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Energy-Momentum Correlation Functions for Transport Coefficients from Lattice QCD

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Motivation: Transport Coefficients

Hydrodynamics:

- Quark-Gluon Plasma modeled by Hydrodynamics
- Used for Early-Universe Physics, Neutron Star Mergers and Heavy-Ion Collisions
- Solve Conservation Law of the Energy-Momentum Tensor (EMT)

$$\partial_\mu T^{\mu\nu} = 0$$

- Gradient-in-u expansion [1] for the EMT

$$T^{\mu\nu} = \underbrace{\epsilon u^\mu u^\nu + P\Delta^{\mu\nu}}_{\text{0th Order}} \underbrace{-\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial_\alpha u^\alpha}_{\text{1st Order}} \underbrace{-\lambda_3\Omega^{\langle\mu}_{\alpha}\Omega^{\nu\rangle\alpha}_{\alpha} + \cdots}_{\text{2nd and higher Orders}}$$

with $\Delta^{\mu\nu} := \eta^{\mu\nu} + u^\mu u^\nu$ and

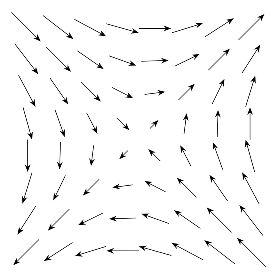
$$\sigma^{\mu\nu} := \Delta^{\mu\alpha}\Delta^{\nu\beta}\left[\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\partial_\gamma u^\gamma\right]$$

$$\Omega^{\mu\nu} := \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}\left[\partial_\alpha u_\beta - \partial_\beta u_\alpha\right]$$

Transport Coefficients couple to different types of flow

Shear Flow

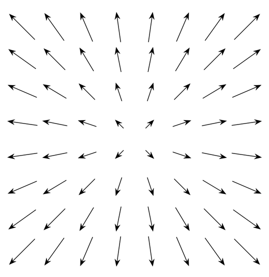
$\eta \leftrightarrow \partial^\mu u^\nu + \partial^\nu u^\mu$



Dissipative

Bulk Flow

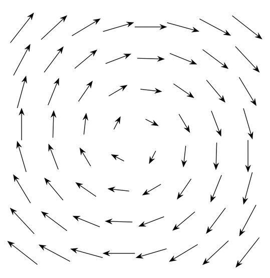
$\zeta \leftrightarrow \partial^\alpha u^\alpha$



Dissipative

Vortical Flow

$\lambda_3 \leftrightarrow \partial^\mu u^\nu - \partial^\nu u^\mu$



Not dissipative

Connection to Transport Coefficients

Kubo Formulae:

- Interrelate Transport Coefficients (η, ζ, λ_3) to EMT Correlators
- Dissipative Transport Coefficients via Analytic Continuation:

$$\eta = \lim_{\omega\to 0}\frac{\rho_\eta(\omega)}{\omega} \qquad \rho_\eta(\omega) = \frac{1}{10}\int \mathrm{d}t \int \mathrm{d}^3 r\ \mathrm{e}^{\mathrm{i}\omega t}\langle[\pi_{ij}(t,\boldsymbol{r}), \pi_{ij}(0,\mathbf{0})]\rangle$$

$$\text{with } \pi_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}T_{kk}$$

$$\zeta = \frac{1}{9}\lim_{\omega\to 0}\frac{\rho_\zeta(\omega)}{\omega} \quad \rho_\zeta(\omega) = \int \mathrm{d}t \int \mathrm{d}^3 r\ \mathrm{e}^{\mathrm{i}\omega t}\langle[T_{\mu\mu}(t,\boldsymbol{r}), T_{\nu\nu}(0,\mathbf{0})]\rangle$$

$\Longrightarrow$ Minkowski-time-dependence needed

$\Longrightarrow$ Lattice only offers Euclidean time ⚠

- Thermodynamic Coefficients without Analytic Continuation [2]:

$$\lambda_3 = 2(f_1' - 4f_3) \qquad f_1' = T\frac{\mathrm{d}}{\mathrm{d}T}f_1$$

$$f_1 = -\frac{1}{2}\int \mathrm{d}\tau \int \mathrm{d}^3 r\ z^2\langle T_{xy}(\tau,\boldsymbol{r})T_{xy}(0,\mathbf{0})\rangle$$

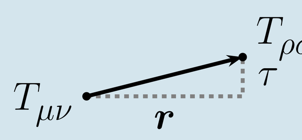
$$f_3 = -\frac{1}{2}f_1 + \frac{1}{4}\int \mathrm{d}\tau \int \mathrm{d}^3 r\ z^2\langle T_{xt}(\tau,\boldsymbol{r})T_{xt}(0,\mathbf{0})\rangle$$

$\Longrightarrow$ Included Euclidean-time Average

$\Longrightarrow$ Directly accessible from the Lattice

Relevant Observable: General EMT Correlator

$$G_{\mu\nu\rho\sigma}(\tau,\boldsymbol{r}) = \langle T_{\mu\nu}(\tau,\boldsymbol{r})T_{\rho\sigma}(0,\mathbf{0})\rangle$$



Problems of EMT Correlators on the Lattice

Different Treatments for the different Transport Coefficients: ⚠

- τ-Dependence (Averaged τ for λ_3, general τ for η, ζ)
- EMT Components (Spatial, traceless for η, trace for ζ, mix for λ_3)
- Extra Integrand (Additional z^2 for λ_3, none for η, ζ)

Properties of EMT Correlators:

- Components of $G_{\mu\nu\rho\sigma}(\tau,\boldsymbol{r})$ are related via spatial rotations around the separation axis $\boldsymbol{r}$
- Terrible large-distance behavior on the Lattice ⚠

Current two Approaches:

- Either store $G_{\mu\nu\rho\sigma}(\tau,\boldsymbol{r})$ for all indices and separations $(\tau,\boldsymbol{r})$
  - $\Longrightarrow$ Store more data than needed ⚠
  - $\Longrightarrow$ Complex dependency between $\boldsymbol{r}$ and (μ,ν,ρ,σ) ⚠
- or store evaluated spatial integrals, e.g. $\int \mathrm{d}^3 r\langle T_{xy}(\tau,\boldsymbol{r})T_{xy}(0,\mathbf{0})\rangle$
  - $\Longrightarrow$ Data is restricted to the chosen Application ⚠
  - $\Longrightarrow$ Bad Signal-to-Noise at large r distorts the Integral ⚠

Key Questions:

- How to handle the **interrelations between the components** $G_{\mu\nu\rho\sigma}(\tau,\boldsymbol{r})$ due to the remaining rotational symmetry and the **complex dependency** between $\boldsymbol{r}$ and (μ,ν,ρ,σ) in an approach that is **applicable to multiple transport coefficients**?

$\Longrightarrow$ Tensor Decomposition

- How to handle the **bad signal-to-noise** for large $\boldsymbol{r}$?

$\Longrightarrow$ Tail Fitting

Tensor Decomposition

General Idea

Decompose $G_{\mu\nu\rho\sigma}$ into all elementary tensors $P^X_{\mu\nu\rho\sigma}$ that respect the relevant symmetries

$$G_{\mu\nu\rho\sigma}(\tau,\boldsymbol{r}) = \sum_{X\in\mathcal{I}} G^X(\tau,r^2)P^X_{\mu\nu\rho\sigma}(\hat{\boldsymbol{r}})$$

Advantages:

- Analyze only the so-called **Component Functions**

$$G^X(\tau,r^2) = \frac{1}{M^X}P^X_{\mu\nu\rho\sigma}(\hat{\boldsymbol{r}})G_{\mu\nu\rho\sigma}(\tau,\boldsymbol{r})$$

$\Longrightarrow$ Only 5, 10 or 14 instead of 55 components

- Separated angular dependence in $P^X_{\mu\nu\rho\sigma}(\hat{\boldsymbol{r}})$ from radial dependence in $G^X(\tau,r^2)$
  - $\Longrightarrow$ Store only r^2-dependent (instead of $\boldsymbol{r}$-dependent) data
  - $\Longrightarrow$ Allows for an easier large-r treatment
- Component functions G^X are either positive or negative semi-definite due to reflection-positivity-like arguments

Connection to previous Relations:

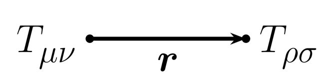
- Every desired integral can be expressed in terms of component functions and projectors and reduced to a one-dimensional integral after evaluating the angular integral, e.g.:

$$\begin{aligned}\lambda_3 &\leftarrow \int \mathrm{d}\tau \int \mathrm{d}^3 r\ z^2\langle T_{xy}(\tau,\boldsymbol{r})T_{xy}(0,\mathbf{0})\rangle \\ &= \int \mathrm{d}^3 r\ z^2\left[\sum_{X\in\mathcal{I}_{\bar{\tau}}} P^X_{xyxy}(\hat{\boldsymbol{r}})G^X(r^2)\right] \\ &= 4\pi\int \mathrm{d}r\ r^4\left[\frac{11}{105}G^{\mathrm{TT}}(r^2) + \frac{1}{21}G^{\mathrm{RT}}(r^2) + \frac{1}{70}G^{\ell\ell}(r^2)\right]\end{aligned}$$

In the Vacuum

Infinite temporal dimension: $t_{\mathrm{E}} \in \mathbb{R}$

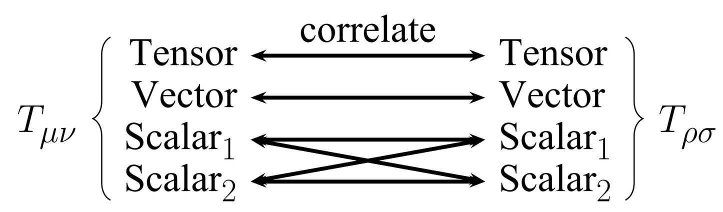
$\Longrightarrow$ Equivalent to any spatial dimension



EMT decomposes into (for $\boldsymbol{r} \parallel \boldsymbol{e}_0$)

$$T_{\mu\nu}\begin{cases}\text{Tensor:} & 5 \text{ components } T_{xy}, T_{xx}-T_{yy}, \dots \\ \text{Vector:} & 3 \text{ components } T_{0x}, \dots \\ \text{Scalar}_1\text{:} & 1 \text{ component } T_{\mu\mu} \\ \text{Scalar}_2\text{:} & 1 \text{ component } T_{00} - \frac{1}{3}T_{ii}\end{cases}$$

Correlations only between same representations



$\Longrightarrow$ Only $1+1+3=5$ Component Functions (instead of 55)

Find 5 Projectors

$$P^{\mathrm{TT}}_{\mu\nu\rho\sigma} := \frac{1}{2}(\delta^{\mathrm{T}}_{\mu\rho}\delta^{\mathrm{T}}_{\nu\sigma} + \delta^{\mathrm{T}}_{\mu\sigma}\delta^{\mathrm{T}}_{\nu\rho}) - \frac{1}{D-1}\delta^{\mathrm{T}}_{\mu\nu}\delta^{\mathrm{T}}_{\rho\sigma}$$

$$P^{\mathrm{RT}}_{\mu\nu\rho\sigma} := \frac{1}{2}\left[\hat{r}_\mu\hat{r}_\rho\delta^{\mathrm{T}}_{\nu\sigma} + \hat{r}_\nu\hat{r}_\rho\delta^{\mathrm{T}}_{\mu\sigma} + (\rho\leftrightarrow\sigma)\right]$$

$$P^{\mathrm{ss}}_{\mu\nu\rho\sigma} := C^{\mathrm{s}}_{\mu\nu}C^{\mathrm{s}}_{\rho\sigma} \qquad P^{\ell\ell}_{\mu\nu\rho\sigma} := C^{\ell}_{\mu\nu}C^{\ell}_{\rho\sigma}$$

$$P^{\mathrm{s}\ell}_{\mu\nu\rho\sigma} := C^{\mathrm{s}}_{\mu\nu}C^{\ell}_{\rho\sigma} + C^{\ell}_{\mu\nu}C^{\mathrm{s}}_{\rho\sigma}$$

with

$$C^{\mathrm{s}}_{\mu\nu} := \frac{1}{\sqrt{D}}\delta_{\mu\nu} \qquad C^{\ell}_{\mu\nu} := \frac{1}{\sqrt{D(D-1)}}(D\hat{r}_\mu\hat{r}_\nu - \delta_{\mu\nu})$$

$$\delta^{\mathrm{T}}_{\mu\nu} := \delta_{\mu\nu} - \hat{r}_\mu\hat{r}_\nu$$

At finite Temperature - τ-Average

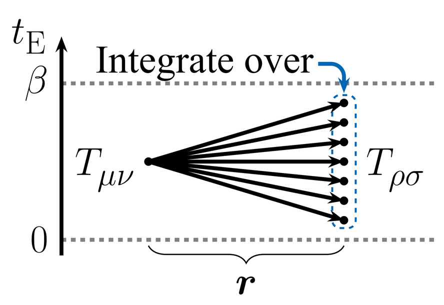
Finite temporal dimension: $t_{\mathrm{E}} \in [0,\beta]$

$\Longrightarrow$ Less Symmetry

$\Longrightarrow$ More Component Functions

But τ-Average restricts them

$\Longrightarrow$ Only even powers in u



EMT decomposes into (for $\boldsymbol{r} \parallel \boldsymbol{e}_z$)

$$T_{\mu\nu}\begin{cases}\text{1 Tensor:} & (T_{xy}, T_{xx}-T_{yy}, \dots) \\ \text{2 Vectors:} & (T_{0x}, T_{0y}) \text{ and } (T_{xz}, T_{yz}) \\ \text{4 Scalars:} & T_{tt}, T_{zz}, T_{0z} \text{ and } T_{xx}+T_{yy}\end{cases}$$

Before τ-Average $\Longrightarrow$ Only $1+3+10=14$ Component Functions

After τ-Average $\Longrightarrow$ Only $1+2+7=10$ Component Functions

Find 14 Projectors:

$$P^{\mathrm{TT}}_{\mu\nu\rho\sigma} := \frac{1}{2}\left(\delta^{\mathrm{sT}}_{\mu\rho}\delta^{\mathrm{sT}}_{\nu\sigma} + \delta^{\mathrm{sT}}_{\mu\sigma}\delta^{\mathrm{sT}}_{\nu\rho} - \delta^{\mathrm{sT}}_{\mu\nu}\delta^{\mathrm{sT}}_{\rho\sigma}\right)$$

$$P^{\mathrm{RT}}_{\mu\nu\rho\sigma} := \frac{1}{2}\left[\hat{r}_\mu\hat{r}_\rho\delta^{\mathrm{sT}}_{\nu\sigma} + \hat{r}_\mu\hat{r}_\sigma\delta^{\mathrm{sT}}_{\nu\rho} + (\mu\leftrightarrow\nu)\right]$$

$$P^{\mathrm{UR}}_{\mu\nu\rho\sigma} := \frac{1}{2}(\hat{r}_\mu u_\nu + \hat{r}_\nu u_\mu)(\hat{r}_\rho u_\sigma + \hat{r}_\sigma u_\rho)$$

$$P^{\mathrm{UT}}_{\mu\nu\rho\sigma} := \frac{1}{2}\left[u_\mu u_\rho\delta^{\mathrm{sT}}_{\nu\sigma} + u_\mu u_\sigma\delta^{\mathrm{sT}}_{\nu\rho} + (\mu\leftrightarrow\nu)\right]$$

$$P^{aa}_{\mu\nu\rho\sigma} := C^{a}_{\mu\nu}C^{a}_{\rho\sigma} \qquad P^{ab}_{\mu\nu\rho\sigma} := C^{a}_{\mu\nu}C^{b}_{\rho\sigma} + C^{b}_{\mu\nu}C^{a}_{\rho\sigma}$$

with $a \neq b \in \{\mathrm{s}, \ell, \mathrm{w}\}$ and

$$C^{\mathrm{s}}_{\mu\nu} := \frac{1}{2}\delta_{\mu\nu} \quad C^{\ell}_{\mu\nu} := \frac{1}{\sqrt{6}}\left(3\hat{r}_\mu\hat{r}_\nu - \delta^{\mathrm{s}}_{\mu\nu}\right) \quad C^{\mathrm{w}}_{\mu\nu} := \frac{1}{\sqrt{12}}\left(3u_\mu u_\nu - \delta^{\mathrm{s}}_{\mu\nu}\right)$$

$$\delta^{\mathrm{s}}_{\mu\nu} := \delta_{\mu\nu} - u_\mu u_\nu \qquad \delta^{\mathrm{sT}}_{\mu\nu} := \delta_{\mu\nu} - u_\mu u_\nu - \hat{r}_\mu\hat{r}_\nu$$

Tail Fitting

The Fit Function

Position-Space Energy-Momentum Conservation implicates *differential* Relations between Component Functions:

- In the Vacuum: 3 Relations for 5 Component Functions
- Finite T, averaged τ: 5 Relations for 10 Component Functions
- Finite T, general τ: 10 Relations for 14 Component Functions

$\rightarrow$ Remaining D.o.F.s imply Spectral Decomposition

Spectral Decomposition (Finite T, averaged τ as an Example):

$$G^X(r^2) = \int_0^\infty \mathrm{d}m\ m\sum_{S\in\mathcal{S}_{\bar{\tau}}}\rho^S(m)\sum_{i=1}^{3}c_i^{S,X}f_i(m,r)$$

- 5 Spectral Functions ρ^S for 10 Component Functions G^X
- Three functions f_i

$$f_1(m,r) = \frac{m}{4\pi}\frac{\mathrm{e}^{-mr}}{mr}$$

$$f_2(m,r) = \frac{m}{4\pi}\frac{\mathrm{e}^{-mr}}{(mr)^3}\left((mr)^2 + 2mr + 2\right)$$

$$f_3(m,r) = \frac{m}{4\pi}\frac{\mathrm{e}^{-mr}}{(mr)^5}\left((mr)^4 + 4(mr)^3 + 12(mr)^2 + 24mr + 24\right)$$

- Coefficients $c_i^{S,X}$ worked out in [3]

Fitting Lattice Data

Procedure:

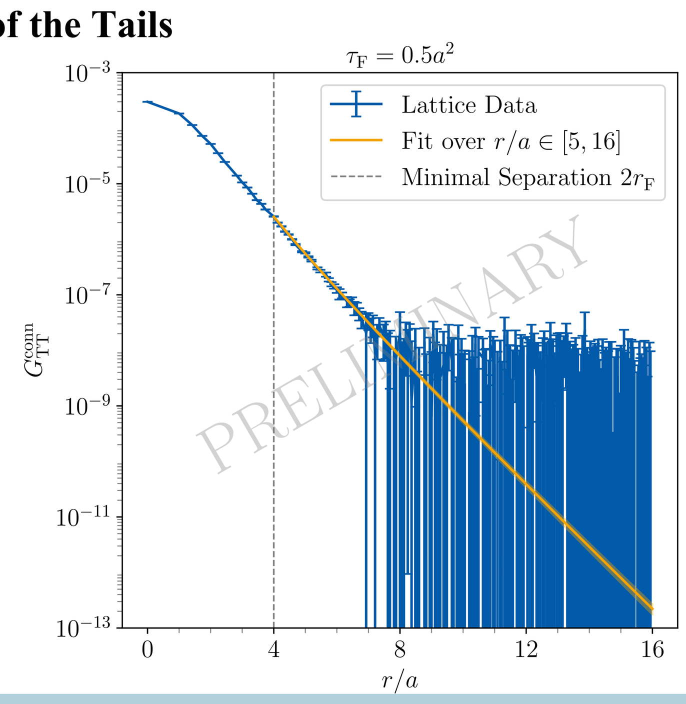
- Perform Gradient Flow on the Fields $\rightarrow$ Reduced UV Fluctuations
- Measure EMT Correlator on the Lattice via Fourier-Method

$$G_{\mu\nu\rho\sigma}(\tau,\boldsymbol{r}) \propto \sum_{\omega_{\mathrm{E}},\boldsymbol{p}} T_{\mu\nu}(\omega_{\mathrm{E}},\boldsymbol{p})T^*_{\rho\sigma}(\omega_{\mathrm{E}},\boldsymbol{r})\mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{E}}\tau + \boldsymbol{p}\cdot\boldsymbol{r})}$$

- Perform Tensor Decomposition $\rightarrow$ Get 10 Functions $G^X(r^2)$
- Subtract disconnected Parts $\rightarrow$ Consider only connected Parts
- Perform **Correlated Fit of the Tails**

Parameter:

- Lattice: 8×32^3
- $\beta = 7.544$
- $T/T_c \approx 6.77$



Conclusions

Transport Coefficients:

- Needed for Hydrodynamical Studies for Heavy-Ion Collisions, Neutron-Star Merges, Early Universe Physics
- Their Calculation requires precise EMT Correlators

Tensor Decomposition $\rightarrow$ Efficient EMT Correlator Calculation:

- Calculate fewer Component Functions (5, 10 or 14 vs. 55)
- Radial dependence r^2 instead of vectorial dependence $\boldsymbol{r}$
- Component Functions are positive or negative semi-definite

Lattice Results exist preliminarily:

- Vacuum Subtraction to be performed
- Zero-Flow-Time Limit to be performed
- Continuum Limit to be performed

$\Longrightarrow$ Work in Progress

References

[1] G. D. Moore and K. A. Sohrabi, **106**, 122302, 1007.5333 .

[2] P. Kovtun and A. Shukla, **2018**, 7, 1806.05774 .

[3] G. D. Moore and J. Winter, “Tensor Decomposition for Energy-Momentum Correlation Functions,” 2606.10526 .

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