Strong coupling constant from the static energy

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*For the TUMQCD collaboration
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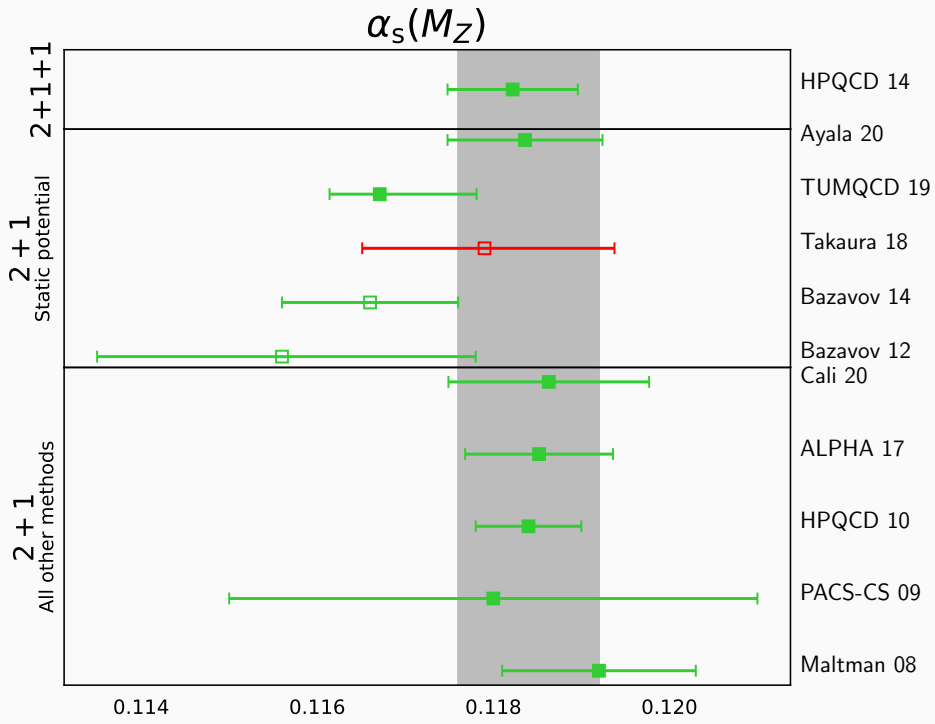
(in various combinations)

Lattice 2025
3rd of November 2025
TIFR, Mumbai

Motivation

- α_s: Fundamental parameter of SM
- Lattice dominates the global average and error
- TUMQCD has long history of determining α_s from the static energy
- Recently we have calculated the 1-loop lattice perturbation theory result (this talk)
    - Allows updating our 2+1 measurement (this talk, preliminary)
    - Important step towards measuring 2+1+1 α_s



Static energy

- Static energy $E_0(r)$ between a static quark and antiquark
- Determined from the large-time behavior of Wilson loops

$$E(r) = -\lim_{T\to\infty} \frac{\ln\langle \mathrm{Tr}(W_{r\times T})\rangle}{T}, \qquad W_{r\times T} = P\left\{ \exp\left(i \oint_{r\times T} dz_\mu \, g A_\mu \right)\right\}$$

- Perturbatively known to $\mathrm{N^3LL}$: [1]

$$E_0(r) = \Lambda_\mathrm{s} - \frac{C_\mathrm{F}\alpha_\mathrm{s}}{r}\left(1 + \#\alpha_\mathrm{s} + \#\alpha_\mathrm{s}^2 + \#\alpha_\mathrm{s}^3 + \#\alpha_\mathrm{s}^3 \ln\alpha_\mathrm{s} + \#\alpha_\mathrm{s}^4 \ln^2\alpha_\mathrm{s} + \#\alpha_\mathrm{s}^4 \ln\alpha_\mathrm{s} \ldots\right)$$

- The ultra-soft scale μ_us gives rise to $\ln\alpha_\mathrm{s}(1/r)$ term
- Scheme independent: can directly compare lattice and $\overline{\mathrm{MS}}$

[1] For review of perturbative results, see: X. Tormo Mod. Phys. Lett. A28 (2013)

Renormalon

- *Dimensional regularization*: requires a renormalon subtraction
- *Lattice regularization*: diverges as $1/a$ towards continuum limit
- Both regularization problems can be absorbed into a constant term
- Two approaches to reduce renormalon effects

Integrated force

- Static force has no renormalon

$$E_0(r) = \int_{r^*}^{r} \mathrm{d}r' F(r') + \mathrm{const}$$

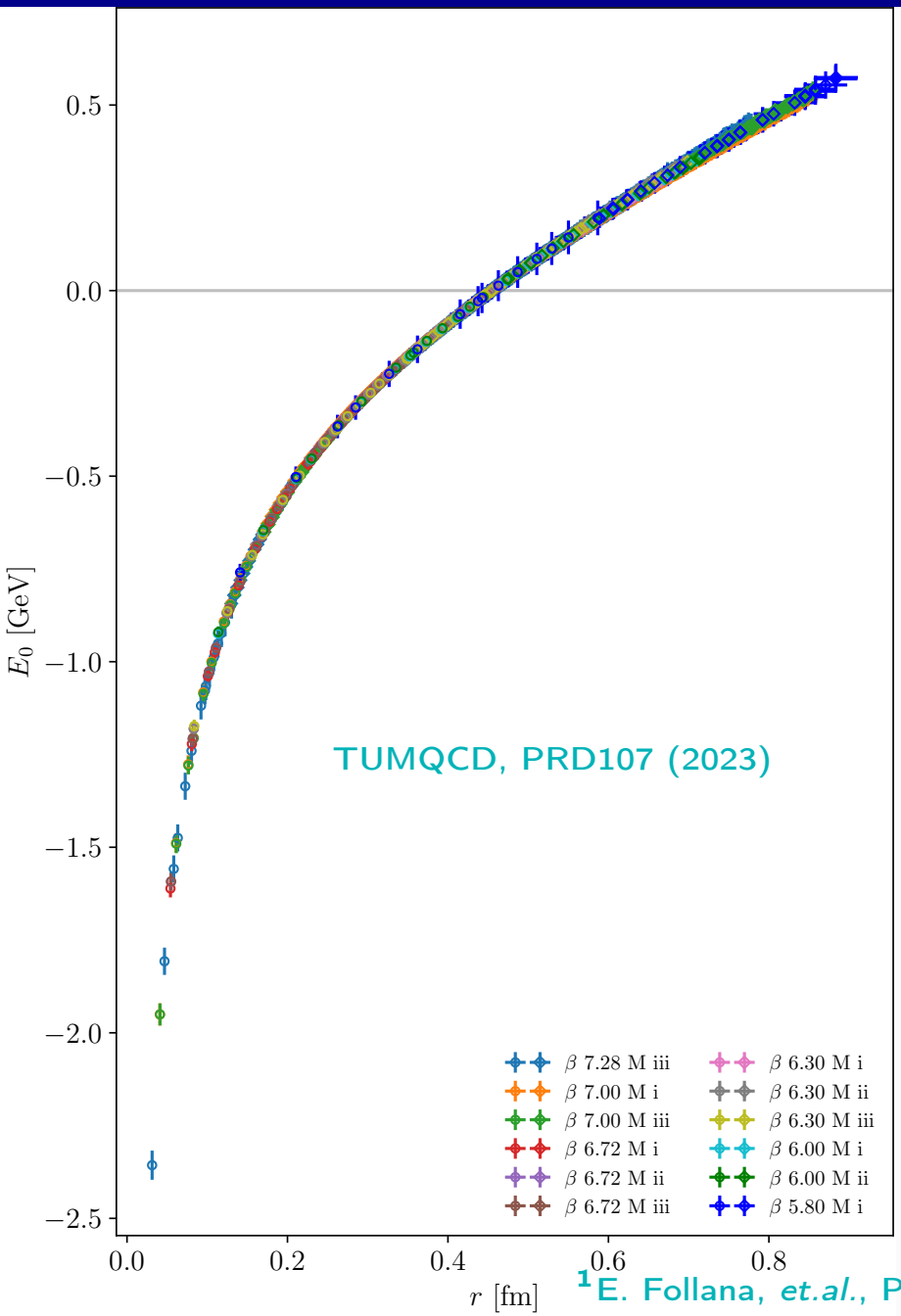
- Now has integration constant

See e.g. Bazavov *et.al.*PRD90 (2014)

Minimal Renormalon Subtraction

- Relate the factorial growth of perturbative series to a power correction
- For more see: Komijami JHEP62 (2017), TUMQCD PRD97 (2018), Kronfeld JHEP12 (2023)

Static energy on the lattice



- HISQ ensembles from MILC and HotQCD[1][2]
- Measured at 2+1 and 2+1+1
- Data ranging from $r \approx 0.025/0.03$–0.09fm
- Coulomb gauge Wilson line correlators allow off-axis directions
- Massive charm at 2+1+1: $1/m_c \sim 0.15fm$

[1] E. Follana, *et.al.*, PRD75 (2007); [2] A. Bazavov, *et.al.*, PRD98 7 (2018)

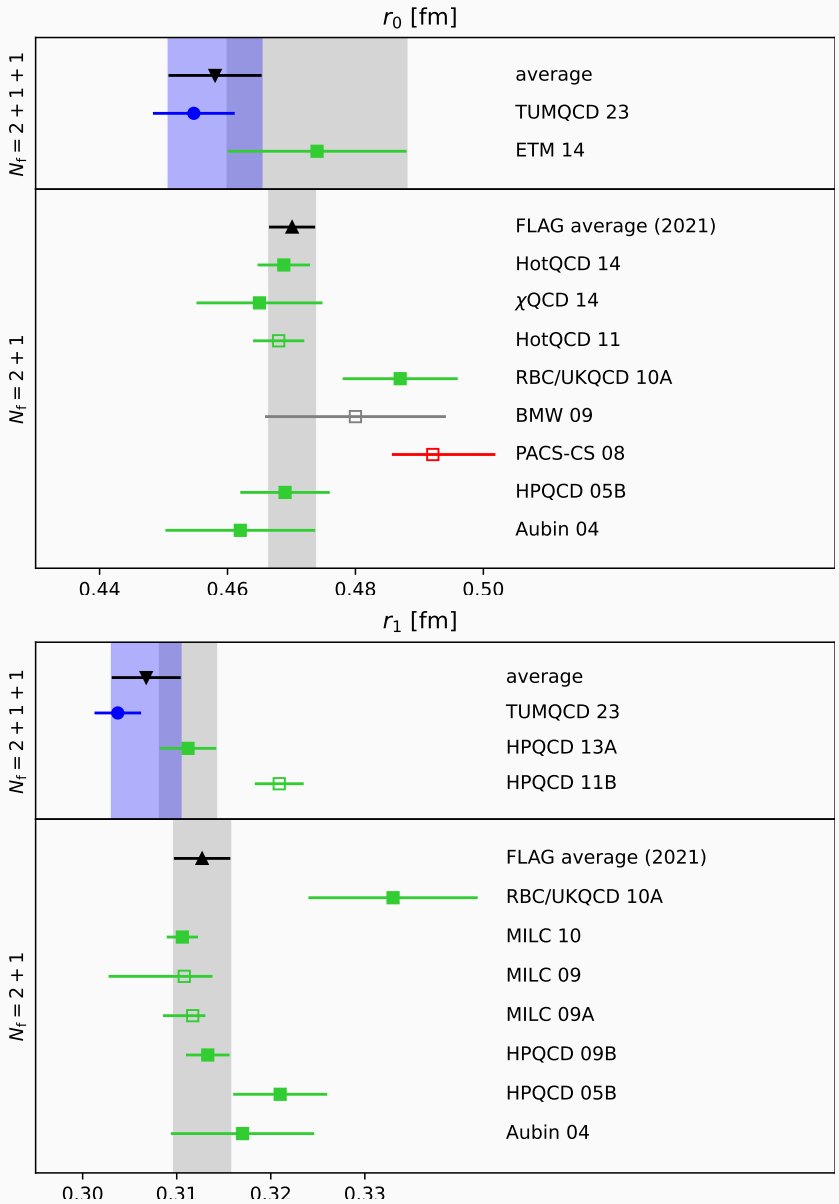
Lattice scales

- Scales r_i measured recently:
  TUMQCD, PRD107 (2023)

$$r_i^2 F(r_i) = \begin{cases} 1.65, & i = 0\,^{1} \\ 1.0\ , & i = 1\,^{2}\ , \\ 0.5\ , & i = 2\,^{3} \end{cases} \quad \begin{array}{l} r_0 = 0.4547(64)\,\mathrm{fm} \\ r_1 = 0.3037(25)\,\mathrm{fm} \\ r_2 = 0.1313(41)\,\mathrm{fm} \end{array}$$

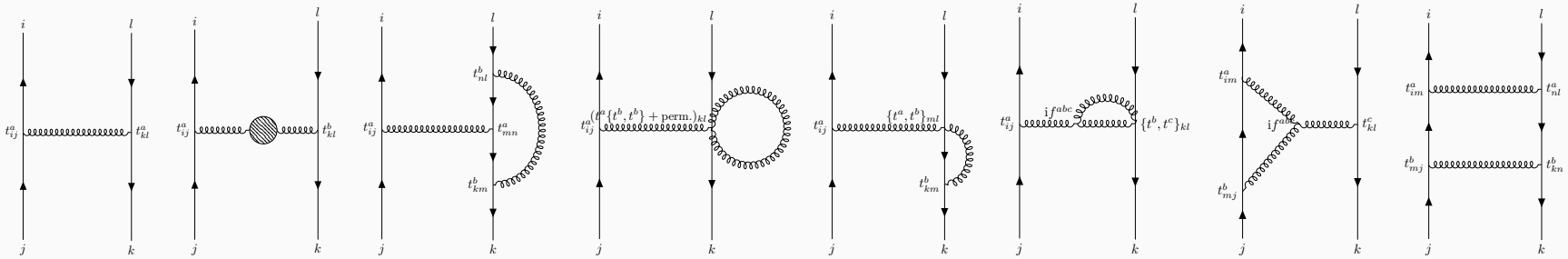
- $r_2 \sim 1/m_c$ affected by charm mass
- For 2+1 we use $r_1 = 0.3106(17)\,\mathrm{fm}$ based on f_π

  (see also the new r_1 by HotQCD 2502.08061)
- For 2+1+1, we use our own determination $r_1 = 0.3037(25)\,\mathrm{fm}$ based on f_{p4s}

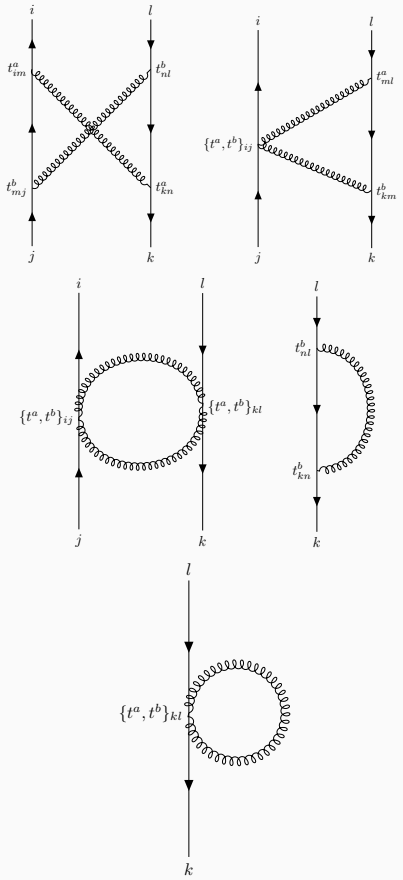


[1] R. Sommer, NPB411 (1994); [2] C. Bernard, *et.al.*, PRD62 (2000); [3] A. Bazavov, *et.al.*, PRD97 (2018)

1-loop lattice perturbation theory

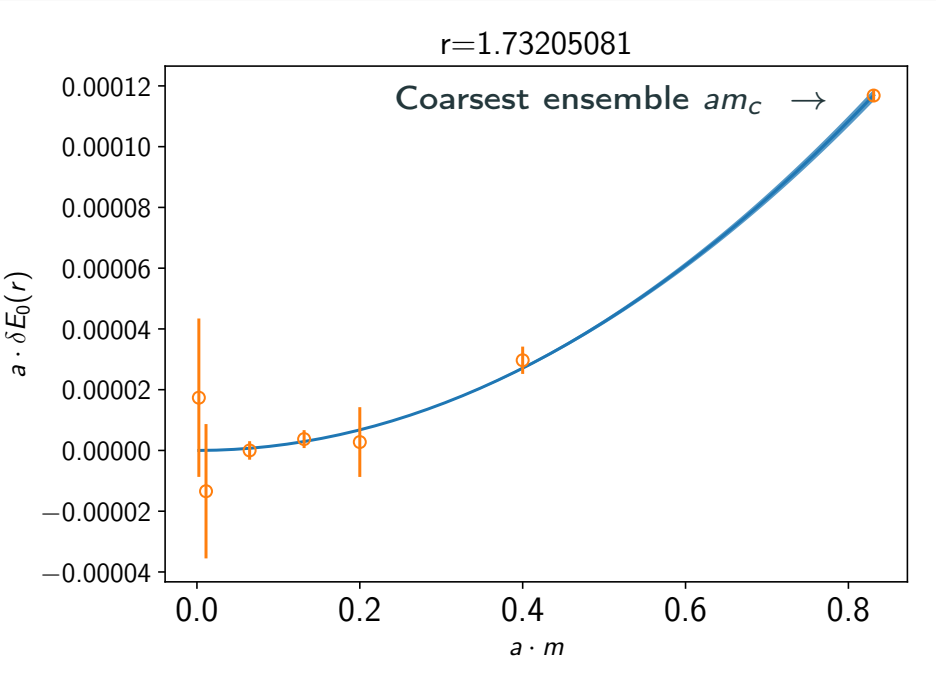


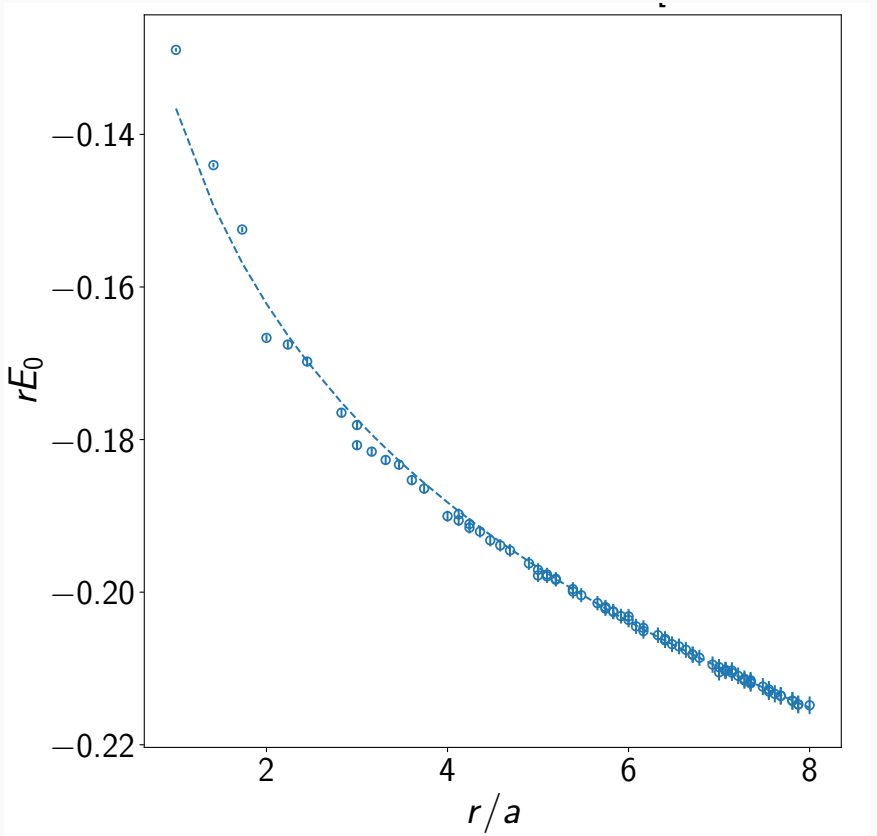
- Lattice breaks rotational symmetry
- $E_0(r, a)$ is discrete and direction dependent
- Calculation of 1-loop diagrams is done numerically with HPsrc and HiPPy programs
- The Wilson loop momenta is discretized and interpolated to the actual lattice volume
- The loop momenta is either integrated at infinite volume with VEGAS or for small loop momenta fermion loops discretized and extrapolated to infinite volume
- Only measure the Wilson line correlators



A. Hart *et.al.* Comput. Phys. Commun. 180 (2009)

1-loop lattice perturbation theory

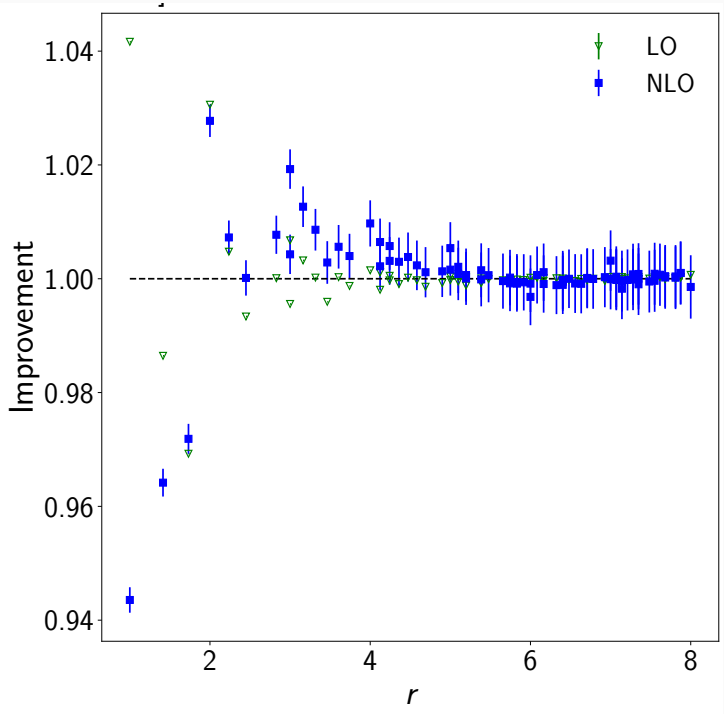


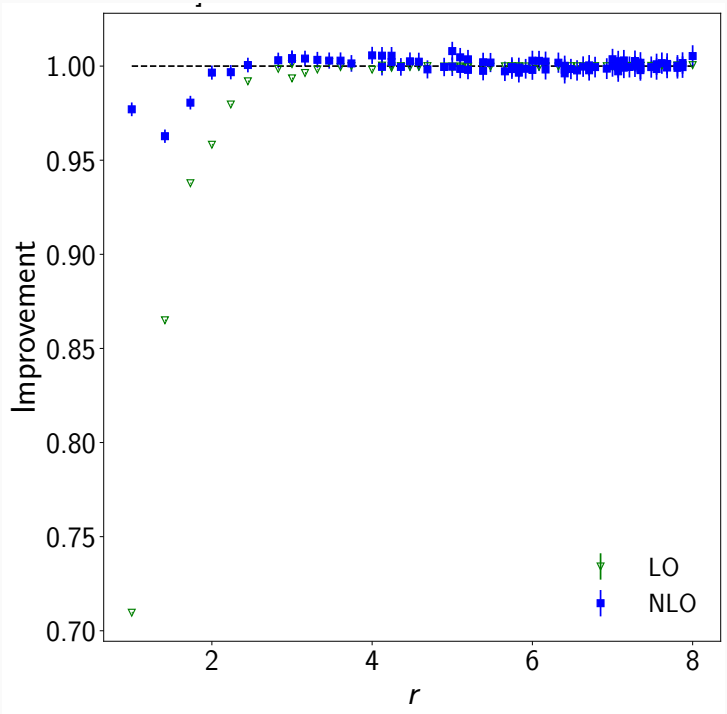


- Measure at a finite set of masses
- Mass dependence of discretization effects small → interpolate in: $\delta E_0(m, r) = E_0(m, r) - E_0(m_{\text{reff}}, r)$, choose $m_{\text{reff}} \simeq 0.06$
- Combine with gluons and fit logarithmic running to get relation to $\overline{V}$-scheme

$$rE_0 = 2\beta_0 \frac{N_c C_F}{(4\pi)^3} (\log(r/a) + B) + r * C \quad \Rightarrow \quad \frac{\Lambda_{\text{lat}}}{\Lambda_{\overline{V}}} = \exp(B)$$

1-loop lattice perturbation theory





- From the lattice perturbation theory, we have

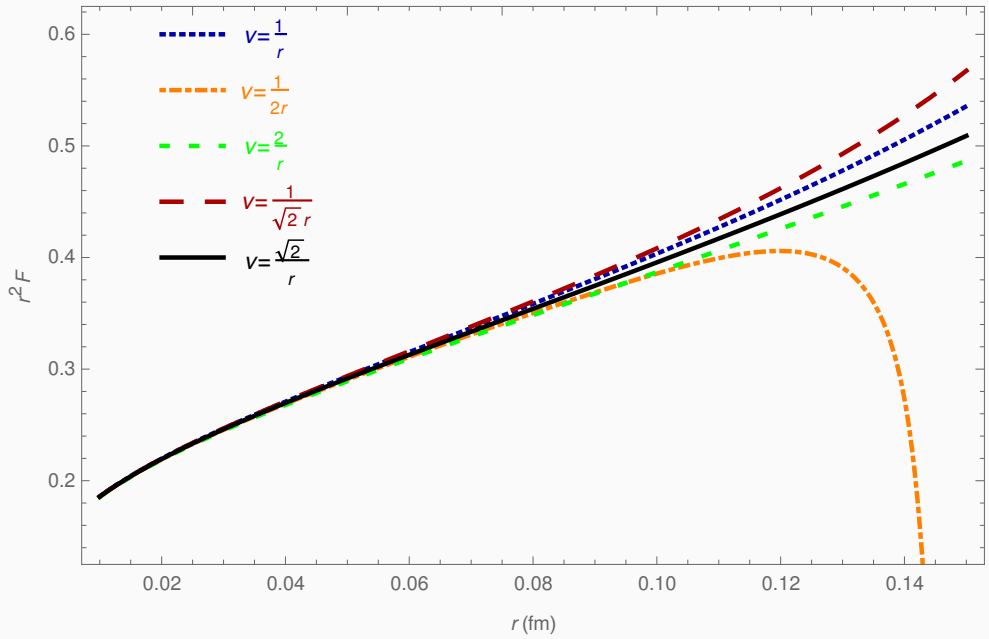
$$E_{0,\mathrm{lat}}^{\text{one loop}} = g_{\mathrm{lat}}^2 E_{0,\mathrm{lat}}^{\mathrm{LO}} + g_{\mathrm{lat}}^4 E_{0,\mathrm{lat}}^{\mathrm{NLO}}$$

- Find r_I such that we match $\overline{V}$-scheme

$$rE_{0,\mathrm{lat}}^{\text{one loop}} = -C_{\mathrm{F}} N_{\mathrm{c}} \alpha_{\overline{V}}(1/r) = -\frac{C_{\mathrm{F}} N_{\mathrm{c}}}{4\pi} g_{\mathrm{lat}}^2 \left[1 + \frac{2\beta_0^{N_{\mathrm{f}}}}{(4\pi)^2} g_{\mathrm{lat}}^2 \left(\log(r/a) - \log(R)\right)\right]$$

- Use the boosted coupling for the lattice coupling $g_{\mathrm{lat}}^2 = \frac{10}{\beta u_0^4}$

Other sources of systematic errors



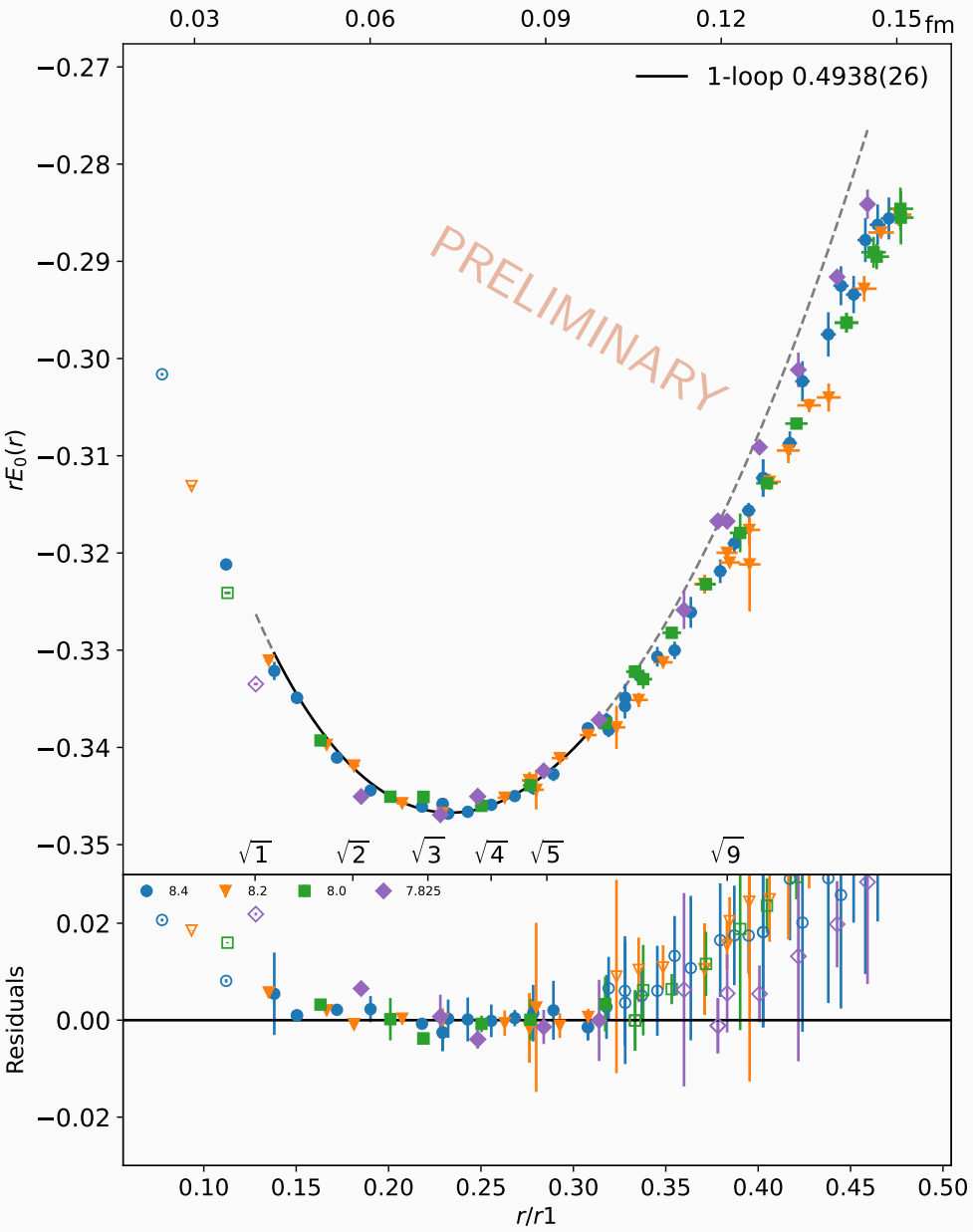
- Soft-scale dependence vary it by factor of $\sqrt{2}$ (or 2)
- Continuum limit: currently just perform joint fit over finest lattices
- US-scale variation, dependence of perturbative expansion order

Figure from A. Bazavov, *et.al.*, PRD100 (2019)

Fit procedure

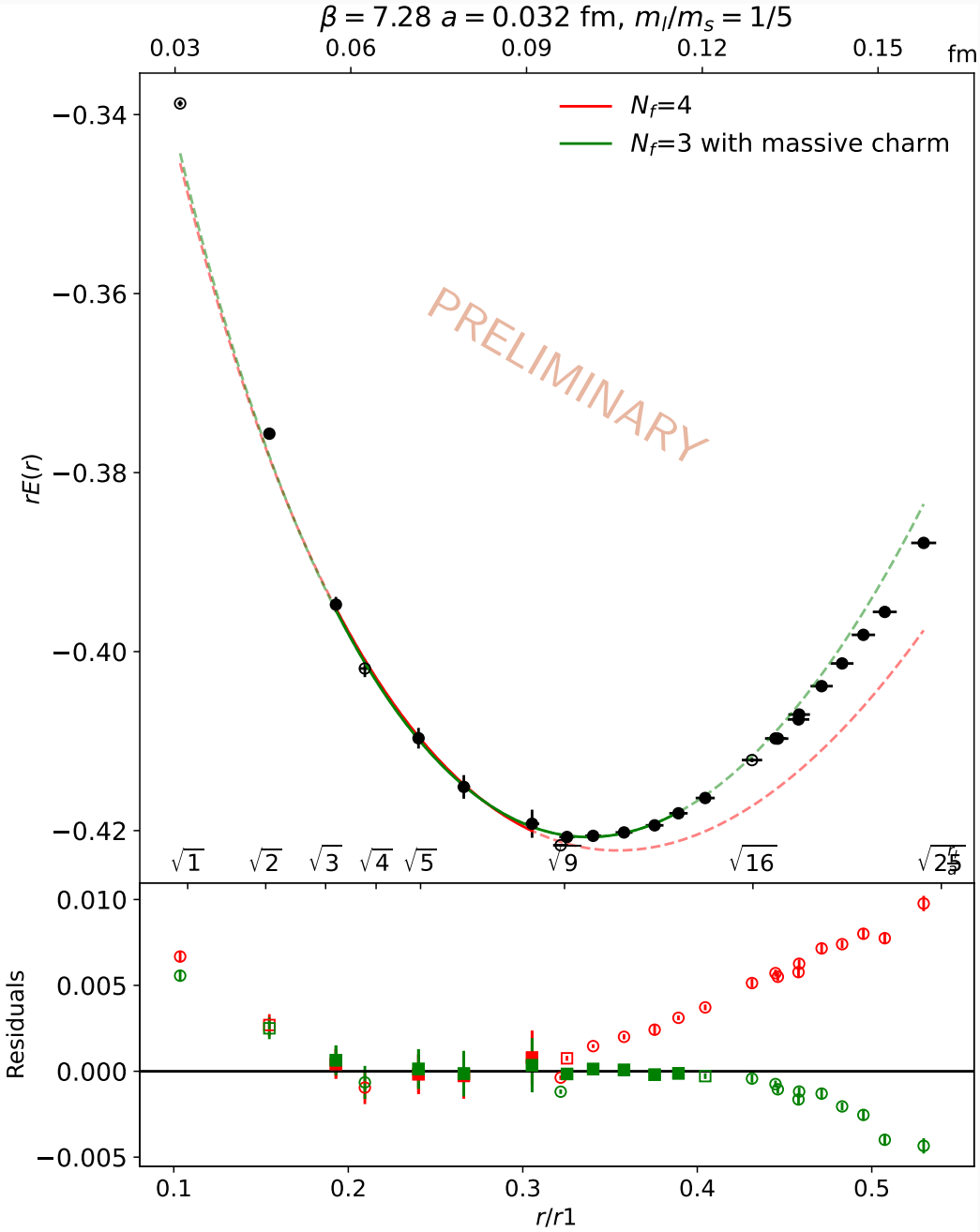
- Fit all ensembles in one combined fit with shared Λ
- Fit parameters Λ and shifts to match all ensembles and fit curve in common scheme
- Perform model average over all fit ranges (AIC)
- Correlated fits with correlation matrix from blocked Jackknife pruned with svd cuts
- Limit fit range to $r < 0.15\,\mathrm{fm}$

Fitting lambda for 2+1



- 1-loop improvement of data
- Missing 2-loop effects → inflate the small r/a errors and ignore first point
- Results agree well with our 2019 determination

Fitting lambda for 2+1+1



- Charm effects noticeable already at $r > 0.1\,\mathrm{fm}$
- Charm effects: limit to 2-loop accuracy
- We get $r_1\Lambda^{(3)} = 0.502(29)$
- This is in agreement with the respective 2-loop result from 2019 2+1 paper

← Example: Finest ensemble, 2-loops no us-resum., MRS

Conclusions

- The 1-loop lattice perturbation theory calculation for HISQ-ensembles finally done
- We can reproduce the 2019 results that used non-perturbative improvement
- Systematic error analysis still pending
- Aim to update the 2019 number. Mean expected to stay the same. 2019 result: $\alpha_{\mathrm{s}}(M_{\mathrm{Z}}) = 0.11660^{+0.00110}_{-0.00056}$
- Next: apply everything properly to 2+1+1 system
- Early results indicate good agreement with also 2+1+1 theory
- Charm effects still need further refinement

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Thank you for your attention!