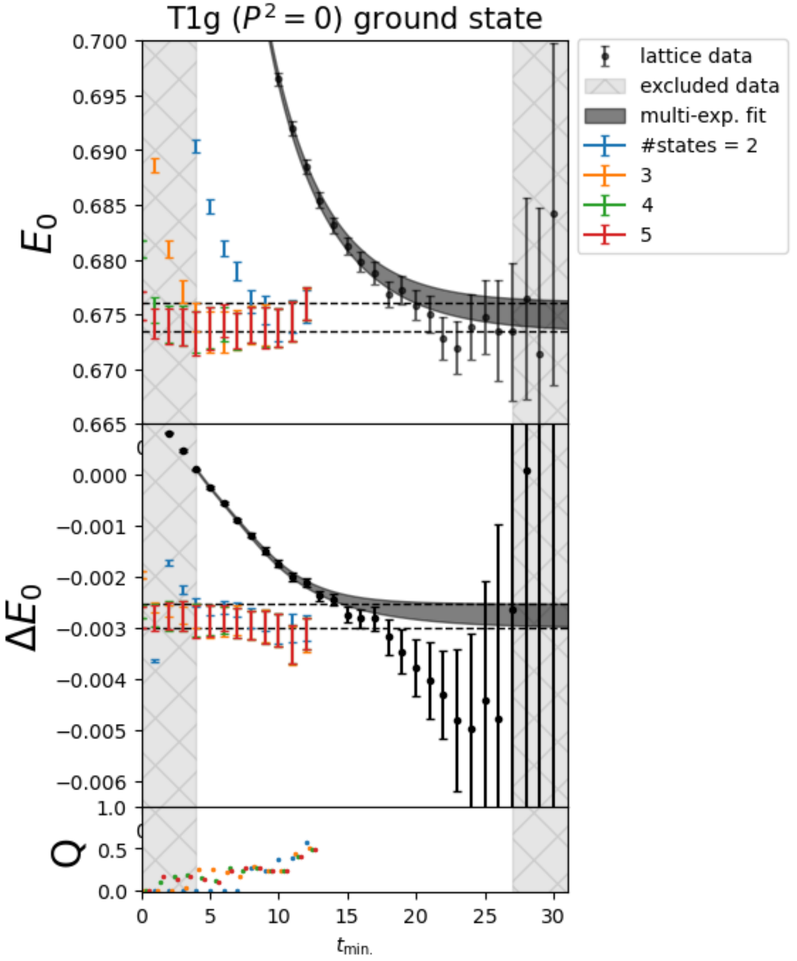

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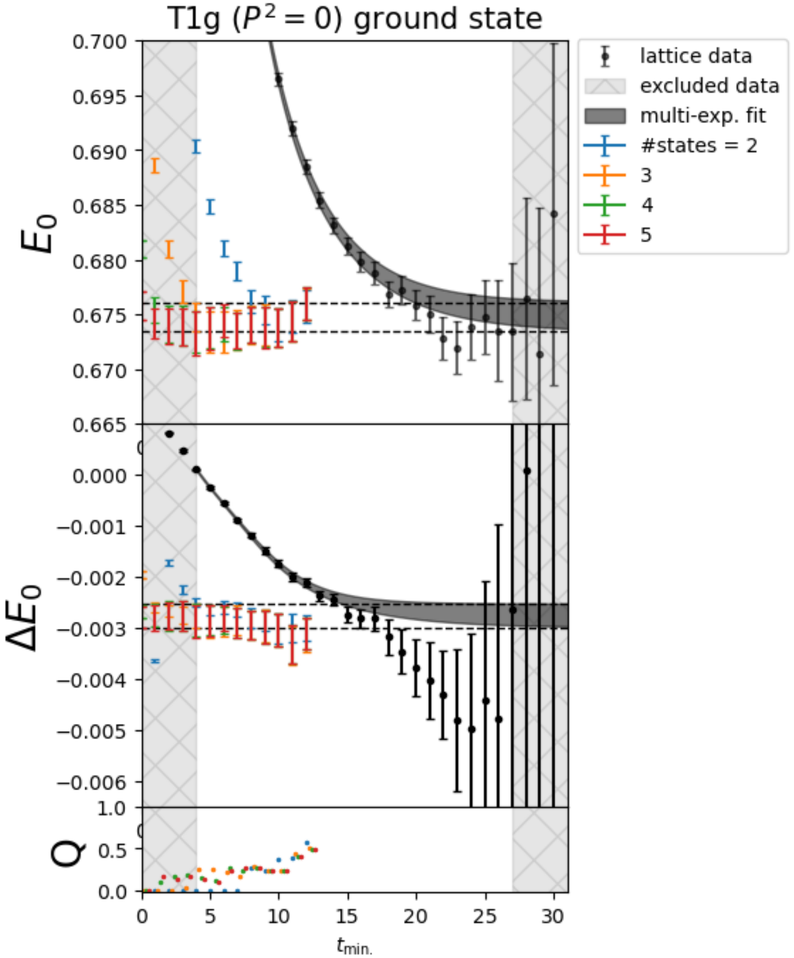
- Irrep averaging to suppress effects of D-wave
  Briceño et al. 2013 [PRD 88, 114507]



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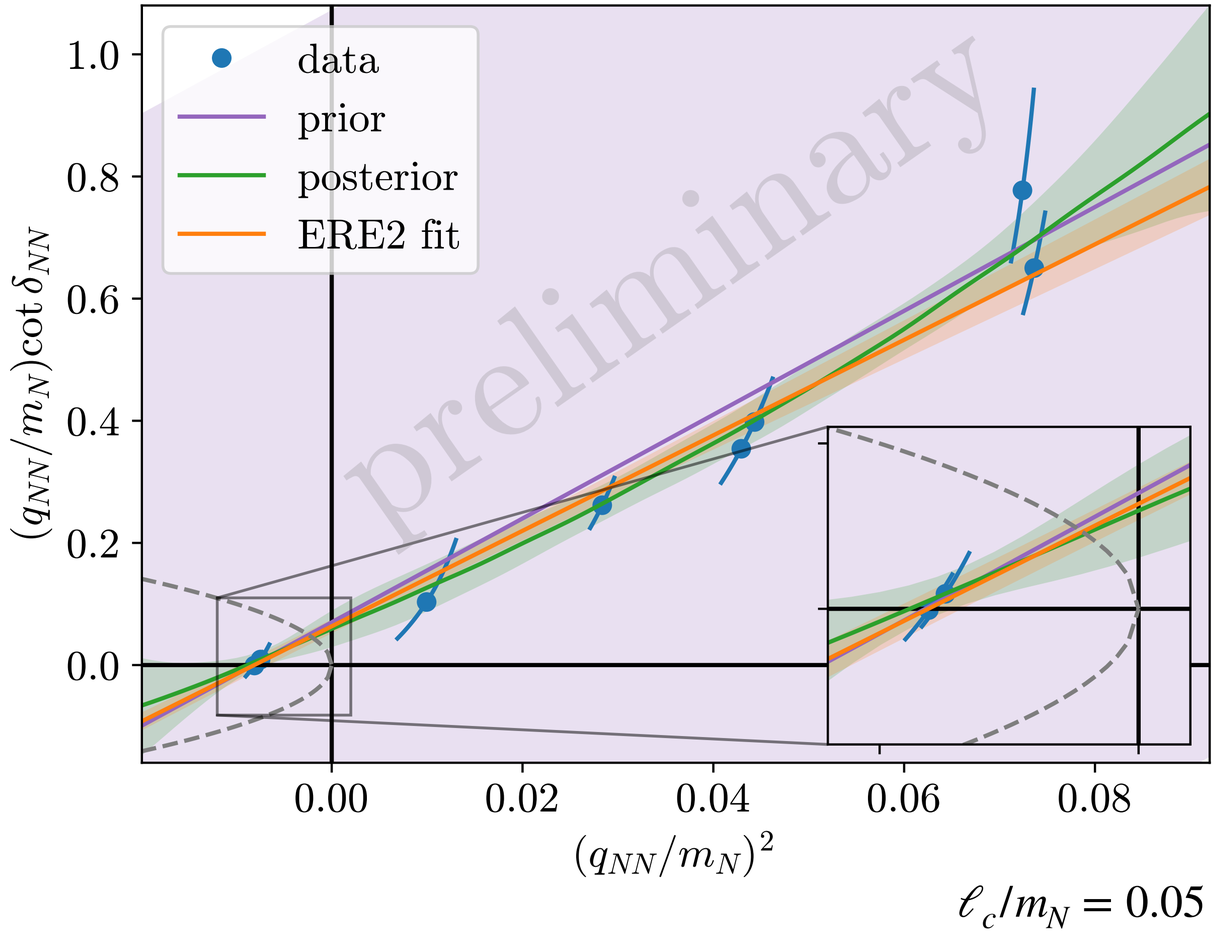
Prior and posterior on the real line

- Good agreement between Bayesian approach and ERE2 fit
- Larger errors since posterior already includes systematic uncertainty



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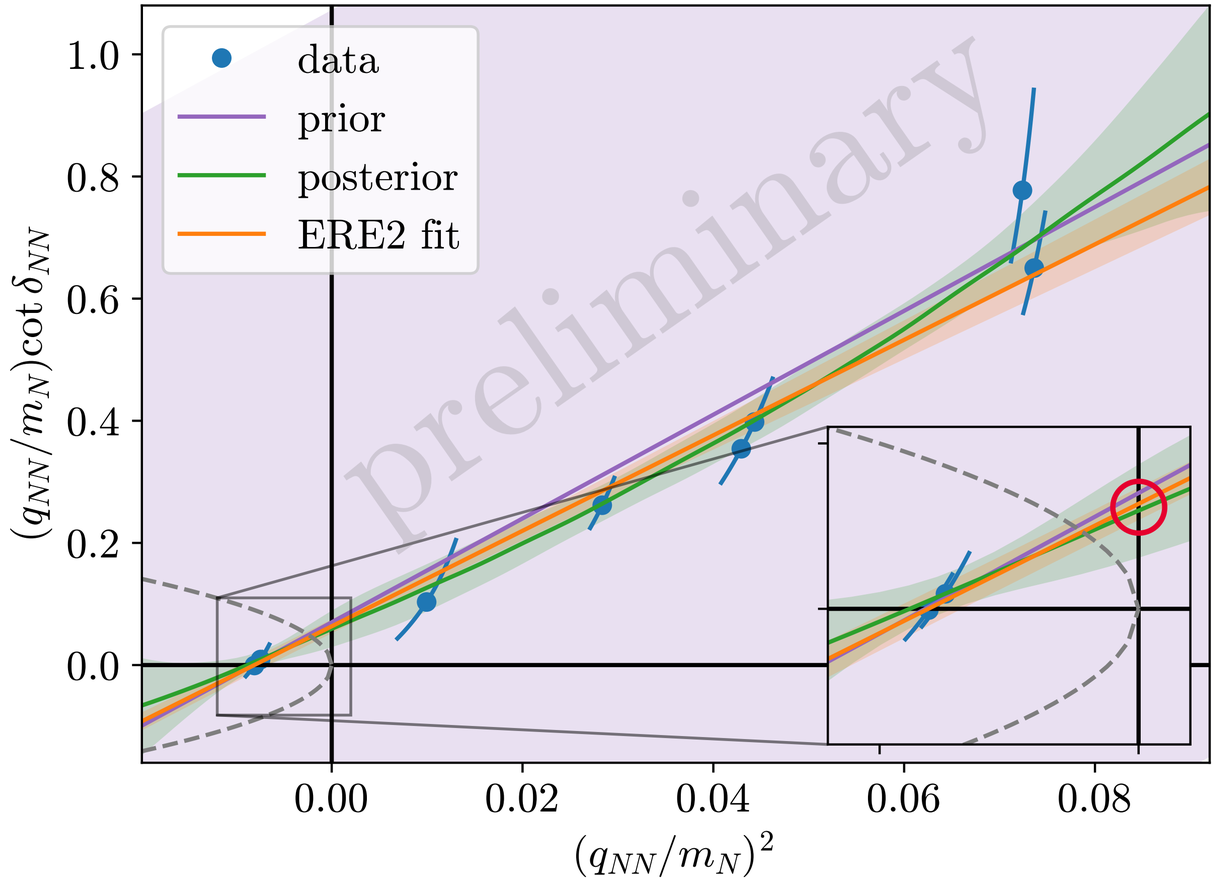
$$m_N a_0 = -\, m_N/(q_{NN}\cot\delta_{NN})\,|_{q=0}$$
$$= -\, 17^{+5}_{-15}$$

(cf. ERE2 fit: $-15.8^{+2.0}_{-2.5}$)



$\ell_c/m_N = 0.05$

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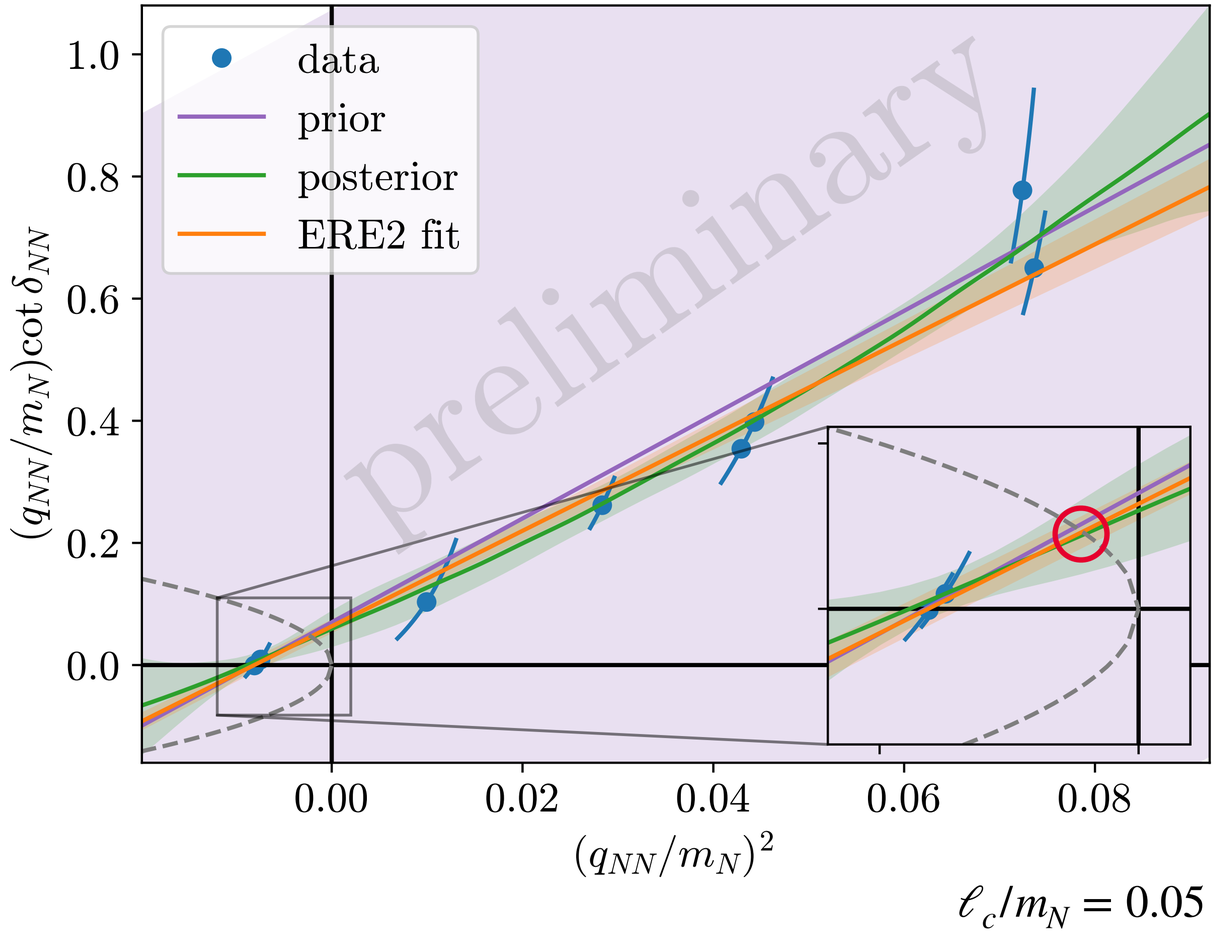
- Most likely virtual state



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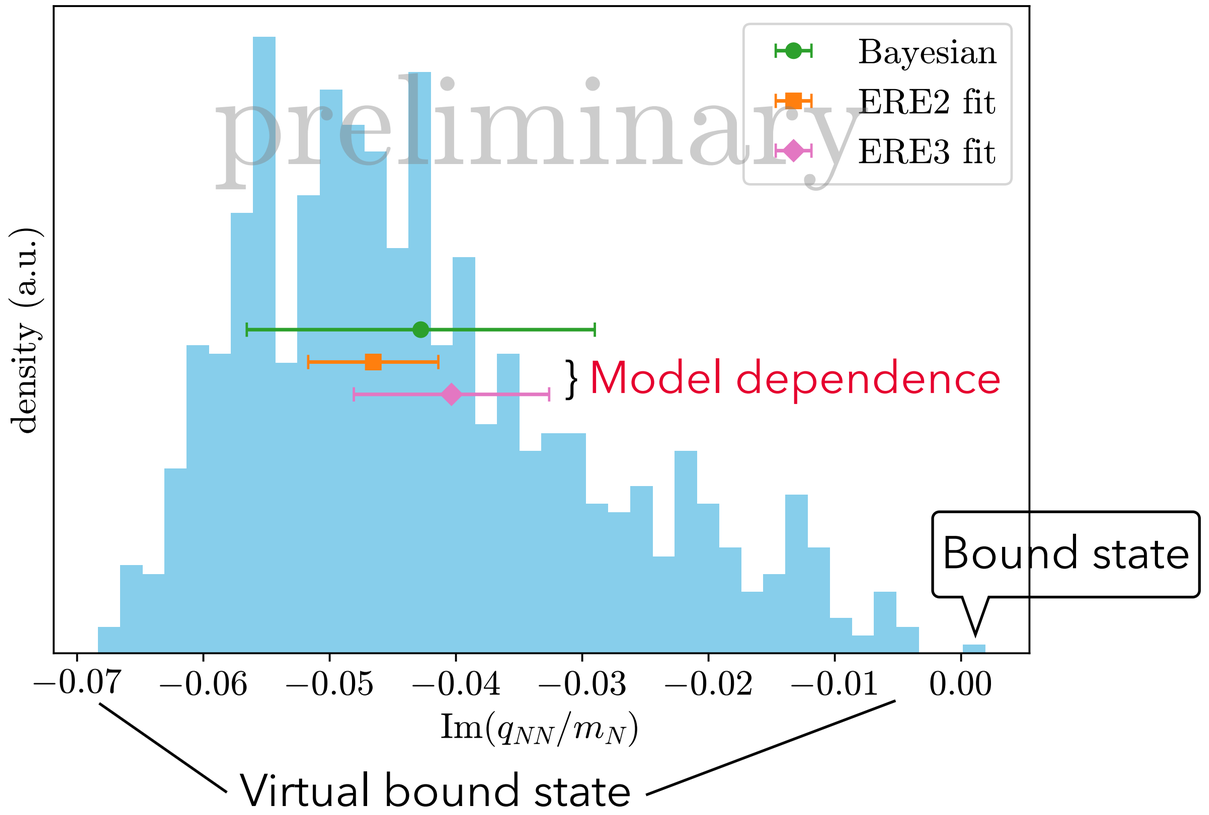
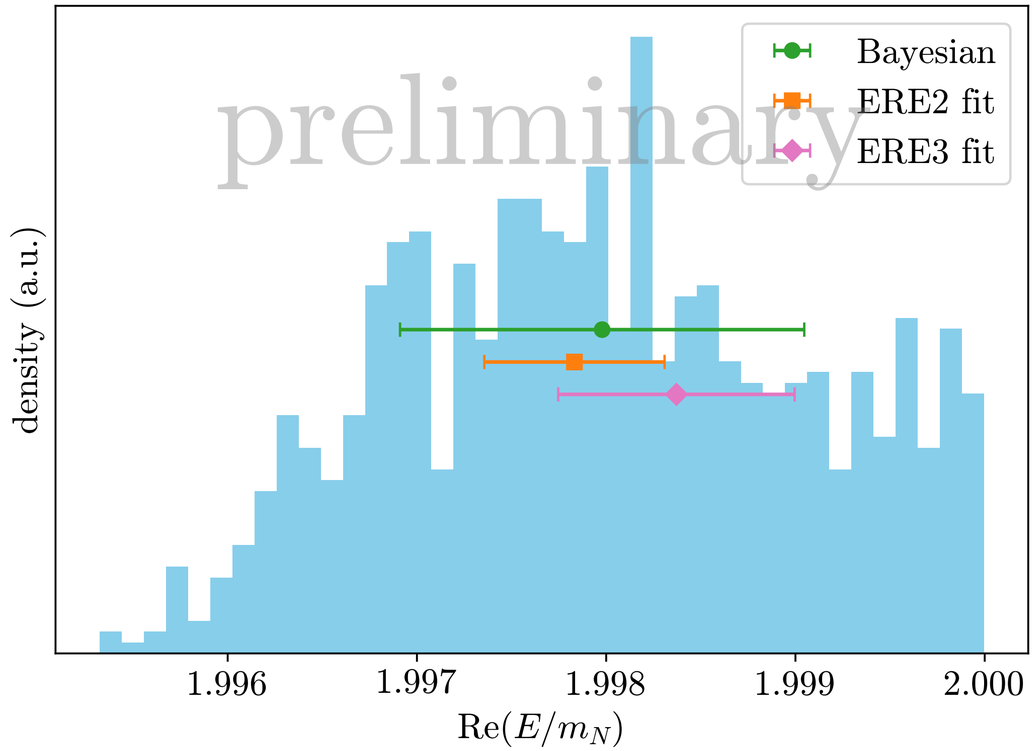
Pole locations





Compares well with fit-based analysis, but quantifies model dependence

Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

$\Lambda(1405)$ and $\Lambda(1380)$ resonances

$M_\pi = 200\,\mathrm{MeV}$

- One or two poles?
- Lattice QCD favours two poles

Dalitz and Tuan 1959 [Annals Phys. 8, 100]
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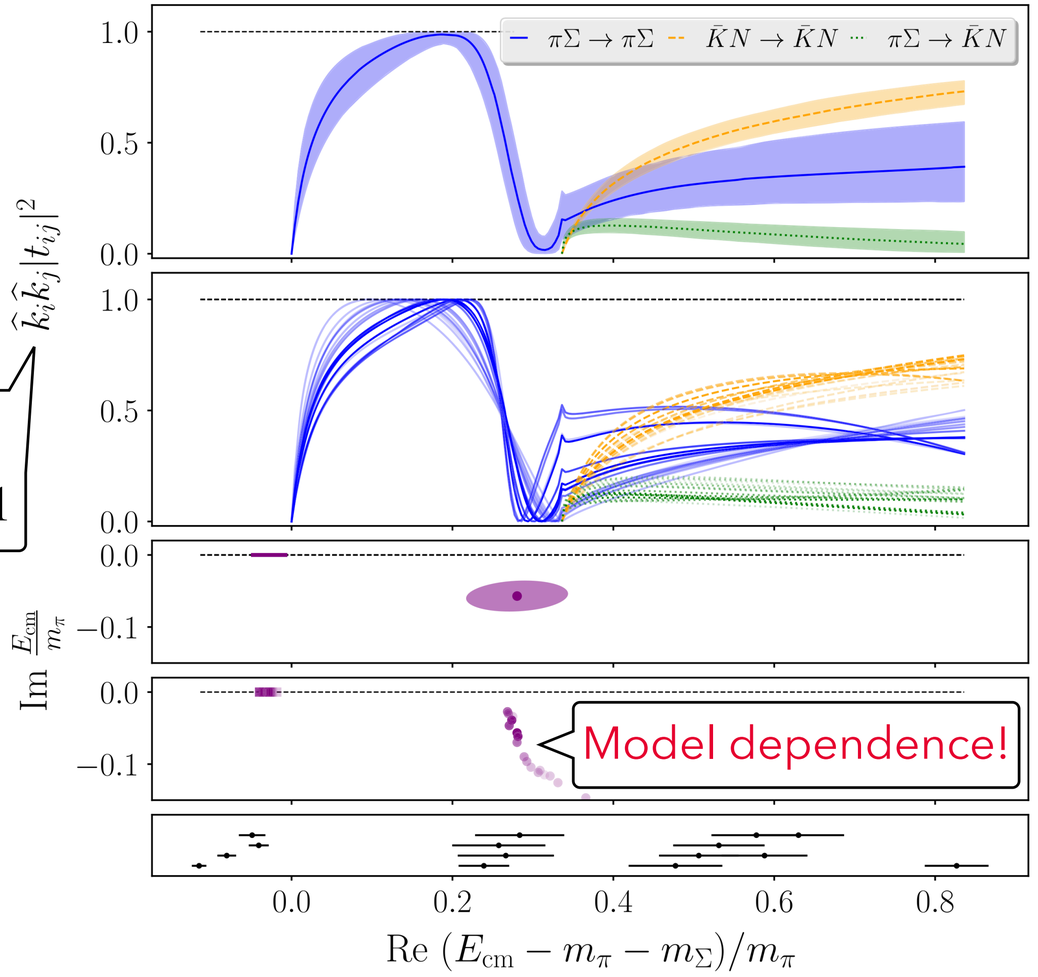

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- Coupled-channel scattering:

$$\mathscr{K}^{-1} = \begin{pmatrix} \mathscr{K}^{-1}_{\pi\Sigma;\pi\Sigma} & \mathscr{K}^{-1}_{\pi\Sigma;\bar{K}N} \\ \mathscr{K}^{-1}_{\bar{K}N;\pi\Sigma} & \mathscr{K}^{-1}_{\bar{K}N;\bar{K}N} \end{pmatrix}$$

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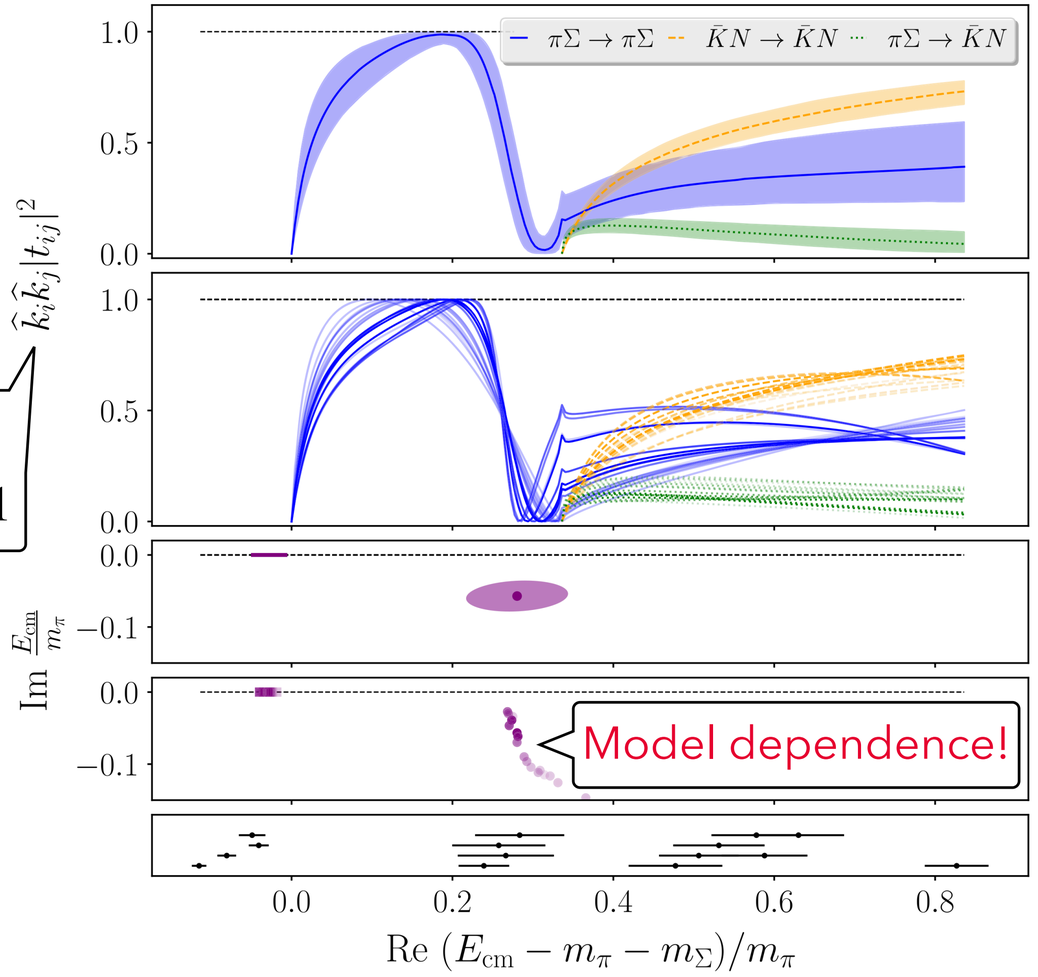

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Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

$\Lambda(1405)$ and $\Lambda(1380)$ resonances

$M_\pi = 200$ MeV

- One or two poles?

Dalitz and Tuan 1959 [Annals Phys. 8, 100]
Oller and Meißner 2001 [Phys. Lett. B 500, 263]
Anisovich et al. 2020 [EPJA 56, 139]
Mai 2021 [EPJ Spec. Top. 230, 1593]

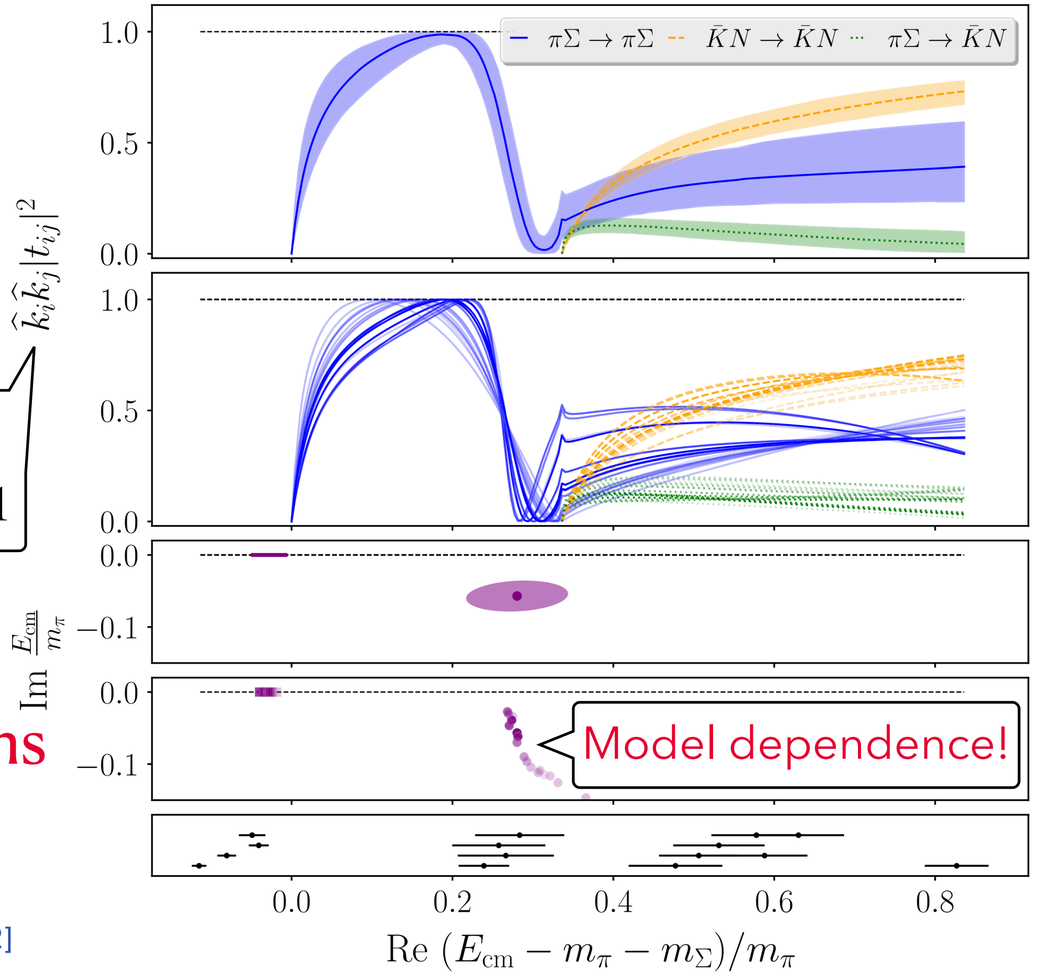
- Lattice QCD favours two poles
- Coupled-channel scattering:

$$\mathscr{K}^{-1} = \begin{pmatrix} \mathscr{K}^{-1}_{\pi\Sigma;\pi\Sigma} & \mathscr{K}^{-1}_{\pi\Sigma;\bar{K}N} \\ \mathscr{K}^{-1}_{\bar{K}N;\pi\Sigma} & \mathscr{K}^{-1}_{\bar{K}N;\bar{K}N} \end{pmatrix}$$

$$t^{-1} \propto \mathscr{M}^{-1}, \quad 0 \le k_i k_j |t_{ij}|^2 \le 1$$

- Symmetries:
  - $(\mathscr{K}^{-1})^T = \mathscr{K}^{-1}$ —> parameterise 3 functions
  - $\mathscr{K}^{-1}_{\pi\Sigma;\bar{K}N} \to -\mathscr{K}^{-1}_{\pi\Sigma;\bar{K}N}$ at each E separately

Oset 2013 [EPJA 49, 32]

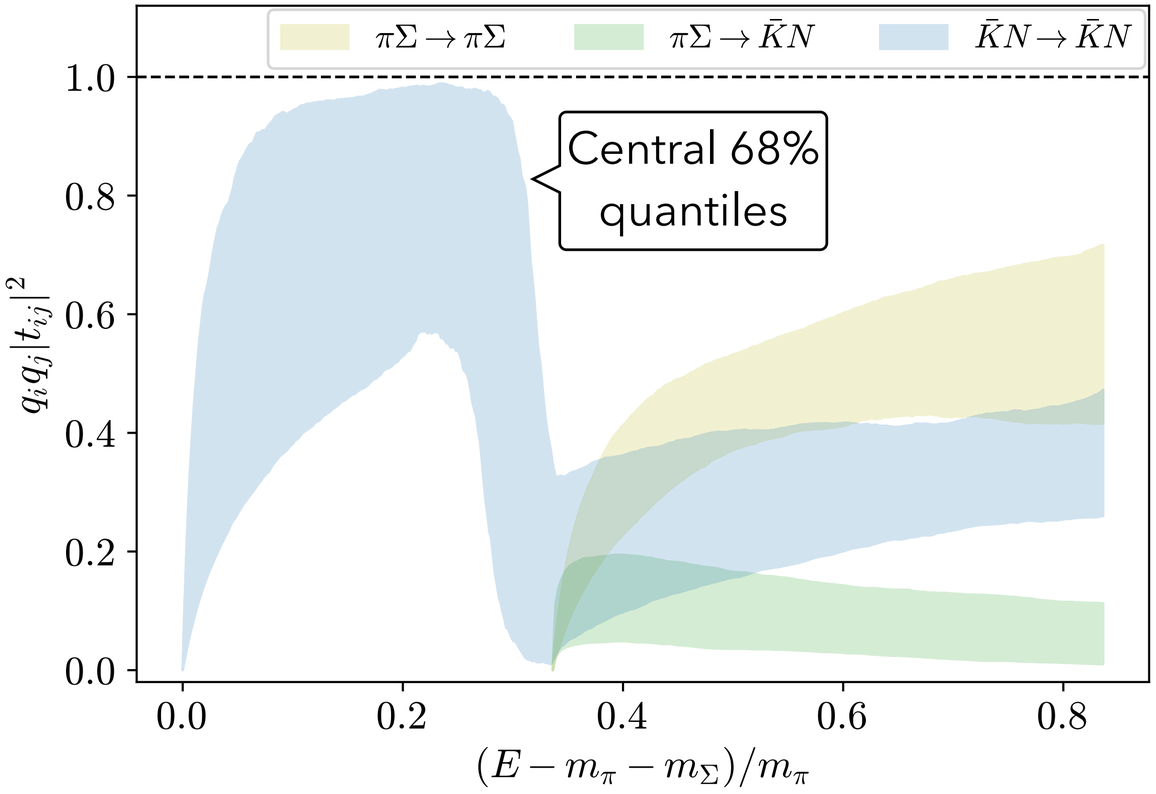


Bulava et al. (BaSc) 2024 [PRL 132, 051901; PRD 109, 014511]

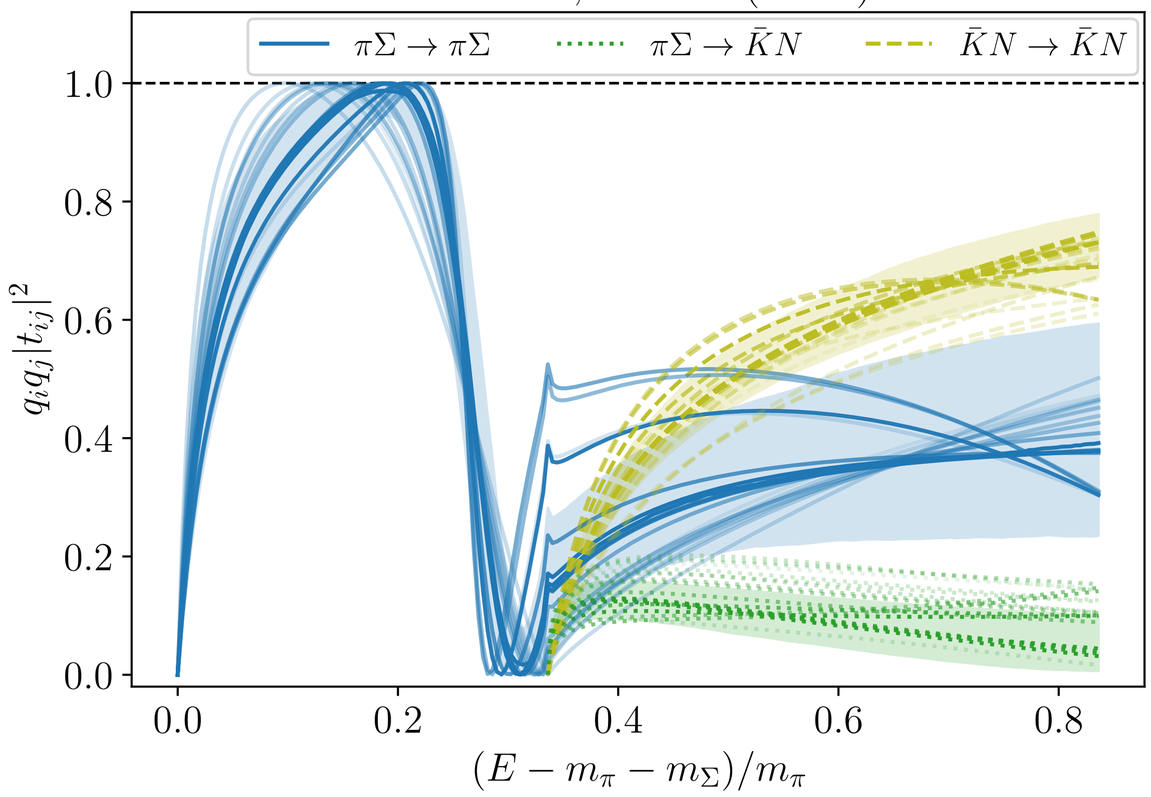
Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

Posterior for the scattering amplitude

This work



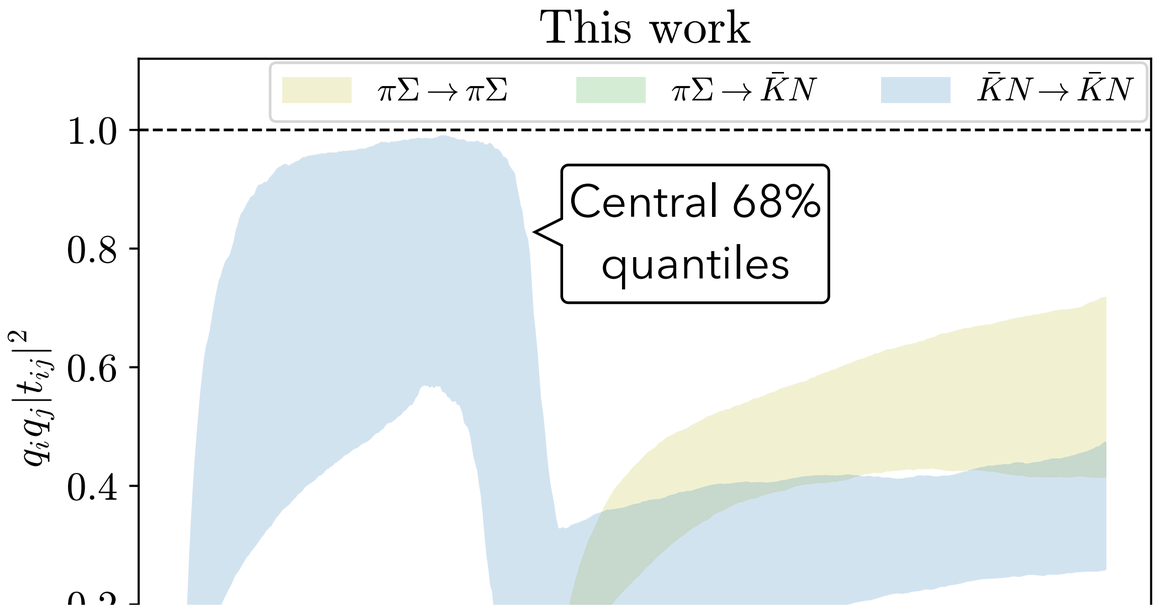
PRL 132, 051901 (BaSc)

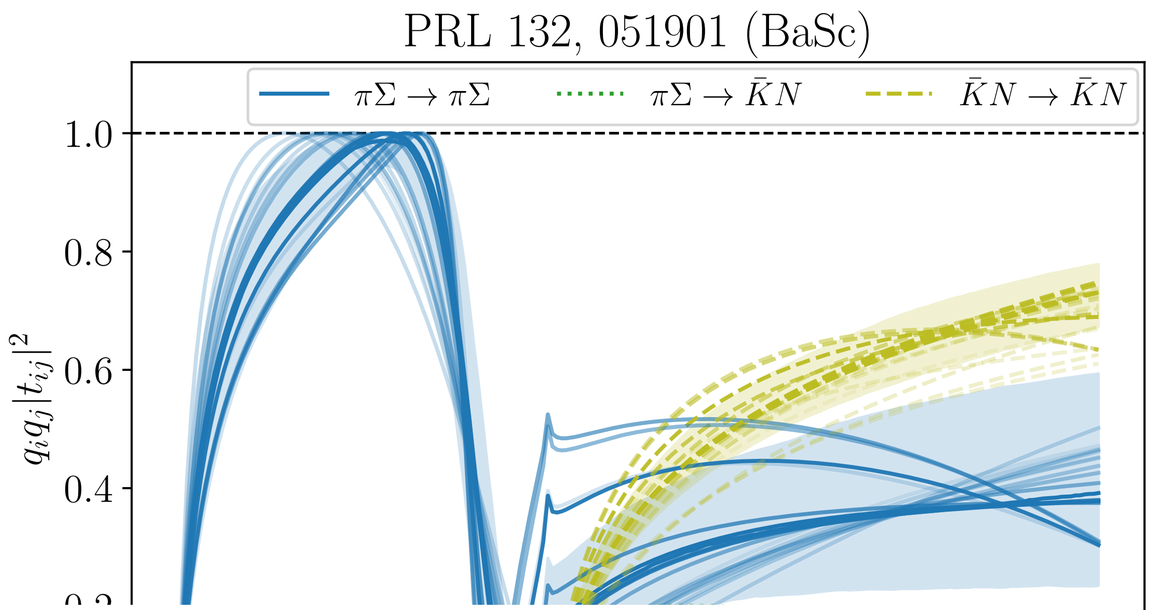


$\boldsymbol{\sigma} = (3\sqrt{2},\ 4\sqrt{2}/5,\ 6\sqrt{2}/5);\ \ell_c/m_\pi = 1$

Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

Posterior for the scattering amplitude

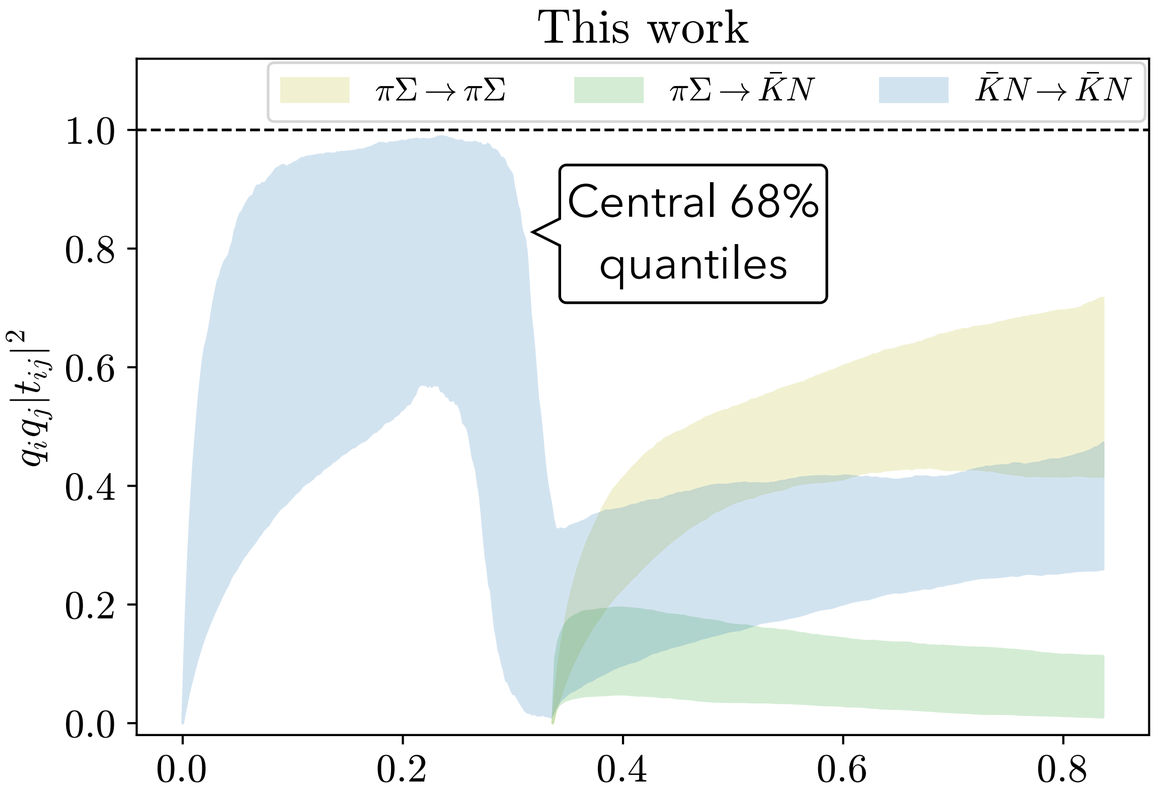


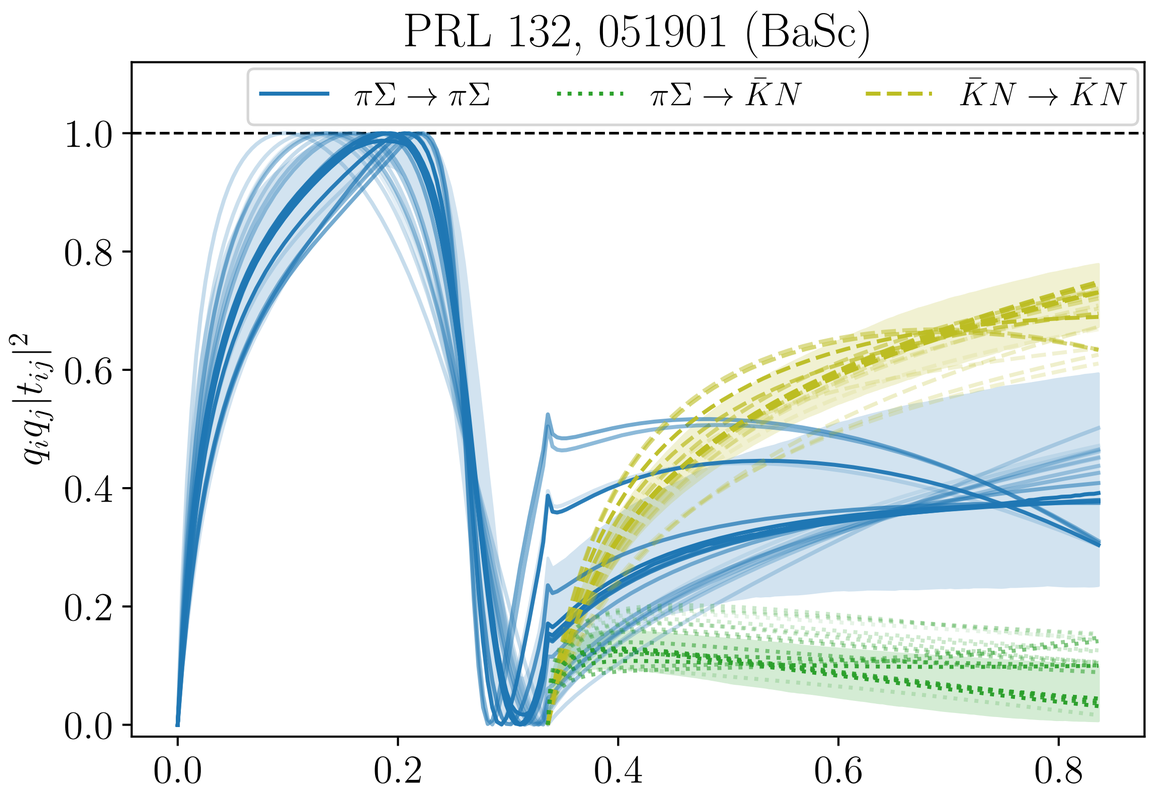


- Bayesian analysis naturally combines statistical error with parameterisation dependence into one total band —> full error budget

Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

Posterior for the scattering amplitude



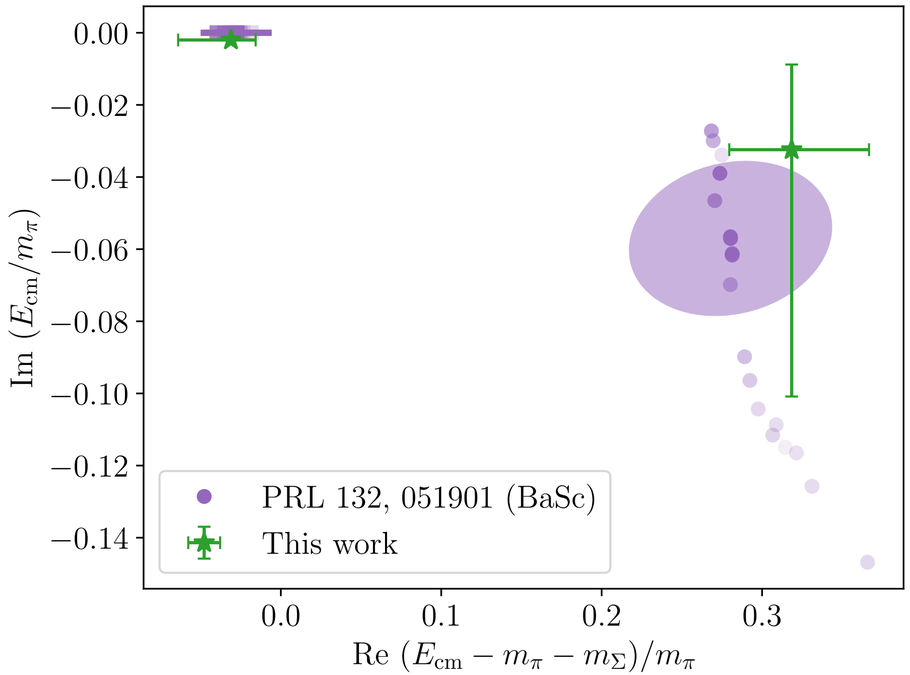


- Good agreement between Bayesian approach and explicit parametric fits

Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

Pole locations

- Two poles found, consistency between priors with different σ or ℓ_c and good agreement with parametric analysis



Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

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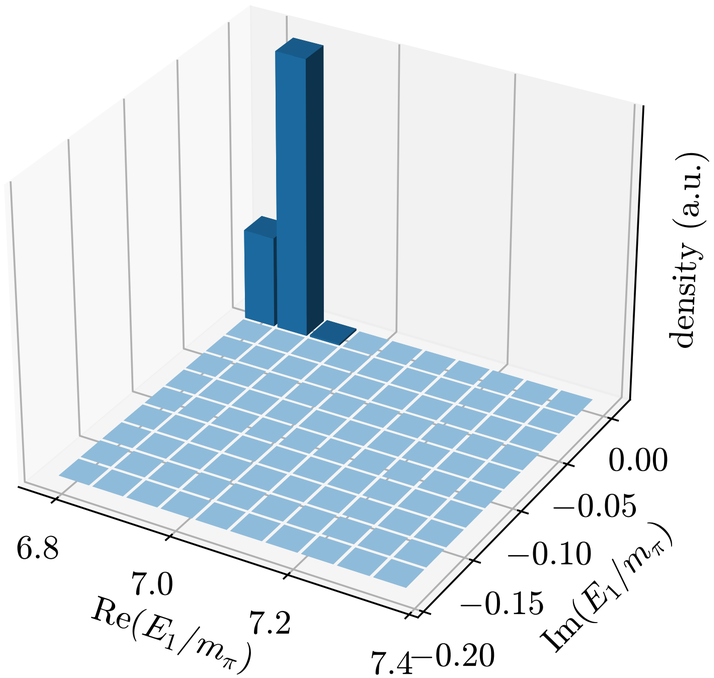
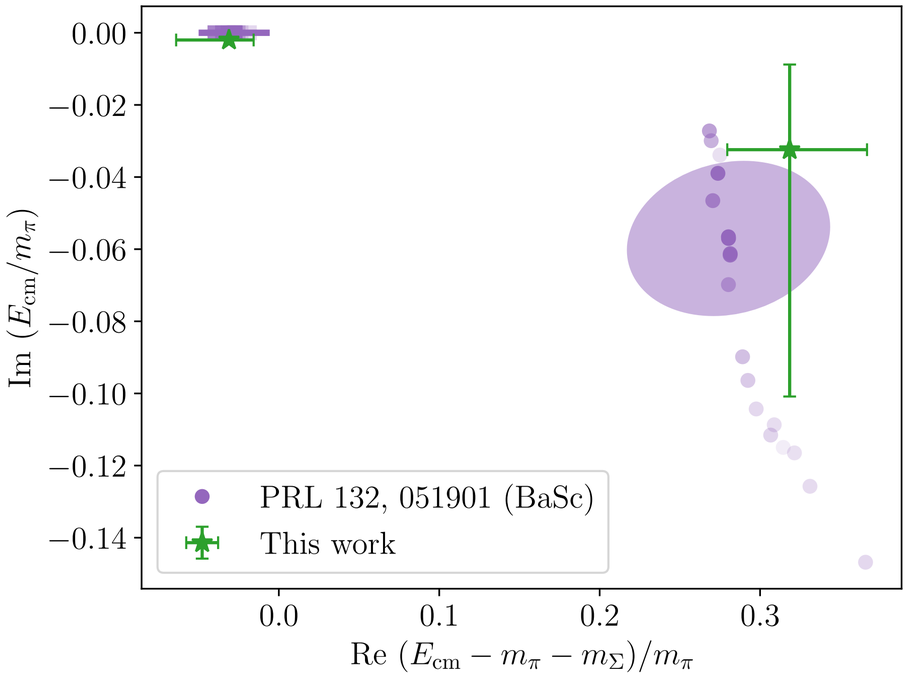
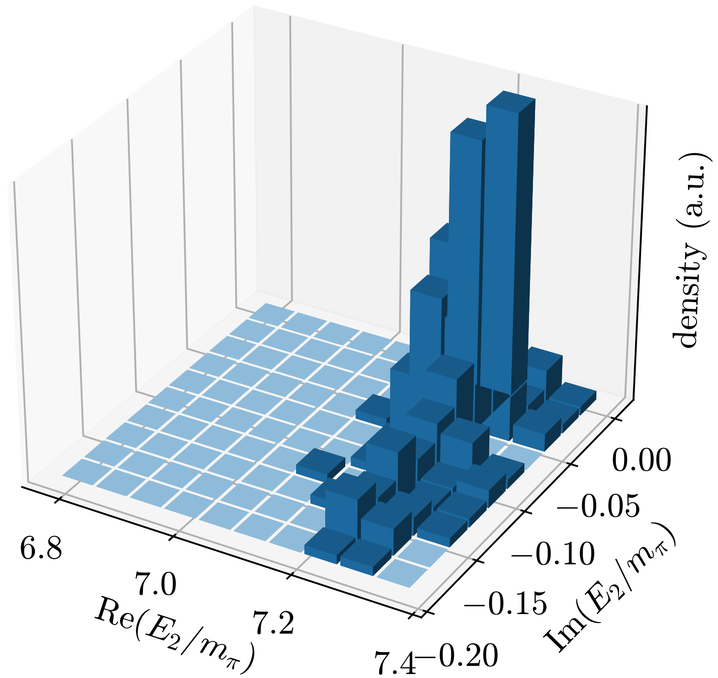






Summary and outlook

Summary and outlook

- Novel analysis procedure for hadron spectroscopy from LQCD using FV formalism —> incorporate systematic error without need for explicit choice of parametrisation
- Key ingredients: Bayesian analysis (real energies) and Nevanlinna-Pick interpolation

Rodas, Dudek, and Edwards (HadSpec) 2023 [PRD 108, 034513]

- Further systems studied, e.g. $I = 0$ $\pi\pi$ scattering by HadSpec @ $M_\pi = 283$ MeV
  —> determine probability that $\sigma/f_0(500)$ is virtual state vs. resonance (—> paper)
- Future directions
  - Three-hadron systems —> parameterise $\mathcal{K}_3$ in addition to $\mathcal{K}_2$

Hansen and Sharpe 2014 [PRD 90, 116003], 2015 [PRD 92, 114509]
Mai and Döring 2017 [EPJA 53, 240]

  - Synergies with other methods (e.g., dispersion relations —> crossing symmetry)

Rodas, Dudek, and Edwards (HadSpec) 2024 [PRD 109, 034513]
Peláez, Rabán, and Ruiz de Elvira 2025 [PRD 111, 074003]

Backup

Bayesian analysis

Sampling the posterior distribution

- Posterior not Gaussian (non-linear QC) —> straightforward sampling impossible
- **Approach #1**: Gaussian approximation
  - Linearise QC around $\mathbf{u}_L$: $\mathbf{E}_{\mathrm{QC}}(\mathbf{u}) - \mathbf{E}_L \simeq J(\mathbf{u} - \mathbf{u}_L)$
  - Sample from approximate posterior = (prior) × (linearised likelihood)
  - Reweight to exact posterior distribution

Del Debbio, Giani, and Wilson 2022 [EPJC 82, 330]

- Advantages: fast, easy access to Bayes factors (= ratio of avg. reweighting factors)
- Challenge: sampling can be inefficient when posterior significantly non-Gaussian

Bayesian analysis

Sampling the posterior distribution

- Posterior not Gaussian (non-linear QC) —> straightforward sampling impossible
- **Approach #2**: Hybrid Monte Carlo Duane et al. 1987 [PLB 195, 216]
- Advantage: robust
- Challenges
  - Computationally more expensive
  - Introduces familiar complications like thermalisation, autocorrelation, etc.
  - Calculation of Bayes factors (= ratio of partition functions) not straightforward

Bayesian analysis

Sampling from posterior distribution using Gaussian approximation

- Linearisation of QC around $\mathbf{u}_L$: $\mathbf{E}_{\mathrm{QC}}(\mathbf{u}) - \mathbf{E}_L \simeq J(\mathbf{u} - \mathbf{u}_L)$
- Approximate posterior distribution:

$$p^*(\mathbf{u}) = \frac{1}{\sqrt{(2\pi)^{n_p} \det \Sigma^*}} \exp\left[-\frac{1}{2}(\mathbf{u} - \mathbf{u}^*)^T(\Sigma^*)^{-1}(\mathbf{u} - \mathbf{u}^*)\right] \text{ with}$$

$$(\Sigma^*)^{-1} = J^T\Sigma_{\mathrm{d}}^{-1}J + \Sigma_p^{-1} \text{ and } \mathbf{u}^* = \Sigma^*\left[\Sigma_p^{-1}\mathbf{u}_0 + J^T\Sigma_{\mathrm{d}}^{-1}(\mathbf{E}_d - \mathbf{E}_L + J\mathbf{u}_L)\right]$$

- Reweighting factors: $w_* = p(\mathbf{u} \,|\, \mathbf{E}_{\mathrm{d}}, \mathbf{u}_0) \big/ p^*(\mathbf{u} \,|\, \mathbf{E}_{\mathrm{d}}, \mathbf{u}_0)$
- Probabilities entering Bayes factors:

$$p(\mathbf{E}_{\mathrm{d}} \,|\, M^{(k)}) = \int d^{n_p}u\, p(\mathbf{E}_{\mathrm{d}} \,|\, \mathbf{u}) p(\mathbf{u} \,|\, \mathbf{u}_0^{(k)}) = \int d^{n_p}u\, p(\mathbf{u} \,|\, \mathbf{E}_{\mathrm{d}}, \mathbf{u}_0^{(k)}) = \langle w_*^{(k)} \rangle_*^{(k)}$$

Bayesian analysis

Prior dependence and analytic continuation

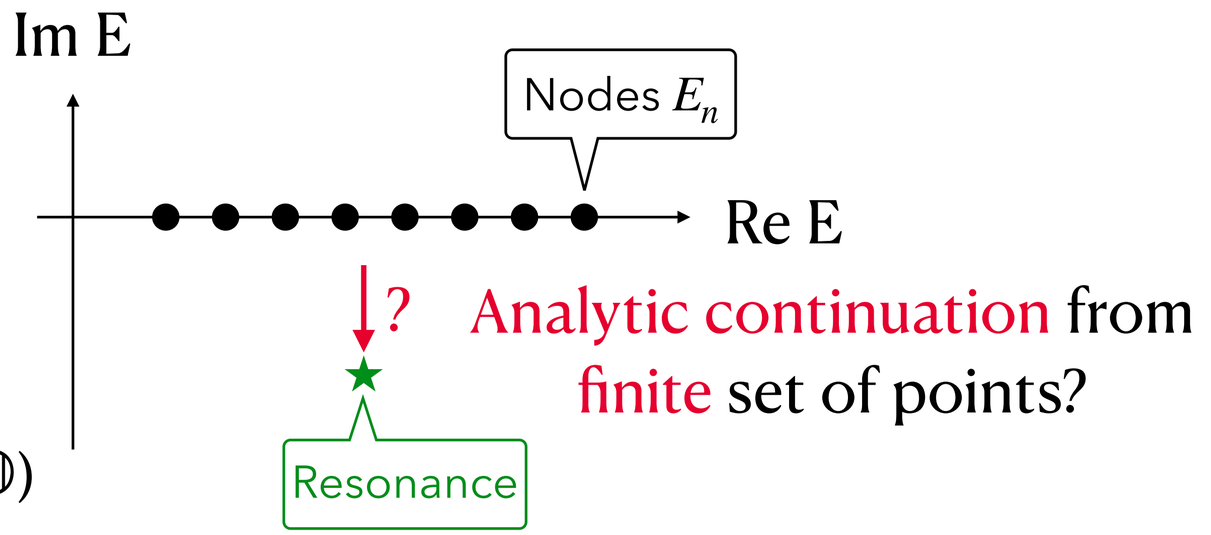
Model $M^{(1)}$ with prior $\mathbf{u}_0^{(1)}$

- Quantify prior dependence by Bayes factor, $B_{12} = \dfrac{p(\mathbf{E}_\mathrm{d} \,|\, M^{(1)})}{p(\mathbf{E}_\mathrm{d} \,|\, M^{(2)})} = \dfrac{Z^{(1)}_\mathrm{post}}{Z^{(2)}_\mathrm{post}}$

  —> which prior is close to values favoured by the data?

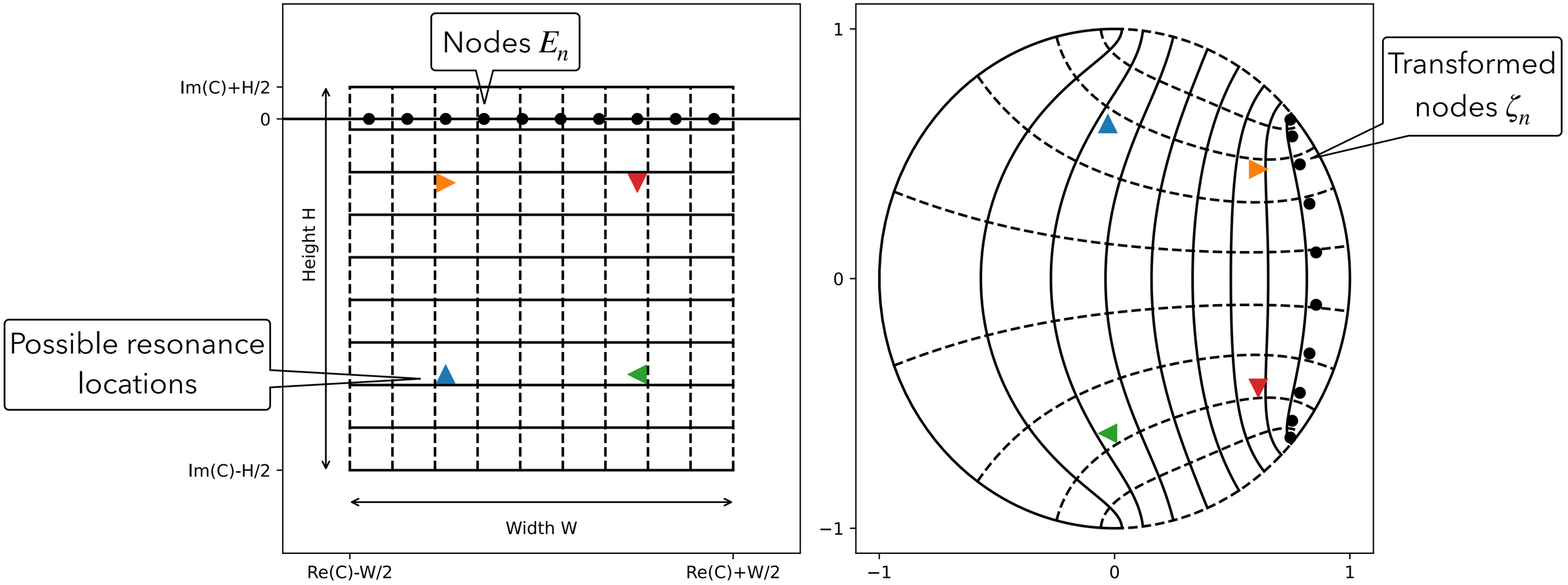
Jeffreys 1935 [Math. Proc. Cambridge Philos. Soc. 31, 203]
Kass and Raftery 1995 [J. Am. Stat. Assoc. 90, 773]

- To be complemented by stability study varying prior + hyperparameters (σ, ℓ_c)
- Bayesian analysis tells us how to compute $\langle \mathbf{u} \rangle_\mathrm{post}$ —> $E_n \in \mathbb{R}$
- Nevanlinna-Pick interpolation: numerical analytic continuation on unit disk ($\mathbb{D} \to \mathbb{D}$)



Nevanlinna-Pick interpolation

Mapping to the unit disk



Nevanlinna-Pick interpolation

Assumptions and limitations

- Assumptions: f analytic in all of U; $f(U) \subset V$ (i.e., f never leaves chosen codomain V)
- Limitation: function to be interpolated ($\mathscr{K}^{-1}(E)$) must be analytic in whole domain U
  - $\mathscr{K}^{-1}(E)$ has simple poles, but no zeros
    —> apply Nevanlinna to its inverse, i.e., $\mathscr{K}(E)$
  - $\mathscr{K}^{-1}(E)$ has both poles and zeros in U, but $\exists a : \mathscr{K}^{-1}(E) \neq a \; \forall E \in U$
    —> apply Nevanlinna to $f(E) = 1/(\mathscr{K}^{-1}(E) - a)$
  - More complicated analytic structure of K matrix (e.g., CDD poles [—> no such a] or sub-threshold left-hand cuts) —> current formalism not directly applicable

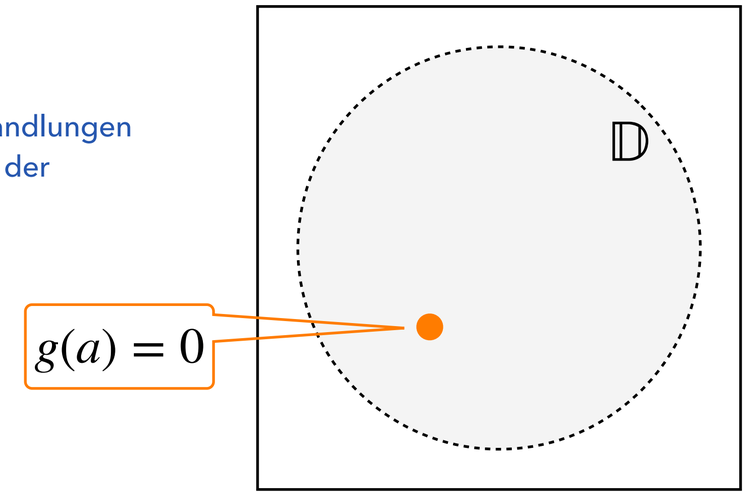
Castillejo, Dalitz, and Dyson 1956 [Phys. Rev. 101, 453]
Krivoruchenko 2010 [PRC 82, 018201]

Nevanlinna-Pick interpolation

Technical details

- Basic idea: “factor out what you know”
- Underlying math: Blaschke factors

Blaschke 1915 [Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig **67**, 194]



Let $g : \mathbb{D} \to \mathbb{D}$ be an analytic function.

Suppose g has a zero at $a \in \mathbb{D}$: $g(a) = 0$.

Then $g(\zeta) = b_a(\zeta)\tilde{g}(\zeta)$, with $\tilde{g} : \mathbb{D} \to \mathbb{D}$ analytic.

Definition Blaschke factors: $b_a(\zeta) = \frac{|a|}{a}\frac{a-\zeta}{1-\bar{a}\zeta}$ for $a \in \mathbb{D}\backslash\{0\}$, and $b_0(\zeta) = \zeta$

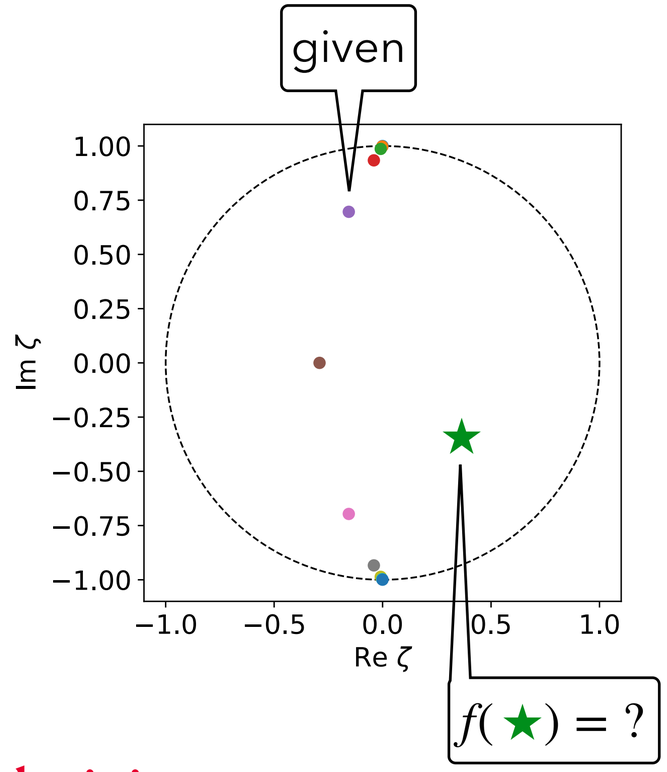
- Repeated application of “factoring” —> Nevanlinna coefficients $P, Q, R, S \leftrightarrow b_a(\zeta)$
- Remainder $\tilde{g}(\zeta) \leftrightarrow$ arbitrary function $h(\zeta)$: encodes freedom in interpolation

Nevanlinna-Pick interpolation

Technical details

- **Question**: What values can the interpolating function take when evaluated away from the input nodes?
- **Answer**: The space of possible values is a disk of radius $r(\zeta)$ centred at $c(\zeta)$. This disk is called the Wertevorrat $\Delta(\zeta)$.

$$r(\zeta) = \frac{|P(\zeta)S(\zeta) - Q(\zeta)R(\zeta)|}{|S(\zeta)|^2 - |R(\zeta)|^2} \qquad c(\zeta) = \frac{Q(\zeta)\bar{S}(\zeta) - P(\zeta)\bar{R}(\zeta)}{|S(\zeta)|^2 - |R(\zeta)|^2}$$



- Rigorous bound of systematic uncertainty, only assuming analyticity
- Additional assumptions when combined with mappings to the unit disk (see above)

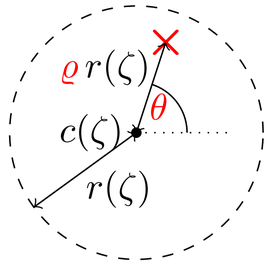
Nevanlinna-Pick interpolation

Uncertainty analysis for pole positions

- Use randomly chosen point within WV as function estimator,
  $w_{\varrho,\theta}(\zeta) = c(\zeta) + \varrho\, r(\zeta)\, e^{i\theta}$ with random numbers $\varrho \in [0,1)$ and $\theta \in [0,2\pi)$

  $\mathscr{K}^{-1}_{\varrho,\theta}(E) = \mathrm{interp}_{\varrho,\theta}(\mathbf{E}, \mathbf{u}_n)(E)$

  —> Find E_0 with $\mathscr{M}^{-1}(E_0) = \mathrm{interp}_{\varrho,\theta}(\mathbf{E}, \mathbf{u}_n)(E_0) - i\rho = 0$



WV in unit disk coordinates for a specific value of ζ corresponding to E

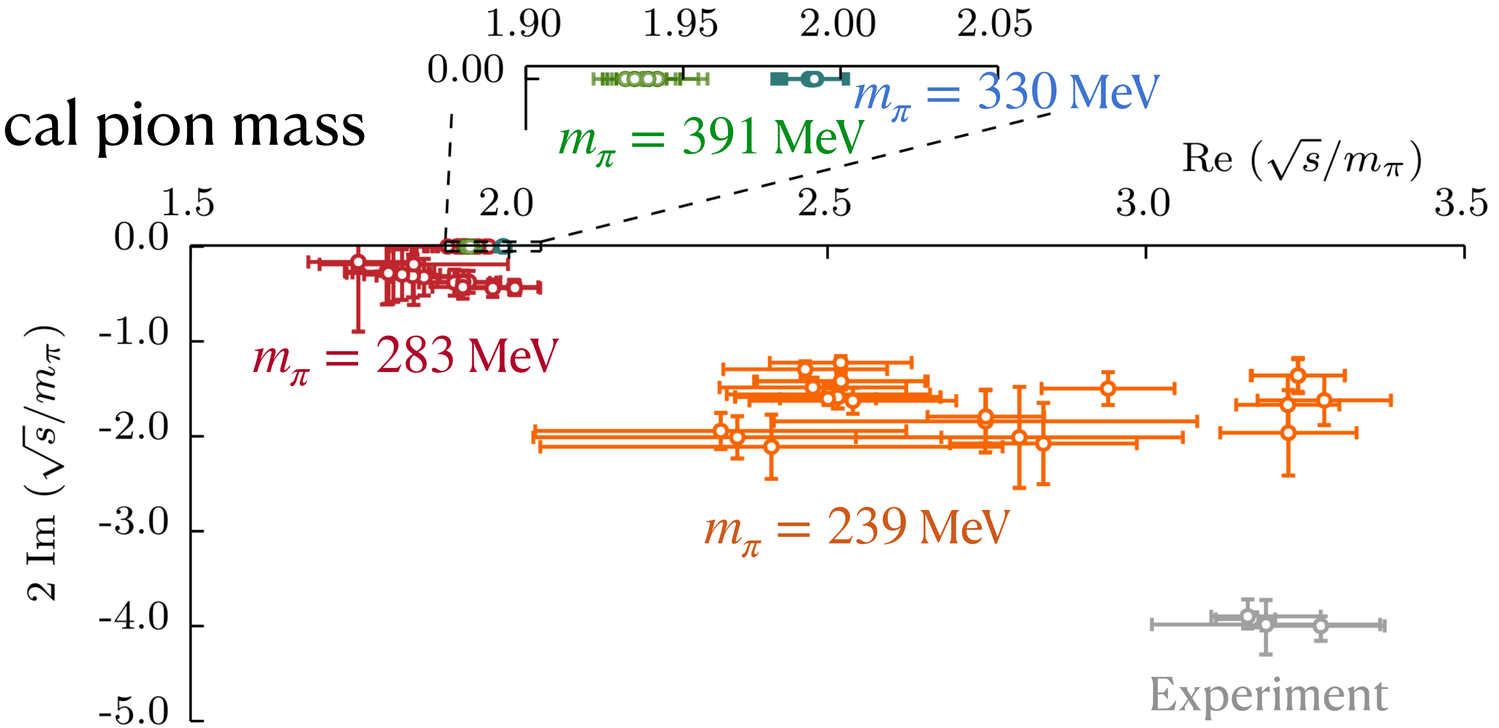
- Use independent random point(s) for each sample from the posterior
- Effectively, the expectation value of the pole location takes the form

  $\langle E_0 \rangle = \frac{1}{\pi Z} \int d^{n_p}u \int_0^1 d\varrho\, \varrho \int_0^{2\pi} d\theta\, E_0(\mathbf{u}, \varrho, \theta) p(\mathbf{u} \,|\, \mathbf{E}_\mathrm{d}, \mathbf{u}_0)$ —> uniform dist. over WV

Isospin-0 $\pi\pi$ system by HadSpec

Pion-mass dependence of the σ pole position

Broad σ meson resonance at physical pion mass



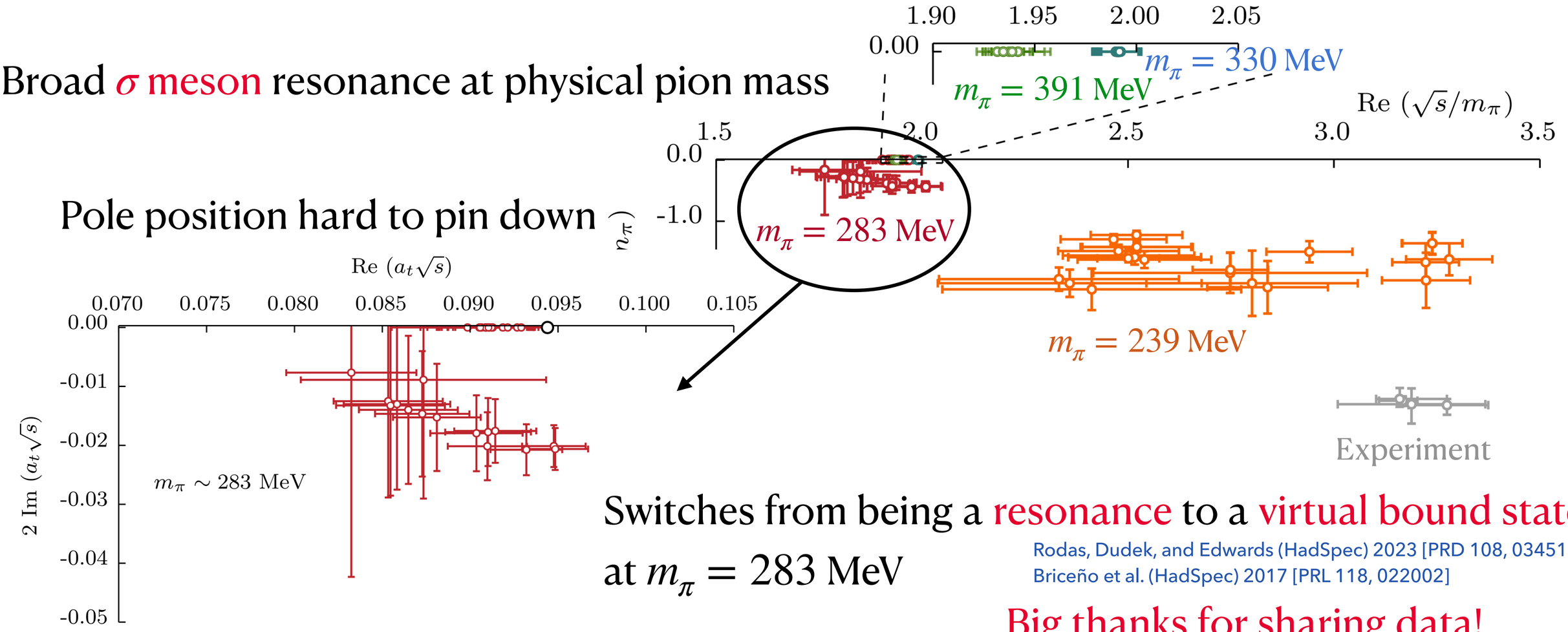
Rodas, Dudek, and Edwards (HadSpec) 2023 [PRD 108, 034513]
Briceño et al. (HadSpec) 2017 [PRL 118, 022002]

Isospin-0 $\pi\pi$ system by HadSpec

Pion-mass dependence of the σ pole position

Broad σ meson resonance at physical pion mass

Pole position hard to pin down



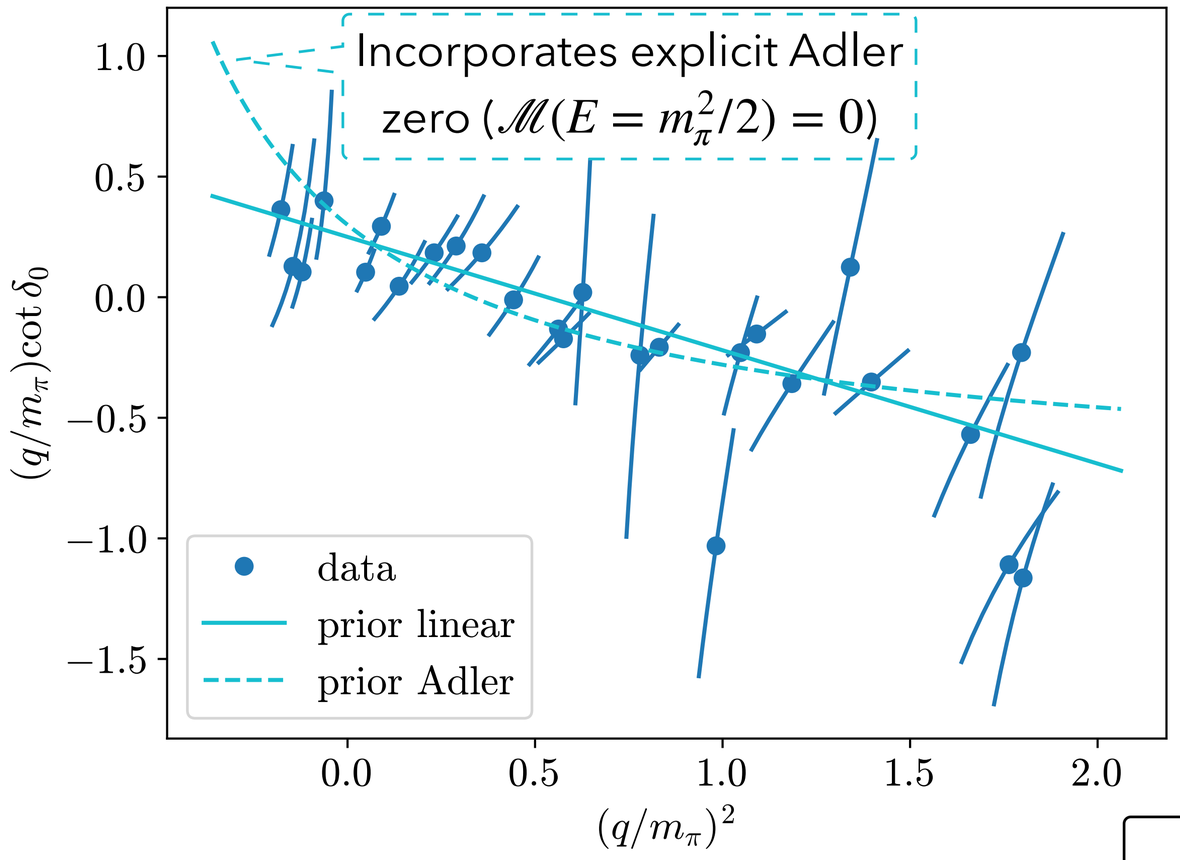
Switches from being a resonance to a virtual bound state at $m_\pi = 283$ MeV

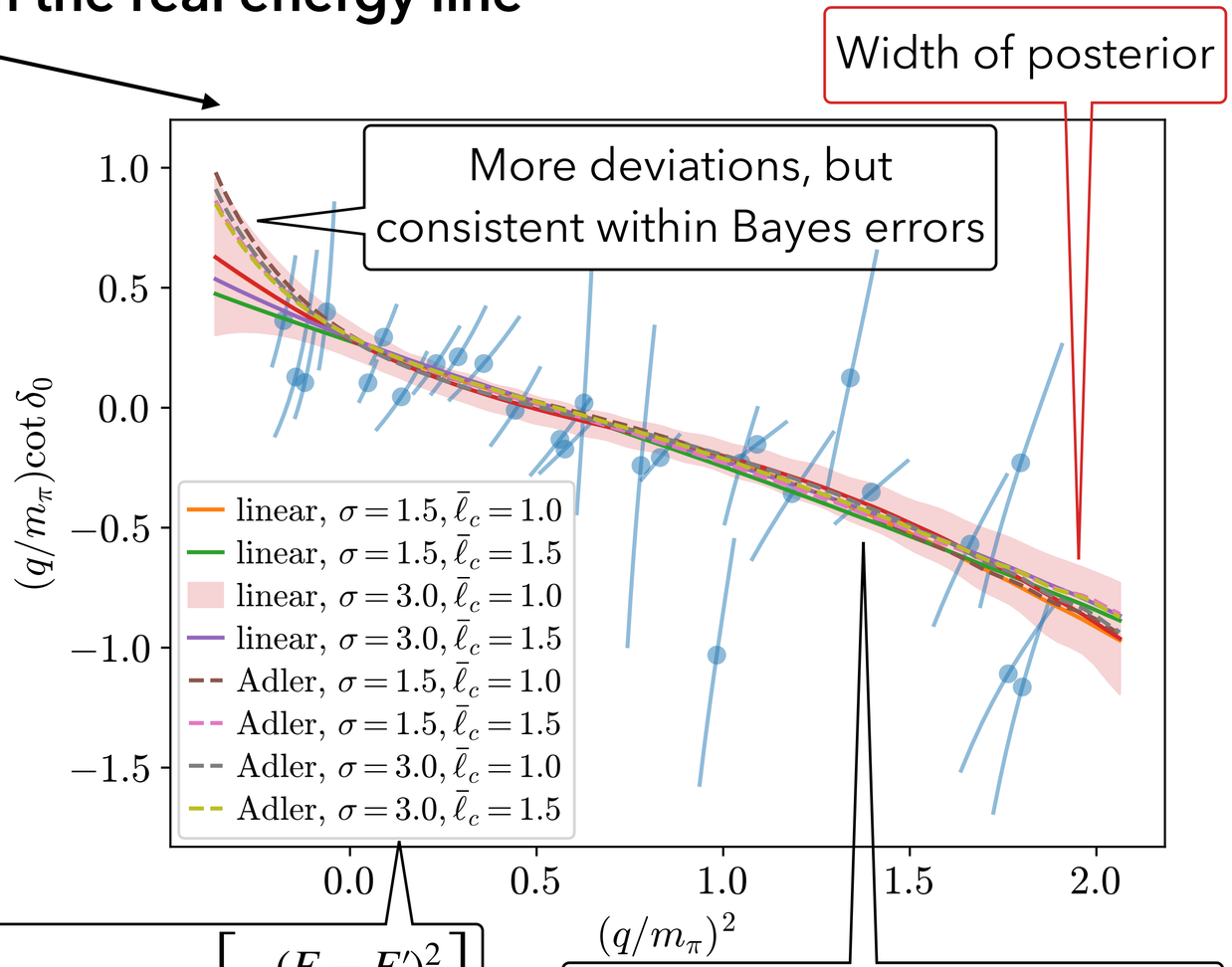
Rodas, Dudek, and Edwards (HadSpec) 2023 [PRD 108, 034513]
Briceño et al. (HadSpec) 2017 [PRL 118, 022002]

Big thanks for sharing data!

Isospin-0 $\pi\pi$ system by HadSpec

Priors and posteriors on the real energy line





$$\Sigma_p(E, E') = \sigma^2 \exp\left[-\frac{(E - E')^2}{2\ell_c^2}\right]$$

Isospin-0 $\pi\pi$ system by HadSpec

σ pole location

Pole position under assumption of…

Prior	Bayes factor	Probability of v.b.s.	E_0/m_π	E_0/m_π (v.b.s.)	E_0/m_π (res.)
Linear	1.000	0.514(54)	$1.883(99) - i0.08(10)$	1.87(11)	$1.901(88) - i0.163(88)$
Adler-zero	0.838	0.328(50)	$1.89(11) - i0.099(91)$	1.84(13)	$1.919(83) - i0.147(72)$

Full error: integration + Bayes + Wertevorrat

$(\sigma = 3, \ell_c/m_\pi = 1)$

- Pole locations consistent within errors between different priors

Isospin-0 $\pi\pi$ system by HadSpec

σ pole location

Pole position under assumption of…

Prior	Bayes factor	Probability of v.b.s.	E_0/m_π	E_0/m_π (v.b.s.)	E_0/m_π (res.)
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Full error: integration + Bayes + Wertevorrat

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- Pole locations consistent within errors between different priors
- Adler-zero prior prefers resonance

Isospin-0 $\pi\pi$ system by HadSpec

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Full error: integration + Bayes + Wertevorrat

$(\sigma = 3, \ell_c/m_\pi = 1)$

- Pole locations consistent within errors between different priors
- Adler-zero prior prefers resonance
- Prior without Adler zero preferred —> consistent with dispersive analysis

Rodas, Dudek, and Edwards (HadSpec) 2024 [PRD 109, 034513]

Isospin-0 $\pi\pi$ system by HadSpec

σ pole location

Pole position under assumption of…

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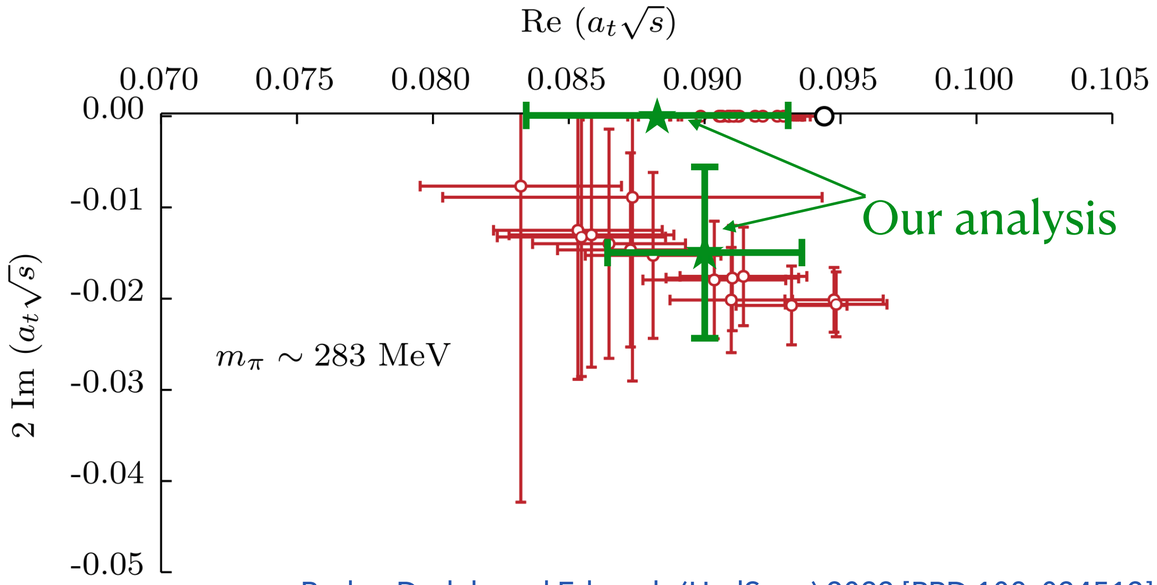
$(\sigma = 3, \ell_c/m_\pi = 1)$

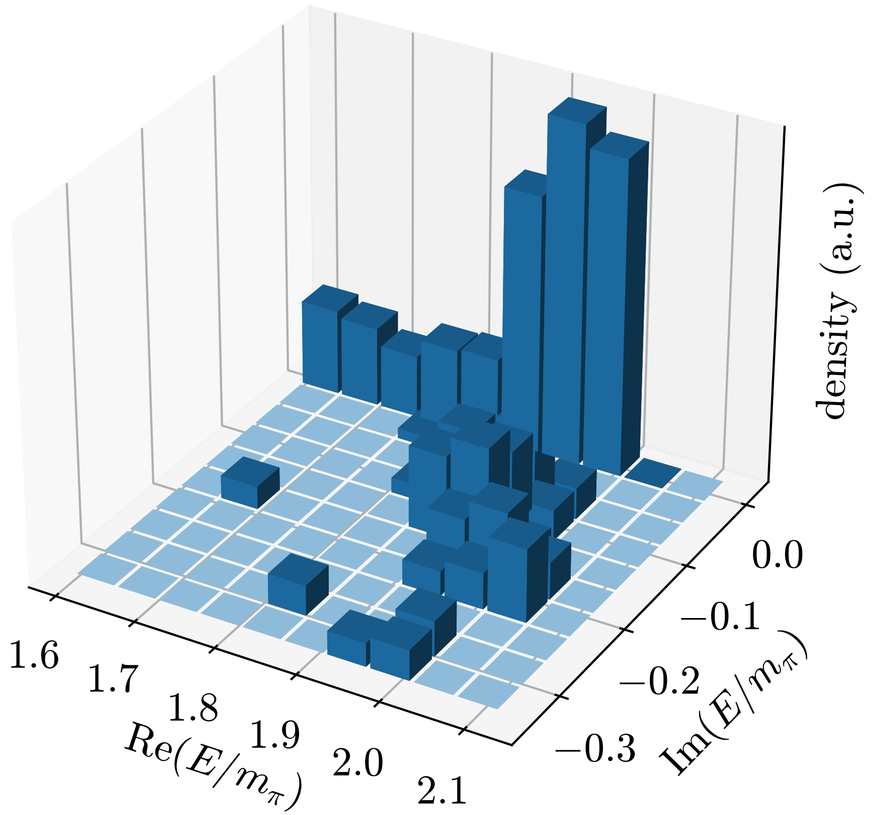
- Pole locations consistent within errors between different priors
- Adler-zero prior prefers resonance
- Compares well with original analysis, but quantifies model dependence



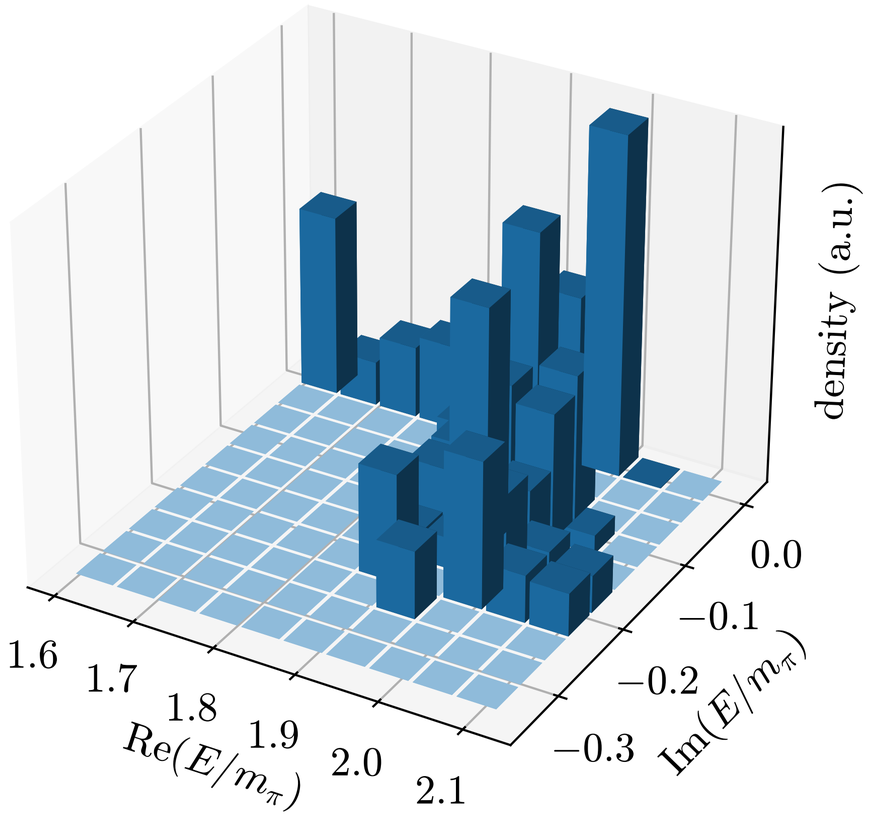
Rodas, Dudek, and Edwards (HadSpec) 2023 [PRD 108, 034513]

Isospin-0 $\pi\pi$ system by HadSpec

σ pole location



Linear prior



Adler-zero prior

Isospin-0 $\pi\pi$ system by HadSpec

Comparison of dispersive and Bayesian analyses

- Dispersive analysis — Rodas, Dudek, and Edwards (HadSpec) 2024 [PRD 109, 034513]
  - Reduce parameterisation dependence by enforcing crossing symmetry
  - Filter parametric fit models —> keep only ones fulfilling dispersion relations
  - Only works for certain systems, e.g. $\pi\pi$

- Bayesian analysis
  - Crossing symmetry not enforced
  - Quantify parameterisation dependence non-parametrically
  - Works for most systems

Systems that can be put in a form compatible with Nevanlinna interpolation

- Future: combine both approaches —> non-parametric description enforcing crossing symmetry —> reduced parameterisation dependence with more robust quantification

Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

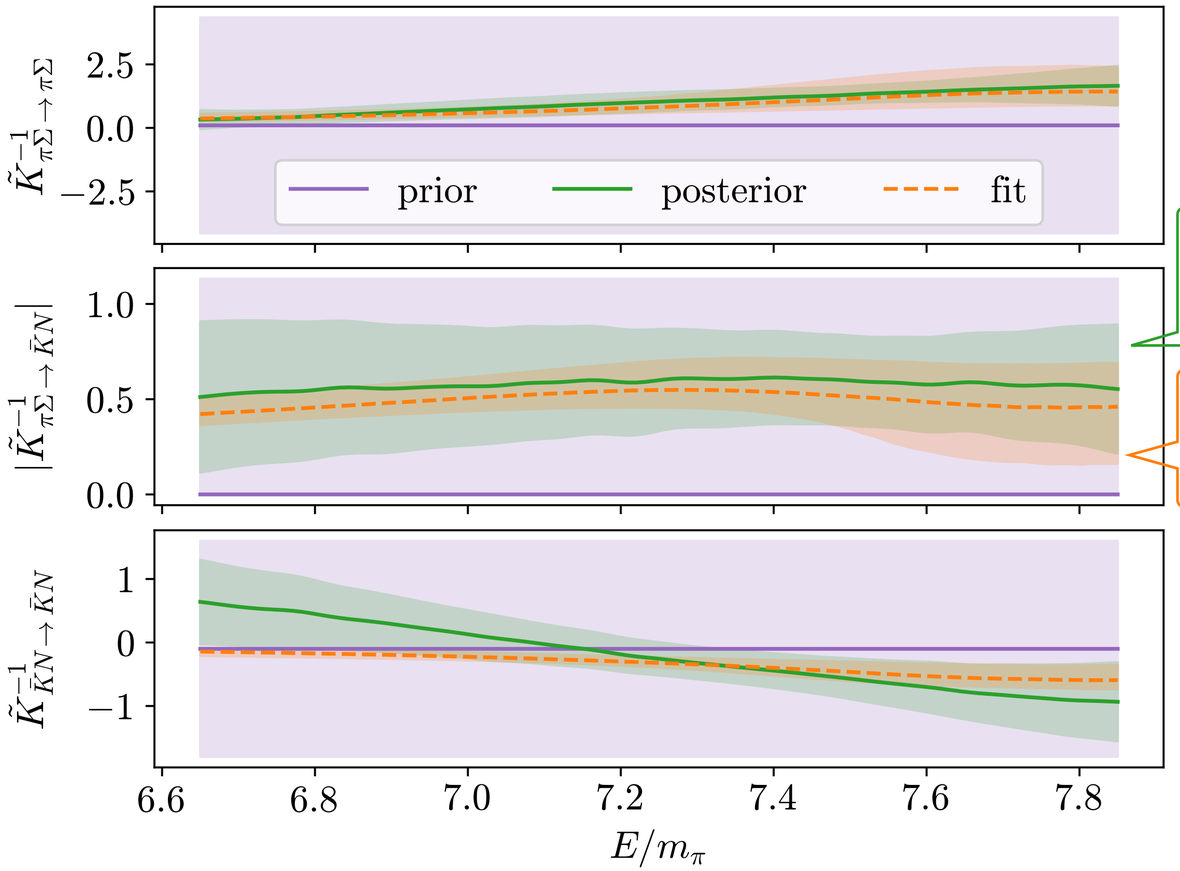
Priors for the K matrix

- Use factorised prior of the form
  $p(\mathbf{u}\,|\,\mathbf{u}_0) = p_{\pi\Sigma;\pi\Sigma}(\mathbf{u}^{(1)}\,|\,\mathbf{u}_0^{(1)}) \times p_{\pi\Sigma;\bar{K}N}(\mathbf{u}^{(2)}\,|\,\mathbf{u}_0^{(2)}) \times p_{\bar{K}N;\bar{K}N}(\mathbf{u}^{(3)}\,|\,\mathbf{u}_0^{(3)})$ —> no prior correlations between different entries of K matrix
- Gaussian priors for diagonal entries
- Design prior for off-diagonal entry to respect sign-flip symmetry,
  $p_{\pi\Sigma;\bar{K}N}(\mathbf{u}^{(2)}\,|\,\mathbf{u}_0^{(2)}) \propto \exp\left[-\frac{1}{2}(\,|\,\mathbf{u}^{(2)}\,|\,-\mathbf{u}_0^{(2)})^T\left(\Sigma_p^{(2)}\right)^{-1}(\,|\,\mathbf{u}^{(2)}\,|\,-\mathbf{u}_0^{(2)})\right]$
  (other choices possible)
- Constant prior close to zero with different widths and correlation lengths

Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

Posterior for the K matrix

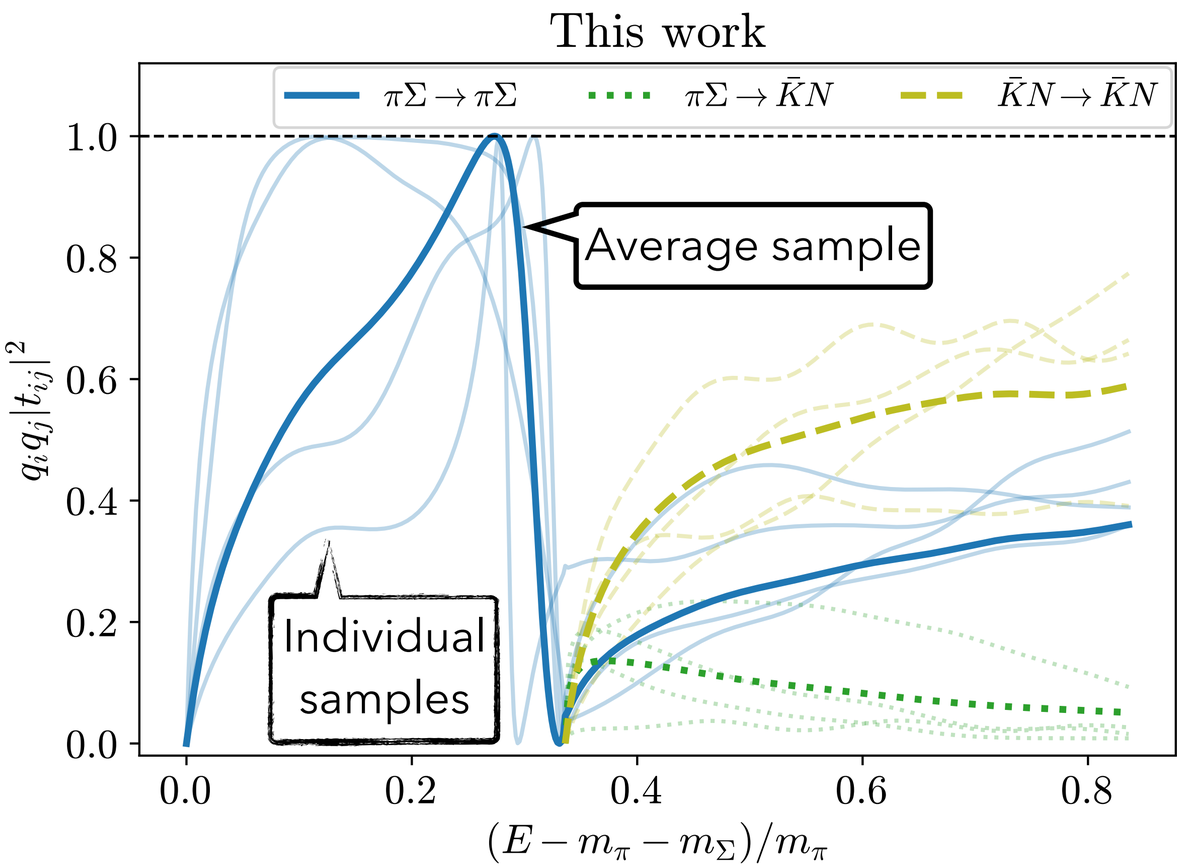
- Gaussian approximation of posterior breaks down <— reweighting: low ESS
- Use HMC for sampling
- Good agreement between Bayesian approach and explicit parametric fits
- Larger errors since posterior already includes systematic uncertainty



Bayesian posterior with full error

Best parametric fit with stat. error

Coupled-channel $\pi\Sigma - \bar{K}N$ scattering on D200

Posterior for the scattering amplitude

- Considerable variation in location of the peak in $\pi\Sigma \to \pi\Sigma$ component
- Peak structure washed out in mean over samples for $q_i q_j |\, t_{ij}\,|^2$ and error band
- Central value: take sample mean of $\mathscr{K}^{-1}$
- Cusp at opening of $\bar{K}N$ threshold present on all samples, sometimes small
- Cusp also washed out in error band