RG running and mixing of FFO from step-scaling matrices

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In collaboration with

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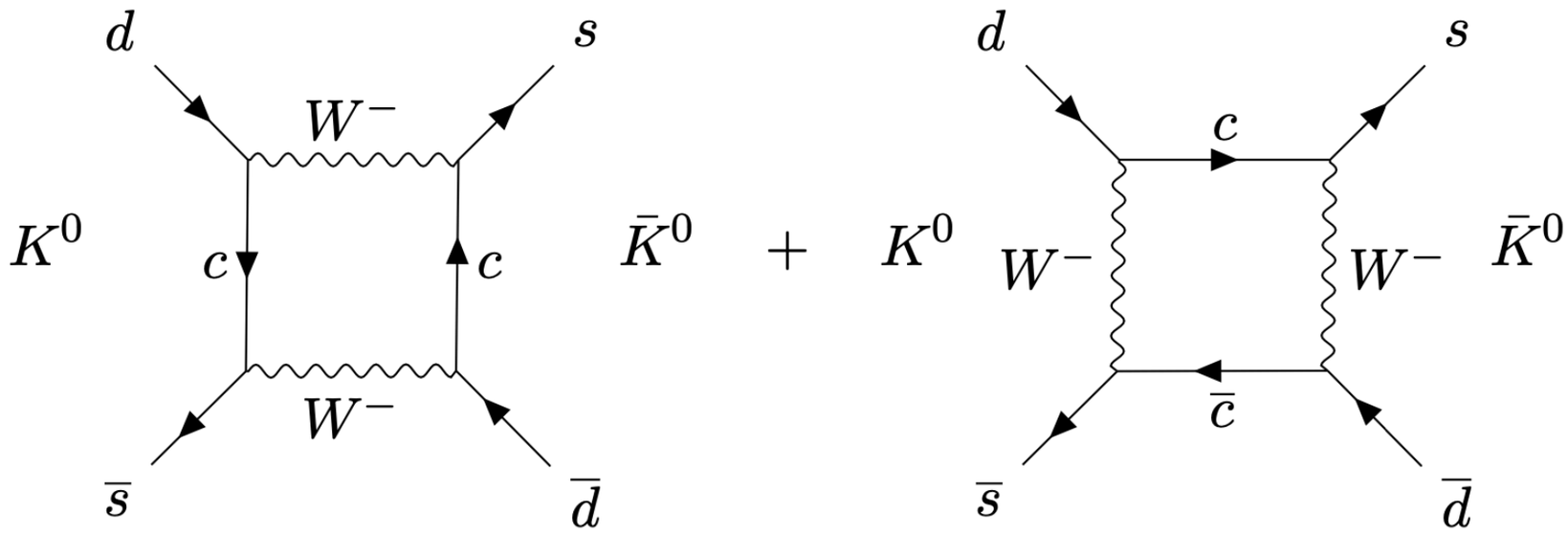
Motivations

- Future goal: accurate evaluation of the CP-violating angle δ of the CKM matrix

- $K^0 - \bar{K}^0$ oscillations in the SM are sensitive to loop effects, and so to BSM contributions



- Indirect investigation of CP violation: ε parameter

$$\varepsilon^{\text{theor}}(\delta) = \frac{A(K_L \to (\pi\pi)_{I=0})}{A(K_S \to (\pi\pi)_{I=0})} \propto \tilde{U}(\mu)\langle \bar{K}^0 | \mathbf{Q}(\mu) | K^0 \rangle F(\delta)$$

To be evaluated non-perturbatively

- Comparing $\varepsilon^{\text{theor}}$ with its experimental estimate we obtain

in the SM:	beyond the SM:
1. new estimate of δ	1. δ kept to the current estimate
2. NP uncertainties	2. bounds to BSM contributions

$K^0 - \bar{K}^0$ oscillations

- Effective Hamiltonian for K oscillations:

SM: $H_{\text{eff}}^{\Delta S=2} = \tilde{U}_1 \mathbf{Q}_1$	**BSM:** $H_{\text{eff}}^{\Delta S=2} = \sum_{i=1}^{5} \tilde{U}_i \mathbf{Q}_i + \sum_{i=1}^{3} \tilde{U}'_i \tilde{\mathbf{Q}}_i$
Only one relevant operator	**An operator basis $\mathbf{Q}_i$**

- Transition amplitudes are calculated with $\langle \bar{K}^0 | H_{\text{eff}}^{\Delta S=2} | K^0 \rangle$

- The renormalisation introduces an energy-scale in the matrix elements and in the Wilson coefficients:

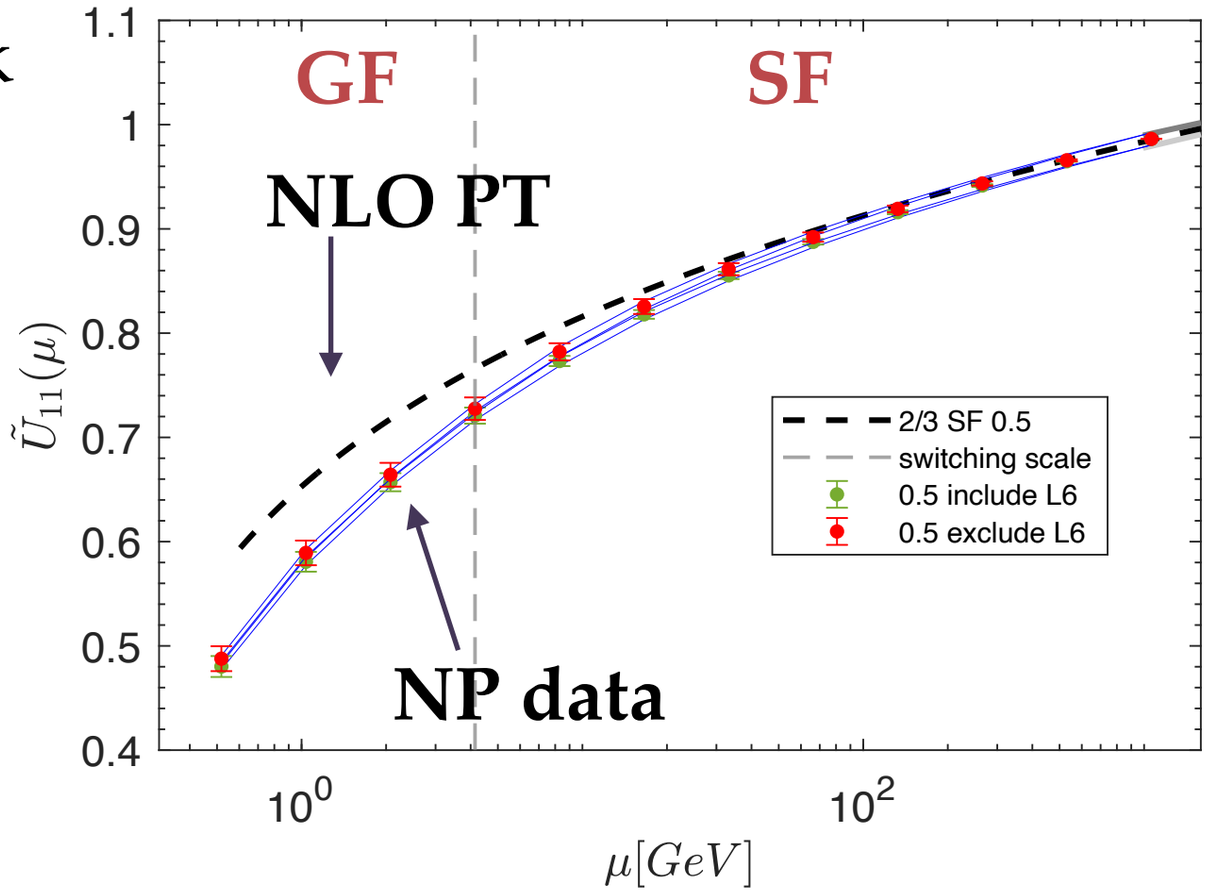
$$\langle \bar{K}^0 | H_{\text{eff}}^{\Delta S=2} | K^0 \rangle_{\text{cons.}}^{\text{parity}} = \tilde{U}_i(\mu) \langle \bar{K}^0 | \mathbf{Q}_i(\mu) | K^0 \rangle$$

Features

- Running evaluation with 3 quark flavours in the sea down to $\sim 4GeV$ with **SF** coupling;

- Running evaluation down to $\sim 500MeV$ with **Gradient Flow** (GF) coupling;

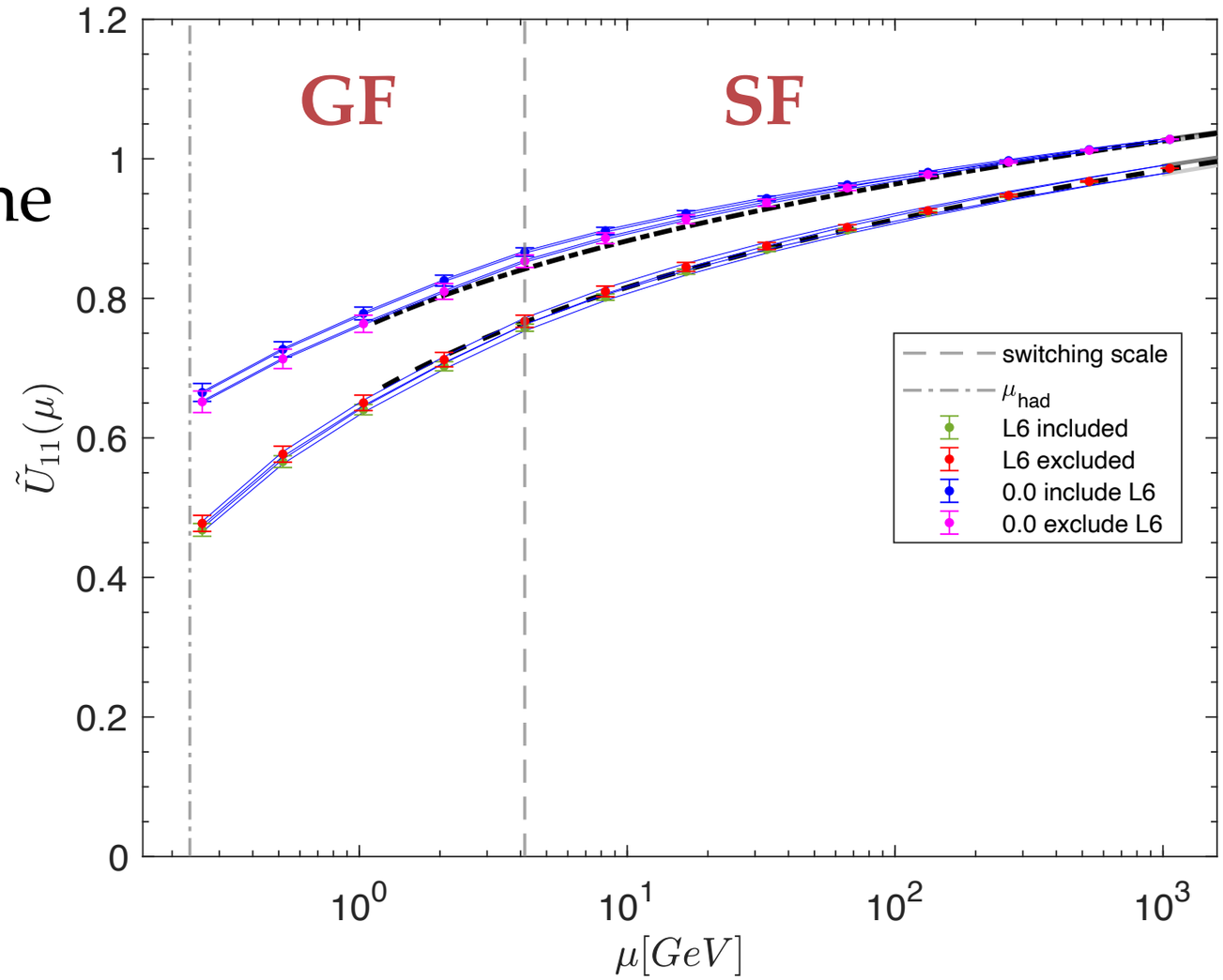
- Updated theoretical formulation of the operator running and mixing in the perturbative regime for $N_f = 3$.



Features

- Results obtained for two different normalisations of the correlation functions

- Difference often observed between PT and non-PT results at 3GeV (the scale at which matrix elements in FLAG are renormalised), could be relevant in the estimate of quantities like $\varepsilon(\delta)$



[FLAG2024]

$$B_i \propto \langle \mathbf{Q}_i(\mu = 3\text{GeV}) \rangle$$

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	B_2	B_3	B_4	B_5
ETM 15	[52]	2+1+1	A	★	○	○	★	a	0.46(1)(3)	0.79(2)(5)	0.78(2)(4)	0.49(3)(3)
RBC/UKQCD 24	[56]	2+1	A	★	★	★	★	b	0.4794(25)(35)	0.746(13)(17)	0.897(02)(10)	0.6882(78)(94)
RBC/UKQCD 16	[58]	2+1	A	○	○	○	★	b	0.488(7)(17)	0.743(14)(65)	0.920(12)(16)	0.707(8)(44)
SWME 15A	[55]	2+1	A	★	○	★	○†	–	0.525(1)(23)	0.773(6)(35)	0.981(3)(62)	0.751(7)(68)
SWME 14C	[451]	2+1	C	★	○	★	○†	–	0.525(1)(23)	0.774(6)(64)	0.981(3)(61)	0.748(9)(79)
SWME 13A‡	[436]	2+1	A	★	○	★	○†	–	0.549(3)(28)	0.790(30)	1.033(6)(46)	0.855(6)(43)
RBC/ UKQCD 12E	[445]	2+1	A	■	○	★	★	b	0.43(1)(5)	0.75(2)(9)	0.69(1)(7)	0.47(1)(6)
ETM 12D	[57]	2	A	★	○	○	★	c	0.47(2)(1)	0.78(4)(2)	0.76(2)(2)	0.58(2)(2)

- Inconsistencies between different estimates
- Some results refer to perturbative renormalisation

The χSF

- In the continuum we map the SF into the χSF with a chiral rotation:

$$\psi' = R\left(\frac{\pi}{2}\right)\psi\,, \quad \bar{\psi}' = \bar{\psi}R\left(\frac{\pi}{2}\right)\,, \quad R(\alpha) = e^{\frac{i}{2}\alpha\gamma_5\tau^3}$$

- Correspondence between correlation functions in the SF and χSF:

$$\langle O[\psi,\bar{\psi}]\rangle^{\text{cont}}_{\text{SF}} = \left\langle O\left[R\left(\pi/2\right)\psi, \bar{\psi}R\left(\pi/2\right)\right]\right\rangle^{\text{cont}}_{\chi\text{SF}}$$

- The boundary rotation removes $\mathcal{O}(a)$ effects in the observables!

$$\left\langle O_{\text{even}}\right\rangle_{\text{c}} = \left\langle O_{\text{even}}\right\rangle^{\text{cont}}_{\text{c}} + \mathcal{O}(a^2)$$

RG evolution

- Evolution matrices between two scales: $\overline{\mathcal{Q}}_i(\mu_2) = U_{ij}(\mu_2, \mu_1)\overline{\mathcal{Q}}_j(\mu_1)$
- Evolution matrices down to a scale $\hat{U}(\mu)$: $\mathbf{U}(\mu_2, \mu_1) =: \left[\tilde{\mathbf{U}}(\mu_2)\right]^{-1} \tilde{\mathbf{U}}(\mu_1)$
- **Problem**: for $N_f = 3$ (and 30) two eigenvalues of $\boldsymbol{\gamma}_0/\beta_0$ accidentally satisfy the **resonance condition** $\lambda_i - \lambda_j = 2$, making it impossible to adopt the usual definition

$$\tilde{\mathbf{U}}(\mu) \equiv \left[\frac{\bar{g}(\mu)}{4\pi}\right]^{-\gamma_0/(2\beta_0)} \mathbf{W}(\mu)$$

- Problem solved in the past for the two-scale operator $U(\mu_2, \mu_1)$, but a single-scale evolution operator is needed to evaluate a Wilson coefficient

Wilson coefficients from the Poincaré-Dulac theorem

- The matrix $\mathbf{A}(g) = \boldsymbol{\gamma}(g)/\beta(g)$ can be set [2013.16220v3] in its **canonical form**

$$\mathbf{A}^{\text{can}}(g) = \frac{1}{g}\left(\boldsymbol{\Lambda} + g^2 \mathbf{N}_2\right)$$

upper-diagonal

diagonal

through a change of operator basis $\quad \mathbf{S}(g) \simeq \left(\mathbb{1} + \sum_{k=1}^{n} \mathbf{H}_{2k} g^{2k}\right) \mathbf{S}_{\text{D}} \equiv \mathbf{s}_n(g) \mathbf{S}_{\text{D}}$

- The evolution operator can be evaluated in the **canonical basis** and then rotated back to the original operator basis:

$$\hat{\mathbf{U}}(u) = \mathbf{S}_{\text{D}}^{-1} \exp\left(-\frac{1}{2}\boldsymbol{\Lambda} \ln(u)\right) \exp\left(-\frac{1}{2}\mathbf{N}_2 \ln(u)\right) \mathbf{s}_n(u) \mathbf{S}_{\text{D}}$$

Perturbative part solved!

Step-scaling functions

- Non-perturbative evolution from the step-scaling functions (SSF):

$$\boldsymbol{\sigma}(u) := \mathbf{U}(\mu/2,\mu)\Big|_{\bar{g}^2(\mu)=u} \longrightarrow \mathbf{U}(u_1,u_N) = \boldsymbol{\sigma}(u_1)\ldots\boldsymbol{\sigma}(u_N)$$

- Discrete step-scaling functions: $\boldsymbol{\Sigma}\left(g_0^2,\frac{a}{L}\right) := \boldsymbol{\mathcal{Z}}\left(g_0^2,\frac{a}{2L}\right)\left[\boldsymbol{\mathcal{Z}}\left(g_0^2,\frac{a}{L}\right)\right]^{-1}$

- $\mathcal{O}(g^2)$ lattice artefacts in the SF energy region removed using subtracted SSF [2112.10606] :

$$\tilde{\boldsymbol{\Sigma}}\left(u,\frac{a}{L}\right) := \boldsymbol{\Sigma}\left(u,\frac{a}{L}\right)[\mathbf{1}+u\log(2)\boldsymbol{\delta}_k(a/L)\boldsymbol{\gamma}_0]^{-1}$$

SSF continuum extrapolations

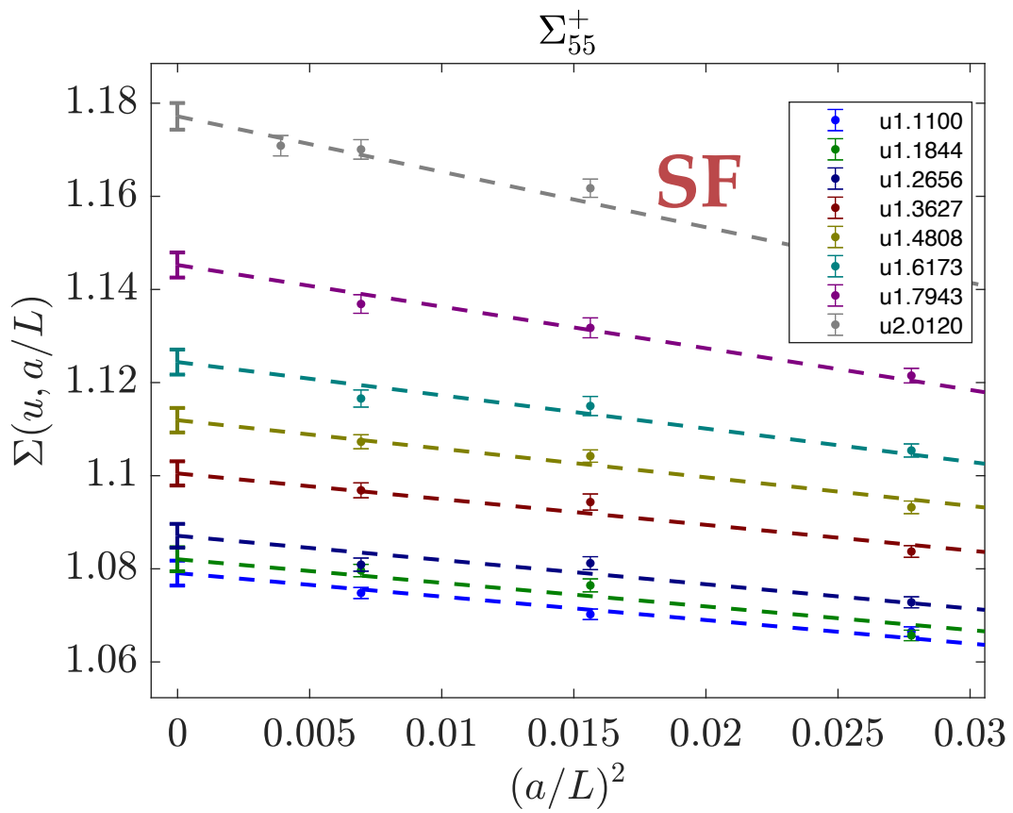
- Global fits are performed with the ansatz

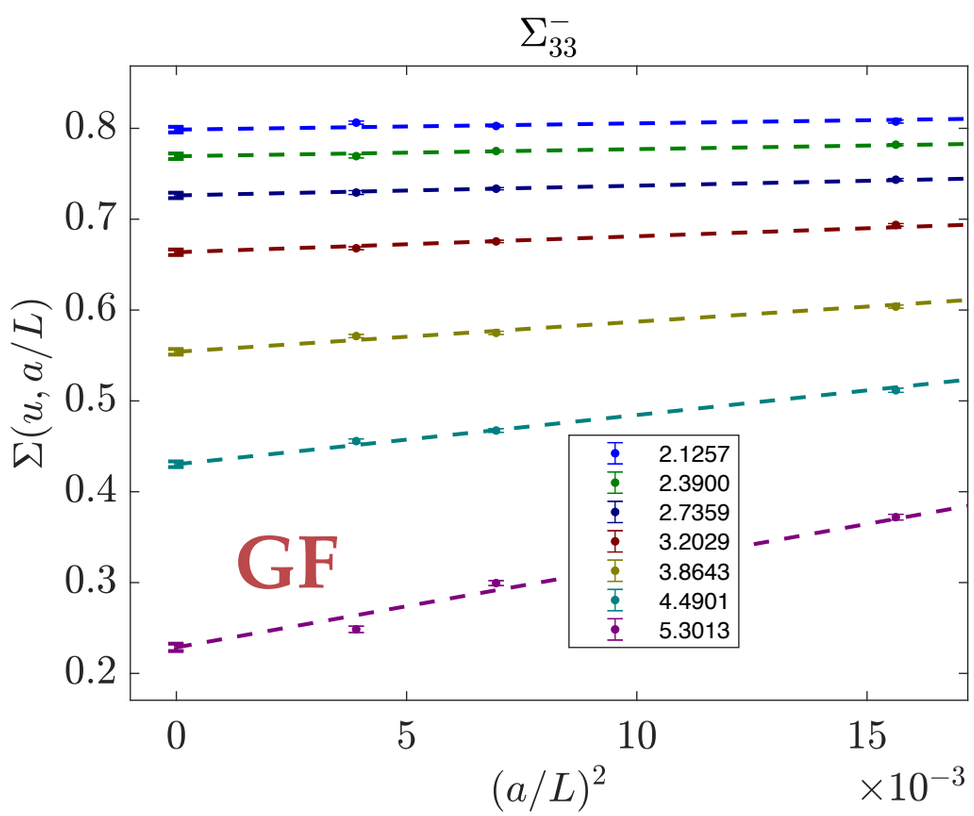
$$\left[\boldsymbol{\Sigma}\left(u_n, a/L\right)\right]_{ij} = \left[\boldsymbol{\sigma}(u_n)\right]_{ij} + \left(\frac{a}{L}\right)^2 \sum_{m=0}^{2} \left[\boldsymbol{\rho}_m\right]_{ij} u_n^m$$

Continuum SSF

- Parameters found by χ^2 minimisation;
- $L/a = 6, 8, 12$ for SF couplings, $L/a = 8, 12, 16$ for GF couplings.

SSF continuum extrapolations





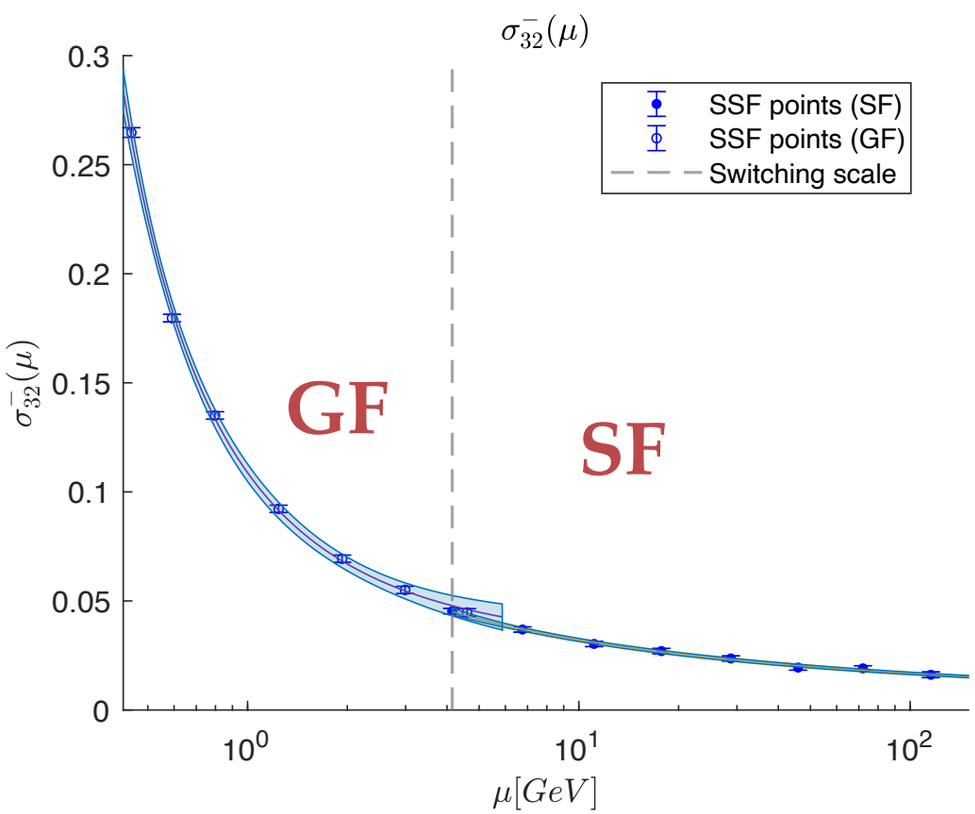
- Fits performed *including or excluding* $L/a = 6$ in the SF region (we will see it later more in detail don't worry)

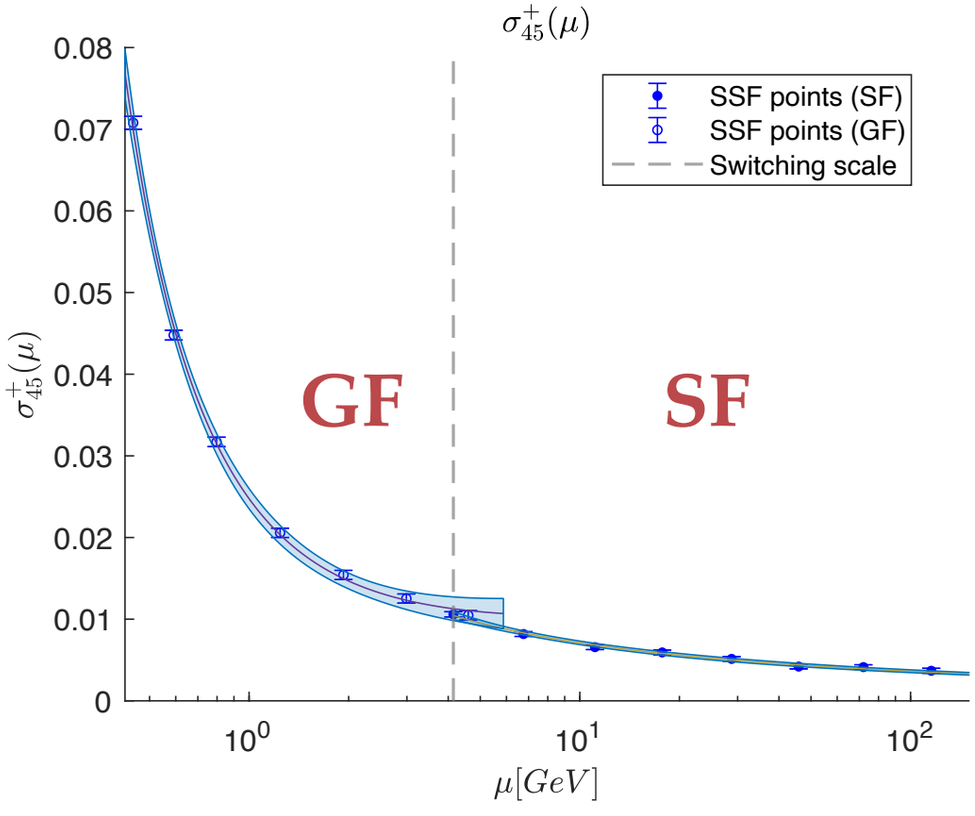
- Full analysis carried on for both choices

SSF continuum fit

- Continuum fit (SF region): $\boldsymbol{\sigma}(u)_{\mathrm{SF}} = \mathbf{1} + \mathbf{r}_1 u + \mathbf{r}_2 u^2 + \mathbf{r}_3 u^3 + \mathbf{r}_4 u^4$

- Continuum fit in the GF region: $\boldsymbol{\sigma}(u)_{\mathrm{GF}} = \mathbf{R}_0 + \mathbf{R}_1 u + \mathbf{R}_2 u^2$





Evolution matrices from SSF

- We now dispose of a matrix function of the scale: $\boldsymbol{\sigma}(\mu)$

- N squared-couplings generated through the coupling SSF:

$$u_n = \sigma^{-1}_{\text{coupling}}(u_{n-1})$$

- Evolution operator between u_1 and u_N obtained from the SSF:

$$\mathbf{U}(u_1, u_N) = \boldsymbol{\sigma}(u_1)\ldots\boldsymbol{\sigma}(u_N)$$

NP part solved!

Final running formula

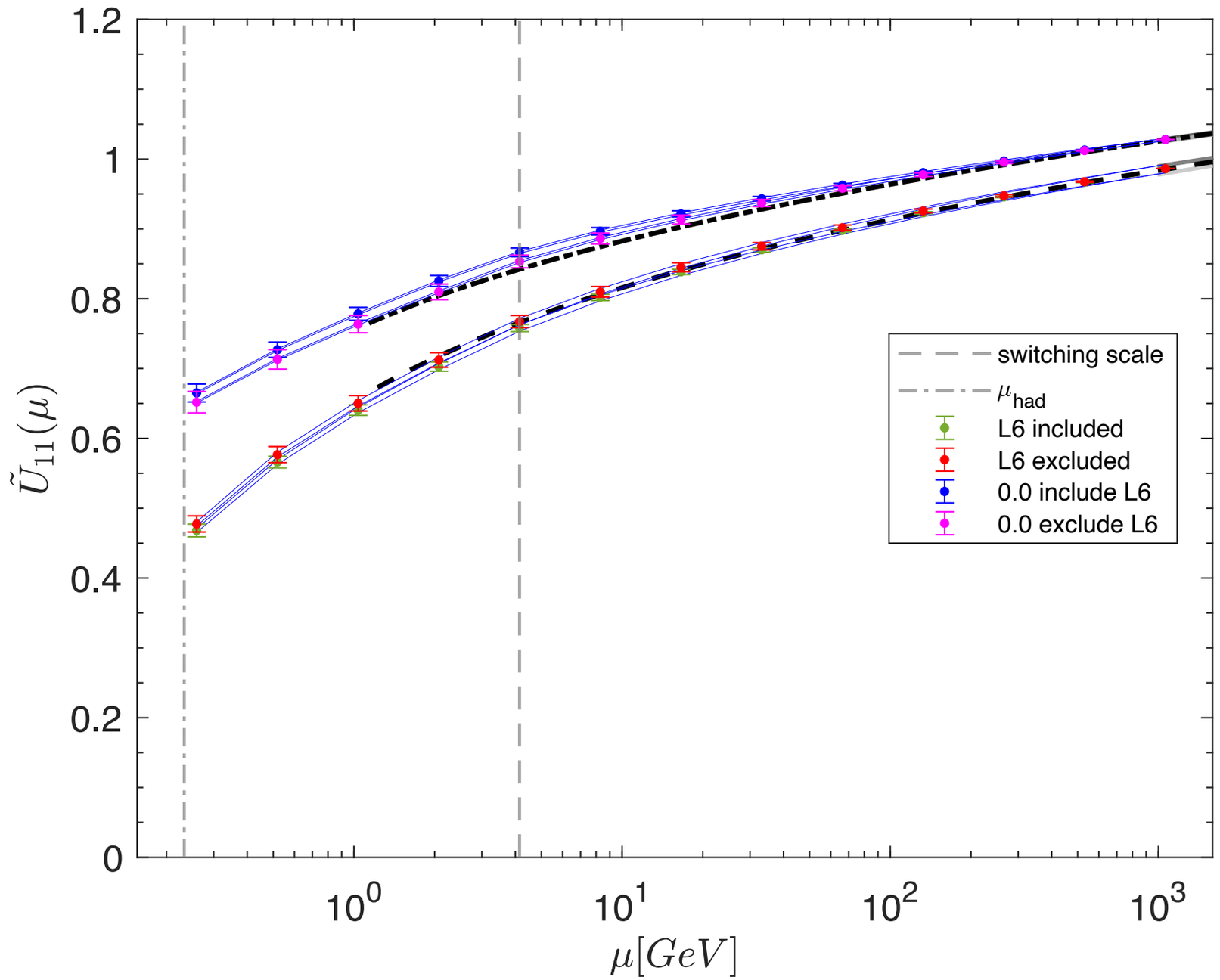
PT and NP results are combined to obtain the Wilson coefficients at NP couplings:

$$\hat{\mathbf{U}}(u) = \mathbf{S}_{\mathrm{D}}^{-1} \exp\left(-\frac{\boldsymbol{\Lambda}}{2} \ln u_{\mathrm{pt}}\right) \exp\left(-\frac{\mathbf{N}_2}{2} \ln u_{pt}\right) \mathbf{s}_n(g)\, \mathbf{S}_{\mathrm{D}}[\mathbf{U}(u, u_{\mathrm{pt}})]^{-1}$$

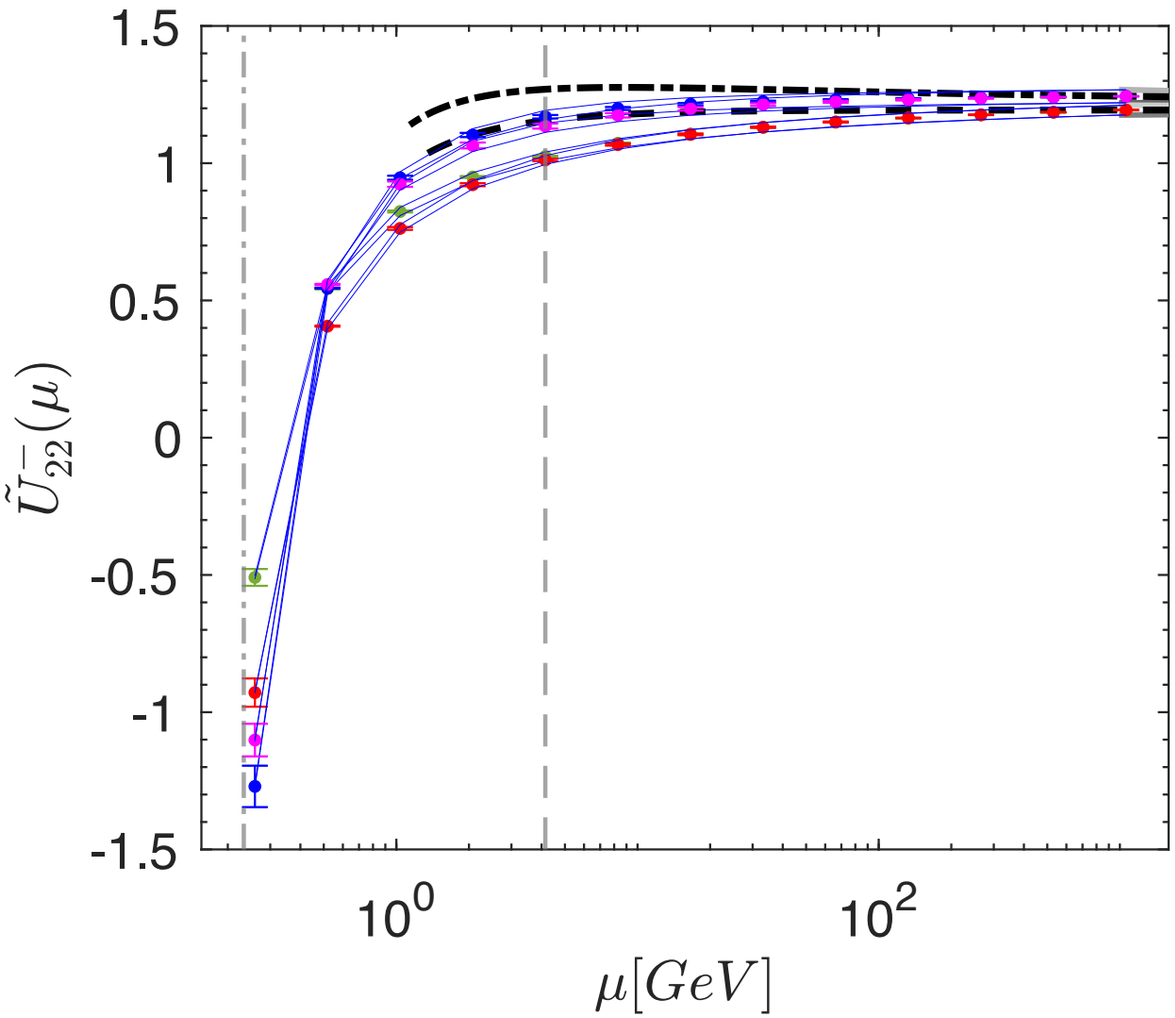
Wilson coefficient evaluated in PT at the matching scale

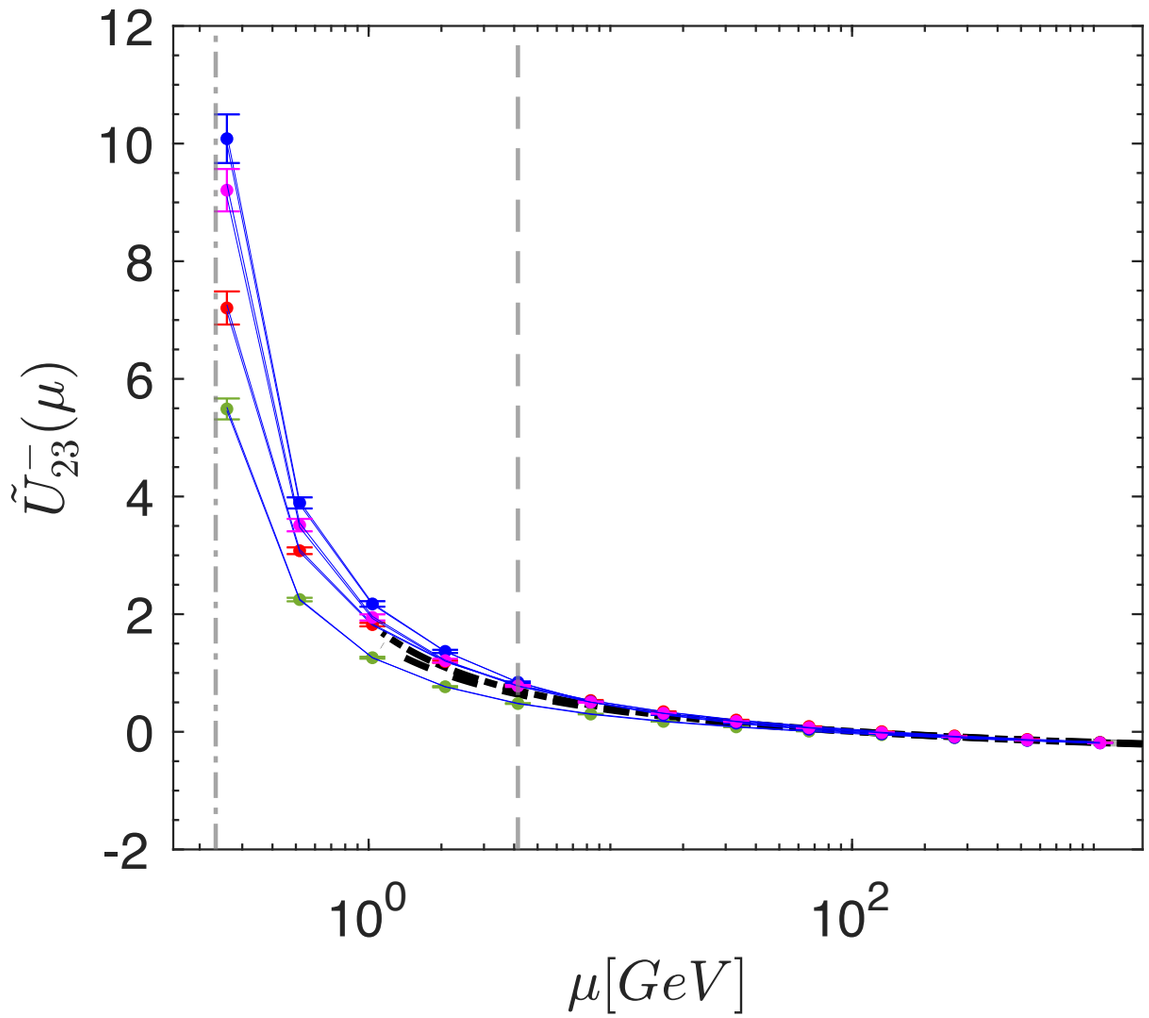
NP running evaluated from SSFs

NP Running: 1|1 SM operator



NP Running: 2|3 BSM operators





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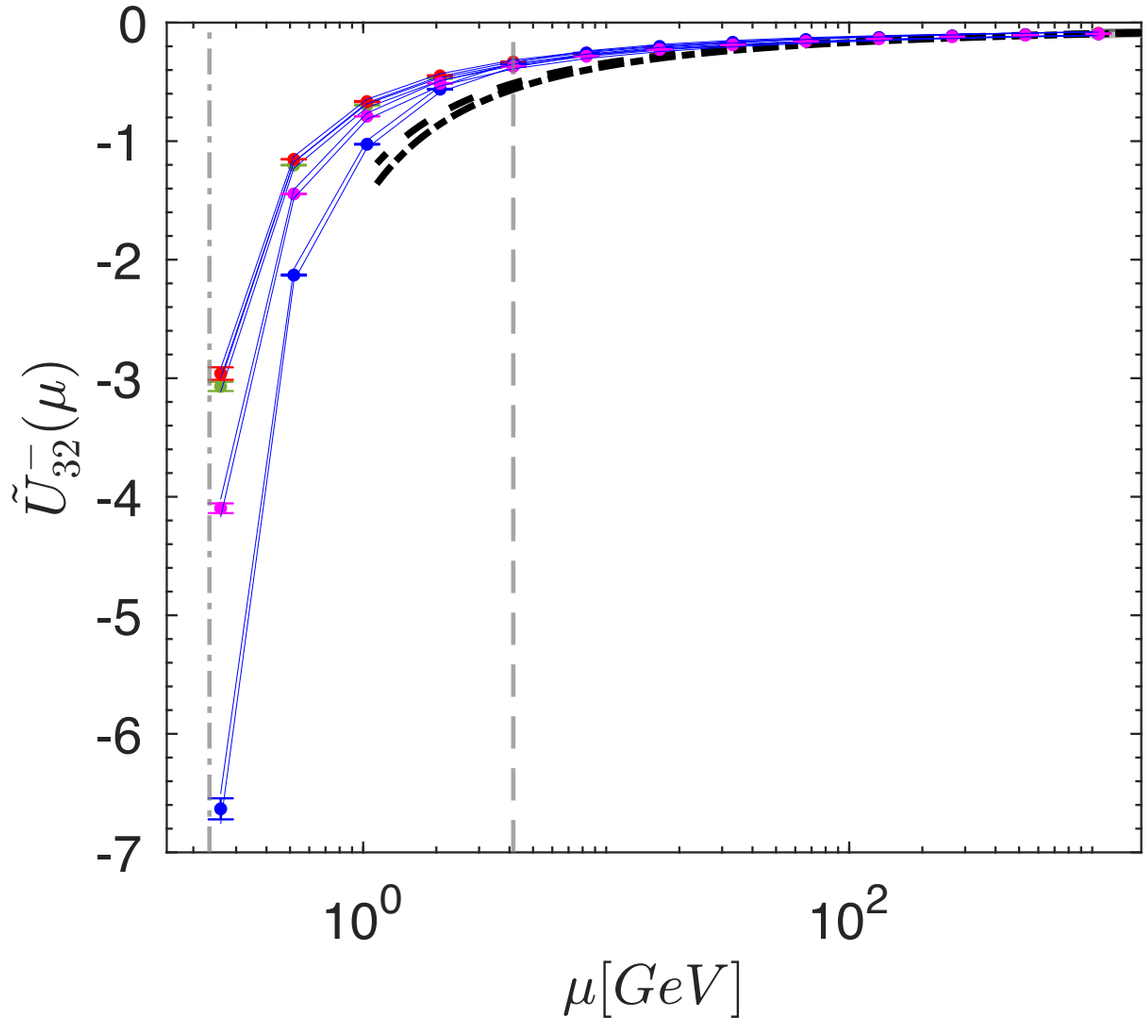
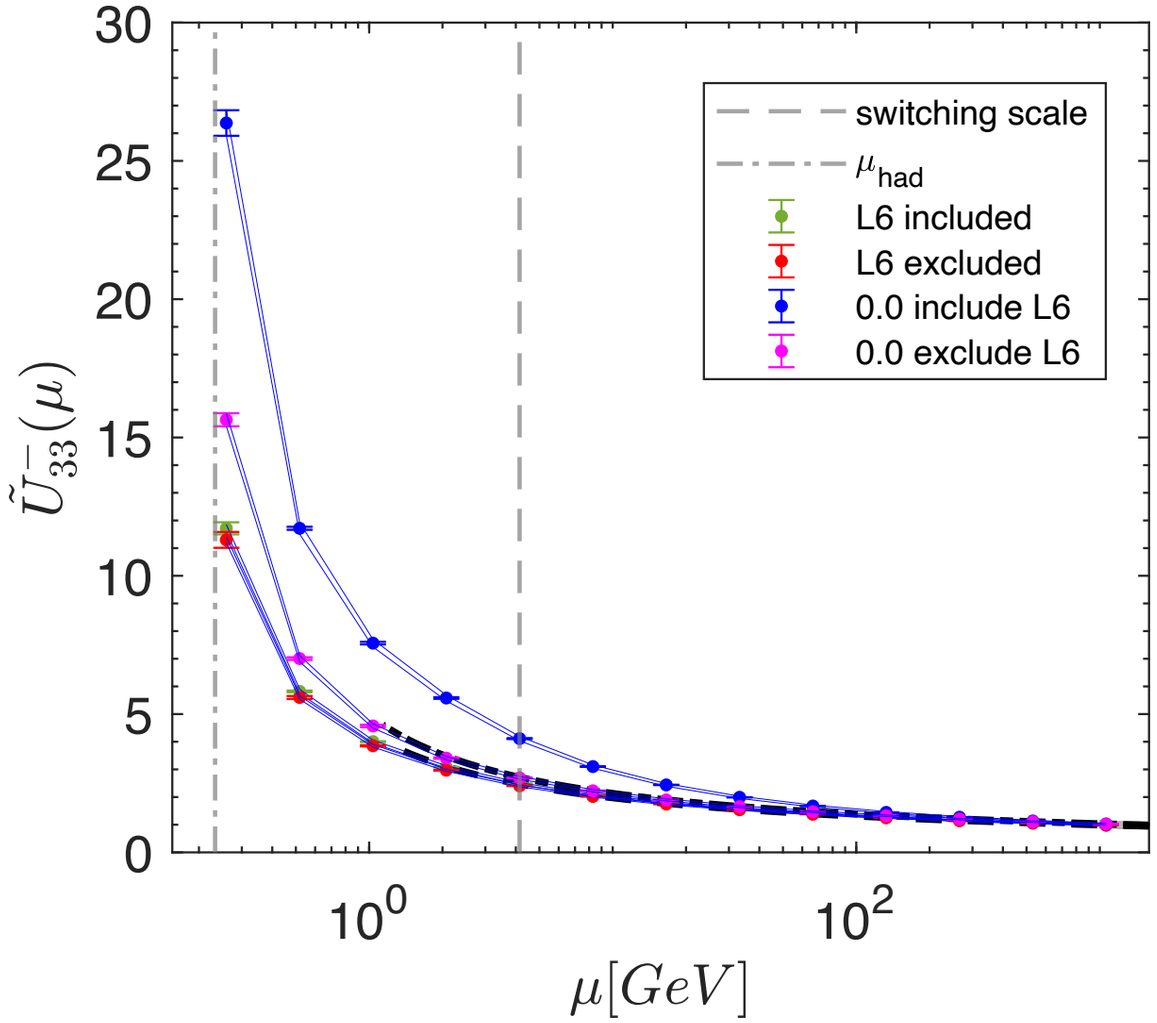
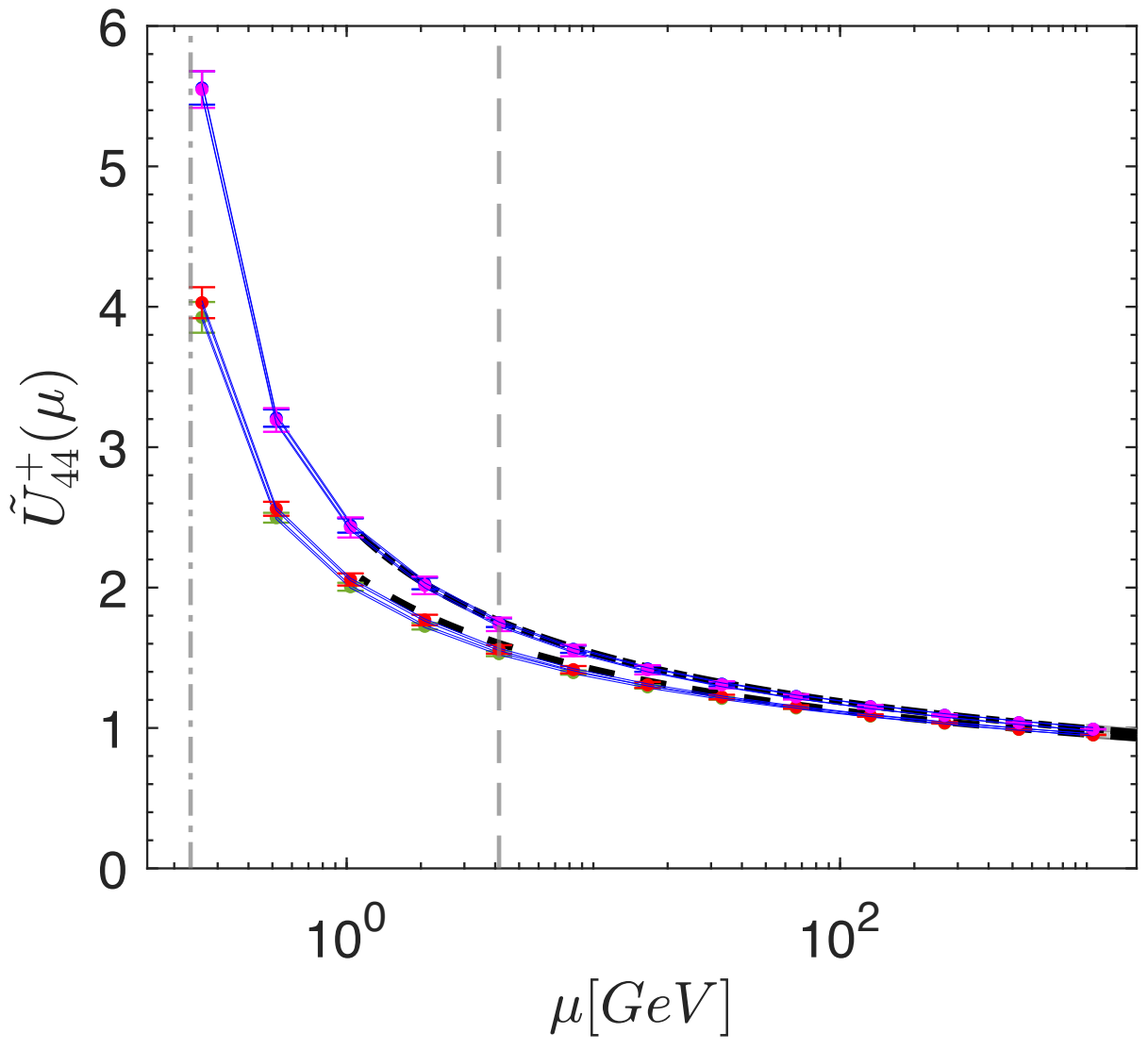
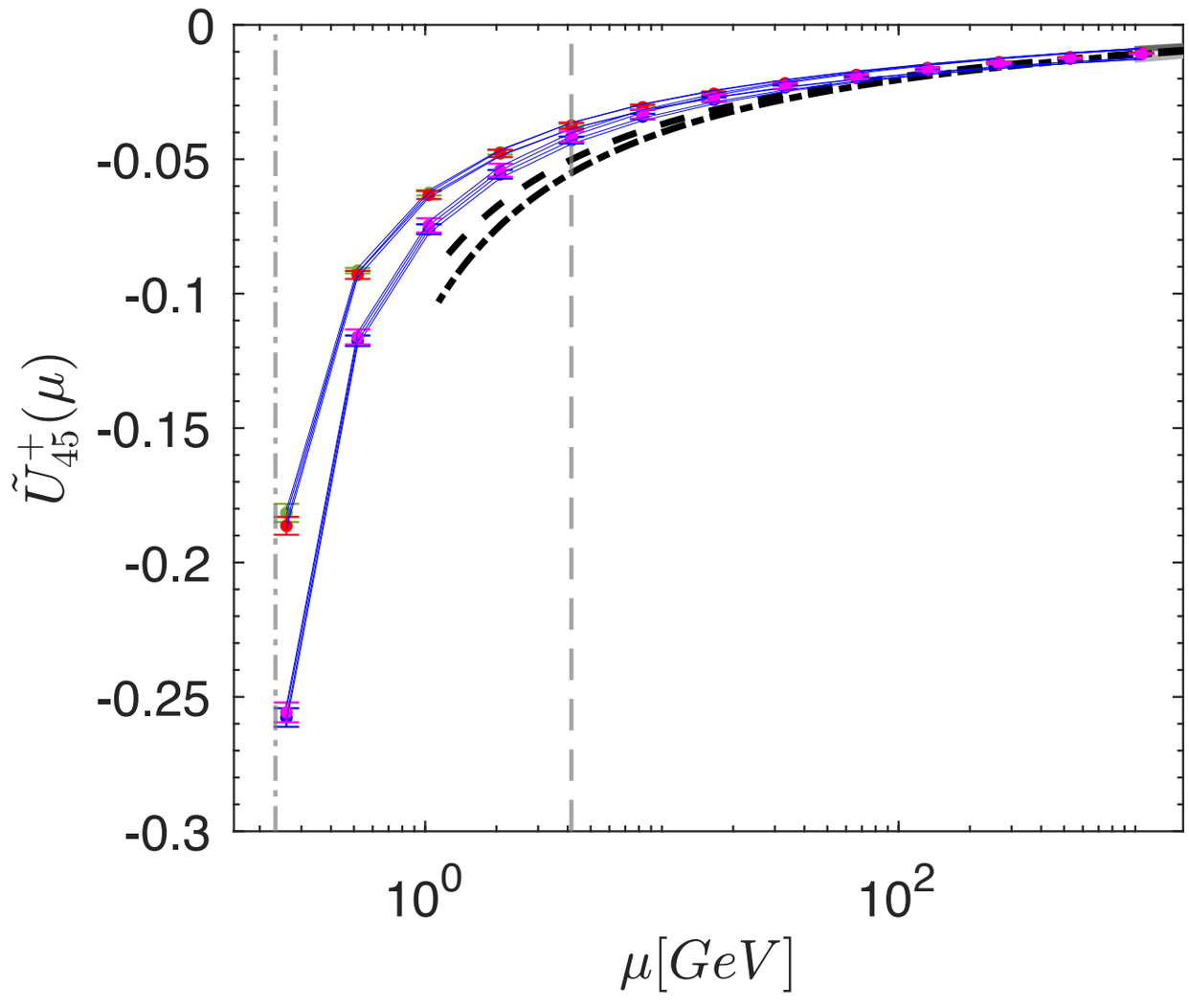


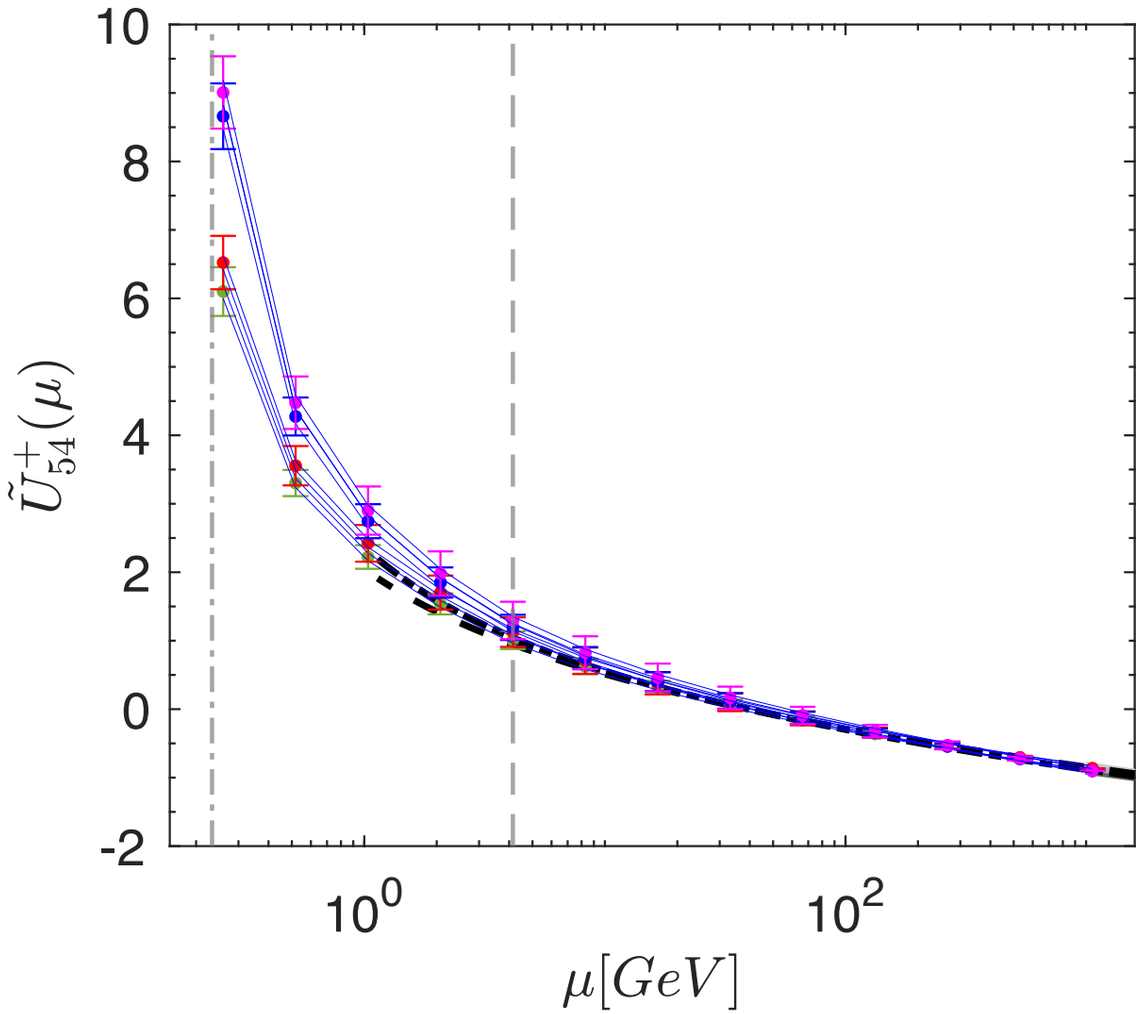


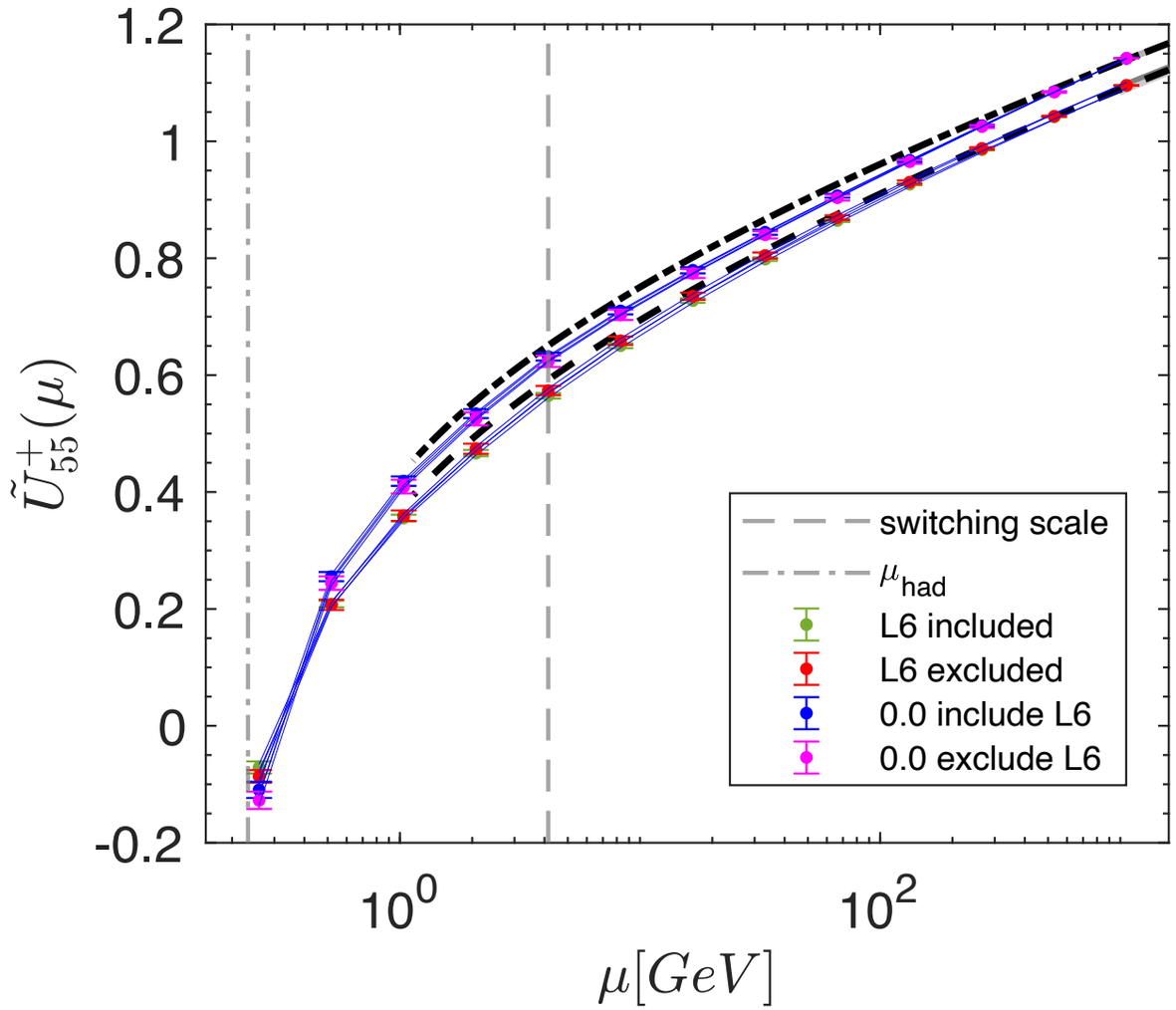
NP Running: 4|5 BSM operators





NP Running: 4|5 BSM operators





Consistency checks

Now it is time to

- Assess the quality of our results;
- Establish whether the $L = 6$ SF data shall be included in the fits;
- Establish if a normalisation scheme can be preferred;

Consistency checks

The following quantities [0712.2429] are computed:

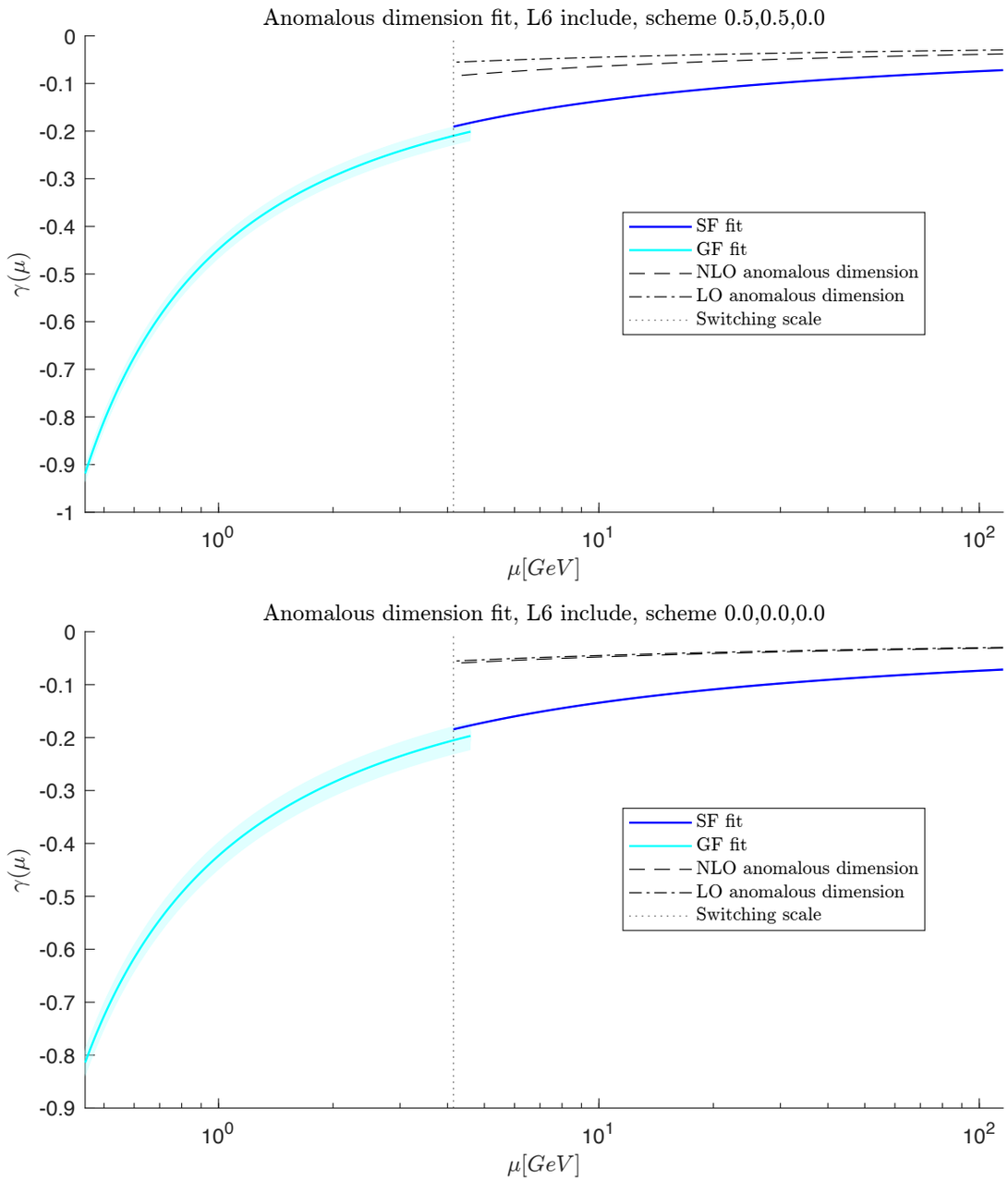
- SM anomalous dimension

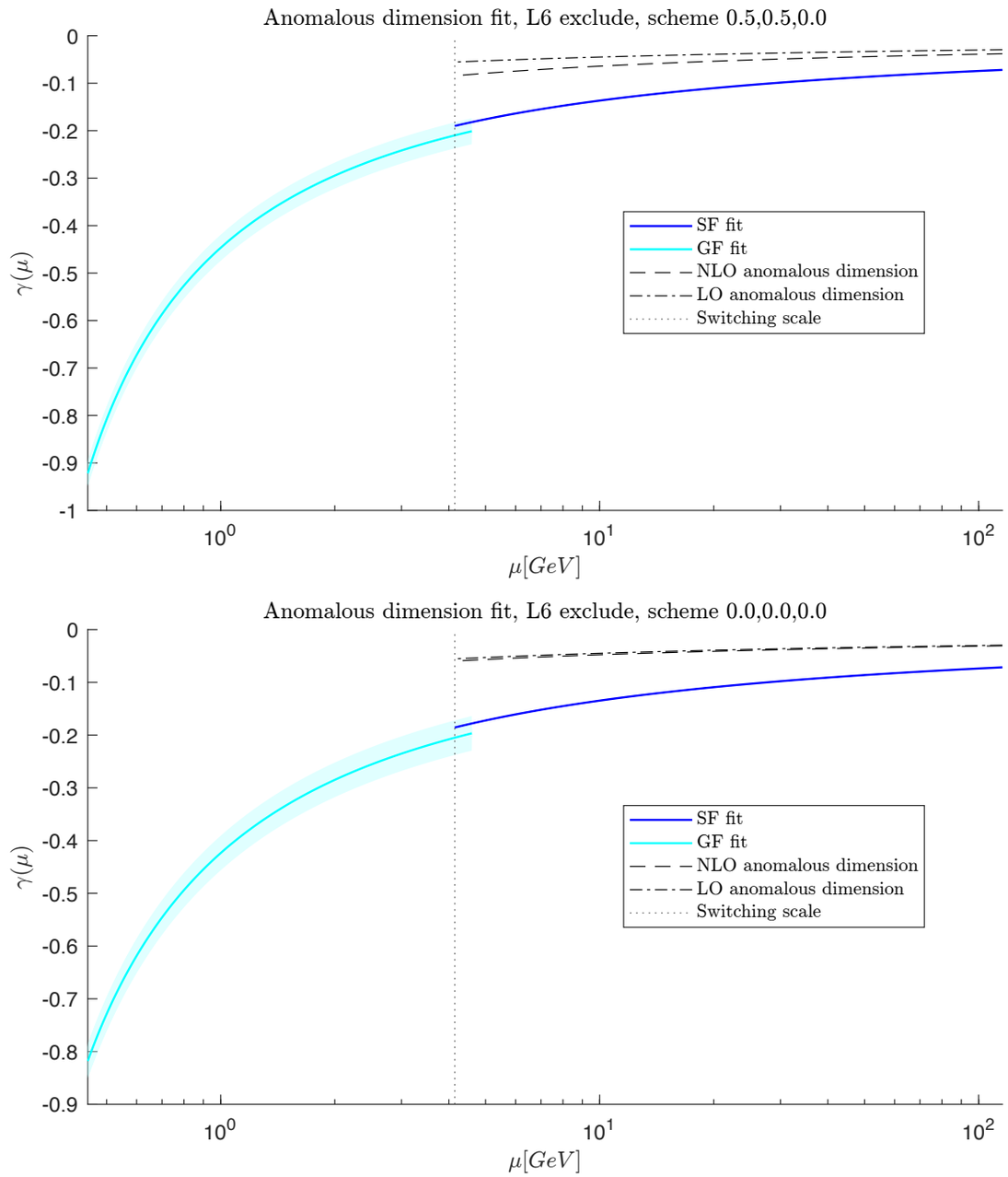
- $\tilde{\mathrm{U}}(\mu_{\mathrm{had}}) = \tilde{\mathrm{U}}(\mu_{\mathrm{pt}})\left[\mathrm{U}(\mu_{\mathrm{had}}, \mu_{\mathrm{pt}})\right] \rightarrow$ constant

- $\mathbf{D}(u_n) \equiv \tilde{\mathbf{U}}(u_n)\boldsymbol{\sigma}(u_{n+1})\left[\tilde{\mathbf{U}}(u_{n+1})\right]^{-1} - \mathbf{1} \rightarrow 0$

At least in principle!

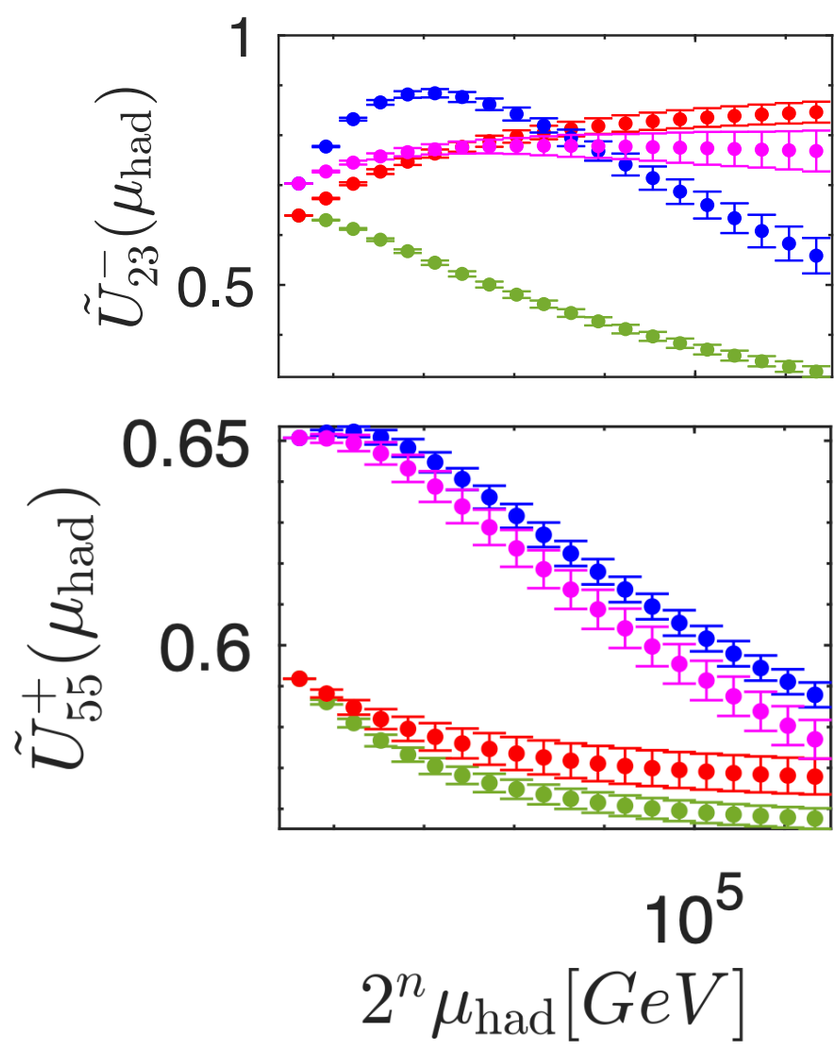
Consistency checks -1

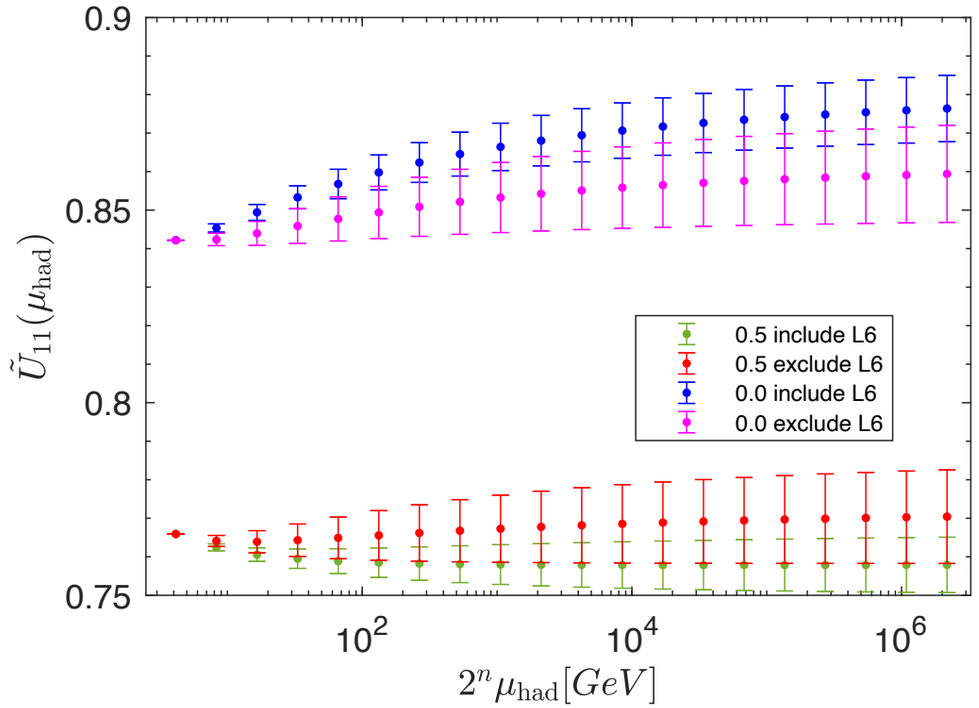




Consistency checks -2

$$\tilde{\mathbf{U}}(\mu_{\rm had}) = \tilde{\mathbf{U}}(\mu_{\rm pt})\Big[\mathbf{U}(\mu_{\rm had},\mu_{\rm pt})\Big] \rightarrow \text{constant}$$

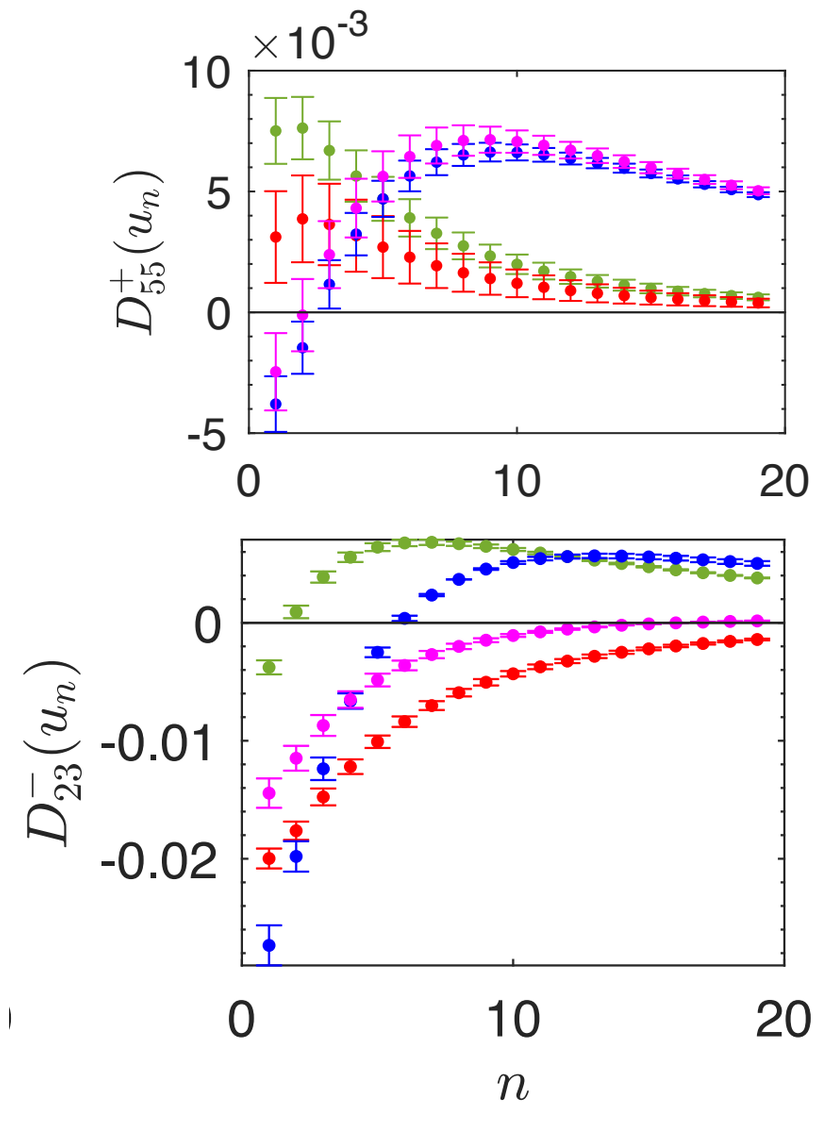


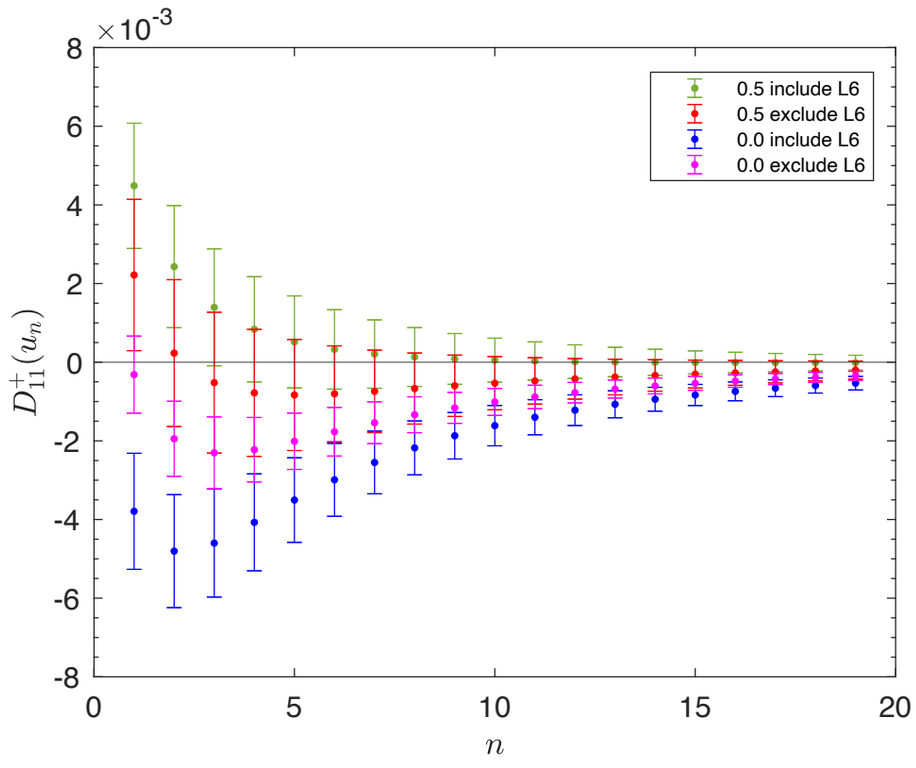


Better behaviour from '0.5' normalisation,
L = 6 data removed

Consistency checks - 3

$$\mathbf{D}(u_n) \equiv \tilde{\mathbf{U}}(u_n)\boldsymbol{\sigma}(u_{n+1})\left[\tilde{\mathbf{U}}(u_{n+1})\right]^{-1} - \mathbf{1} \to 0$$





Better behaviour (again) from '0.5' normalisation, L = 6 data removed

Final comments, future

- Preliminary analysis computing non-perturbatively the running down to a scale $\mathcal{O}(200\,\mathrm{MeV})$ incorporating the NLO in the perturbative part

- Preliminary determination of a 'favourite' normalisation choice

Planned computations to obtain $\varepsilon^{\mathrm{theor}}$:

- (ongoing) bare tm-QCD matrix elements estimated on Wilson gauge configurations (CLS);

- NP evaluation of the renormalisation constants in the χSF at the hadronic scale ($\sim$ 200MeV) at the lattice spacings of the CLS ensembles.

Backup

Four-fermion operators

- Four-Fermion Operators (FFO):

$$\mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2\pm\Gamma_2\Gamma_1]} := \mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2]} \pm \mathcal{O}^{\pm}_{[\Gamma_2\Gamma_1]} \,,$$
$$\mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2]} := \frac{1}{2}\left[\left(\bar{\psi}_1\Gamma_1\psi_2\right)\left(\bar{\psi}_3\Gamma_2\psi_4\right) \pm \left(\bar{\psi}_1\Gamma_1\psi_4\right)\left(\bar{\psi}_3\Gamma_2\psi_2\right)\right]$$

- Parity-odd operators:

$$\mathcal{Q}^{\pm}_1 = \mathcal{O}^{\pm}_{[VA+AV]} \quad \mathcal{Q}^{\pm}_3 = \mathcal{O}^{\pm}_{[PS-SP]} \quad \mathcal{Q}^{\pm}_5 = -2\mathcal{O}^{\pm}_{[T\tilde{T}]}$$
$$\mathcal{Q}^{\pm}_2 = \mathcal{O}^{\pm}_{[VA-AV]} \quad \mathcal{Q}^{\pm}_4 = \mathcal{O}^{\pm}_{[PS+SP]}$$

- Behaviour under renormalisation as in a regularisation with exact chiral symmetry:

$$\begin{pmatrix} \bar{\mathcal{Q}}^{\pm}_1 \\ \bar{\mathcal{Q}}^{\pm}_2 \\ \bar{\mathcal{Q}}^{\pm}_3 \\ \bar{\mathcal{Q}}^{\pm}_4 \\ \bar{\mathcal{Q}}^{\pm}_5 \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22} & \mathcal{Z}_{23} & 0 & 0 \\ 0 & \mathcal{Z}_{32} & \mathcal{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ 0 & 0 & 0 & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}^{\pm} \begin{pmatrix} \mathcal{Q}^{\pm}_1 \\ \mathcal{Q}^{\pm}_2 \\ \mathcal{Q}^{\pm}_3 \\ \mathcal{Q}^{\pm}_4 \\ \mathcal{Q}^{\pm}_5 \end{pmatrix}$$

W(μ) definition issues

- W(μ) must satisfy the equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\mathbf{W}(\mu) = [\boldsymbol{\gamma}[\bar{g}(\mu)], \mathbf{W}(\mu)] - \beta[\bar{g}(\mu)]\left(\frac{\boldsymbol{\gamma}[\bar{g}(\mu)]}{\beta[\bar{g}(\mu)]} - \frac{\boldsymbol{\gamma}^{(0)}}{\bar{g}(\mu)b_0}\right)\mathbf{W}(\mu)$$

and must admit the perturbative expansion

$$\mathbf{W}(\mu) = \mathbf{1} + \bar{g}^2(\mu)\mathbf{J}_1 + \bar{g}^4(\mu)\mathbf{J}_2 + \bar{g}^6(\mu)\mathbf{J}_3 + \dots$$

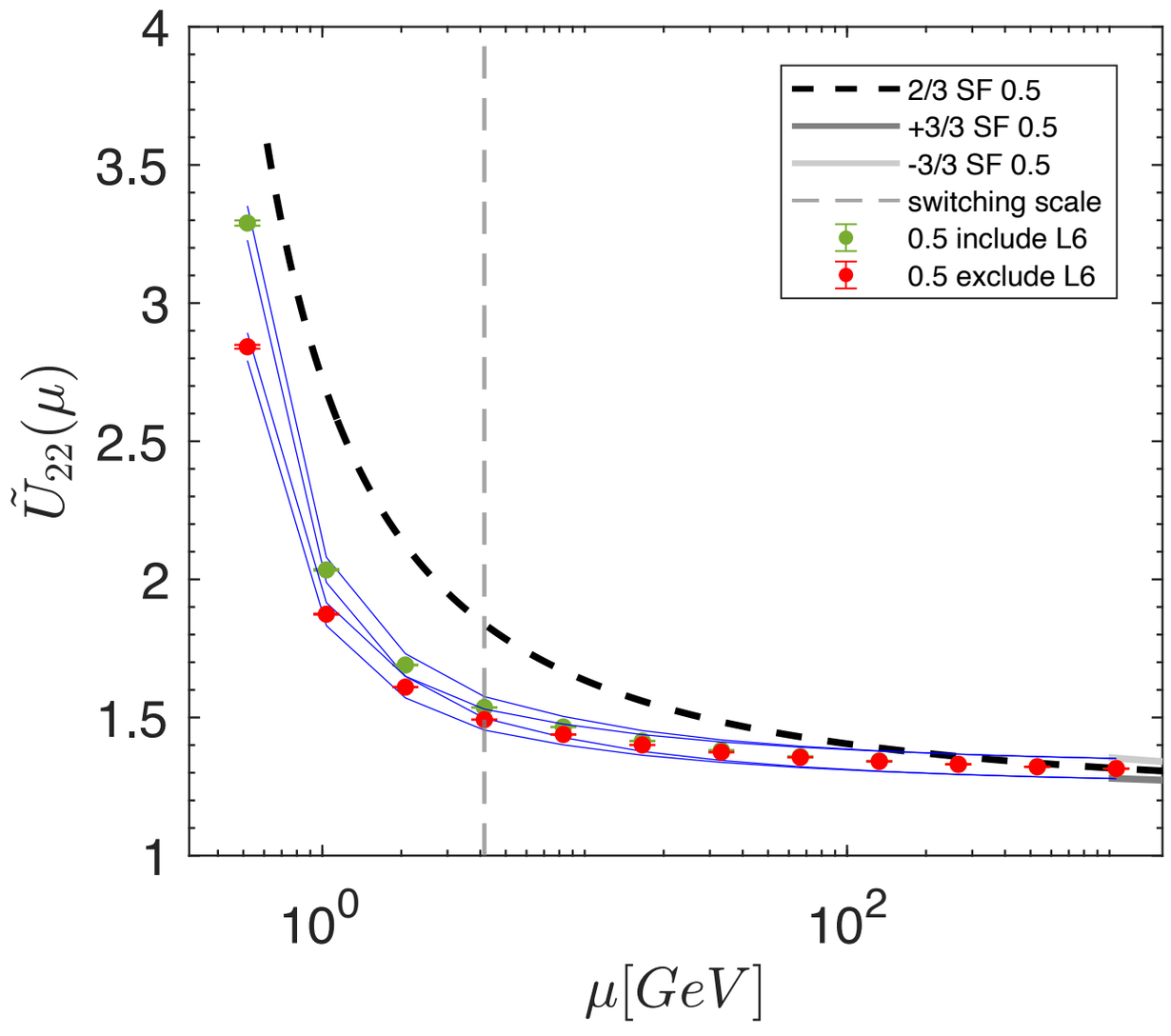
that implies $$2\mathbf{J}_1 - \left[\frac{\boldsymbol{\gamma}_0}{b_0}, \mathbf{J}_1\right] = \frac{b_1}{b_0}\frac{\boldsymbol{\gamma}_0}{b_0} - \frac{\boldsymbol{\gamma}_1}{b_0}$$

- Non-invertible system of equations if $N_f = 3$ (or 30) !

Error estimates

- Statistical errors: uncertainties from simulations propagated in the fits

- Systematic errors (guess) : Lack of knowledge on higher orders of the anomalous dimension



Normalisation schemes

Correlation functions in the χSF for the i-th FFO:

$$G_i(x_0) = \left\langle \mathcal{O}_5'^{uu} \mathcal{Q}_i^{uuud}(x_0) \mathcal{O}_5^{du} \right\rangle ,$$

$$L_i(x_0) = \frac{1}{3} \sum_{k=1}^{3} \left\langle \mathcal{O}_k'^{uu} \mathcal{Q}_i^{uuud}(x_0) \mathcal{O}_k^{du} \right\rangle$$

Normalisation that removes fermionic renormalisation constants:

$$\mathcal{G}_i := \frac{G_i}{\mathcal{N}(\alpha,\beta,\gamma)} , \quad \mathcal{L}_i := \frac{L_i}{\mathcal{N}(\alpha,\beta,\gamma)}$$

"Schemes" considered:

'0.0': $\alpha = 0.0, \beta = 0.0, \gamma = 0.0$
'0.5': $\alpha = 0.5, \beta = 0.5, \gamma = 0.0$

Every choice of normalisation requires a separate evaluation of the NLO anomalous dimension, affecting the NP running