

# Two-particle matrix elements in a box



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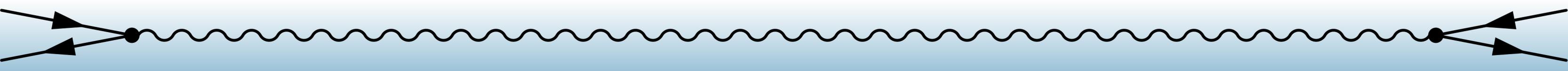
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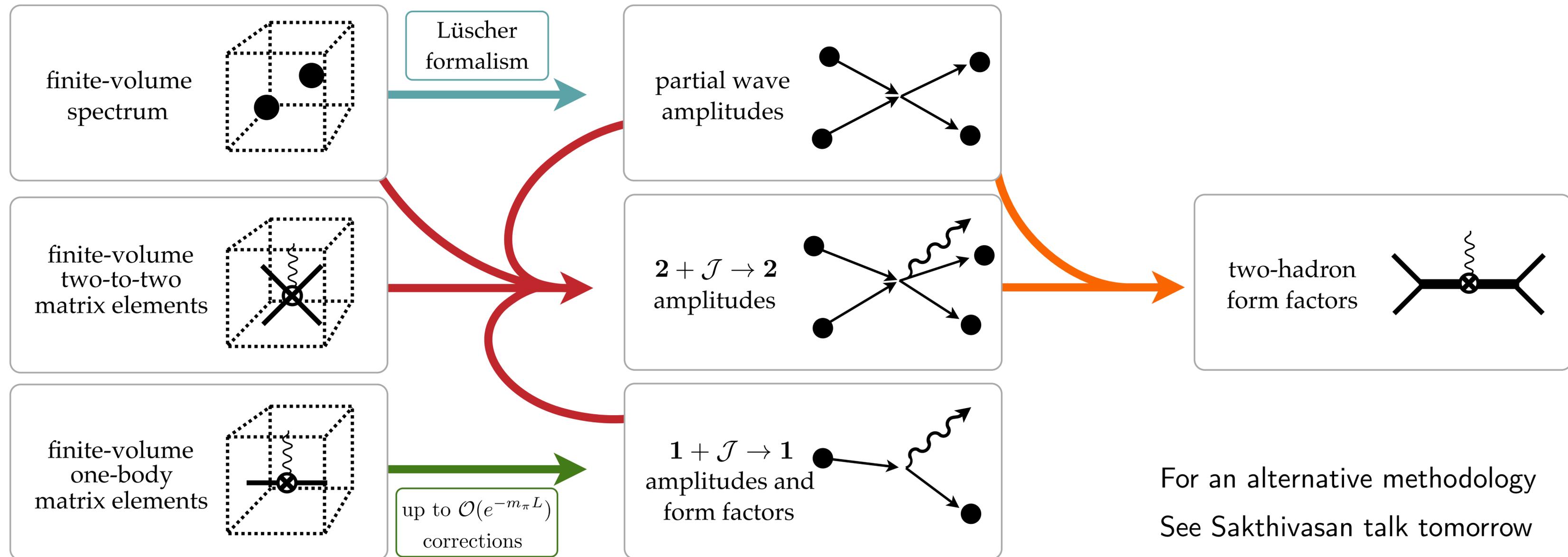
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Briceño UCB, Jackura W&M, Nicholson UNC



# Bound state matrix elements



For an alternative methodology  
See Sakthivasan talk tomorrow  
9:40 AM @ AG80 Structure

# Tunable two-particle interaction model

## Continuum, infinite volume



## Discrete, finite volume

- Non-relativistic system fermions
- Contact two-particle interaction

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2M} \right) \psi + g_0 (\psi^\dagger \psi)^2$$

$\psi$  : 2 flavors + 2 spin components

- Deuteron-like two particle states

$$|pq\rangle \propto (|p^+ \uparrow, p\rangle \otimes |n^0 \downarrow, q\rangle + |p^+ \downarrow, p\rangle \otimes |n^0 \uparrow, q\rangle)$$

- Finite volume transfer operator

$$\langle pq | \mathcal{T} | p' q' \rangle = \frac{\delta_{pp'} \delta_{qq'} + \frac{g_0}{V} \delta_{p+q, p'+q'}}{\sqrt{\xi(p)\xi(q)\xi(q')\xi(p')}}}$$

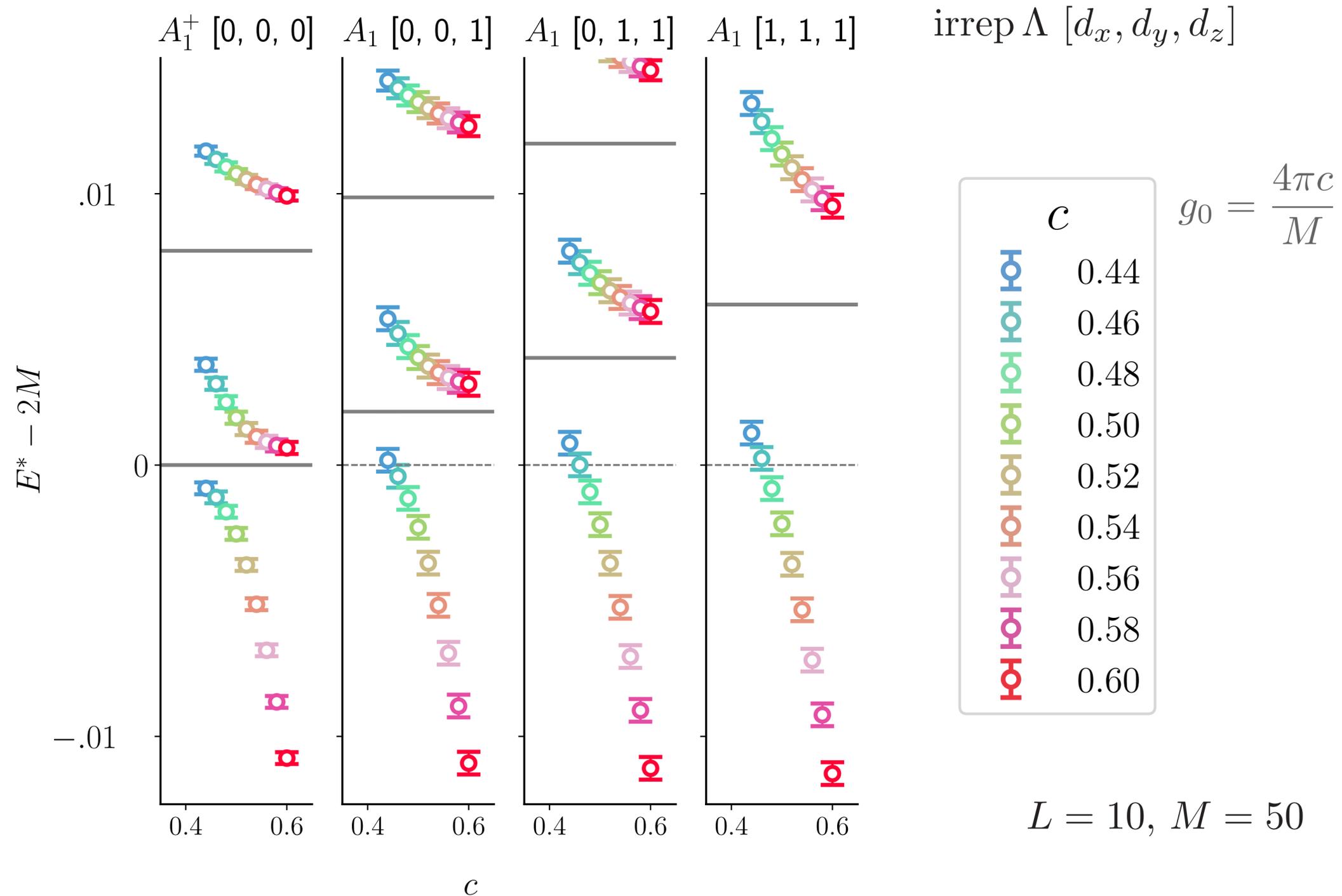
- Single particle dispersion fixed to continuum values  $\xi(p)^{-1} = \exp(-p^2/(2M))$
- Momentum is quantized to pbc

$$p = \frac{2\pi}{L} [n_x, n_y, n_z] \quad n_i \in \{(L-1)/2, \dots, L/2\}$$

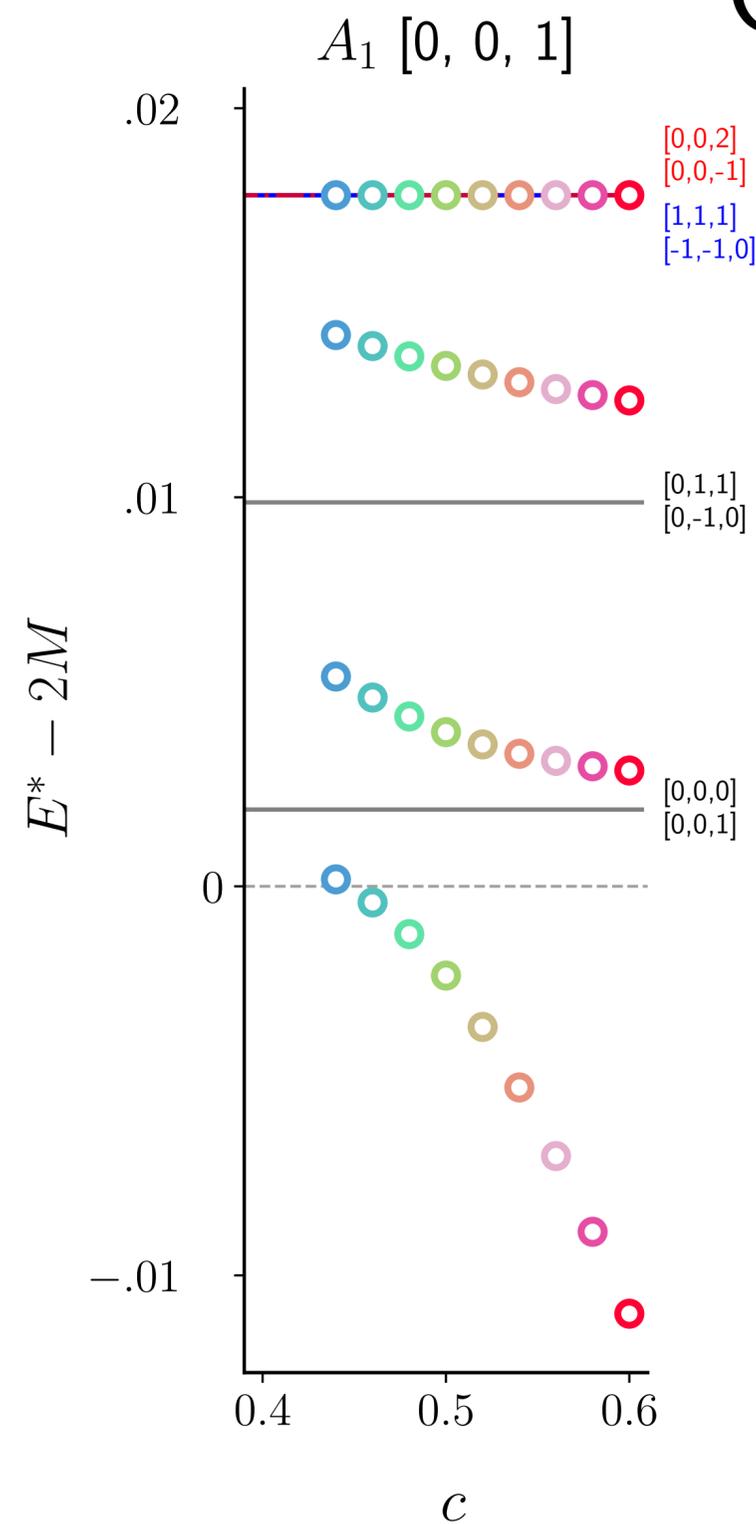
For modest volumes  $\sim 10$ , transfer operator can be diagonalized exactly!

# Finite Volume Spectrum

- Spectrum for various lattice momenta  $\mathbf{P} = \frac{2\pi}{L} \mathbf{d}$  as a function of  $c$ .
- With increasing  $c$ , ground state energy decreases.
- Added  $\sim 1\%$  errors to account for discretization

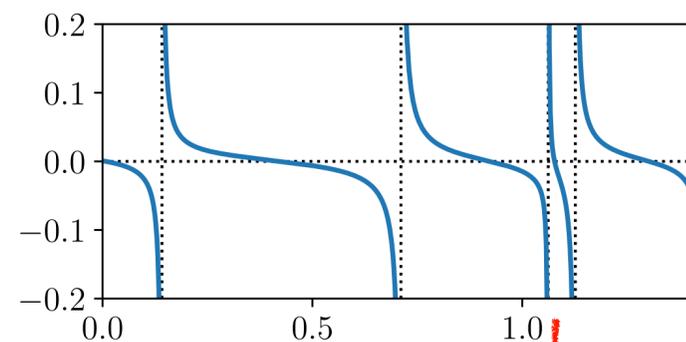


# Coupling independent energy levels

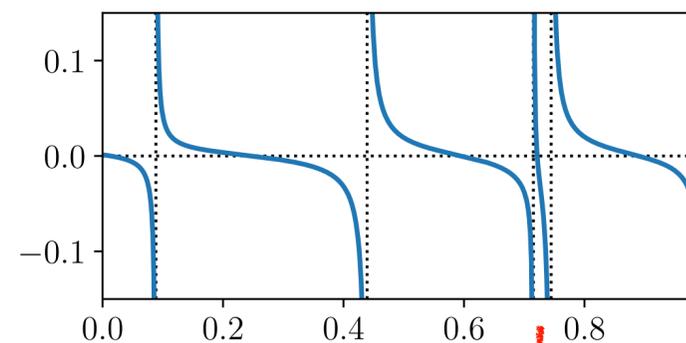


$M = 0.5$

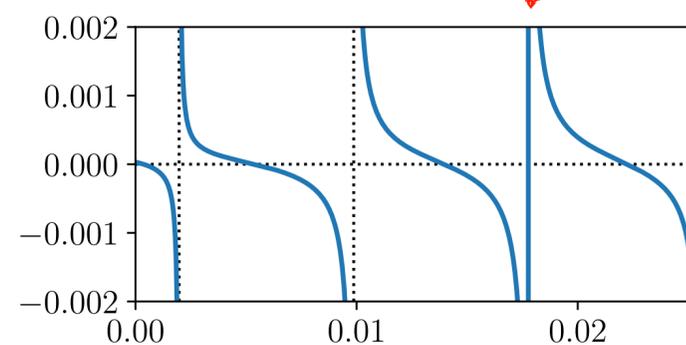
$\text{Re}[F_{00,00}(L = 10, [0, 0, 1])]$



$M = 1$



$M = 50$



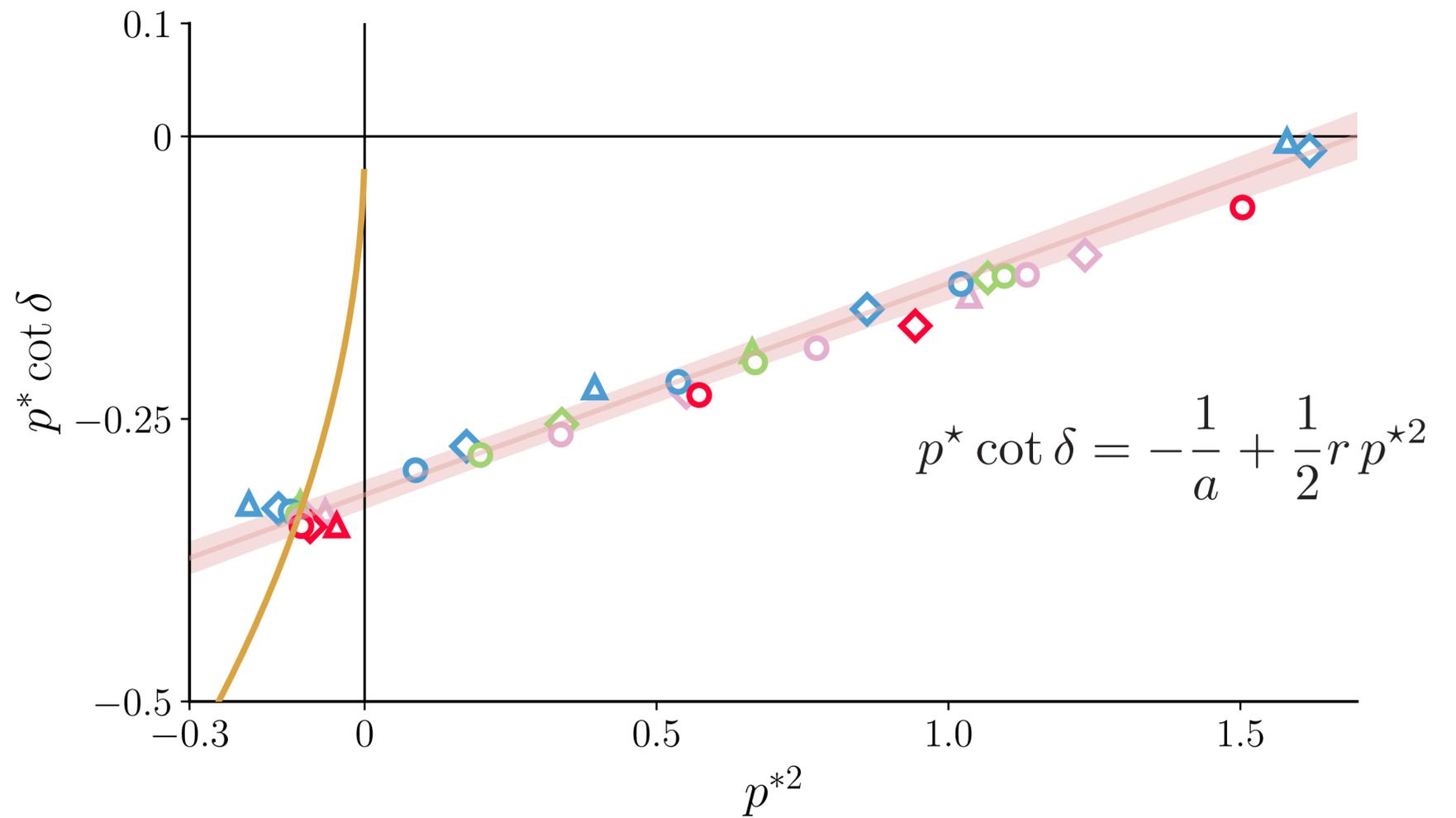
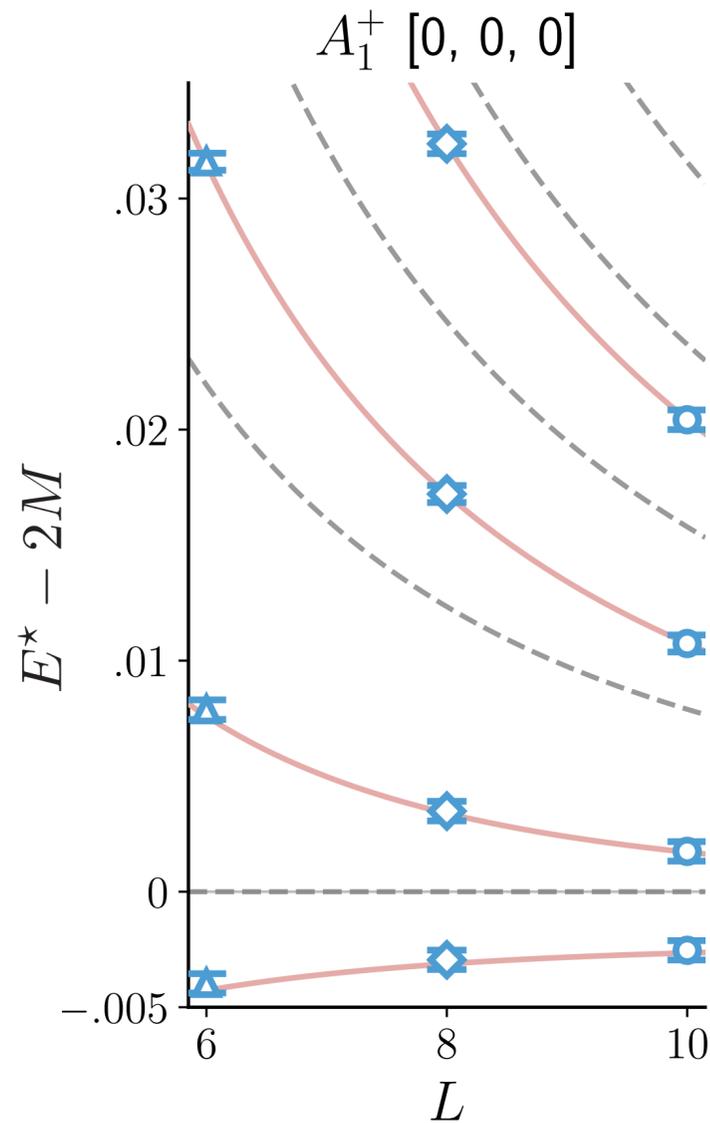
Non-relativistic systems have non-interacting degeneracies that lead to FV states with fixed energy.

$$\det(\mathcal{M}^{-1} + F) = 0$$

# Lüscher Quantization Condition $\det(\mathcal{M}^{-1} + F) = 0$

$$p^* \cot \delta = -8\pi E^* \text{Re}[F_{00,00}(L, \vec{P})]$$

$$\mathcal{M} = 8\pi E^* \frac{1}{p^* \cot \delta - ip^*}$$



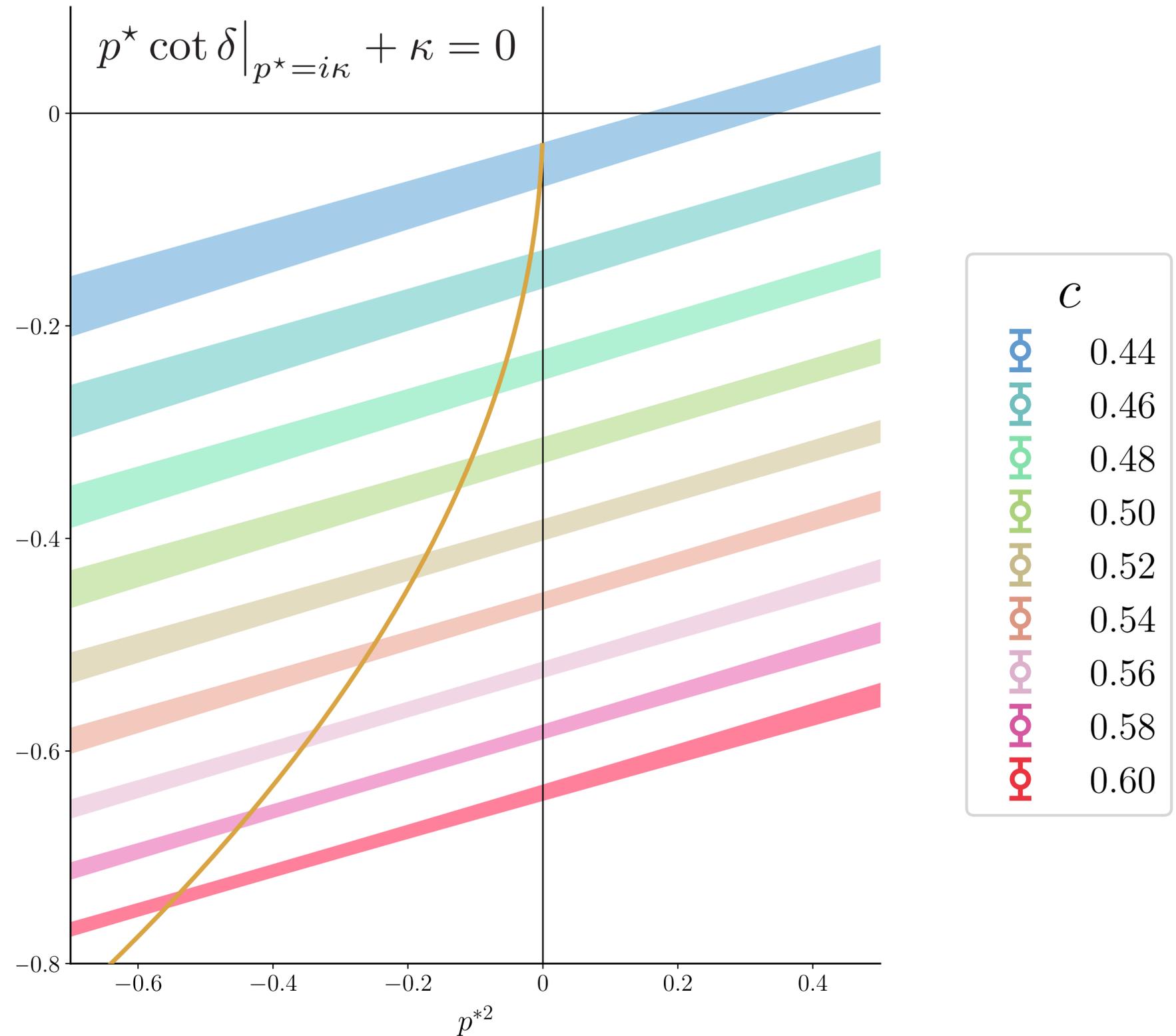
$$c = 0.5$$

# Bound state pole

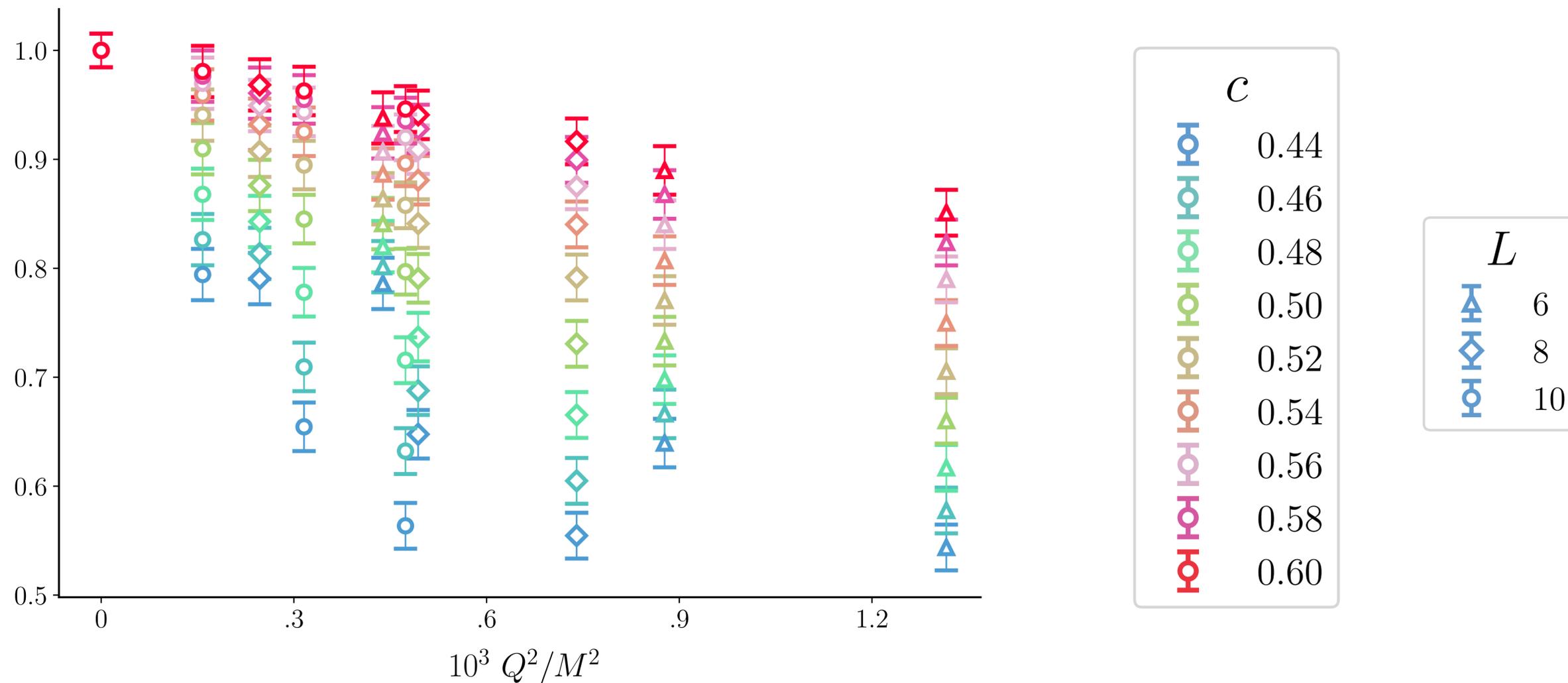
- Infinite volume phase-shift as a function of  $c$ .
- With increasing  $c$ , the bound state has an increasing binding energy.

$$\mathcal{M}(s) \sim \frac{(ig)^2}{s - s_0}$$

$$\mathcal{M} = 8\pi E^* \frac{1}{p^* \cot \delta - ip^*}$$



# Two-body Finite Volume Matrix Elements



$$\langle p' | J(q) | p \rangle = \delta_{p'+q,p}$$

$$\langle n | J(q) | m \rangle = \sum_{i,j} v_i^{(n)*} v_j^{(m)} \langle p_i | J(q) | p_j \rangle \langle p'_i | p'_j \rangle$$

w.f. from transfer op. diagonalization

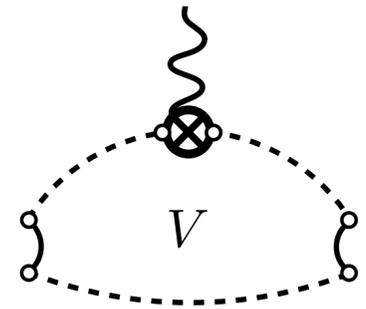
# Finite volume to infinite volume

$$\langle E_f, L | \mathcal{J} | E_i, L \rangle = \tilde{\mathcal{R}}^{1/2}(E_f, L) \tilde{\mathcal{W}}_L(E_f, E_i, L) \tilde{\mathcal{R}}^{1/2}(E_i, L)$$

ISI / FV norm 

FSI/ FV norm 

**Short-range  
dynamics**

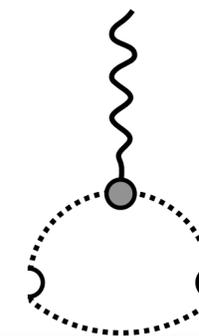
$$\tilde{\mathcal{W}}_L(E_f, E_i, L) = A_{22}(E_f, E_i) + f \cdot G(E_f, E_i, L)$$




**Infinite volume  
form factor:**

$$f_B(Q^2) = g^2 \left( A_{22}(Q^2) + f \cdot \underbrace{\left( \mathcal{G}(Q^2) - \frac{2\mathcal{G}^0(Q^2)}{E_f + E_i} \right)}_{\text{Triangle function}} \right)$$

Triangle function:



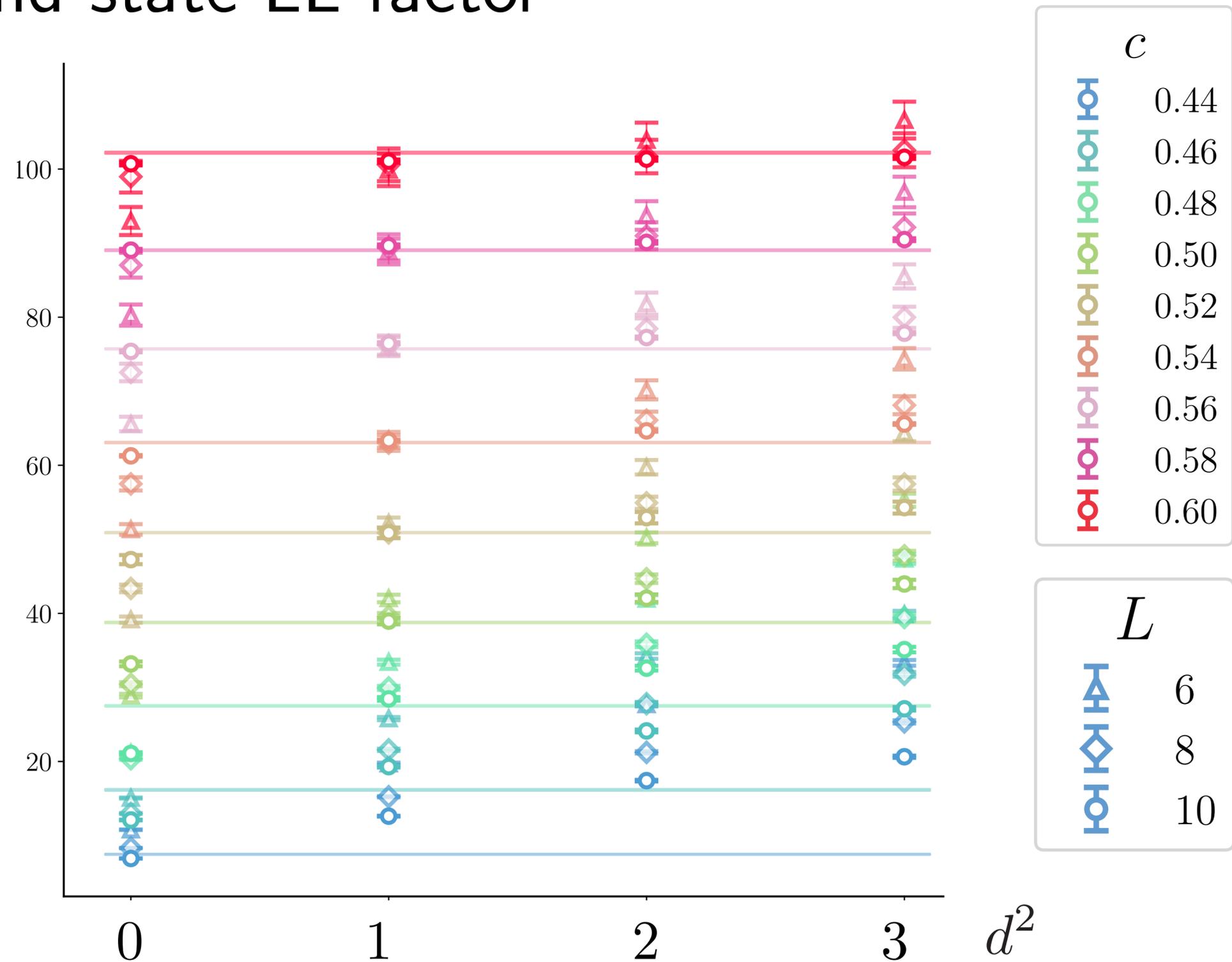
# Ground state LL factor

Residue of the QC  $\det(\mathcal{M}^{-1} + F) = 0$

$$\tilde{\mathcal{R}}(E_n, \mathbf{P}, L) = - \lim_{E \rightarrow E_n} \frac{E - E_n}{\mathcal{M}(\sqrt{E^2 - \mathbf{P}^2})^{-1} + F(E, \mathbf{P}, L)}$$

Lines show the large volume limit:

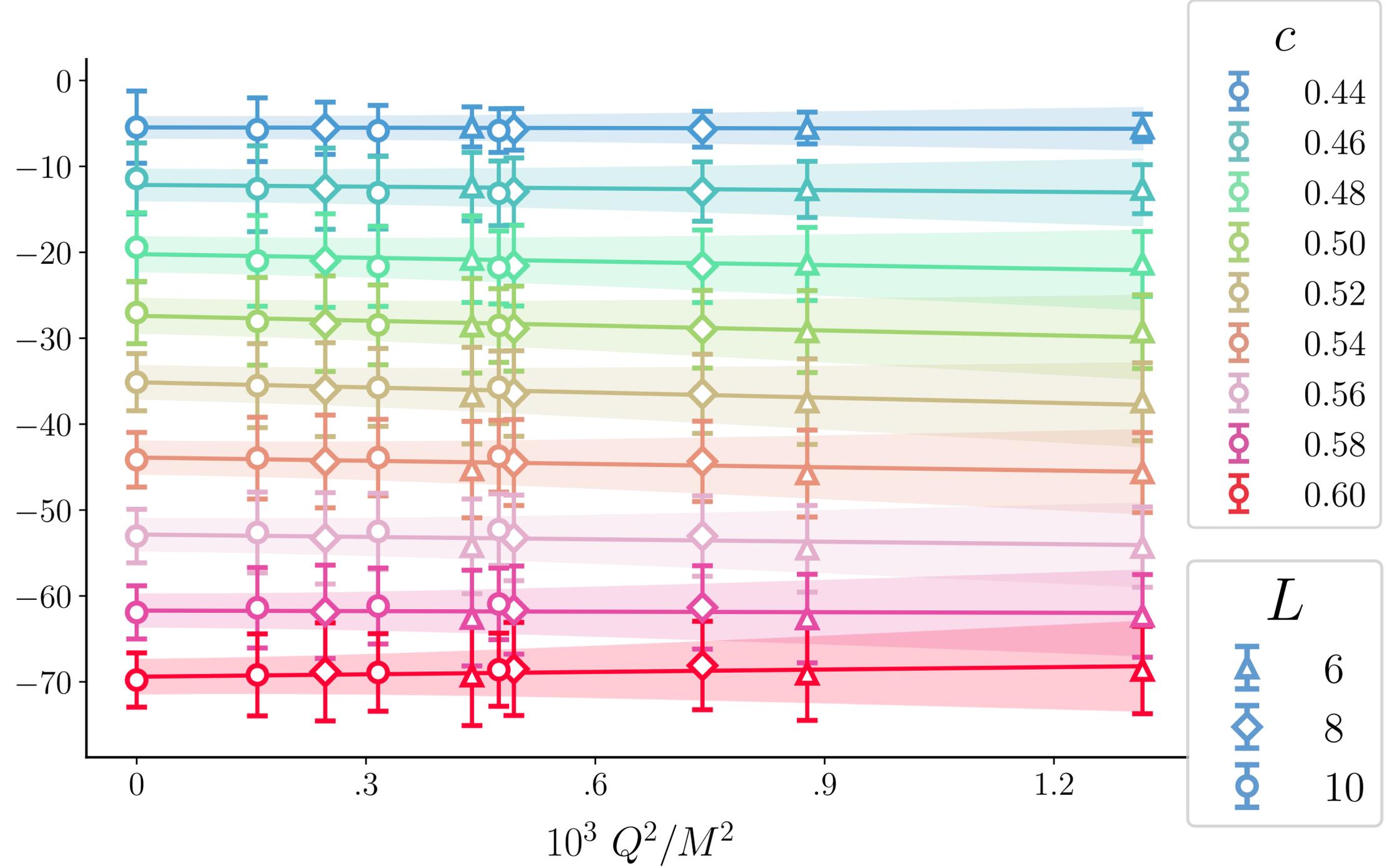
$$\tilde{\mathcal{R}}(E_B, L \rightarrow \infty) = - \left[ \frac{2E_B}{g^2} + \mathcal{O}(e^{-\kappa L}) \right]^{-1}$$



# Short-range dynamic function

$$\tilde{A}_{22} = g^2 A_{22}$$

- Mild virtuality dependence in sharp contrast to FV matrix elements.
- Appears to tend toward zero as the state becomes unbound.



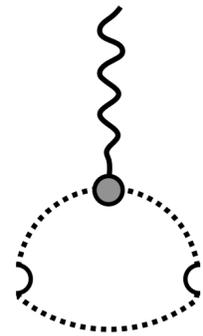
# Bound state form factor:

$$f_B(Q^2) \text{ vs. } \langle E_f, L | \mathcal{J} | E_i, L \rangle$$

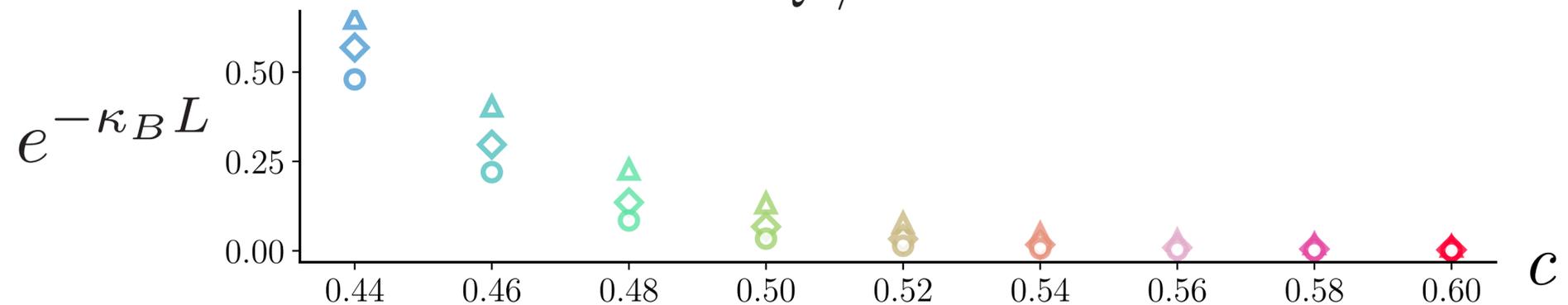
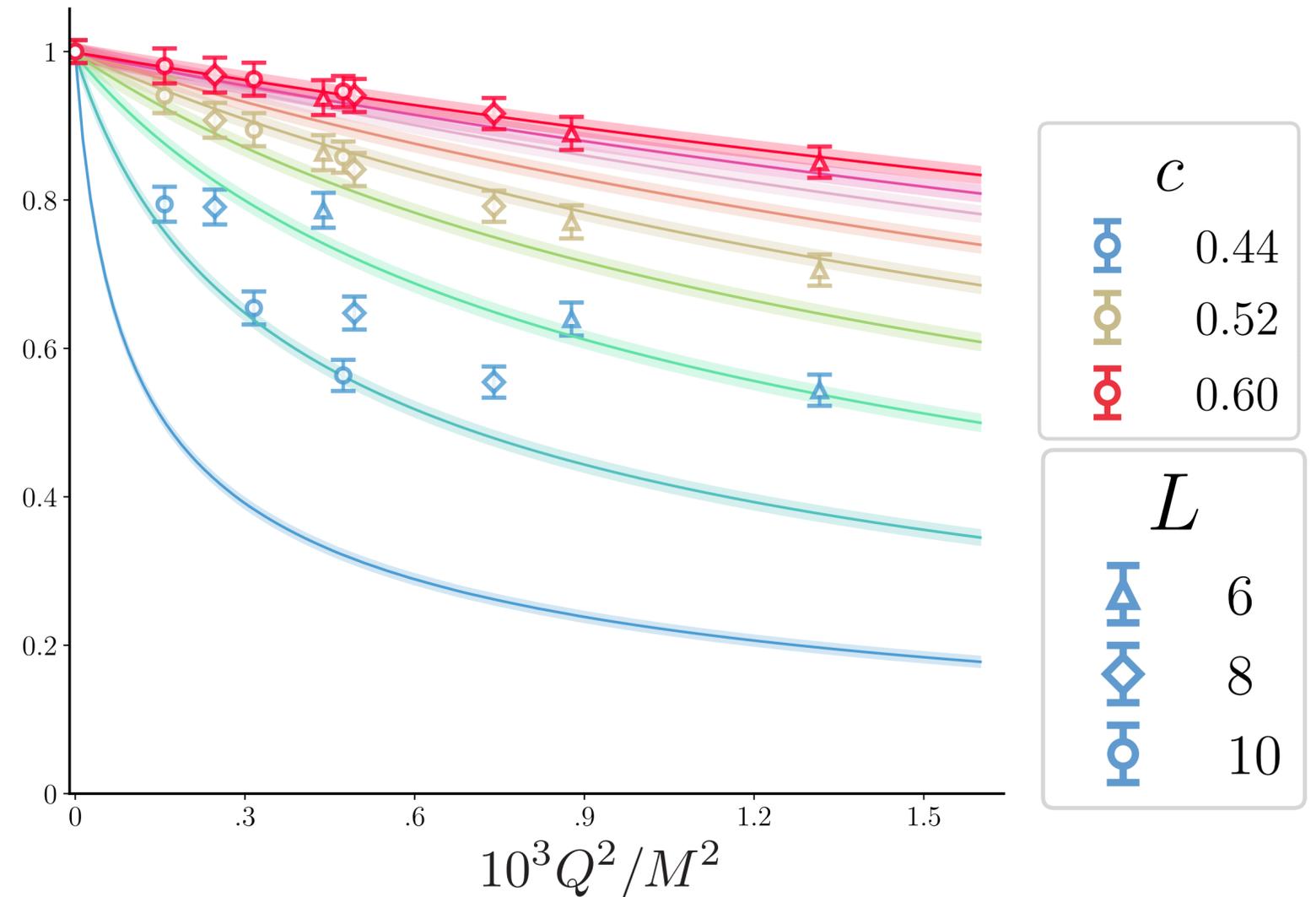
Finally we can extract the bound state ff:

$$f_B(Q^2) = \tilde{A}_{22}(Q^2) + g^2 \left( \mathcal{G}(Q^2) - \frac{2\mathcal{G}^0(Q^2)}{E_f + E_i} \right)$$

Triangle function:



- The finite and infinite volume form factors match best for deepest bound state

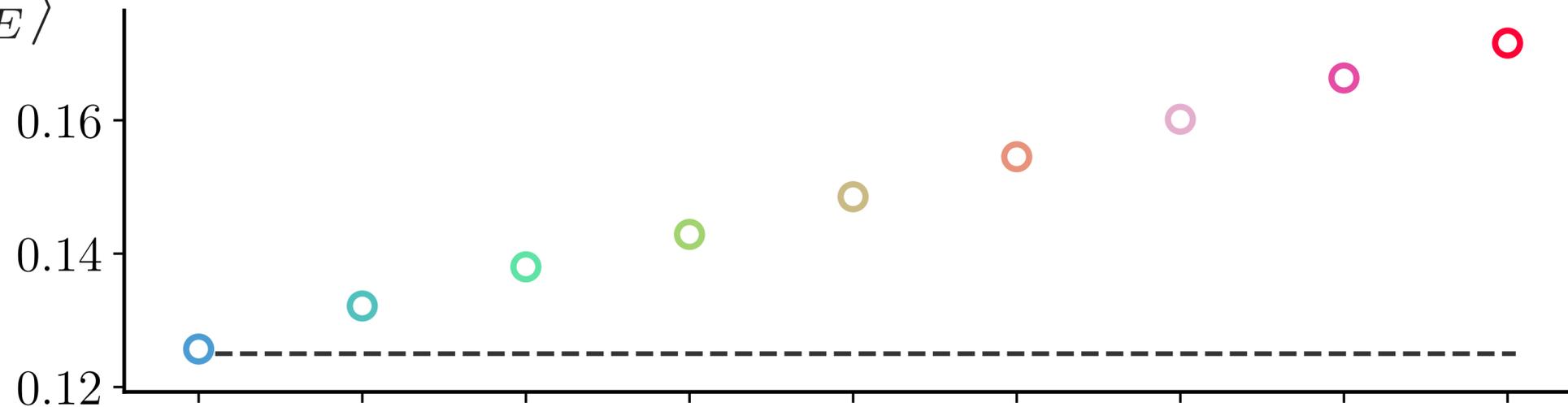


# Bound state charge radius

$$\langle r_E^2 \rangle = -6 \frac{d}{dQ^2} f_B(Q^2) \Big|_{Q^2=0}$$

$$\gamma = \sqrt{M(2M - E_B)}$$

$$\gamma^2 \langle r_E^2 \rangle$$

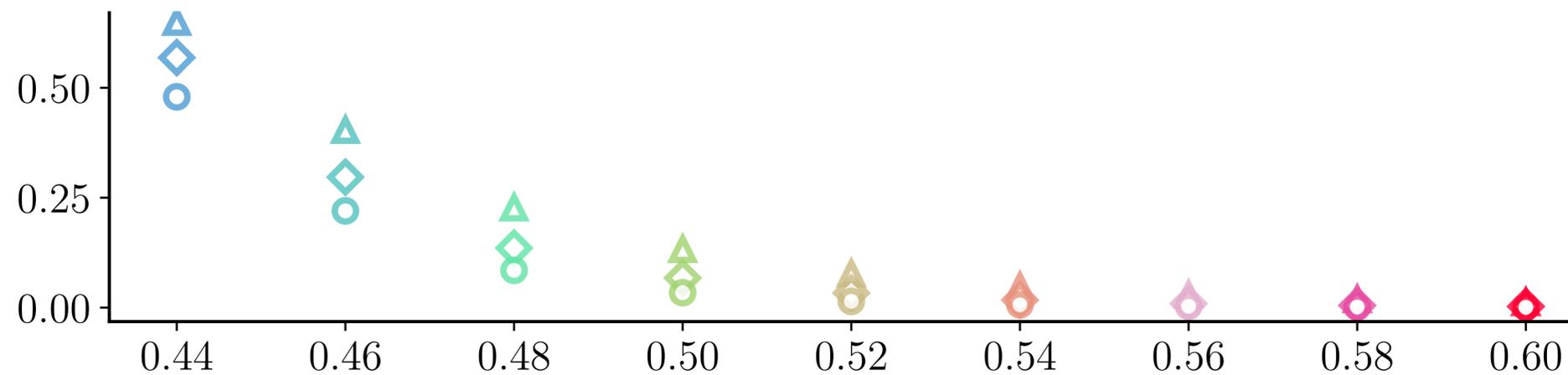


$$\langle r_E^2 \rangle^{\text{LO}} = \frac{1}{8\gamma^2}$$

[KSW, PRC **59**, 617 (1999)]

- As a check we recover the charge radius in the small binding-energy limit

$$e^{-\kappa_B L}$$



# Summar and Conclusion

- ❖ We calculated finite volume energies and matrix elements for a system with a tunable bound state.
- ❖ We showed the the impact of finite volume effects as a function of binding energy, recovering the expected result of negligible effects for a deeply bound state.
- ❖ Finite volume effects can be substantial at low binding energy. This is relevant, for example, for the analysis of nuclear matrix elements in the lattice.

$$f_B(Q^2) \text{ vs. } \langle E_f, L | \mathcal{J} | E_i, L \rangle$$

