Towards the time-like pion form factor beyond the elastic regime using domain-wall QCD

G. Morandi[a,b]

in collab. with M. Bruno[a,b] and F. A. Bresciani[a,b].
For the **RBC/UKQCD collaborations**.

[a] *Università degli Studi di Milano - Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*

[b] *INFN, Sezione di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*

42nd International Symposium on Lattice Field Theory (Lattice 2025)

The RBC & UKQCD collaborations

BNL & RBRC
Peter Boyle
Taku Izubuchi
Chulwoo Jung
Christopher Kelly
Patrick Oare
Amarjit Soni (emeritus)
Xin-Yu Tuo
Tianle Wang
Shuhei Yamamoto

CERN
Matteo Di Carlo
Felix Erben
Andreas Jüttner (Southampton)

Columbia University
Ryan Abbott
En-Hung Chao (MIT)
Norman Christ
Jonah Eick
Rajiv Eranki
Sarah Fields
Ceran Hu
Yikai Huo
Erik Lundstrum
Robert Mawhinney

University of Connecticut
Thomas Blum
Jonas Hildebrand
Luchang Jin
Vaishakhi Moningi
Anton Shcherbakov
Douglas Stewart
Joshua Swain
Masaaki Tomii (BNL)

DESY Zeuthen
Raoul Hodgson

University of Edinburgh
Matthew Black
Luigi Del Debbio
Vera Gülpers
Maxwell T. Hansen
Nils Hermansson Truedsson
Ryan Hill
Rajnandini Mukherjee
Antonin Portelli
Teseo San Jose

Johannes Gutenberg University of Mainz
Alessandro Barone

KEK
Shigemi Ohta (emeritus)

Liverpool Hope
Nicolas Garron (Liverpool)

University of Liverpool
Tobias Tsang
Aurelia Zuchanke

University of Milano Bicocca
Francesca A. Bresciani
Mattia Bruno
Gabriele Morandi

University of Regensburg
Daniel Knüttel
Christoph Lehner
Julian Parrino
Andreas Völklein

University of Siegen
Anastasia Boushmelev
Oliver Witzel

University of Southampton
Bipasha Chakraborty
Ahmed Elgaziari
Jonathan Flynn
Ollie Heald
Joe McKeon
Chris Sachrajda

Stony Brook University
Sergey Syritsyn (RBRC)

Outline

1. Introduction
2. Methodology and Results
3. Conclusions

Motivations

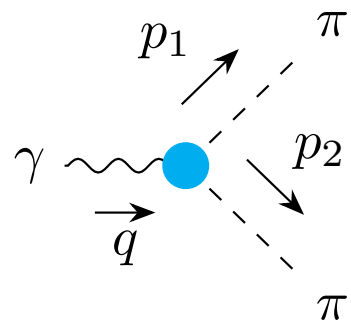
The pion form factor

- The pion form factor F_π (PFF) is a physical quantity which describes the coupling between the photon and two pions:

Focus on **Time-like** regime, defined by:

$$\langle 0|J_\mu(q)|\pi\pi, \mathbf{p_1 p_2}\rangle = (p_1 - p_2)_\mu F_\pi(q^2)\,,$$

with $q = p_1 + p_2$, $J_\mu =$ em current.



- Phenomenologically relevant quantity for the HVP contribution to the muon $g-2$, the pion charge radius, properties of ρ meson to name a few.
- *Lattice QCD simulations*: solid, rigorous and systematically improvable non-perturbative framework to compute hadronic quantities from first principles numerical simulations.

$$\Downarrow$$

Aim: calculate the **time-like** pion form factor **beyond the inelastic threshold**.

State of the art determinations

traditional approach à la Lüscher

Calculations in the time-like region performed using finite volume methods

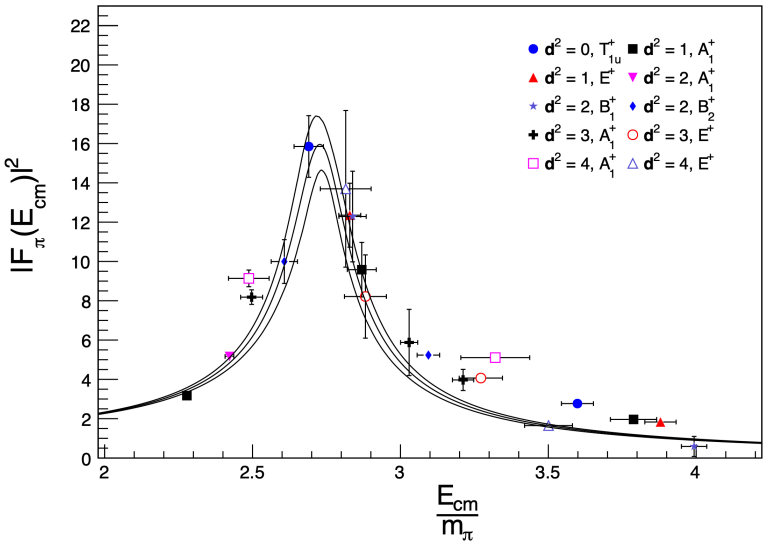
[Lüscher, 1986] [Lüscher, 1991] [Lellouch and Lüscher, 2001] [Meyer, 2011].

Limited to the **elastic regime**, $2m_\pi \leq \sqrt{s} \leq 4m_\pi$ (isospin limit)

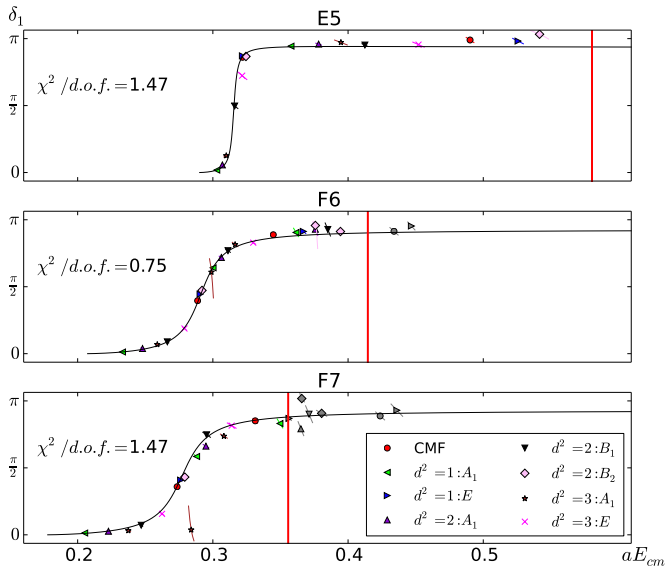
(e.g. [Feng et al., 2015] [Bulava et al., 2016] [Erben et al., 2020] . . .)

$\Longrightarrow$ Quantization condition known for two and three particles

(e.g. [Polejaeva and Rusetsky, 2012] [Briceno and Davoudi, 2013] [Hansen and Sharpe, 2014])



Taken from [Bulava et al., 2016].



Taken from [Erben et al., 2020].

Long-term goal

towards hadronic decays with multi-hadron final states

- The pion form factor serves mostly as a **test bed** for inverse-like methods in QCD.

- **Far-reaching goal**: study hadronic decays with two-hadron final states $\Rightarrow$ extraction of CKM matrix elements.

- From long-time behavior of suitable 4pt-correlator [Bulava and Hansen, 2019] [Bruno and Hansen, 2021], extract matrix elements of the form

$$\langle B/D|\mathcal{H}_w(0)|\pi\pi\rangle \ ,$$

with $\mathcal{H}_w(0)$ the effective weak Hamiltonian.

- Many particle states are relevant $\Rightarrow$ quantization condition is challenging. Try inverse-method techniques.

Lattice setup

ensemble 48I at physical pion mass

- Analysis on the 48I ensemble previously generated by the RBC/UKQCD collaborations [Blum et al., 2016].

- $N_f = 2 + 1$ Möbius domain wall fermions, Iwasaki gauge action, **physical pion mass** $m_\pi \simeq 139.2$ MeV.

Size	β	am_l	am_h	a^{-1}[GeV]	L[fm]	$m_\pi L$
$48^3 \times 96 \times 24$	2.13	0.00078	0.0362	1.730(4)	5.476(12)	3.863(6)

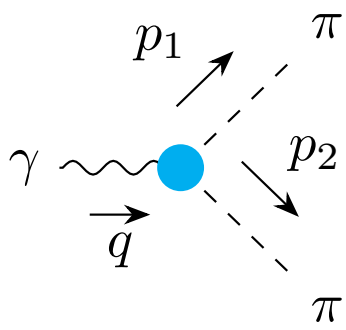
Time-like form factor

center-of-mass frame

We consider

$$\langle 0|V_\mu(q)|\pi\pi, \mathbf{p}_1\mathbf{p}_2\rangle = (p_1 - p_2)_\mu F_\pi(q^2)\,,$$

with $q = p_1 + p_2$ and $V_\mu =$ **isospin-1 local** em current.



Center-of-mass frame: pions have back-to-back spatial momenta, vanishing total momentum $\Longrightarrow$ $\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$, therefore only spatial components survive:

$$\langle 0|V_k(q)|\pi\pi, \mathbf{0}\rangle = 2\mathbf{p}_k F_\pi(4\omega_{\mathbf{p}}^2)\,, \quad k = 1, 2, 3\,;$$

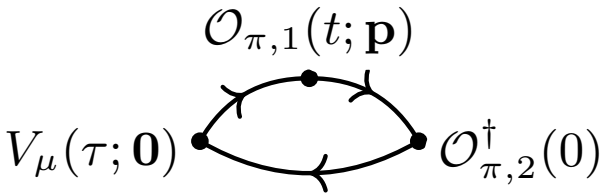
where $\omega_{\mathbf{p}} = \sqrt{m_\pi^2 + |\mathbf{p}|^2}$.

3pt correlation function

spectral decomposition

- $C_\mu^{3\mathrm{pt}}(t,\tau;\mathbf{p}) \equiv \left\langle V_\mu(\tau;\mathbf{0})\mathcal{O}_{\pi,1}(t;\mathbf{p})\mathcal{O}^\dagger_{\pi,2}(0)\right\rangle$,

  $\mathcal{O}_{\pi,i}$: single pion interpolators.



- Inserting a complete set of states in $C_\mu^{3\mathrm{pt}}(t,\tau;\mathbf{p})$ yields

$$C_\mu^{3\mathrm{pt}}(t,\tau;\mathbf{p}) = \sum_n e^{-E_n(\mathbf{p})t}\,\langle 0|V_\mu(\tau)\mathcal{O}_{\pi,1}(t)|n,\mathbf{p}\rangle\,\langle n,\mathbf{p}|\mathcal{O}^\dagger_{\pi,2}(0)|0\rangle\ .$$

see e.g. [Bulava and Hansen, 2019] [Bruno and Hansen, 2021]

- Limit $\tau, t \to \infty$ to isolate $|n,\mathbf{p}\rangle \to |\pi,\mathbf{p}\rangle$ ($Z^{1/2}_{\pi,i}(\mathbf{p}) \equiv \langle \pi,\mathbf{p}|\mathcal{O}^\dagger_{\pi,i}(0)|0\rangle$):

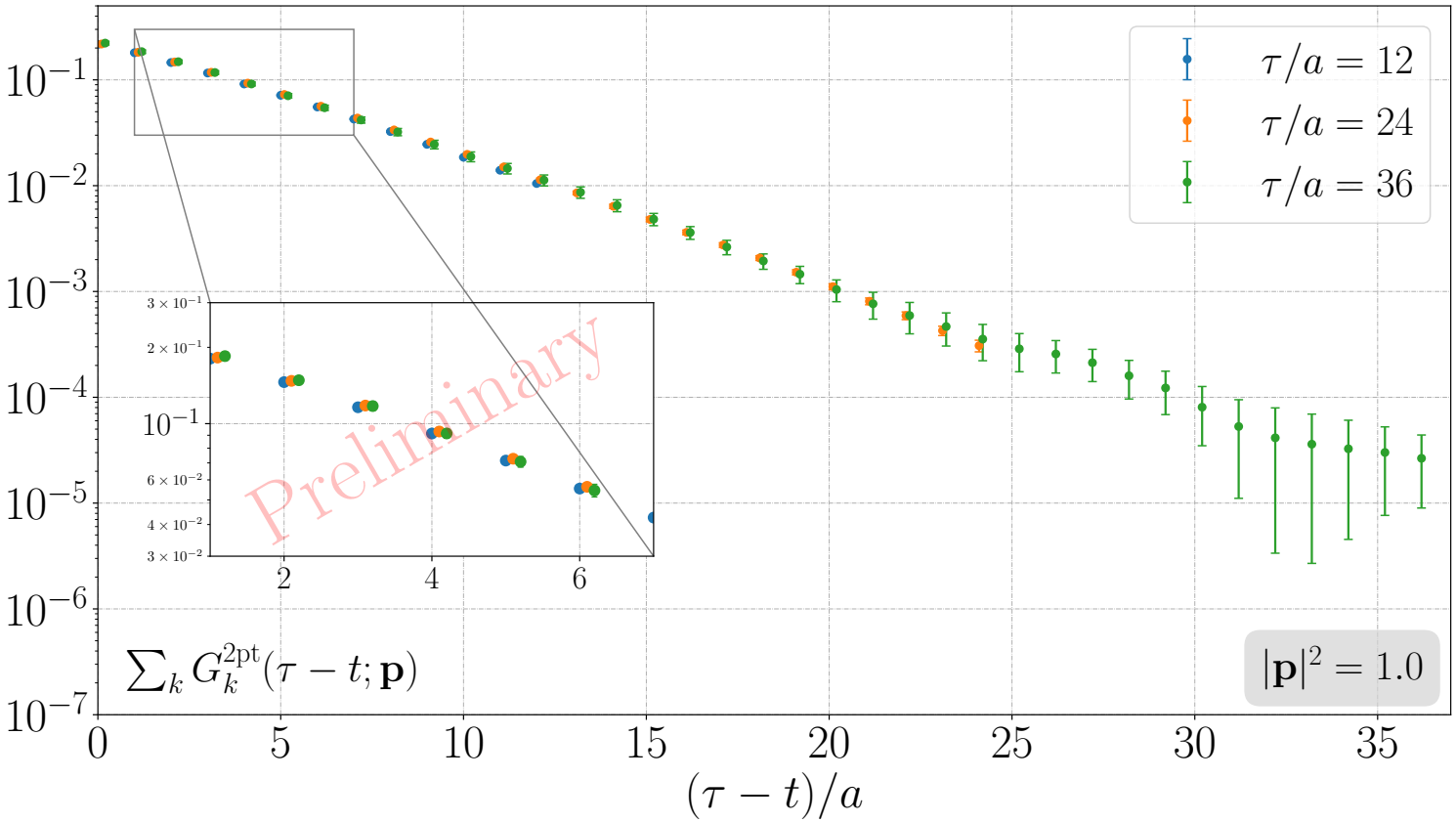
$$\lim_{t\to\infty} C_\mu^{3\mathrm{pt}}(t,\tau;\mathbf{p}) = Z^{1/2}_{\pi,2}(\mathbf{p})e^{-E_\pi(\mathbf{p})t}G_\mu^{2\mathrm{pt}}(\tau-t;\mathbf{p})\,,$$
$$G_\mu^{2\mathrm{pt}}(\tau-t;\mathbf{p}) \equiv \langle 0|V_\mu(\tau)\mathcal{O}_{\pi,1}(t)|\pi,\mathbf{p}\rangle\ .$$

$G_k^{2\mathrm{pt}}(\tau - t; \mathbf{p})$

τ-INDEPENDENCE

Dependence on $(\tau - t)$ up to excited states when $t \sim 0$.

($|\mathbf{p}|^2$ in units of $(2\pi/L)^2$ in plot)



GEVP STRATEGY

ENERGIES AND MATRIX ELEMENTS OF V_k

Based on [Blum et al., 2025], perform GEVP analysis on the $\langle V_3(t)V_3(0)\rangle$ correlator to extract $\langle 0|V_3(0)|n,\mathbf{0}\rangle$ and $E_n(\mathbf{0})$:

$$C(t) = \left(\begin{array}{c|c} V(t)V(0) & V(t)\mathcal{O}_{\pi\pi}(0) \\ \hline \mathcal{O}_{\pi\pi}(t)V(0) & \mathcal{O}_{\pi\pi}(t)\mathcal{O}_{\pi\pi}(0) \end{array}\right)$$

7 operators used on ensemble `48I`:

- $V(t) \rightarrow$ 3 ops: 1 **local** current and 2 smeared currents polarized along $k = 3$.
- $\mathcal{O}_{\pi\pi}(t) \rightarrow$ 4 ops: smeared and polarized two-pion operators with vanishing total momentum but different relative momenta $\{\mathbf{p}, -\mathbf{p}\}$ s.t. $|\mathbf{p}|^2 = \left(\frac{2\pi}{L}\right)^2 \{1,2,3,4\}$.

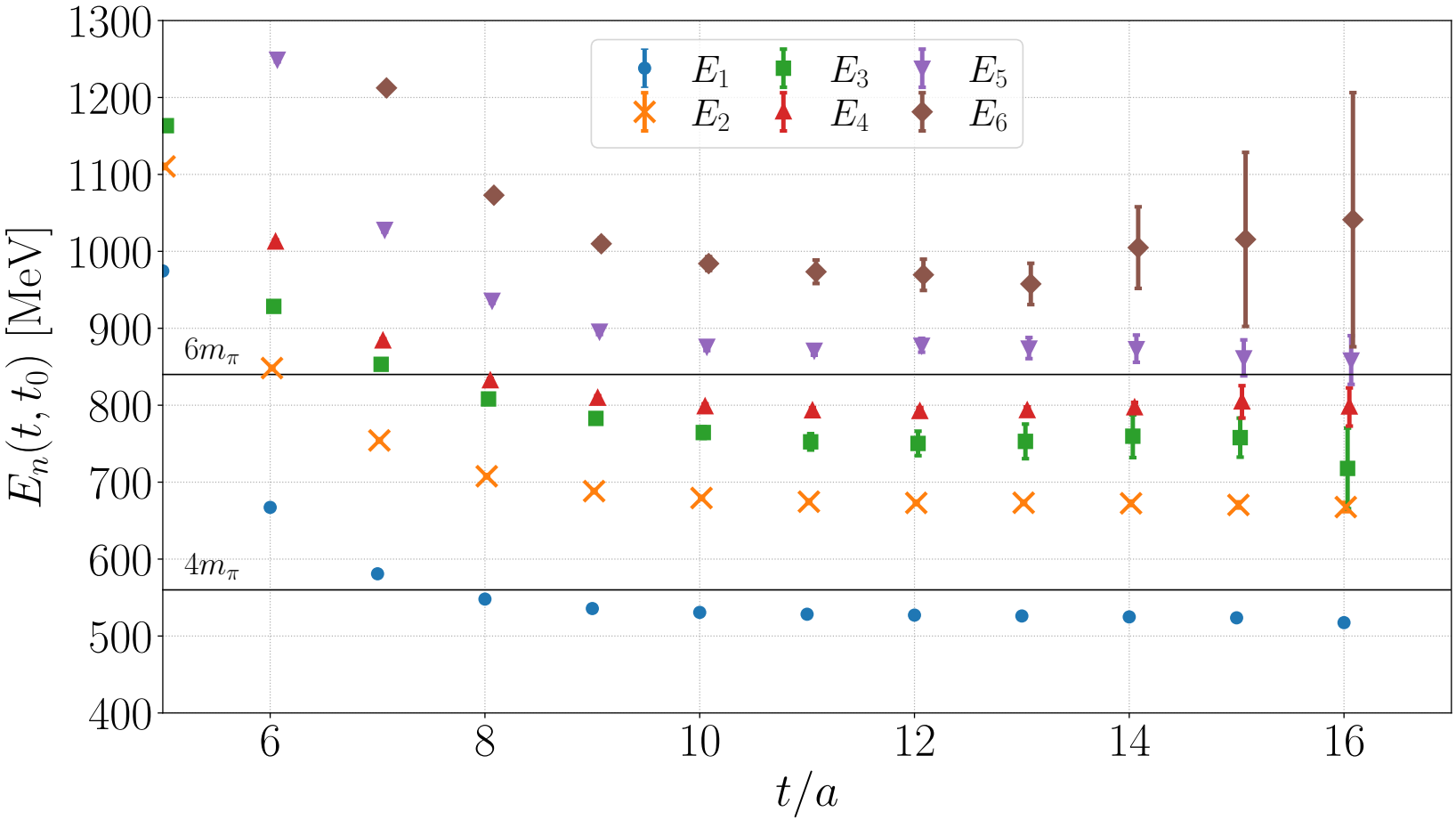
$$G_k^{\mathrm{2pt}}(t;\mathbf{p}) = \sum_n \langle 0|V_k(0)|n,\mathbf{0}\rangle \langle n,\mathbf{0}|\mathcal{O}_{\pi,1}(0)|\pi,\mathbf{p}\rangle\, e^{-E_n(\mathbf{0})t},$$

$\Longrightarrow$ **constrained fit** on $G_k^{\mathrm{2pt}}(t;\mathbf{p})$ to extract $\langle n,\mathbf{0}|\mathcal{O}_{\pi,1}(0)|\pi,\mathbf{p}\rangle$.

GEVP

SPECTRUM

Published in ref. [Bruno et al., 2019], repeated analysis here.



$$G_k^{2\mathrm{pt}}(t;\mathbf{p}) = \sum_n \langle 0|V_k(0)|n,\mathbf{0}\rangle \times \\ \times \langle n,\mathbf{0}|\mathcal{O}_{\pi,1}(0)|\pi,\mathbf{p}\rangle\, e^{-E_n(\mathbf{0})t}\,.$$

Spectral analysis of $G^{2\mathrm{pt}}$

calculating the form factor from LSZ

- $\langle 0|V_k(0)|n,\mathbf{0}\rangle, E_n$ from GEVP, $\langle n,\mathbf{0}|\mathcal{O}_{\pi,1}(0)|\pi,\mathbf{p}\rangle$ from fit.
- Construct the spectral density (from $G^{2\mathrm{pt}}$)

$$\rho_{k,\mathbf{p}}(\mathcal{E}\,|\,\sigma,\mathcal{O}_{\pi,1}) \equiv \sum_{n=1}^{N} \Delta(\mathcal{E}-E_n;\sigma)\,\langle 0|V_k(0)|n,\mathbf{0}\rangle\,\langle n,\mathbf{0}|\mathcal{O}_{\pi,1}(0)|\pi,\mathbf{p}\rangle\,.$$

- The kernel Δ from the LSZ reduction formula $\Rightarrow$ $\Delta(\mathcal{E};\sigma) = \frac{i}{\mathcal{E}+i\sigma}$

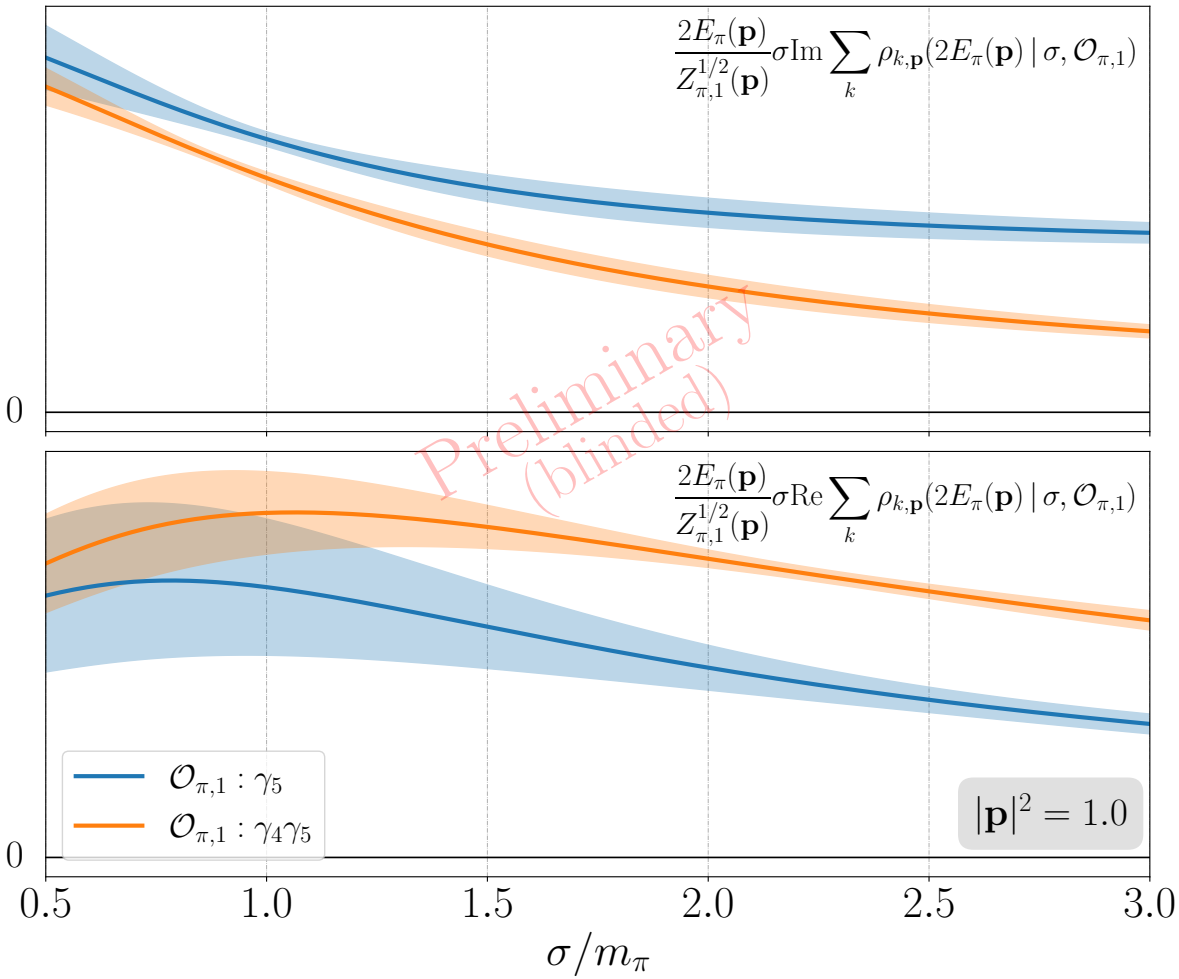
$$\lim_{\sigma\to 0^+}\lim_{L\to\infty} \frac{2E_\pi(\mathbf{p})}{Z_{\pi,1}^{1/2}(\mathbf{p})}\sigma\rho_{k,\mathbf{p}}(\mathcal{E}\,|\,\sigma,\mathcal{O}_{\pi,1})\big|_{\mathcal{E}=2E_\pi(\mathbf{p})} = 2\mathbf{p}_k F_\pi(4E_\pi^2(\mathbf{p}))\,,$$

following [Bulava and Hansen, 2019] ([Bruno and Hansen, 2021]).

- By looking at $\mathrm{Re}\,\Delta$ and $\mathrm{Im}\,\Delta \Longrightarrow$ extract $\mathrm{Re}\,F_\pi$ and $\mathrm{Im}\,F_\pi$.

Spectral reconstruction

A non-perturbative demonstration of LSZ



- Blinded results.
- LSZ residue is unique once $Z^{1/2}_{\pi,1}(\mathbf{p})$ is removed.

  $\Longrightarrow$ Strong check with different $\mathcal{O}_{\pi,1}$'s.
- Ongoing analysis for extrapolation to $\sigma \to 0$.

  Window problem: $m_\pi \gg \sigma \gg 1/L$.
  * $\sigma \gg 0$: good finite-size, bad LSZ.
  * $\sigma \simeq 0$: bad finite-size, good LSZ.

Conclusions

...and outlooks

- First preliminary tests of inverse problem approach to QCD amplitudes
  * infinite-volume method via LSZ reduction formula;
  * at physical pion mass, with difficult window problem.

- Encouraging results from consistency between different operators as $\sigma \to 0$.

- **Outlooks:**
  * Direct spectral reconstruction of $G^{2\mathrm{pt}}$ using inverse problem techniques (no GEVP), e.g. [Hansen, Lupo, and Tantalo, 2019] [Bruno, Giusti, and Saccardi, 2024].
  * Cross check with finite-volume formalism.
  * Ongoing computation of $G^{2\mathrm{pt}}$ at larger $|\mathbf{p}|^2$ values + increase stats.
  * Extend the calculation to other ensembles.

Thanks for your attention!

Backup slides

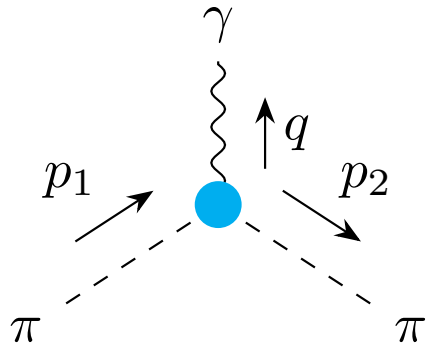
Space-like form factor $F_\pi(0)$

extraction from $C_0^{3\mathrm{pt}}(t,\tau;\mathbf{p})$

Space-like form factor:

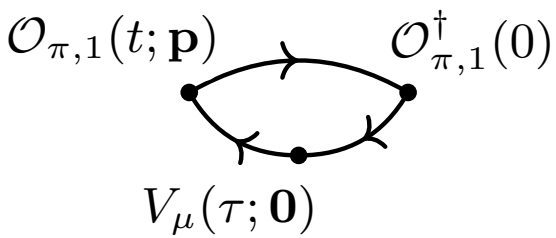
$$\langle \pi, \mathbf{p}_1 | V_\mu(q) | \pi, \mathbf{p}_2 \rangle = (p_1 + p_2)_\mu F_\pi(q^2)\,,$$

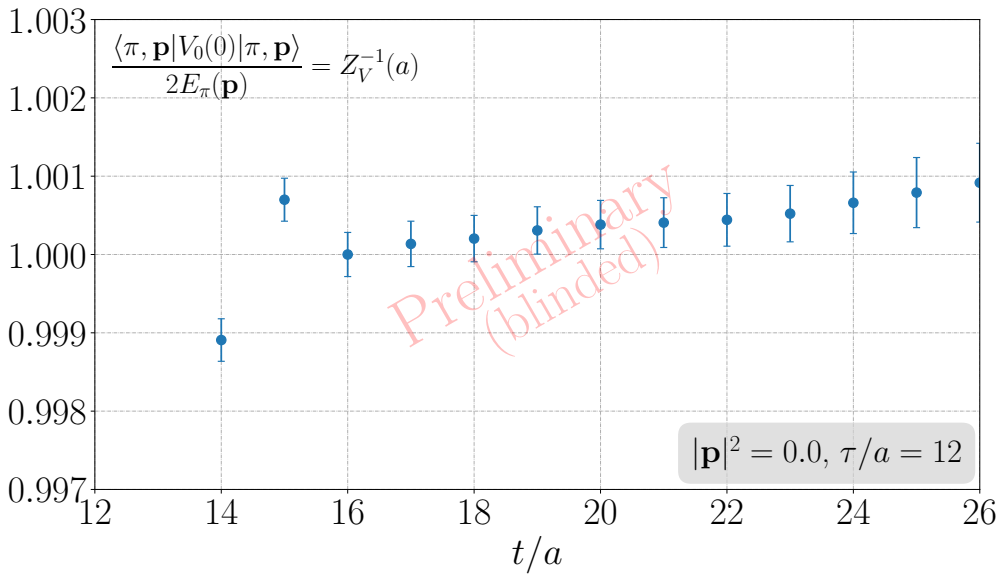
with $q = p_1 - p_2$, $V_\mu =$ isospin-1 local em current, which needs renormalization: $V_\mu^R(x) = Z_V(a) V_\mu(x)$.

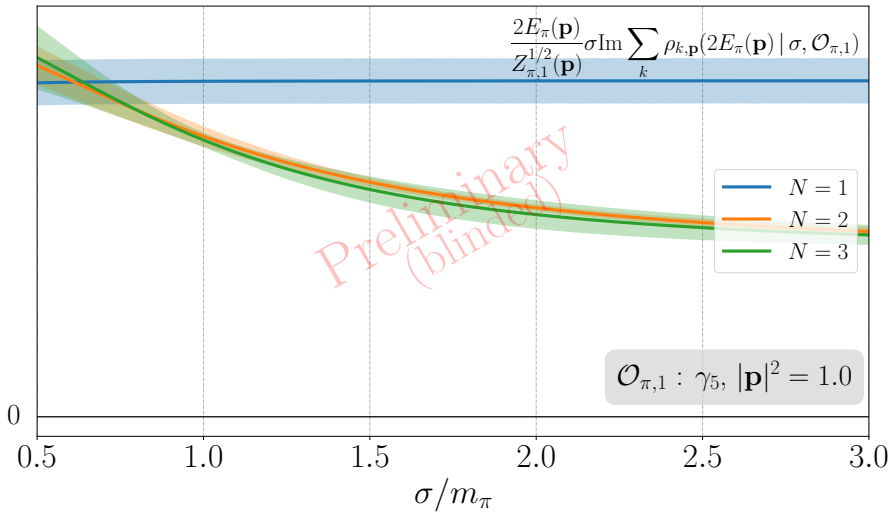
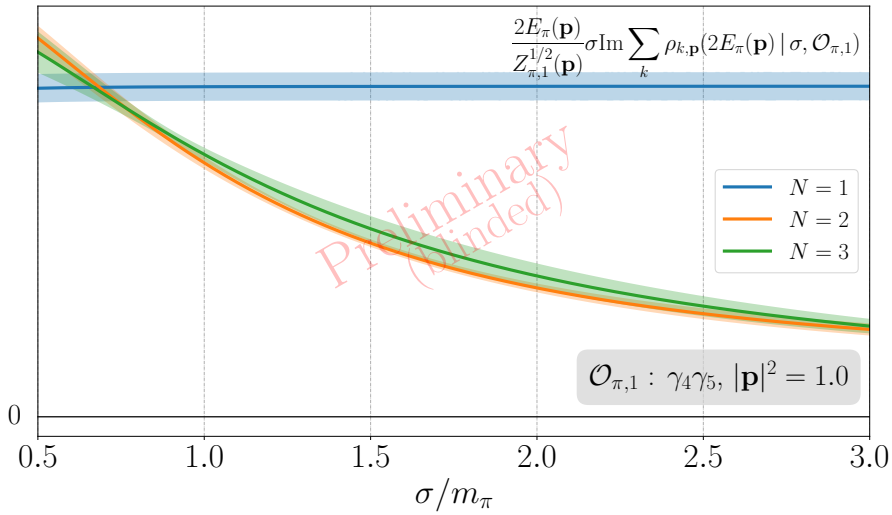
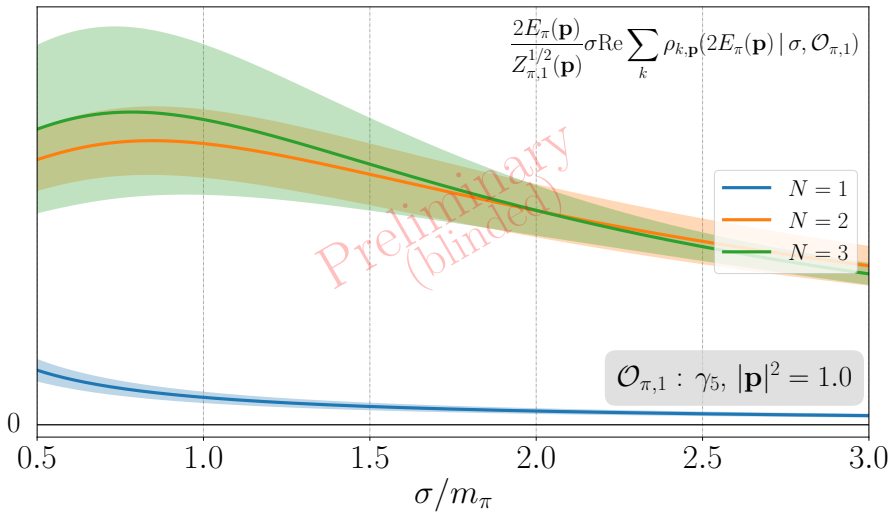
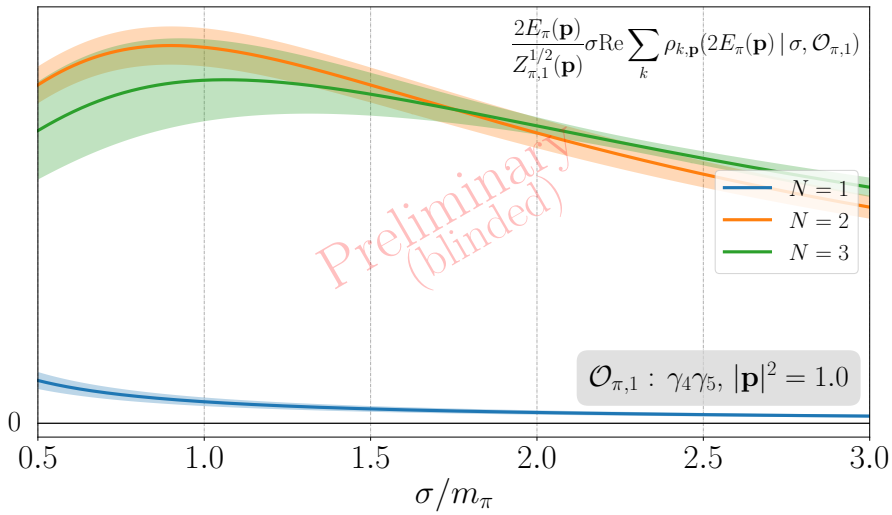


Extraction from $C^{3\mathrm{pt}}$ for $t > \tau$:

$$\left\langle \mathcal{O}_{\pi,1}(t;\mathbf{p}) V_\mu(\tau;\mathbf{0}) \mathcal{O}^\dagger_{\pi,1}(0) \right\rangle$$
$$\simeq |Z_{\pi,1}(\mathbf{p})|^2 \langle \pi, \mathbf{p} | V_\mu(0) | \pi, \mathbf{p} \rangle e^{-E_\pi(\mathbf{p}) t}\,.$$





Spectral reconstruction

stability w.r.t. N