Real-time stochastic dynamics near a QCD critical point

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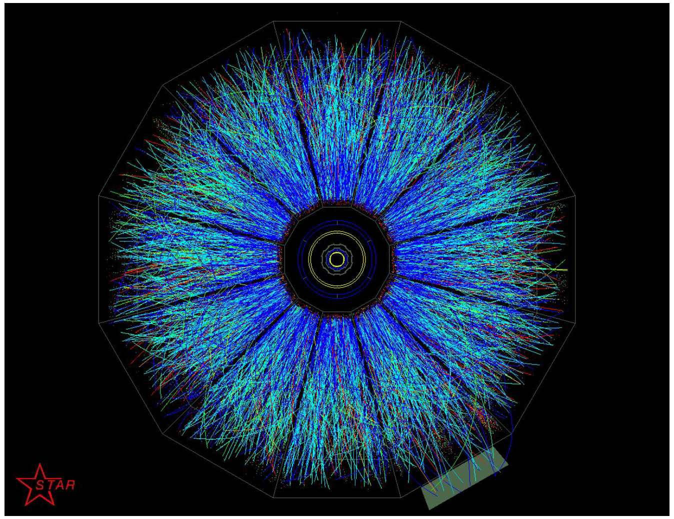
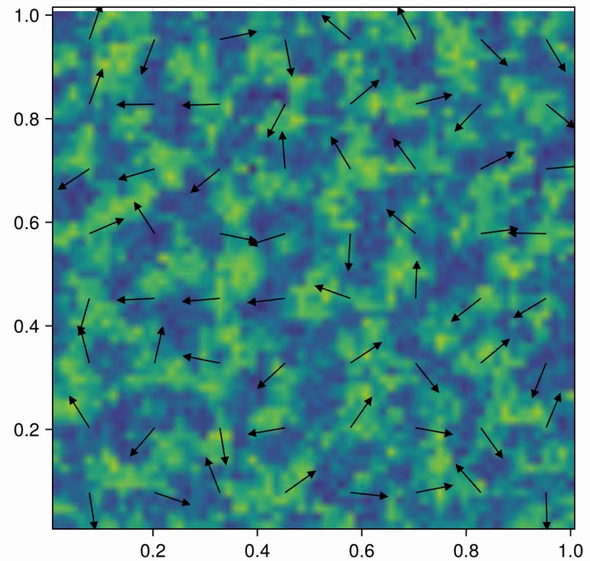
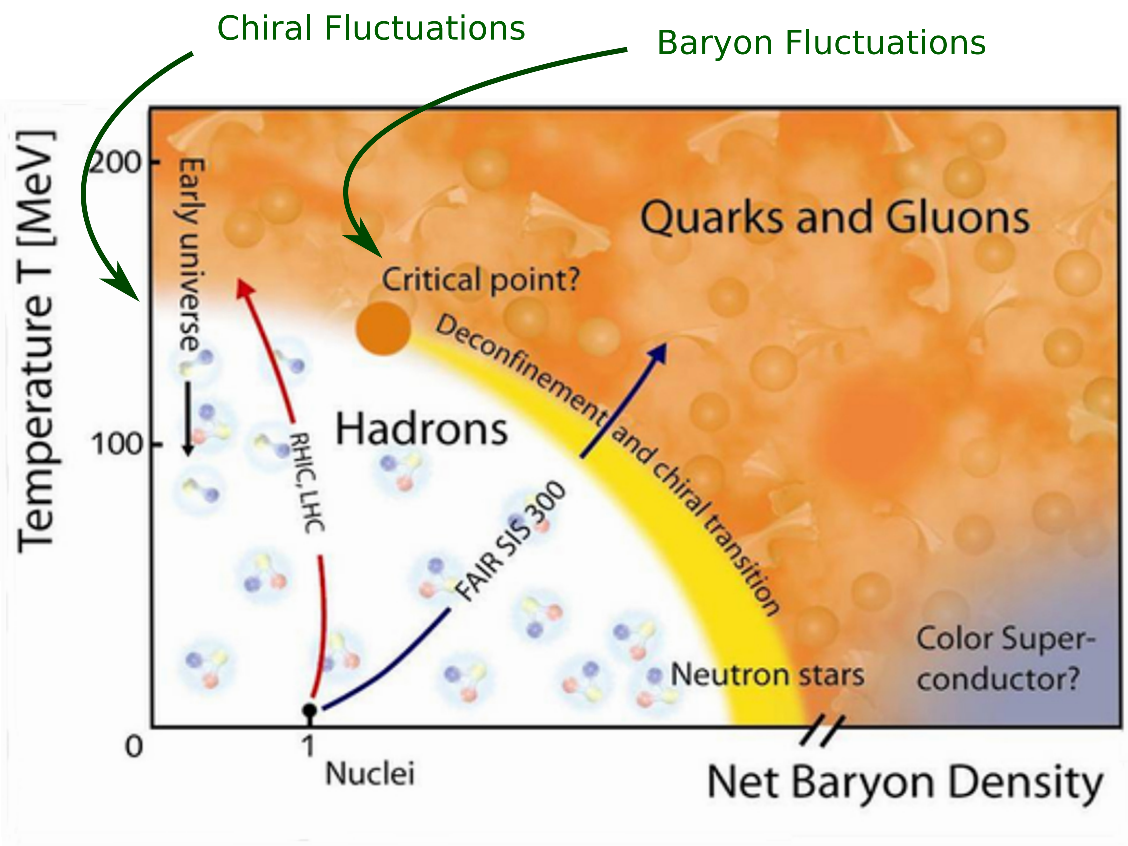




With C. Chattopadhyay, J. Ott, V. Skokov

References: 2403.10608, 2411.15994, 2510.12557, 2603.21479

Motivation



Can we locate the chiral phase transition, or the endpoint of a first-order QGP-hadron gas transition?

Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD)

Chiral phase transition: Model G (Rajagopal & Wilzcek)

Possible critical endpoint: Model H (Son & Stephanov)

Digression: Diffusion

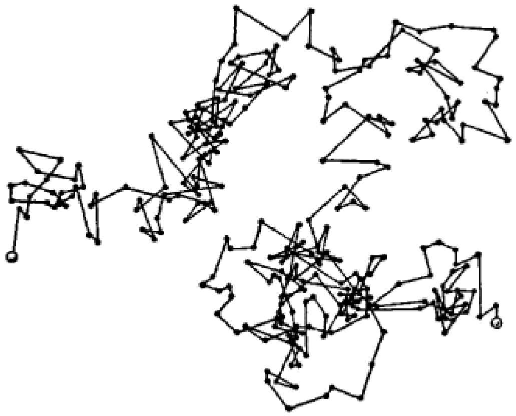
Consider a Brownian particle

$$\dot{p}(t) = -\gamma_D p(t) + \zeta(t) \qquad \langle \zeta(t)\zeta(t') \rangle = \kappa \delta(t - t')$$

drag (dissipation) white noise (fluctuations)

For the particle to eventually thermalize

$$\langle p^2 \rangle = 2mT$$



drag and noise must be related

$$\kappa = \frac{mT}{\gamma_D}$$

Einstein (Fluctuation-Dissipation)

Relation to Effective Field Theory

This theory is equivalent to a 0+1 EFT, the KMS (or MSR) theory

$$Z = \int \mathcal{D}x_a \mathcal{D}x_r \, e^{i\int dt L} \qquad L = M\dot{x}_r\dot{x}_a - cx_a x_r - \nu x_a \dot{x}_r + \frac{i}{2}\sigma x_a^2$$

with $2\nu/\sigma = 1/T$. Follows from

$$Z = \int \mathcal{D}x_r \mathcal{D}\xi \; \delta(M\ddot{x}_r - \nu\dot{x}_r + cx_r - \xi) e^{-\int dt\, \xi^2/(2\sigma)}$$

Fluctuation-Dissipation relation encoded in KMS symmetry

$$x_r(t) \to x_r(-t)$$
$$x_a(t) \to x_a(-t) + i\beta\partial_t x_r(-t)$$

Glorioso, Liu, arXiv:1805.09331

Hydrodynamic equation for critical mode

Equation of motion for critical mode ϕ ("model H")

$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \frac{\delta \mathcal{F}}{\delta \phi} - g \left(\vec{\nabla}\phi\right) \cdot \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} + \zeta \qquad\qquad (g = 1)$$

Diffusion Advection Noise

Equation of motion for momentum density π

$$\frac{\partial \vec{\pi}^T}{\partial t} = \eta \nabla^2 \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} + g \left(\vec{\nabla}\phi\right) \cdot \frac{\delta \mathcal{F}}{\delta \phi} - g \left(\frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \cdot \vec{\nabla}\right) \vec{\pi}^T + \vec{\xi}$$

Free energy functional: Order parameter ϕ, momentum density $\vec{\pi} = w\vec{v}$

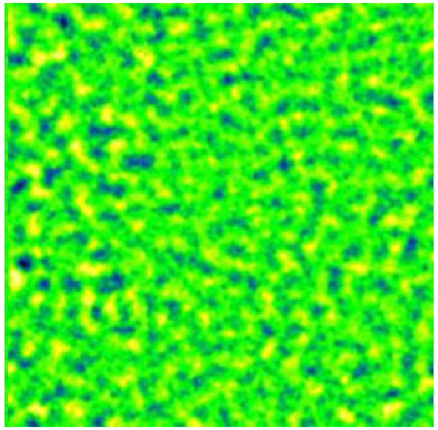
$$\mathcal{F} = \int d^3x \left[\frac{1}{2w}\vec{\pi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4\right] \qquad\qquad D = m^2\kappa$$

Fluctuation-Dissipation relation

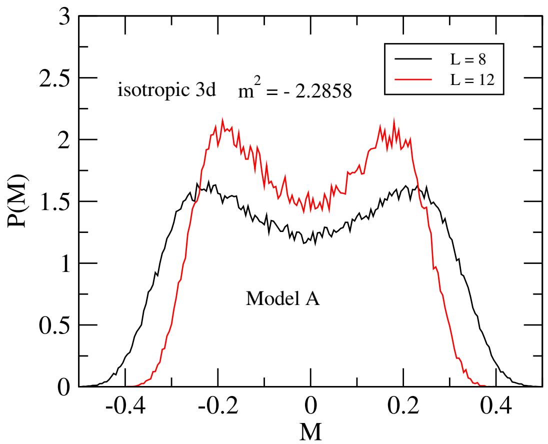
$$\langle\zeta(x,t)\zeta(x',t')\rangle = -2\kappa T\nabla^2\delta(x-x')\delta(t-t')$$

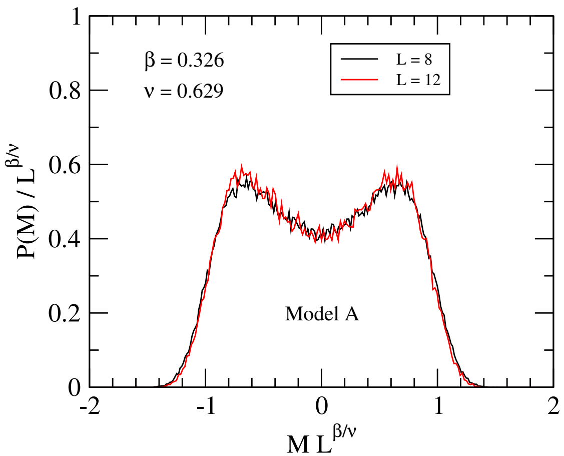
$$\langle\xi_i(x,t)\xi_j(x',t')\rangle = -2\eta T\nabla^2 P^T_{ij}\delta(x-x')\delta(t-t')$$

ensures $P[\phi,\vec{\pi}] \sim \exp(-\mathcal{F}[\phi,\vec{\pi}]/T)$



Tune m^2 to critical point $m^2 = m_c^2$ (Ising critical point)





Why direct numerical simulation?

Critical dynamics is non-perturbative.

Need non-perturbative approach capable of dealing with non-trivial background.

Numerical realization

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = \Gamma T \delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{\Gamma T}{(\Delta t) a^3}}\theta\right] \qquad \langle \theta^2 \rangle = 1$$

Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

Idea: Use Metropolis update

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\theta \qquad p = min(1, e^{-\beta \Delta \mathcal{F}})$$

Numerical realization

Central observation

$$
\begin{aligned}
\langle \psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x}) \rangle &= -(\Delta t)\,\Gamma\,\frac{\delta \mathcal{F}}{\delta \psi} + O\left((\Delta t)^2\right) \\
\langle [\psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})]^2 \rangle &= 2(\Delta t)\,\Gamma\,T + O\left((\Delta t)^2\right) \,.
\end{aligned}
$$

Metropolis realizes both diffusive and stochastic step. Also

$$
P[\psi] \sim \exp(-\beta \mathcal{F}[\psi])
$$

Note: Still have short distance noise; need to adjust bare parameters such as Γ, m^2, λ to reproduce physical quantities.

Numerical realization: Model H

Model H: Conserving update

$$\pi_\nu^{trial}(\vec{x}, t+\Delta t) = \pi_\nu(\vec{x}, t) + r_{\nu\mu}\,,$$
$$\pi_\nu^{trial}(\vec{x}+\hat{\mu}, t+\Delta t) = \pi_\nu(\vec{x}+\hat{\mu}, t) - r_{\nu\mu}\,,$$

$$r_{\nu\mu} = \sqrt{2\eta T(\Delta t)}\,\zeta_\nu\,.$$

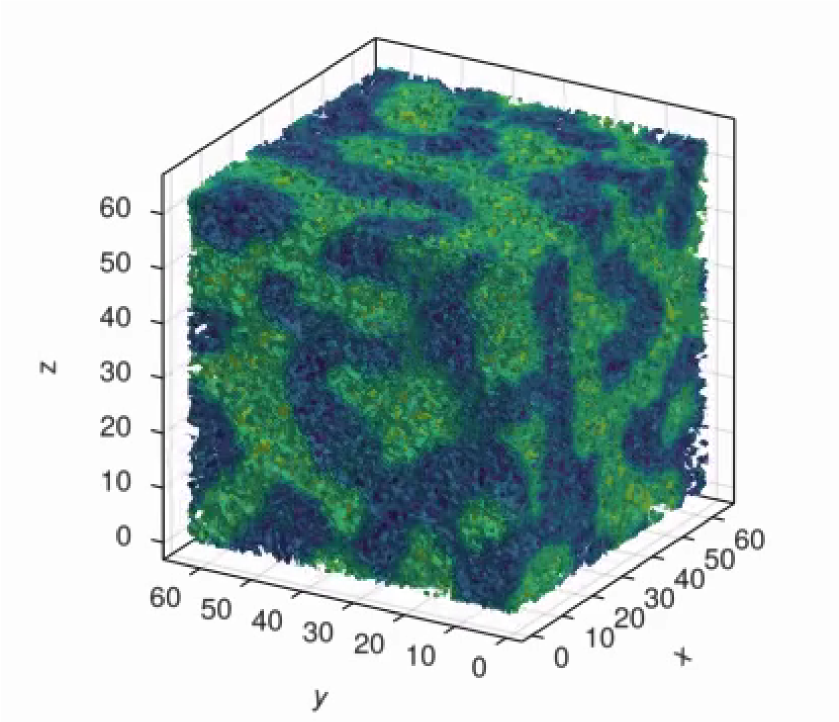
Advection (PB terms) conserves $\mathcal{H}$. On the lattice use "skew" discretized derivatives

$$\dot{\phi} = -\frac{1}{\rho}\,\pi_\mu^T\,\nabla_\mu^c\phi\,,$$
$$\dot{\pi}_\mu^T = -\left[\frac{1}{2}\nabla_\nu^c\left(\frac{1}{\rho}\pi_\nu^T\pi_\mu^T\right) + \frac{1}{2\rho}\pi_\nu^T\,\nabla_\nu^c\pi_\mu^T + \left(\nabla_\mu^c\phi\right)\left(\nabla_\nu^c\nabla_\nu^c\phi\right)\right]\,,$$

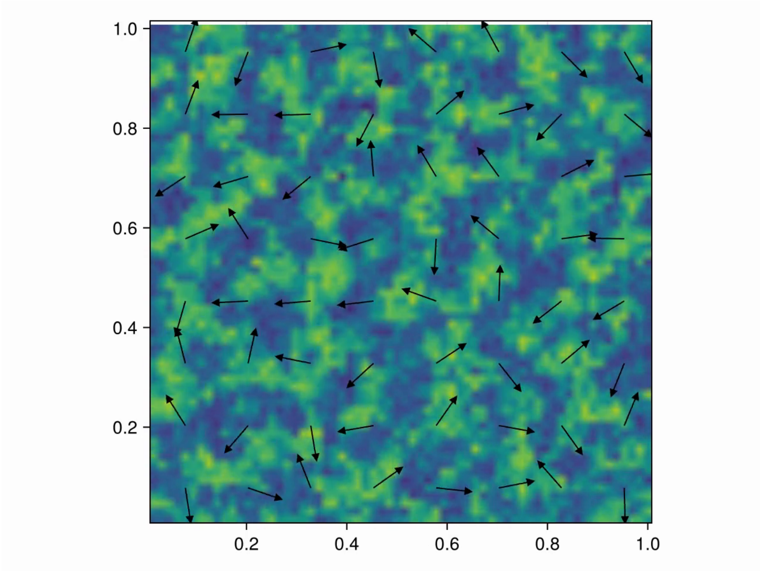
and project on π_μ^T using Fourier transforms.

Numerical results (critical Navier-Stokes)

Order parameter (3d)



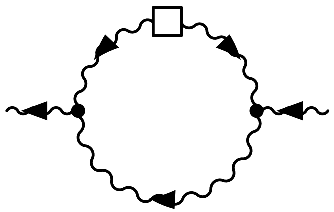
Order parameter/velocity field (2d)



Renormalized viscosity

Renormalization of η

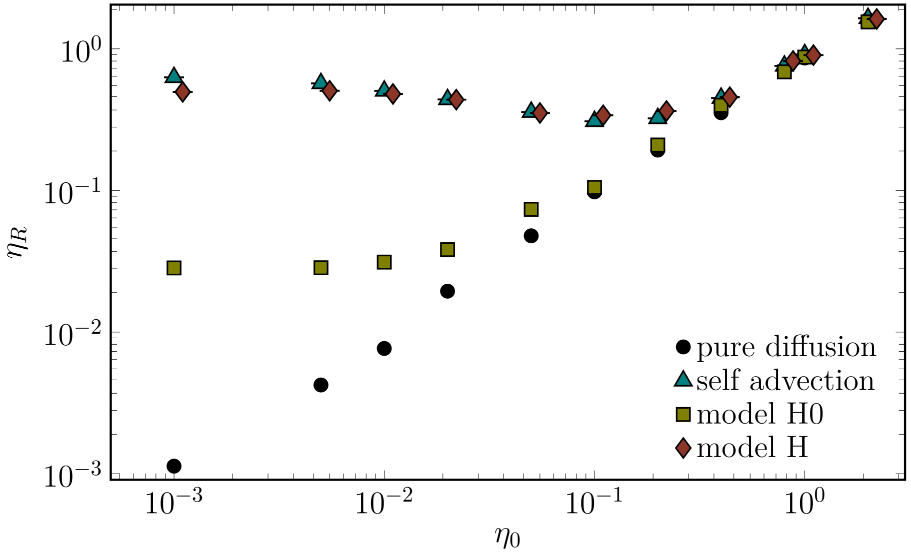
"Stickiness of shear waves"



$$\eta_R = \eta + \frac{7}{60\pi^2}\frac{\rho T \Lambda}{\eta}$$

Leads to minimum viscosity

Top: Model H

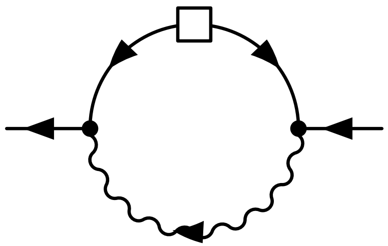


Middle: No self-advection

Bottom: No advection

Relaxation Rate

Order parameter relaxation rate

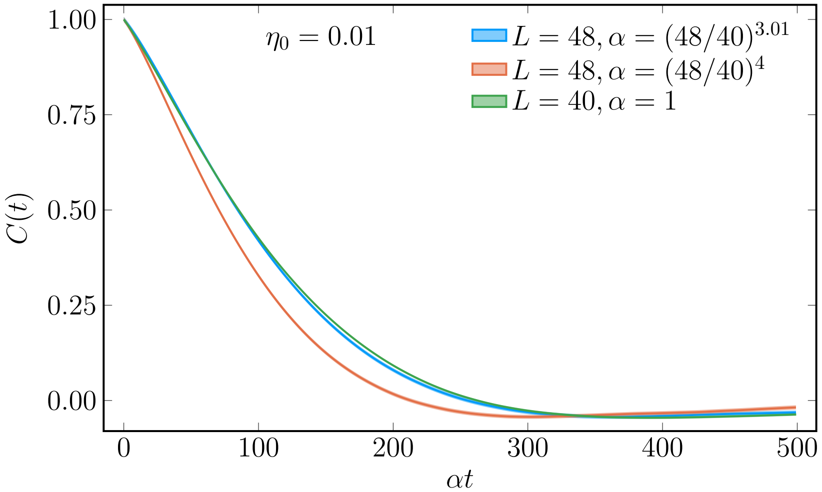


$$\Gamma_k = \frac{\Gamma}{\xi^4}(k\xi)^2\left(1+(k\xi)^2\right) + \frac{T}{6\pi\eta_R\xi^3}K(k\xi)$$

Crossover from $\tau_R \sim \xi^4$ at large η_R

to $\tau_R \sim \xi^3$ for small η_R

$$C(t) = \langle \phi_k(0)\phi_{-k}(t)\rangle$$

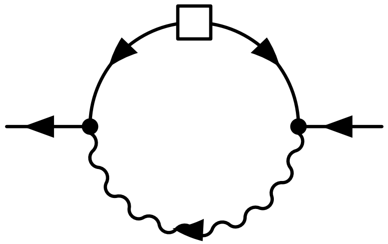


Dynamic Scaling:

$$z(\eta = 0.01) = 3.07$$

Relaxation Rate

Order parameter relaxation rate



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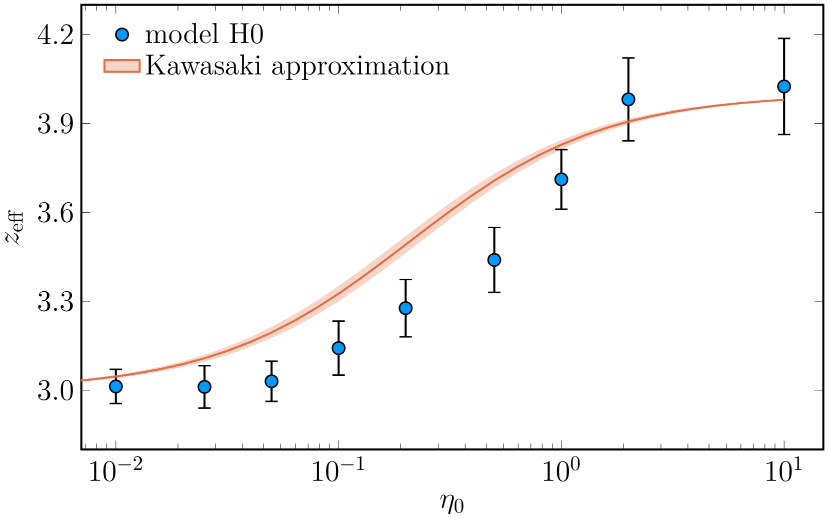
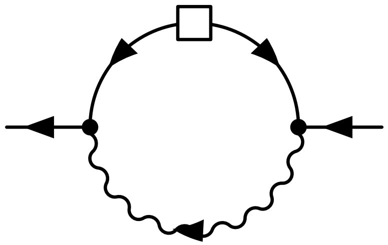


Crossover from $\tau_R \sim \xi^4$ at large η_R

to $\tau_R \sim \xi^3$ for small η_R

Universality: model H/H0

Order parameter relaxation rate



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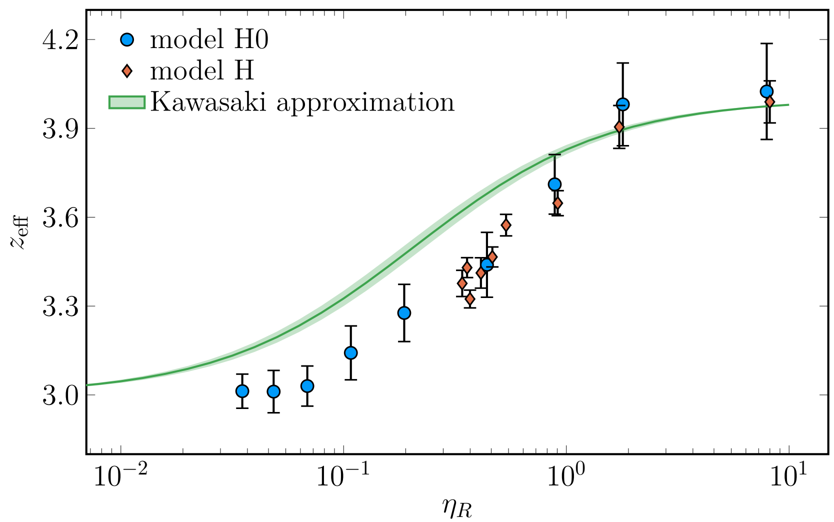


Crossover from $\tau_R \sim \xi^4$ at large η_R

to $\tau_R \sim \xi^3$ for small η_R

Relativistic Fluid Dynamics in the Density Frame

Idea: Give up manifest covariance, define (using lab-frame densities)

$$T^{0\mu} = (\mathcal{E}, M^i) \equiv \mathcal{T}^{0\mu}(\beta_\mu) \qquad \mathcal{T}^{\mu\nu} = [e(\beta) + P(\beta)]\, u^\mu u^\nu + P(\beta)\eta^{\mu\nu}$$

Ideal equations of motion

$$\partial_t \mathcal{E} + \partial_i M^i = 0$$

$$\partial_t M^i + \partial_j \mathcal{T}^{ij} = 0$$

Match dissipative stresses to the Landau frame

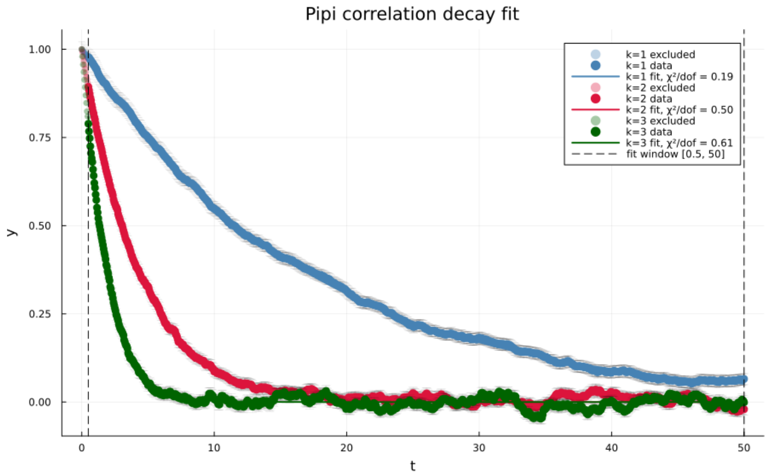
$$T^{ij} = \mathcal{T}^{ij} + \Pi^{ij} \qquad \Pi^{ij} = -T\kappa^{ijkl}\partial_{(k}\beta_{l)}$$

Can be sampled via Metropolis.

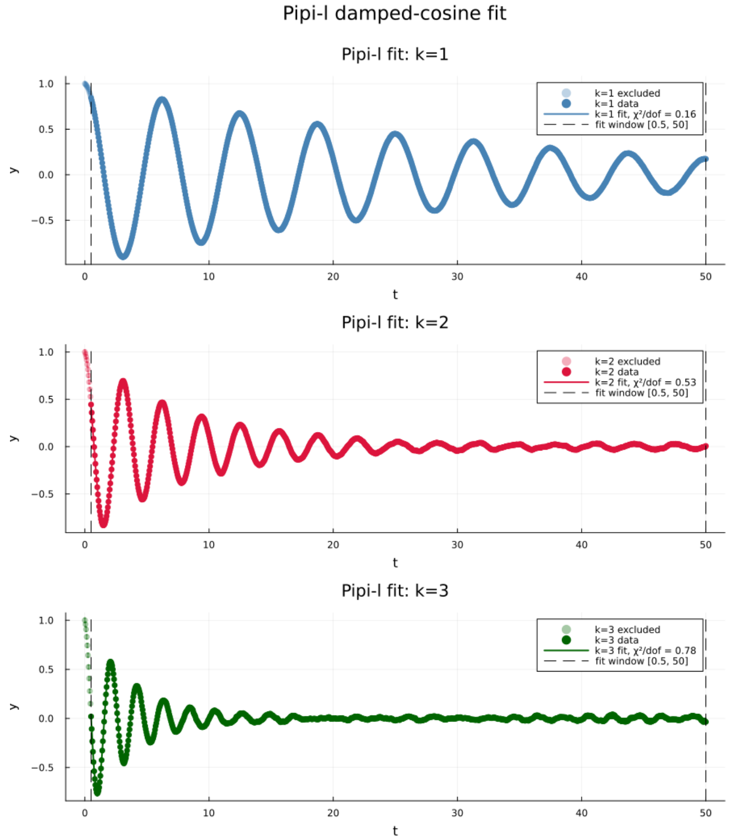
Arman & Jain, 2010.15782. Bhambure, Sing & Teaney, 2412.10306.

Teaser: Stochastic compressible Navier-Stokes

shear: $\langle \pi_x(k_y,0)\pi_x(-k_y,t)\rangle$



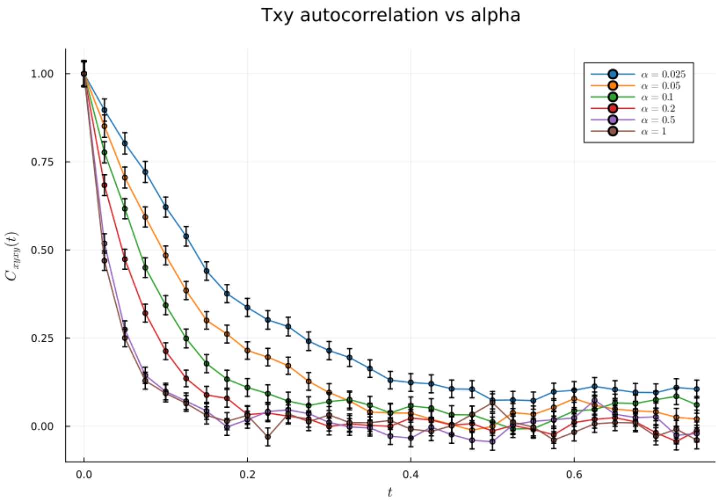
sound: $\langle \pi_x(k_x,0)\pi_x(-k_x,t)\rangle$

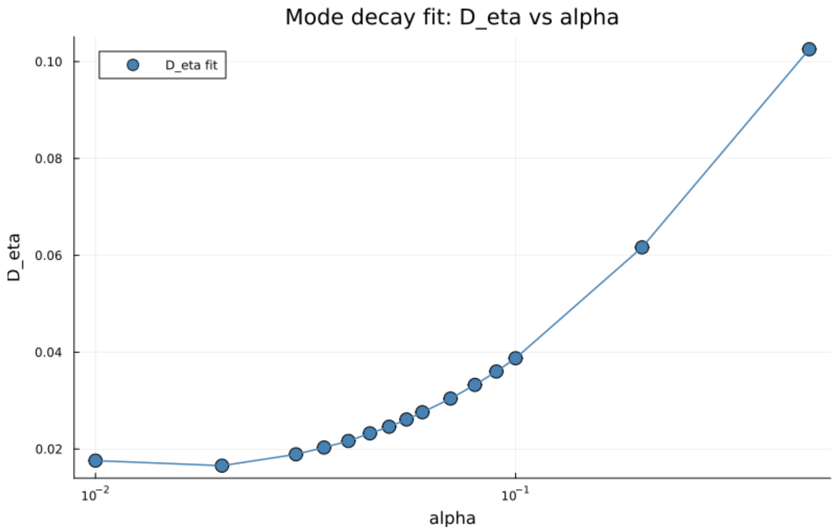


Find hydrodynamic scaling $D \sim k^2$ and $\Gamma_s = \frac{4}{3}\Gamma_\eta$.

Combine Lagrange-step $\Phi^I(x(x'),t) = x'$ with Metropolis.

Teaser: Stochastic compressible Navier-Stokes





Stress tensor correlation function

$$\langle \Pi_{xy}(0)\Pi_{xy}(t)\rangle \sim 1/(D_\eta t^{3/2})$$

Renormalization of shear mode decay rate D_η.

Summary and Outlook

Numerical simulation of stochastic fluid dynamics, observed renormalization of shear viscosity, dynamical scaling, and critical transport.

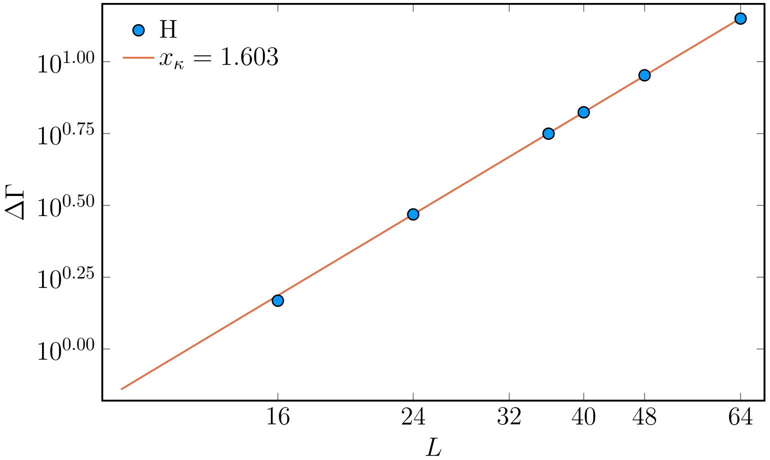
Other models: Chiral dynamics (model G), see Derek, Aleksas, Jannis, and friends. Superfluid dynamics (model F).

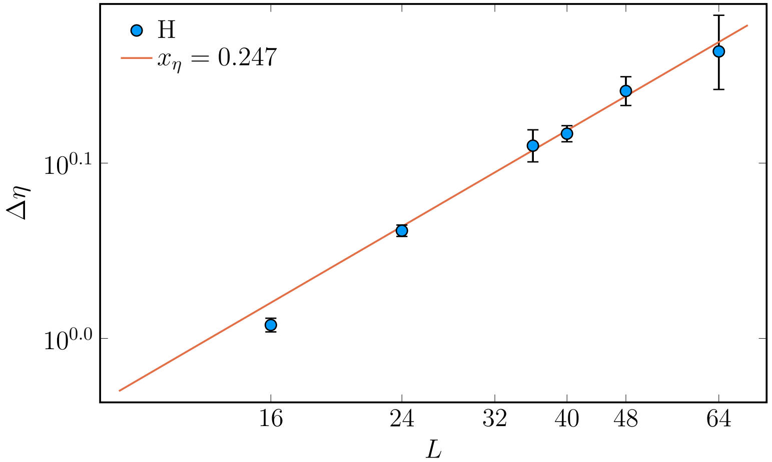
Compressible (full Navier-Stokes theory) and relativistic fluid dynamics in progress. Density frame approach seems promising. Goal is to do full dynamics of critical and non-critical fluctuations in a heavy ion collision.

Critical behavior of transport coefficients (Here: 2d)

$$\kappa = \frac{1}{3VT} \int dt\, d^3x_{1,2}\, \langle \vec{\jmath}(0,x_1)\vec{\jmath}(t,x_2)\rangle$$

$$\eta = \frac{1}{VT} \int dt\, d^3x_{1,2}\, \langle \Pi_{xy}(0,x_1)\Pi_{xy}(t,x_2)\rangle$$





Consistent with $z = 4 - x_\kappa - \eta^*$ ($\eta^* = 0.25$ correlation function exponent):

$$z = 2.11 \qquad\qquad 4 - x_\kappa - \eta^* = 2.15$$