Heavy Quark Masses from Step-Scaling

Simon Kuberski with A. Conigli, P. Fritzsch, A. Gérardin, J. Heitger, G. Herdoíza, C. Pena, H. Simma, R. Sommer

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ALPHA Collaboration

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Heavy quark masses

Motivation

- Higgs branching ratios to b and c quarks depend directly on m_{b} and m_{c}.
- Essential inputs to CKM phenomenology and global SM fits.

Challenges

- Large cutoff effects because $am_h = \mathrm{O}(1)$ on conventional lattices.
- Precise RG running remains demanding.

Step-Scaling Strategy

Quark mass determinations

- Conventionally, using Wilson fermions, we fix the heavy quark mass hopping parameter κ_h from a heavy hadron mass m_H via

$$am_H(\kappa_h) = am_H^{\mathrm{PDG}}$$

  and then compute the renormalized (RGI) quark mass.

→ E.g. from the bare subtracted quark mass $m_{\mathrm{q},h} = \frac{1}{2\kappa_h} - \frac{1}{2\kappa_{\mathrm{crit.}}}$ via

$$m_h^{\mathrm{RGI}} = \frac{M}{m_R(\mu_{\mathrm{had}})} Z_m(\mu_{\mathrm{had}}) \left[m_{\mathrm{q},h} + (r_m - 1)\frac{\mathrm{Tr}[M_{\mathrm{q}}]}{N_{\mathrm{f}}} + b_m\, am_{\mathrm{q},h}^2 + \mathrm{O}(a^2, a\mathrm{Tr}[M_{\mathrm{q}}]) \right]$$

- Leading cutoff effects are $\mathrm{O}(a^2 m_h^2)$ where $a_{\max} m_{\mathrm{c}} \approx 0.4$.

→ What about the bottom quark?

Quark mass determinations: alternative route

- Different route: Compute m_h^{RGI} in **small volumes** on **fine lattices**:
  $\rightarrow L_1 = 0.5\,\mathrm{fm}$ such that $a_{\max} m_{\mathrm{c}} < 0.1$ and $a_{\max} m_{\mathrm{b}} \approx 0.4$.

- In the continuum, make contact to infinite volume via

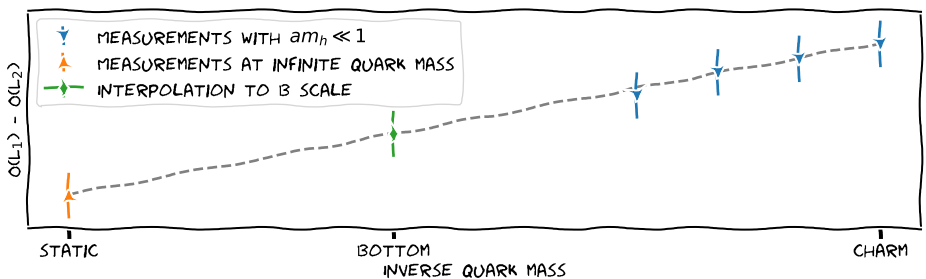
$$
\begin{aligned}
L_{\mathrm{ref}} m_h^{\mathrm{RGI}} &= \Big(L_{\mathrm{ref}} m_{\mathrm{H}}^{\mathrm{PDG}} - L_{\mathrm{ref}}[m_{\mathrm{H}} - m_{\mathrm{H}}(L_2)] - L_{\mathrm{ref}}[m_{\mathrm{H}}(L_2) - m_{\mathrm{H}}(L_1)] \Big) \frac{m_h^{\mathrm{RGI}}}{m_{\mathrm{H}}(L_1)} \\
&\equiv \frac{L_{\mathrm{ref}} m_{\mathrm{H}}^{\mathrm{PDG}} - \rho_m(L_2) - \sigma_m(L_1)}{\pi_m(L_1)} \,,
\end{aligned}
$$

  where again tuning of κ_h ensures $a m_H(\kappa_h) = a m_H^{\mathrm{PDG}}$.

- What did we gain?
  - Can simulate heavy (bottom) quarks with controlled cutoff effects in L_1.
  - Suppressed cutoff effects in step-scaling functions $\rho_m(L_2)$ and $\sigma_m(L_1)$.
  - Step-scaling functions can be reliably computed in **HQET on the lattice**.

Combining relativistic heavy quarks with HQET

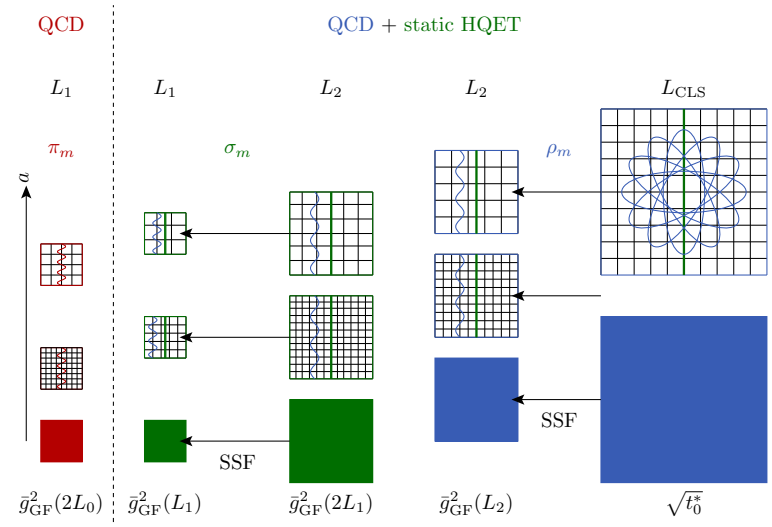
- We cannot evaluate step-scaling functions $\rho_m(L_2)$ and $\sigma_m(L_1)$ directly at m_b.
- Accessible in the static limit of non-perturbative Heavy Quark Effective Theory: Additive renormalization and matching with QCD cancels in the difference!
- Our plan: Interpolate between static and relativistic heavy quarks.



[Guazzini et al., 0710.2229][ALPHA, 2312.09811, 2312.10017] inspired by [de Divitiis et al., hep-lat/0305018]

Lattice Setup

Lattice setup: step-scaling



- Simulation of 32 ensembles in the massless Schrödinger functional (SF) [SK et al., 1811.02591]:
  - $L_2 = 2L_1 = 1.0\,\mathrm{fm}$
  - $L_1 = 2L_0 = 0.5\,\mathrm{fm}$
  - Renormalization in L_0 where $\bar{g}^2_{\mathrm{GF}}(L_0) = 3.949$
- Correlation functions with heavy and static valence quarks.
- Contact to the $2+1$ flavor CLS ensembles.

Lattice setup: Large volume

- Determine the light-quark mass dependence on CLS ensembles,

$$\rho_m(L_2) = L_{\mathrm{ref}}[m_{\mathrm{H}} - m_{\mathrm{H}}(L_2)] = L_{\mathrm{ref}}\left[m_{\mathrm{H}} - m_{\mathrm{H}}^{\mathrm{sym}}\right] + L_{\mathrm{ref}}\left[m_{\mathrm{H}}^{\mathrm{sym}} - m_{\mathrm{H}}(L_2)\right]$$
$$\equiv \tau_m(m_\pi, m_K) + \rho_m^{\mathrm{sym}}(L_2)$$

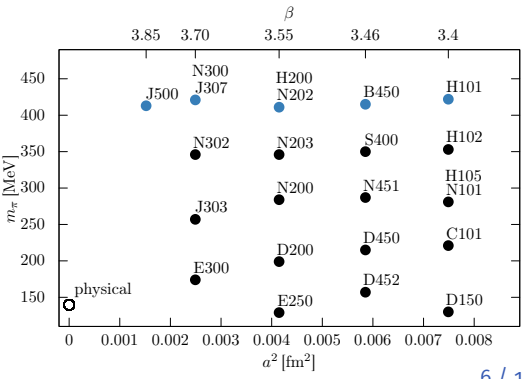
where $m_{\mathrm{H}}^{\mathrm{sym}}$ is the heavy hadron mass at the SU(3) flavor-symmetric point.

- Chiral trajectory with $2m_{\mathrm{l}} + m_{\mathrm{s}} = \mathrm{const.}$
  $\rightarrow$ set the heavy hadron mass through flavor averaged mesons:

$$m_H \in \left\{\frac{2}{3}m_D + \frac{1}{3}m_{D_{\mathrm{s}}}, \quad \frac{2}{3}m_B + \frac{1}{3}m_{B_{\mathrm{s}}}\right\}$$

- Reach physical point via

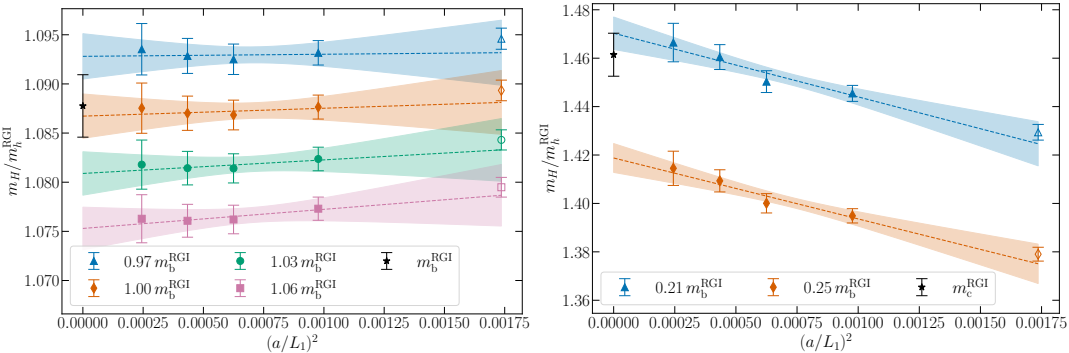
$$\lim_{m_\pi \to m_\pi^{\mathrm{phys}}} \tau_m(m_\pi, m_K)$$



The calculation

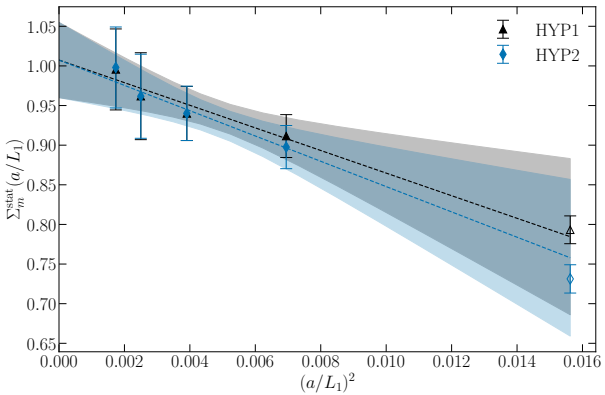
Quark mass determination in small volume

$0.0078\,\mathrm{fm} \leq a \leq 0.021\,\mathrm{fm}$



- Renormalize the quark mass at $\mu = 1/L_0 \approx 790\,\mathrm{MeV}$.
- Continuum extrapolation of $\frac{m_{\mathrm{H}}(L_1)}{m_h^{\mathrm{RGI}}}$ at the bottom (*left*) and charm (*right*) scales.
- Exploration of systematics: • Cuts in (a/L) and am_h^{RGI}.
  • Explore also fits with $\alpha_s(1/a)^\Gamma a^2$ and a^3 effects.

Determination of $\sigma_m(L_1) = \lim_{a\to 0} \Sigma_{\mathrm{m}}(L_1, a/L_1) = L_{\mathrm{ref}}[m_{\mathrm{H}}(L_2) - m_{\mathrm{H}}(L_1)]$



- Combined continuum extrapolation with two actions for the static quark.

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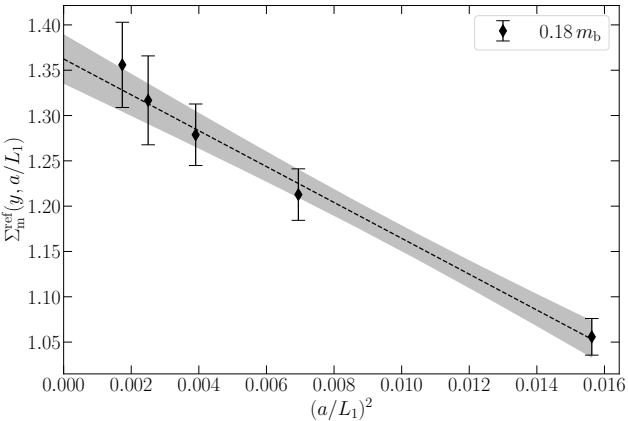

- Combined continuum extrapolation with two actions for the static quark.
- For relativistic heavy quark masses:
  - Extrapolate $\Sigma_\mathrm{m}^\mathrm{ref}$ directly below the charm scale.

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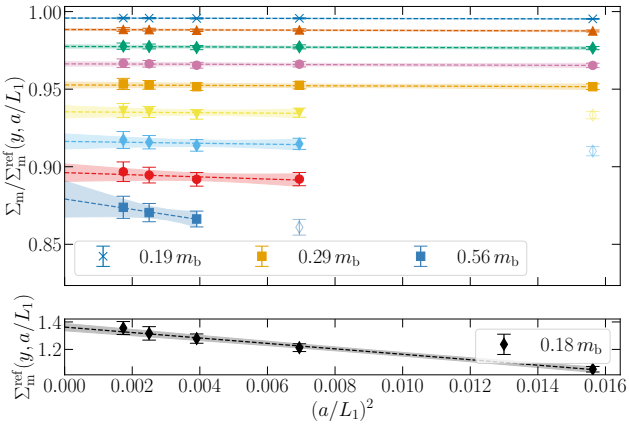

- Combined continuum extrapolation with two actions for the static quark.
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  - For larger masses: Extrapolate ratio of heavy step-scaling function to $\Sigma_\mathrm{m}^\mathrm{ref}$.

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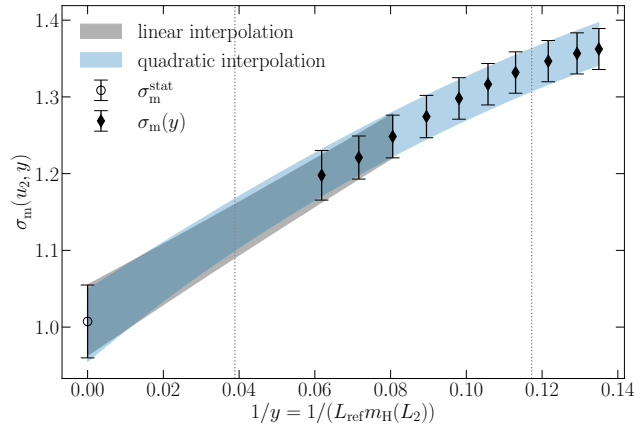

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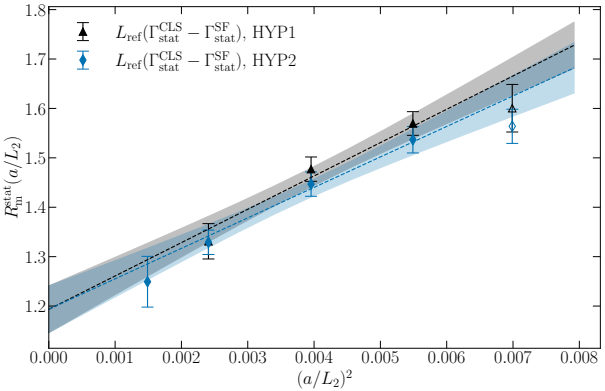
  - For larger masses: Extrapolate ratio of heavy step-scaling function to $\Sigma_\mathrm{m}^\mathrm{ref}$.
- Interpolate to the bottom and charm scales and check for systematics.

Determination of $\rho_m^{\mathrm{sym}}(L_2) = \lim_{a\to 0} R_{\mathrm{m}}(L_2, a/L_2) = L_{\mathrm{ref}}\left[m_{\mathrm{H}}^{\mathrm{SU}(3)} - m_{\mathrm{H}}(L_2)\right]$



- Combined continuum extrapolation in the static limit

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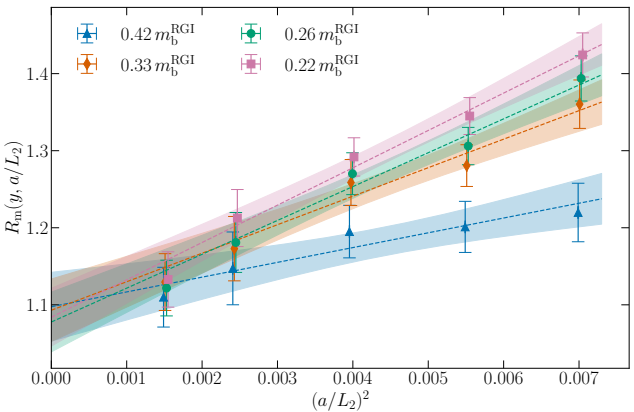

- Combined continuum extrapolation in the static limit
- Four relativistic heavy quark masses $m_h \geq m_c$.

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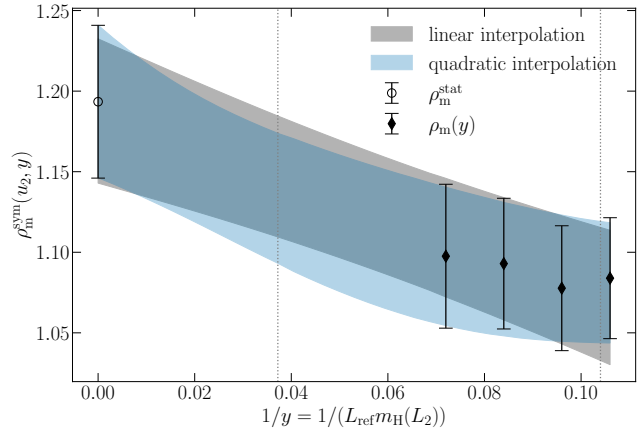

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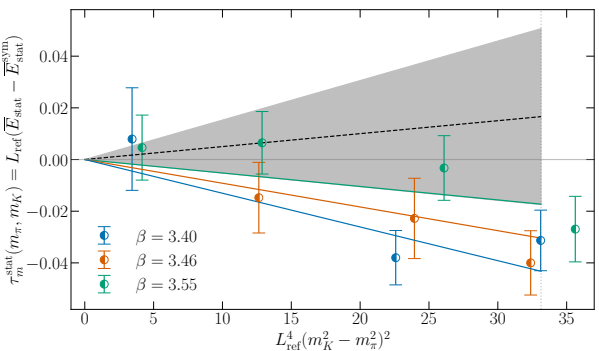
- Interpolation to the bottom and charm scales.

Quark mass dependence in large volume

Determination of $\tau_m(m_\pi, m_K) = L_{\mathrm{ref}}[m_{\mathrm{H}}(m_\pi, m_K) - m_{\mathrm{H}}^{\mathrm{sym}}]$



- **Flavor averaged** meson masses: Splitting scales with $(m_{\mathrm{s}} - m_{\mathrm{l}})^2$.
- Chiral-continuum extrapolation in the static limit [ALPHA, 2111.04693] including $m_\pi < m_\pi^{\mathrm{phys}}$…

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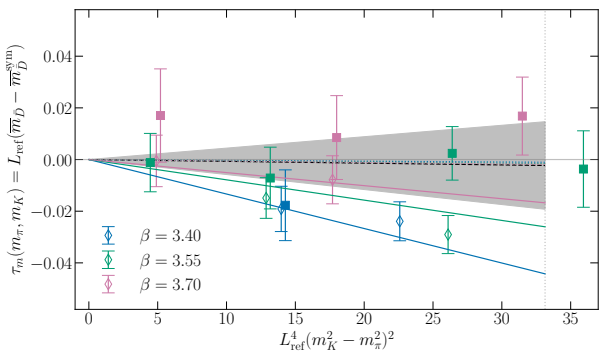

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- …and at the charm scale using two valence actions
  - Wilson on CLS (*open*) [ALPHA, 2101.02694]
  - Twisted-mass on CLS (*filled*) [ALPHA, 2309.14154]

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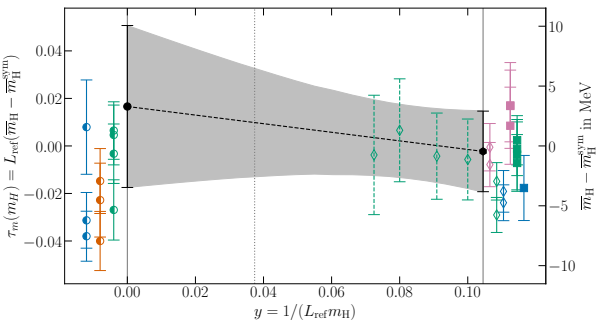

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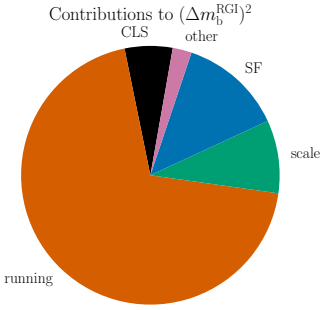
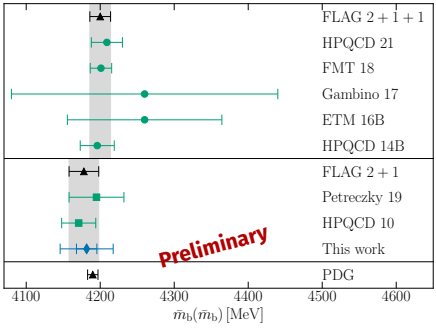
→ Heavy quark mass dependence at the physical point.

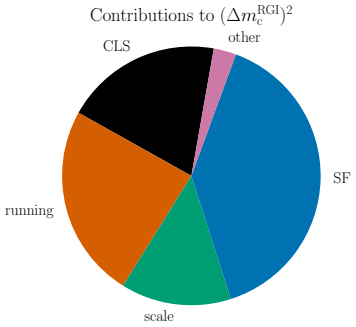
Results: Bottom quark mass

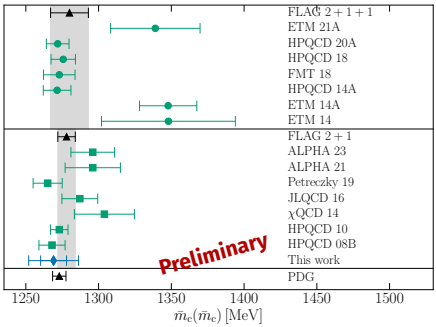




- Uncertainty of $m_{\rm b}^{\rm RGI}$ limited by $\frac{M}{m_R(1/L_0)}$ [ALPHA, 1802.05243]: will be improved.
- Systematics (continuum extrapolations, quark mass interpolations) subleading: Renormalisation is non-perturbative, uncertainties are **statistics dominated**.
- Entirely different method compared to existing work.

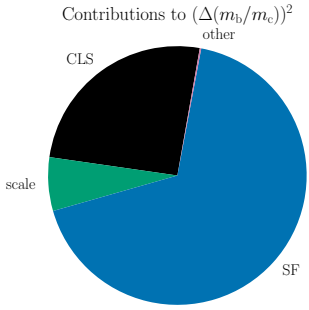
Results: Charm quark mass

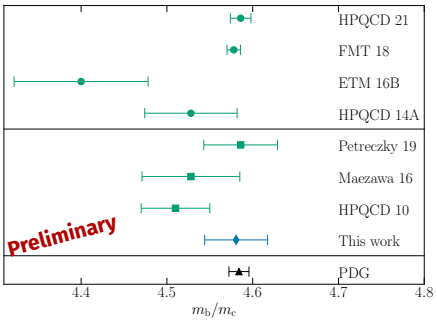




- Step-scaling not as efficient for the charm quark mass:
  SF contributes significantly to the uncertainty.
- Cross-check of the step-scaling chain.
- Result is only weakly correlated with existing CLS-based determinations.

Results: Quark mass ratio





- Contribution from the RG running cancels in ratio $m_b^{\mathrm{RGI}}/m_c^{\mathrm{RGI}} = \bar{m}_b(\mu)/\bar{m}_c(\mu)$.
- Charm and bottom quark masses are correlated but residual uncorrelated uncertainties lead to a non-negligible total uncertainty in their ratio.

Outlook

Step-scaling enables a precise and systematically controlled determination of the bottom quark mass.

- Last bits of data for $\rho_m^{\mathrm{sym}}(L_2)$ are being computed — results will soon be final.
- Next step: Form-factors of semi-leptonic B-meson decays and $f_{B_{(\mathrm{s})}}$:
  - Strategy is clear, step-scaling functions can be computed.
    [ALPHA, 2312.09811, 2312.10017]
  - Computation of light quark mass and momentum dependence ongoing.
    [A. D'Anna, Thu 3:30].
  - Ground-state form factors are challenging.