DISCRETISATION EFFECTS IN YM-THEORY

A study of cutoff effects in different lattice actions

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UNIVERSALITY AND THE CONTINUUM LIMIT

- Lattice discretisations of operators are not unique
  - Example: Plaquette and clover-leaf are both discretisations of the field-strength tensor
  - Universality: After renormalisation and extrapolation to the continuum different discretisations should agree
- The continuum approach of lattice observables can be quite complicated due to modifications of a^n scaling by logarithms [Husung et al, 2019]
- We would like to know for a given set of lattice actions:
  - that universality holds indeed
  - how fast the continuum limit is approached (what is the error of working at non-zero lattice spacing?)

HOW TO MEASURE SCALING VIOLATIONS

- To determine scaling / universality violations we need a "physical" quantity with a well-defined continuum limit that can be measured with high precision

- Similar requirements faced in scale-setting - some choices are:

  - Spectral quantities (hadron masses…):

    - Classical Symanzik expansion:

$$\lim_{a\to 0}\langle O\rangle \sim \langle O\rangle_{\text{cont.}} + a^2\langle OS_2\rangle + \ldots$$

    - downsides: expensive, depends on fitting plateaus, signal-to-noise etc.

  - "Intermediate quantities" based on quark-antiquark potential, e.g. Sommer scale:
    Not that easy to compute, involves fits/extrapolations

  - Flow scales (this talk)

THE GRADIENT FLOW

Flow in field space with respect to a fictitious time;
defined by differential equation and initial conditions:

$$\dot{B}_\mu(x,t) = D_\nu G_{\nu\mu}(x,t)$$
$$B_\mu(x,0) = A_\mu(x)$$

where

$$G_{\nu\mu}(x,t) = \partial_\nu B_\mu(x,t) - \partial_\mu B_\nu(x,t) + [B_\nu(x,t), B_\mu(x,t)]$$

- Drives the field $B_\mu(x,t)$ towards the stationary points of the action - smoothing effect on gauge field

- Originally explored in context of trivialising maps [Lüscher, year?]

THE OBSERVABLE

Define "action density" in terms of flowed fields:

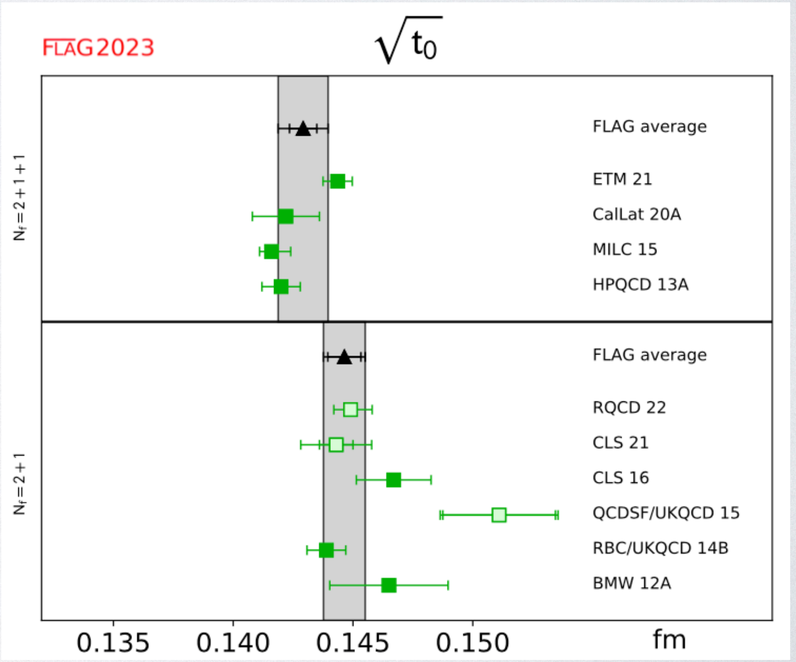
$$E(x,t) = \mathrm{tr}\{G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\}$$

Form the dimensionless combination:

$$t^2\langle E(t)\rangle = \text{const.}$$

Various scales given by:

$$t^2\langle E(t)\rangle = \begin{cases} 0.3 & \text{for } t = t_0 \\ 0.5 & \text{for } t = t_1 \\ 0.15 & \text{for } t = t_2 \end{cases}$$



GF: 5D LOCAL FIELD THEORY

The gradient flow can be treated as a local field theory in 4+1 dimensions where the 5th dimension is the fictitious time $t \geq 0$

$$S_{5d} = S_{\text{bnd}} + S_{\text{flow}}$$

where

$$S_{\text{bnd}} = \int d^4x \frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu,a} \; ; \qquad S_{\text{flow}} = \int_0^t dt' \int d^4x L^a_\mu(x,t') \{ \dot{B}_\mu(x,t) - D_\nu G_{\nu\mu}(x,t) \}$$

$L^a_\mu(x,t')$ is a Lagrange-multiplier field

SYM-EFT AND THE CONTINUUM LIMIT

- Any lattice action can be expanded in terms of actions in the continuum:

$$\lim_{a\to 0} S_{\text{lat.}} \sim S_{\text{cont.}} + \sum_{n=2}^{\infty} a^n S_n$$

- Observables receive corrections from the expansion of the action and the discretisation of the operator:

$$\langle O \rangle_{\text{lat}} \sim \langle O \rangle_{\text{cont}} + a^2 \left\{ \langle O_2 \rangle + \langle O S_2 \rangle \right\} + \dots$$

- Dimensionless renormalised observables from different discretisations should agree in the continuum → Universality

EXPANDING FLOW OBSERVABLES

In the 5D theory we get the expansion

$$S^{5d}_{\text{lat}} \underset{a\to 0}{\sim} S^{5d}_{\text{cont}} + a^2 S_{\text{bnd},2} + a^2 S_{\text{flow},2} + \cdots$$

Recalling the definition of the flow-time scale

$$t^2\langle E(t)\rangle = \text{const.}$$

the $\mathcal{O}(a^2)$ contribution to t_c is given by

$$\Rightarrow \left.\frac{dt}{d(a^2)}\right|_{t=t_c} = -\underbrace{\left[\left.\frac{d}{dt}\right|_{t=t_c} t^2\langle E(t)\rangle\right]^{-1}}_{D_c} t^2\frac{dE}{d(a^2)}$$

Note: flow quantities are affected by an additional $\mathcal{O}(a^2)$ counter term which can be removed by changing the initial condition of the flow equation. We will not further discuss this here.

EXPANSION OF FLOW-TIME RATIOS

$$t_c^{\text{lat}} \sim t_c - \frac{a^2 t_c^2}{D_c} \left\{ \langle E(t_c) S_{\text{bnd},2} \rangle + \langle E(t_c) S_{\text{flow},2} \rangle + \langle E_2(t_c) \rangle \right\}$$

Interesting bit: depends on (boundary) action

Depends on integration of flow equation

Depends on discretisation of observable

Caution:

$$\frac{t_0^{\text{pl}}}{t_0^{\text{cl}}} \sim 1 - \frac{a^2 t_0^2}{D_0} \left\{ \langle E_2^{\text{pl}}(t_0) \rangle - \langle E_2^{\text{cl}}(t_0) \rangle \right\}$$

Only classical cutoff effects, no dependence on action

IMPROVED FLOW-TIME RATIOS

Use improved action density:

$$E_{\text{imp}} = \frac{1}{3}\left(4E_{\text{pl}} - E_{\text{cl}}\right) \to E_2 = 0$$

Use improved flow equation: Zeuthen flow [Ramos & Sint 2016]

$$\to S_{\text{flow},2} = 0$$

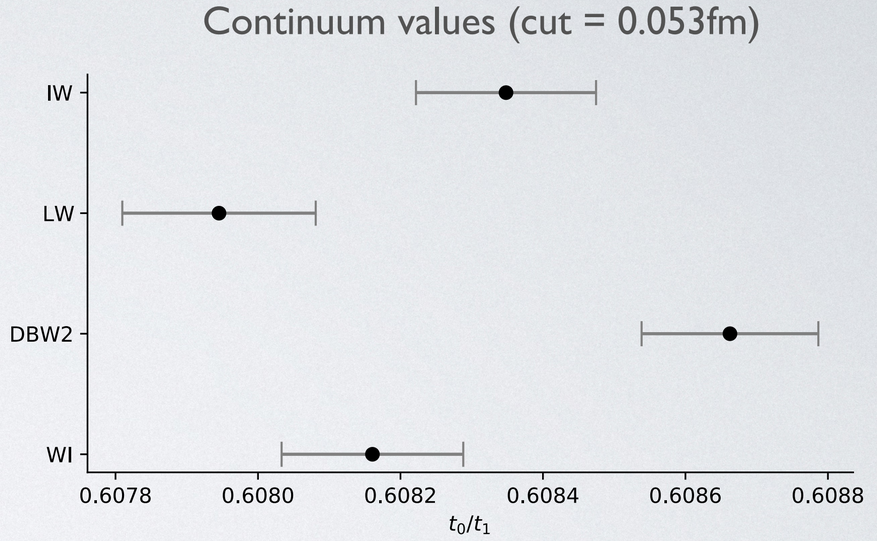
Compute a ratio of two different flow times:

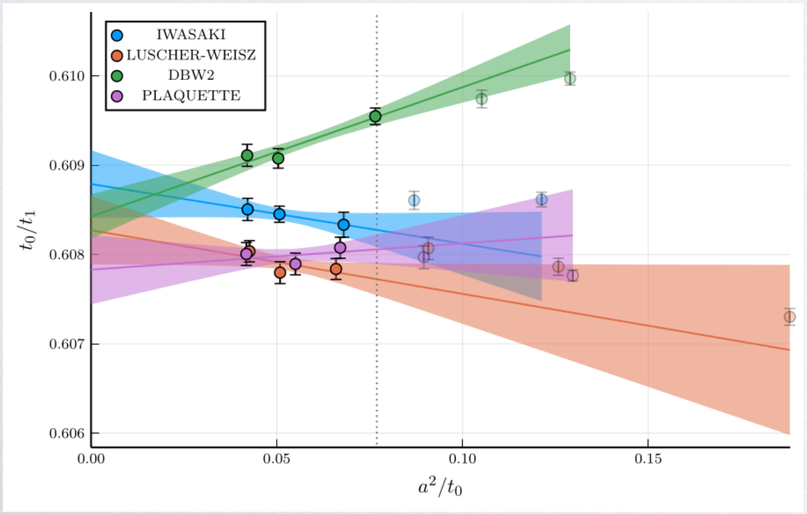
$$\frac{t_c^{\text{imp}}}{t_{c'}^{\text{imp}}} \sim 1 - a^2 \left\{ \frac{t_c^2}{D_c}\langle E(t_c) S_{\text{bnd},2}\rangle - \frac{t_{c'}^2}{D_{c'}}\langle E(t_{c'}) S_{\text{bnd},2}\rangle \right\}$$

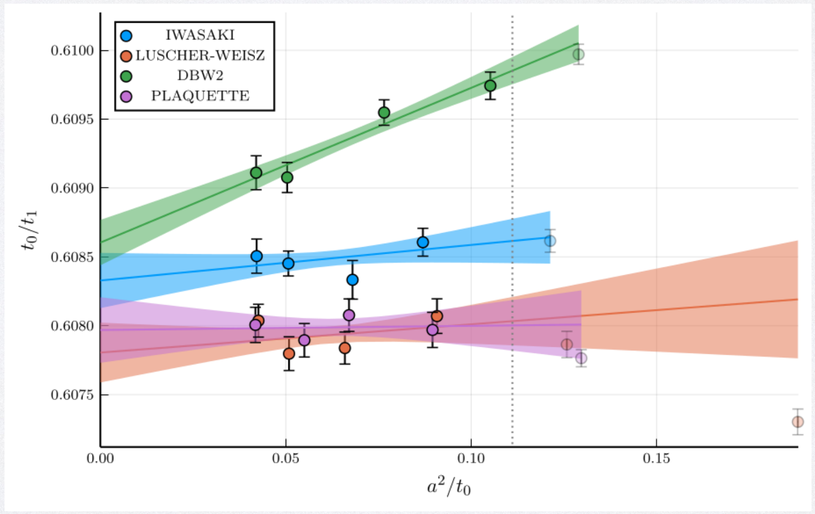
RESULTS: "SHORT RANGE"

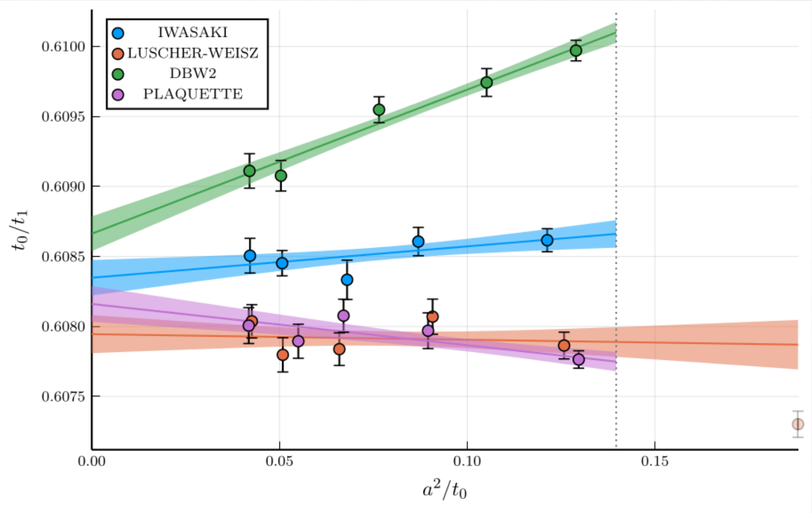
Continuum extrapolations: $\chi^2/\langle\chi^2\rangle$

Action \ Cut	~ 0.039 fm	~ 0.047 fm	~ 0.053 fm
Wilson	1.026	0.63	0.73
Lüscher-Weisz	1.24	1.75	1.39
Iwasaki	0.00065	1.029	0.69
DBW2	0.92	0.88	0.70





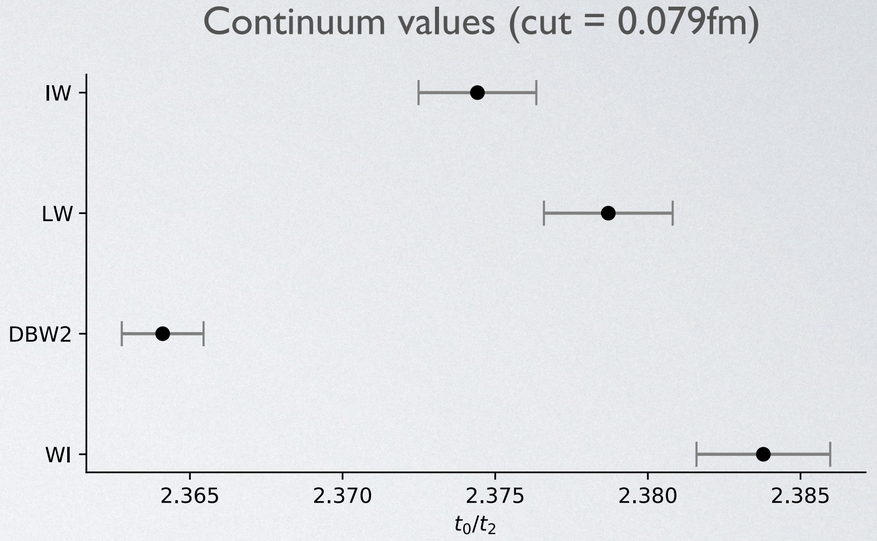


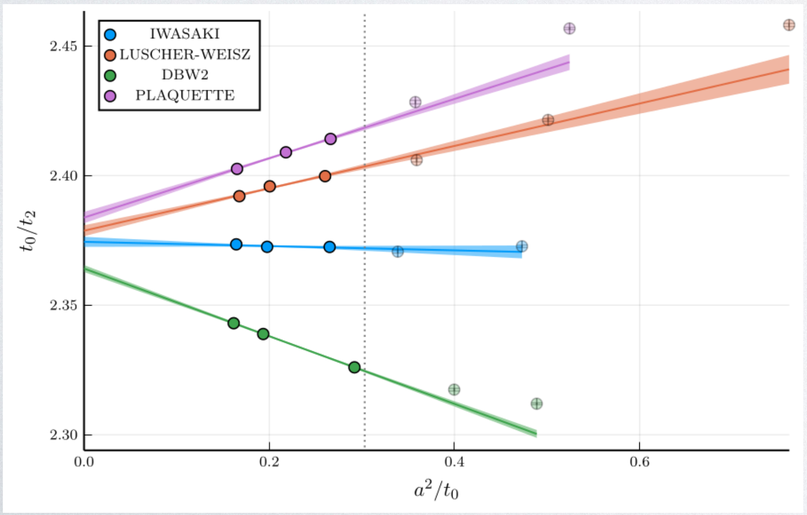


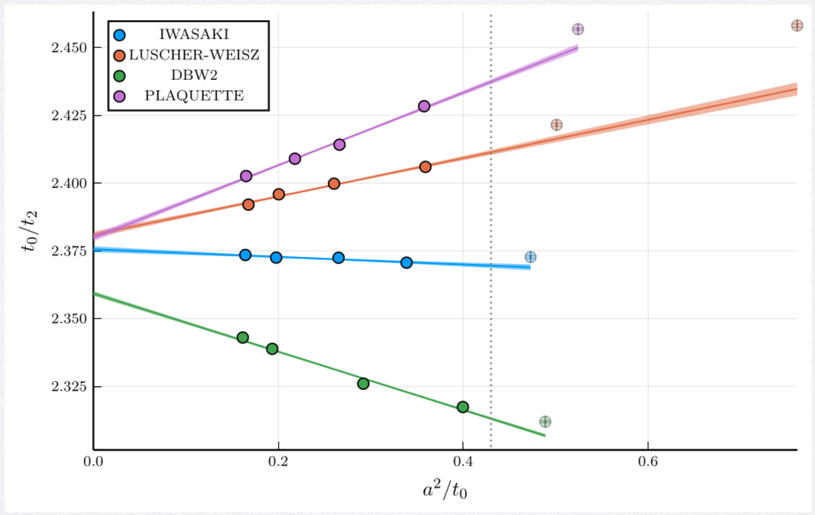
RESULTS: "LONG RANGE"

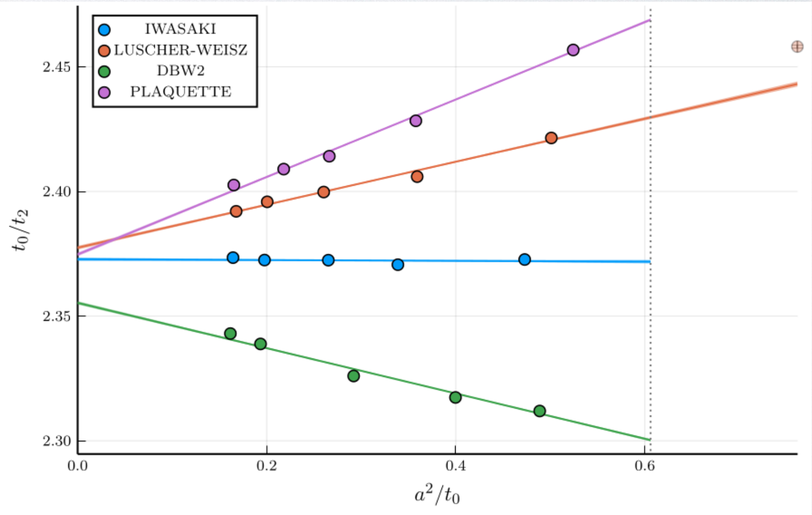
Continuum extrapolations: $\chi^2/\langle\chi^2\rangle$

Action \ Cut	~ 0.079 fm	~ 0.094 fm	~ 0.11 fm
Wilson	0.17	2.78	9.48
Lüscher-Weisz	1.60	1.61	5.91
Iwasaki	0.82	0.66	3.86
DBW2	0.000017	11.3	22.8



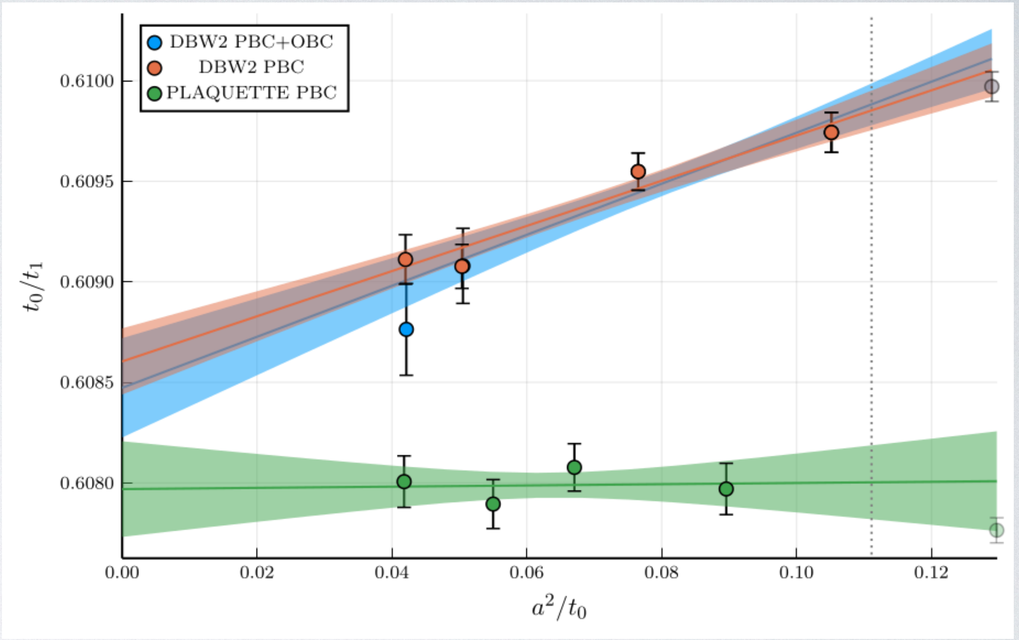


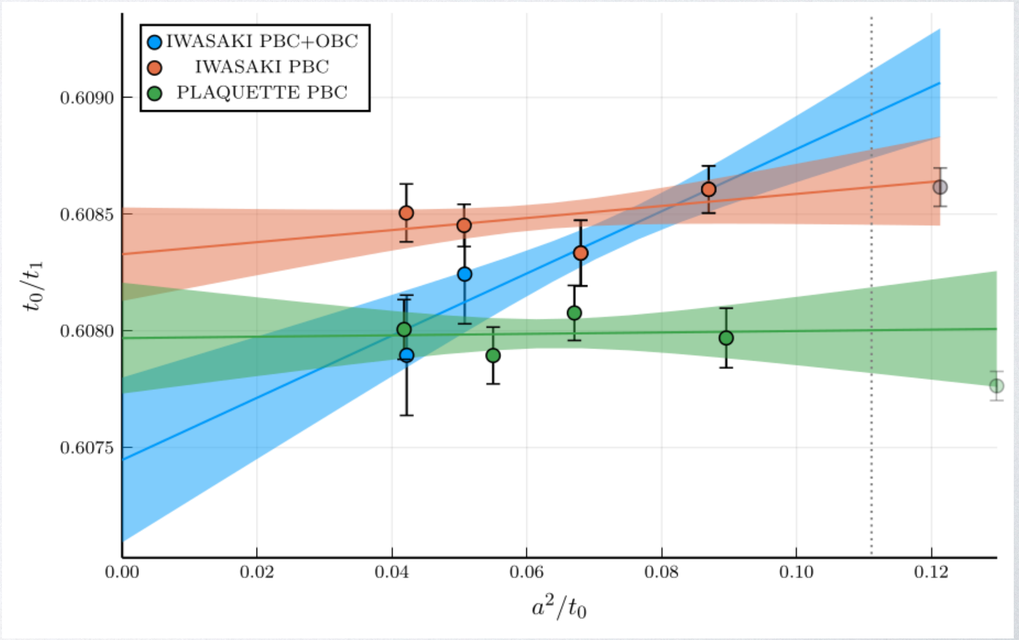




PRELIMINARY RESULTS: OPEN BOUNDARY CONDITIONS

- Topology frozen on finest lattices with DBW2 and Iwasaki actions
- We recompute these points with open boundary conditions as a cross check
- Some tension between PBC/OBC for Iwasaki but may just be statistics
- Work in progress!





CONCLUSIONS

- We argued that flow-time ratios based on the improved action density and the Zeuthen flow are a good probe for asymptotic scaling and universality
- We measured such flow-time ratios with sub-percent precision for 4 different pure-gauge lattice actions
- Continuum values are largely compatible between the actions we test except for DBW2, which is an outlier for the long-range t_0/t_2-ratio
- Asymptotic scaling in a^2 appears to break beyond $a \sim 0.08$fm