

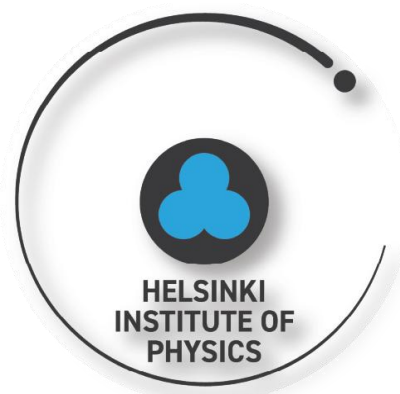
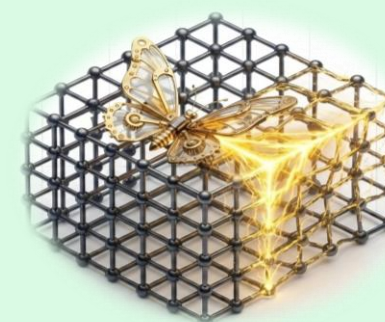


Lyapunov Exponent:

Measure of Chaotic Dynamics in Boost-Invariant Glasma

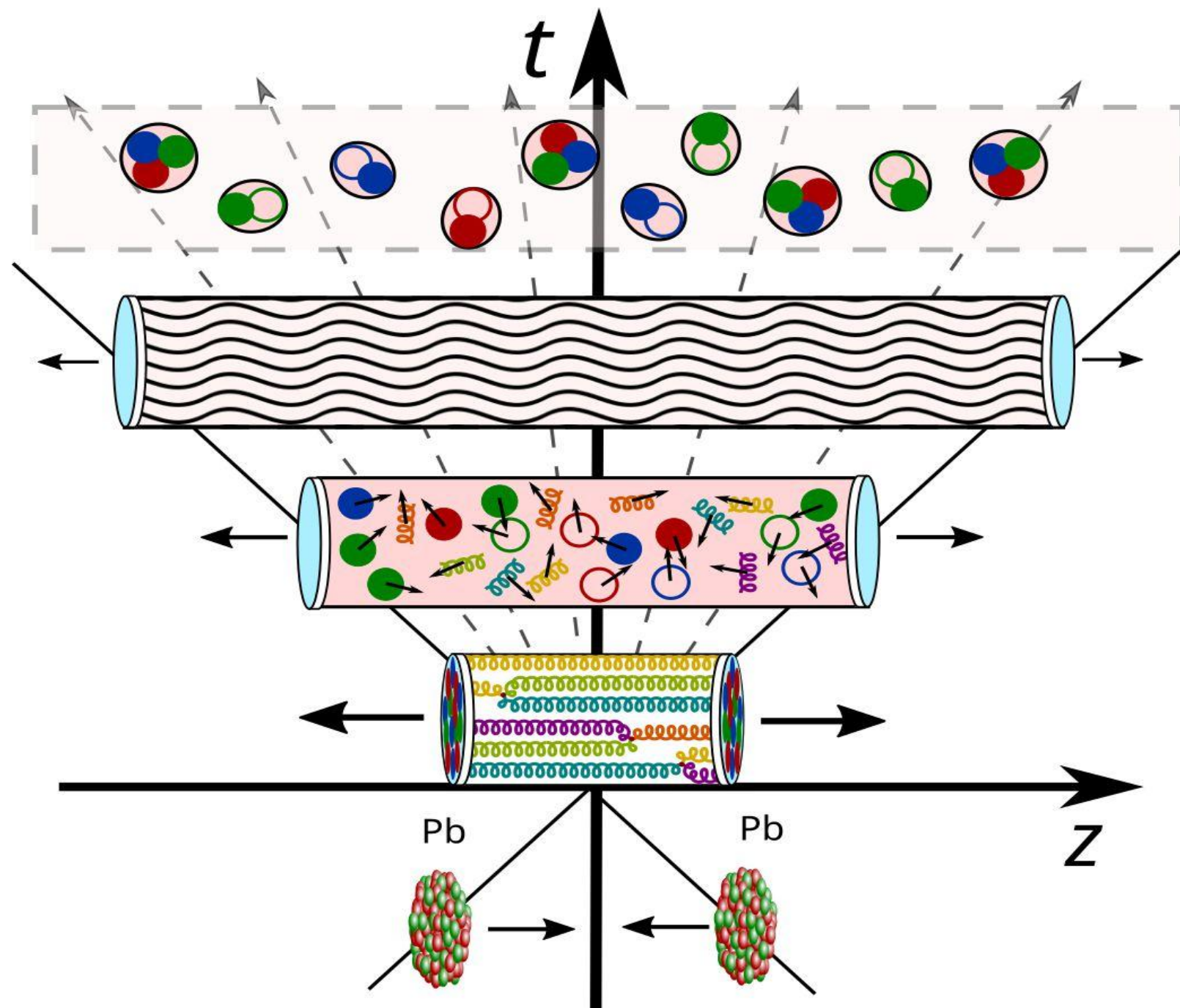


Leading Lyapunov exponent in the Glasma
Pooja, Dana Avramescu, Tuomas Lappi
Phys. Rev. D 114, 014010 (2026) 



Pooja
University of Jyväskylä
Center of Excellence in Quark Matter

Different stages of HICs

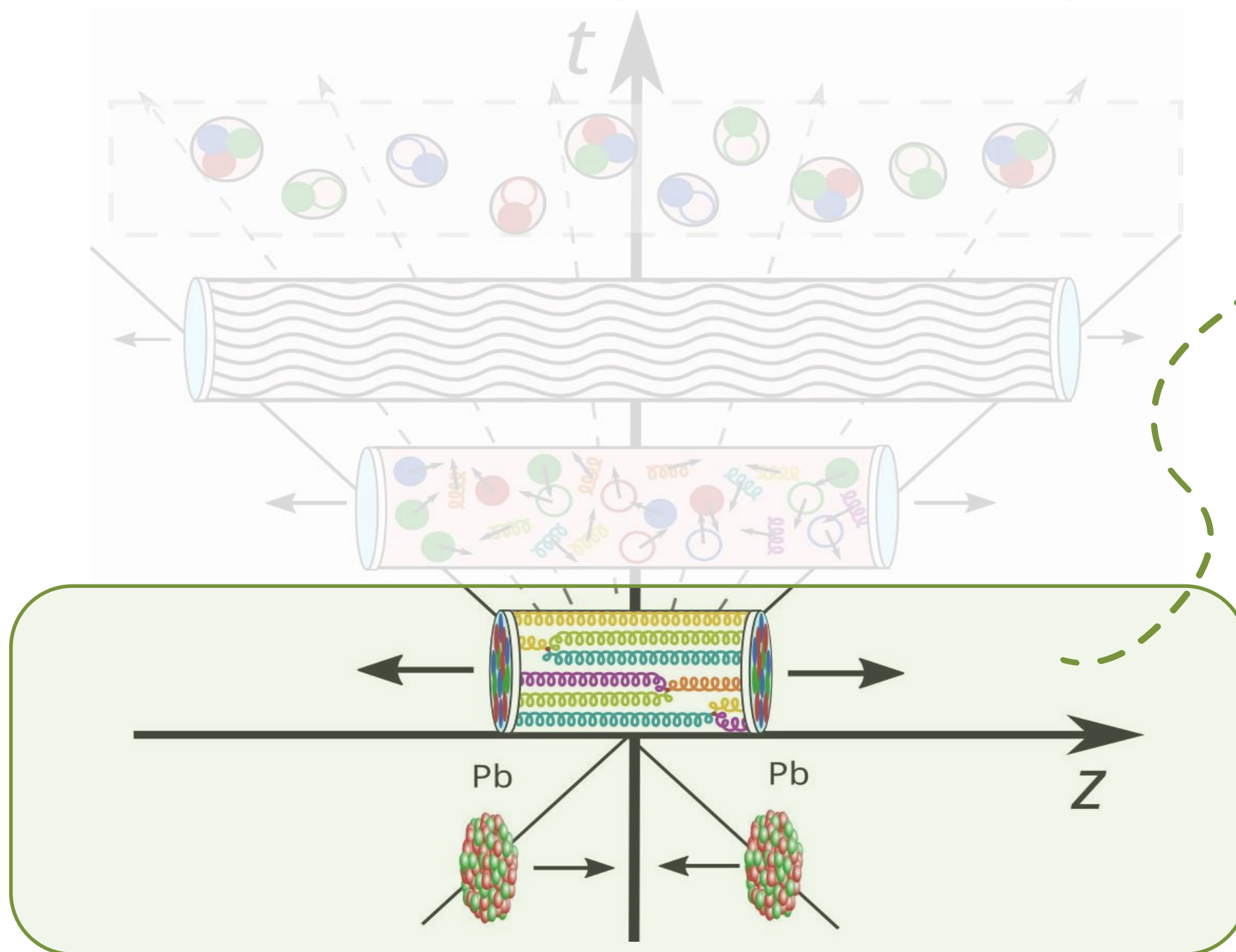


J. Berges, M. P. Heller, A. Mazeliauskas, R. Venugopalan, [Rev. Mod. Phys. 93 \(2021\) 3, 035003](#)

- ❖ **Before collision:** $\tau \leq 0 \text{ fm}/c$
Gluon field of high-energy nucleus
- ❖ **Initial stage:** $\tau \lesssim 0.4 \text{ fm}/c$
Glasma: strong classical gluon fields
- ❖ **Thermalization:** $\tau \lesssim 1 \text{ fm}/c$
Effective kinetic theory
- ❖ **Equilibration:** $\tau \lesssim 10 \text{ fm}/c$
Relativistic hydrodynamics
- ❖ **Final stages:** $\tau \geq 10 \text{ fm}/c$
Hadronization

Initial stage of HICs

3



Glasma

- ❖ **Color glass condensate**
Effective field theory for high-energy QCD
- ❖ **Initialization**
McLerran-Venugopalan (MV) model ($g^2\mu \sim Q_s$)
- ❖ **Classical gauge fields**
High occupation number
- ❖ Longitudinal flux tubes of gluon color fields
- ❖ **Out-of-equilibrium** medium

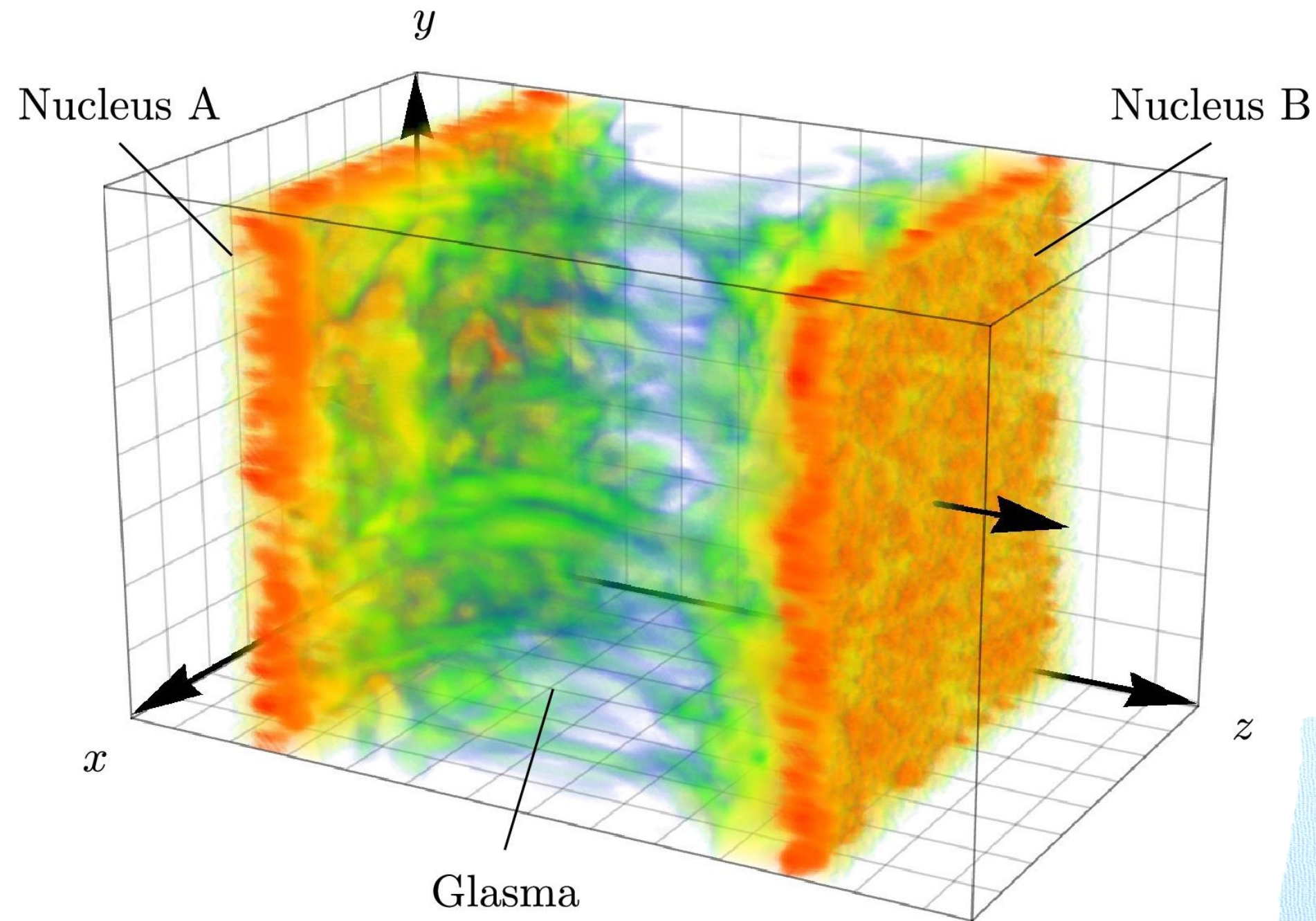
T. Lappi, Phys. Lett. B 643 (2006) 11-16

F. Gelis, Int. J. Mod. Phys. A 28 (2013) 1330001

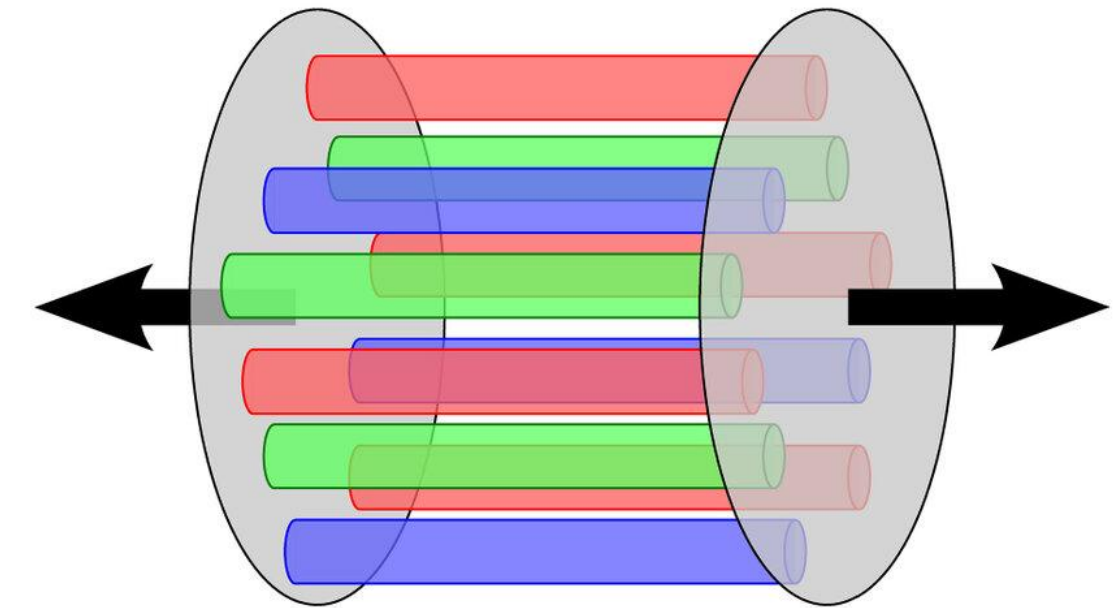
Features of Glasma

4

Dana's talk 🎤
17 Aug, 9 AM



A. Ipp, D. Müller, *Phys. Lett. B* 771 (2017) 74-79



K. Fukushima, *Acta Phys.Polon.B* 42 (2011) 2697-2715

- Relevant scale: Saturation momentum, Q_s
- Initial longitudinal color flux tubes of $\delta x_T \simeq Q_s^{-1}$
- Fields dilute after $\tau \simeq Q_s^{-1}$

Evolution of expanding Glasma: CYM equations

5

Due to the large density,
the gluon field behaves
like a classical field.



Dynamics is governed
by classical EoMs,
namely the **Classical
Yang-Mills (CYM)**
equations.

**Classical
Yang-Mills
(CYM)
Equations**

$$\begin{aligned}\dot{A}_i &= \frac{1}{\tau} E^i \\ \dot{A}_\eta &= \tau E_\eta\end{aligned}$$

$$\begin{aligned}\dot{E}^i &= \tau [D_j, F_{ji}] - \frac{ig}{\tau} [A_\eta, [D_i, A_\eta]] \\ \dot{E}_\eta &= \frac{1}{\tau} [D_i, [D_i, A_\eta]]\end{aligned}$$

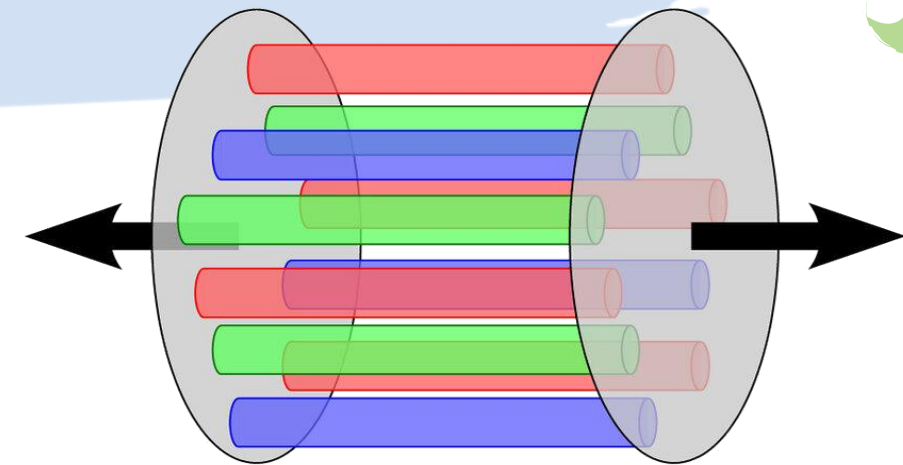
*Evolution of the system is studied assuming Glasma as the initial condition,
and evolving this condition by virtue of the **CYM** equations.*

Instability of boost-invariant Glasma fields

Leading Lyapunov exponent in the glasma

Pooja, D. Avramescu, T. Lappi

Phys. Rev. D 114, 014010 



B. Muller, A. Trayanov, Phys. Rev. Lett. 68 (1992) 3387

U. W. Heinz, C. R. Hu, S. Leupold, S. G. Matinyan, B. Muller, Phys. Rev. D 55 (1997) 2464

P. Romatschke, R. Venugopalan, Phys. Rev. D 74 (2006) 045011

P. Romatschke, R. Venugopalan, Phys. Rev. L 96 (2006) 062302

T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, T. T. Takahashi, A. Yamamoto, Phys. Rev. D 82 (2010) 114015

S. Bazak, S. Mrówczyński, Phys. Rev. C 109 (2024) 024903

Is Glasma chaotic?

7

Butterfly Effect:

Small changes in initial conditions lead to unpredictable, significant results.

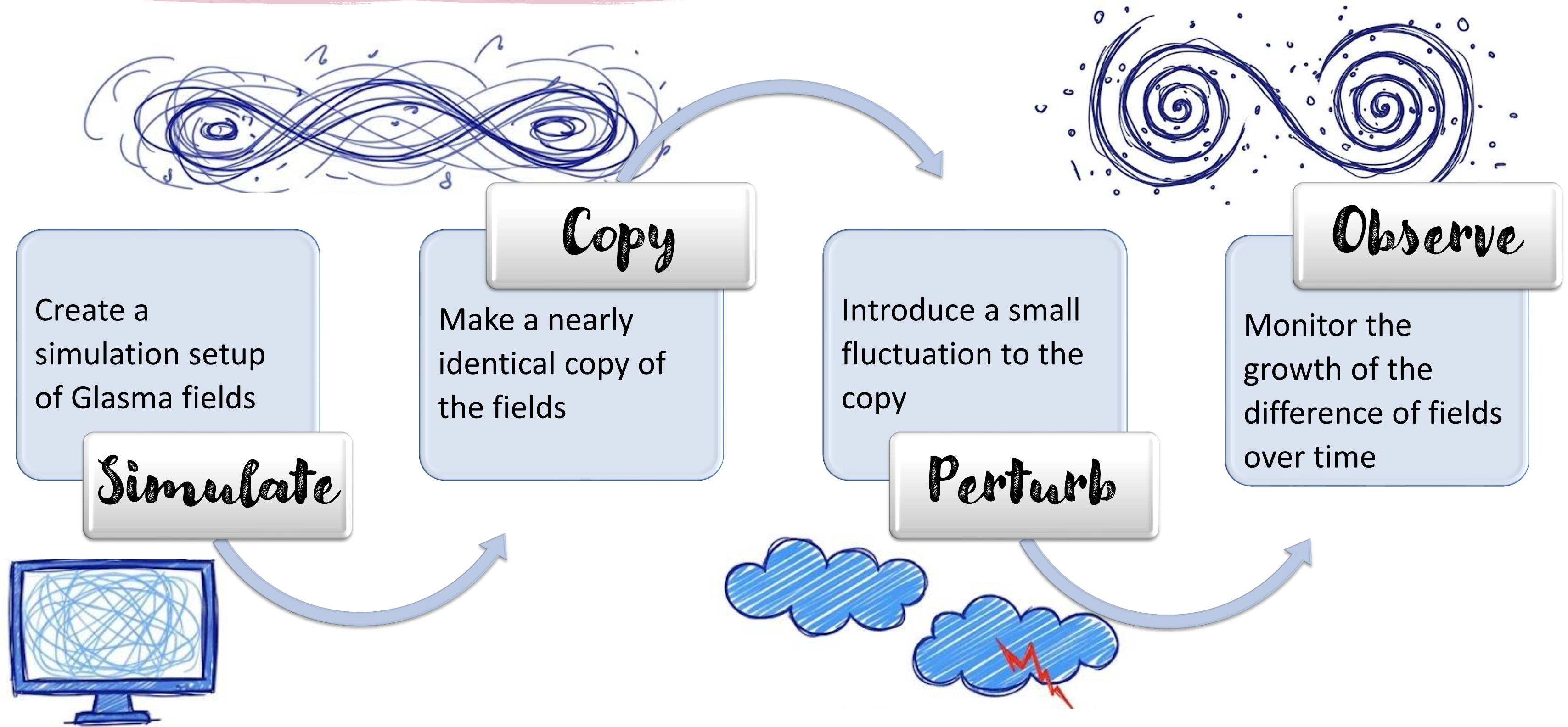


Ingredients for chaos in Glasma:

- ❖ **Non-Abelian Nature:** Gluon fields self-interact. This non-linear interaction is a source of complexity and chaos.
- ❖ **Classical Field Dynamics:** In the early, dense stage, gluons can be treated as classical Yang-Mills fields, highly non-linear.
- ❖ **Sensitivity to Initial Conditions:** The initial color charges in the nuclei are random. Small changes here should propagate.

Seeding the chaos

8



Seeding the chaos in initial classical fields

9

Perturb any one of the initial fields:

Perturbed field	Unperturbed field	Small perturbation
$E'_\eta{}^a(\tau = 0, \mathbf{x}_\perp)$	$E_\eta{}^a(\tau = 0, \mathbf{x}_\perp)$	$+\frac{2\chi^a(\mathbf{x}_\perp)}{a_\perp^2}$
$A'_i{}^a(\tau = 0, \mathbf{x}_\perp)$	$A_i{}^a(\tau = 0, \mathbf{x}_\perp)$	$+\frac{2\xi_i^a(\mathbf{x}_\perp)}{a_\perp}$

$$\begin{aligned}\langle \chi^a(\mathbf{x}_\perp) \rangle &= 0 \\ \langle \chi^a(\mathbf{x}_\perp) \chi^b(\mathbf{y}_\perp) \rangle &= \alpha^2 \delta^{ab} \delta_{\mathbf{x}_\perp, \mathbf{y}_\perp}\end{aligned}$$

$$\begin{aligned}\langle \xi_i^a(\mathbf{x}_\perp) \rangle &= 0 \\ \langle \xi_i^a(\mathbf{x}_\perp) \xi_j^b(\mathbf{y}_\perp) \rangle &= \beta^2 \delta^{ab} \delta_{ij} \delta_{\mathbf{x}_\perp, \mathbf{y}_\perp}\end{aligned}$$

leads to perturbed B'_η

$$\begin{aligned}U'_i(\tau = 0, \mathbf{x}_\perp) &= U_i(\tau = 0, \mathbf{x}_\perp) \cdot \exp[ig a_\perp \delta A_i^a(\tau = 0, \mathbf{x}_\perp)] T^a \\ B_\eta(\tau_n, \mathbf{x}_\perp) &= -F_{xy}(\tau_n, \mathbf{x}_\perp) \approx -\frac{1}{g a_\perp^2} [U_{i\hat{j}}(\tau, \mathbf{x}_\perp)]_{ah}\end{aligned}$$

anti-Hermitian

α, β : Perturbation amplitude

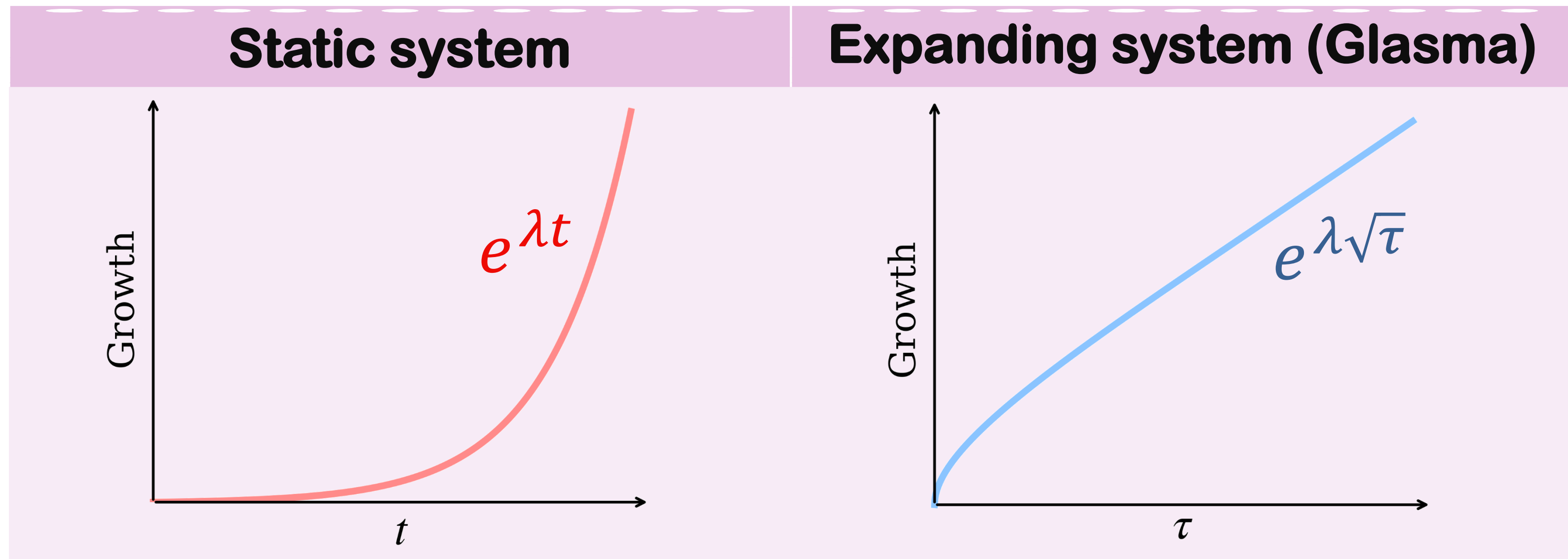
***Solve CYM
equations of
motion
separately for
 E, A , and E', A'***



***Measure
 $\left(\delta E_\eta(\tau)\right)^2$ and
 $\left(\delta B_\eta(\tau)\right)^2$ vs τ***

Impact of longitudinal expansion

11



Growth in the Glasma is not exponential in time, but exponential in the square root of the proper time, τ .

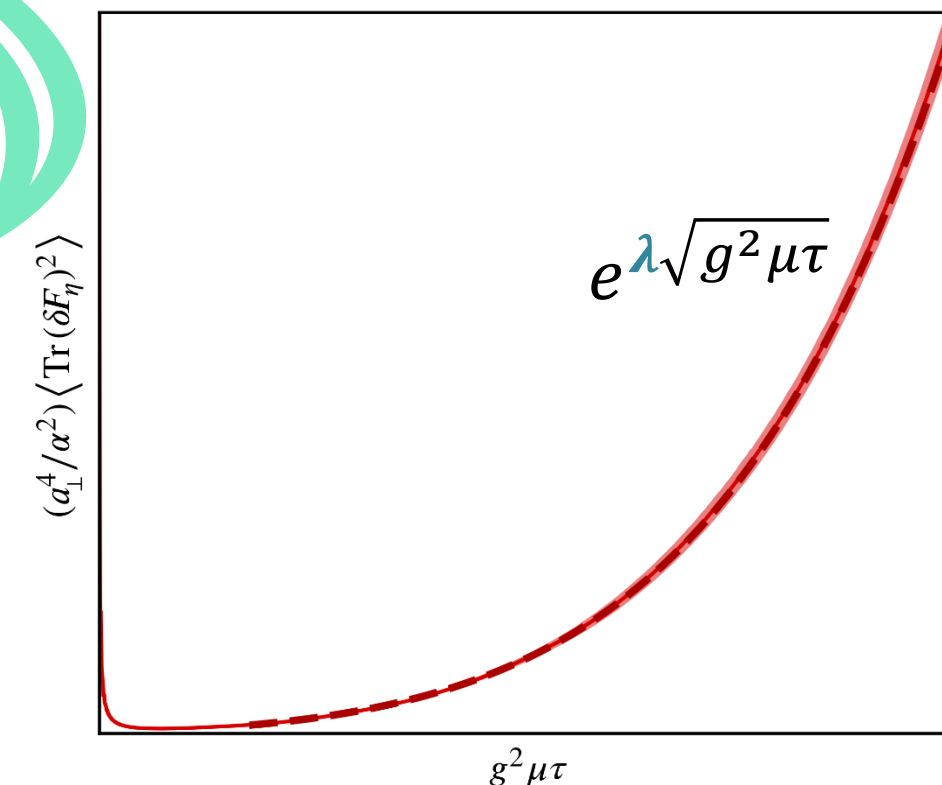
As the field perturbations attempt to grow, the boost invariant longitudinal expansion of the colliding system stretches physically and dilutes the growth.

Impact of longitudinal expansion of Glasma

12

In a static box, chaotic perturbations grow exponentially with linear time ($e^{\lambda t}$). However, in the expanding system, longitudinal expansion stretches the system. This dilutes the energy density, modifying the chaotic growth into a square-root trajectory.

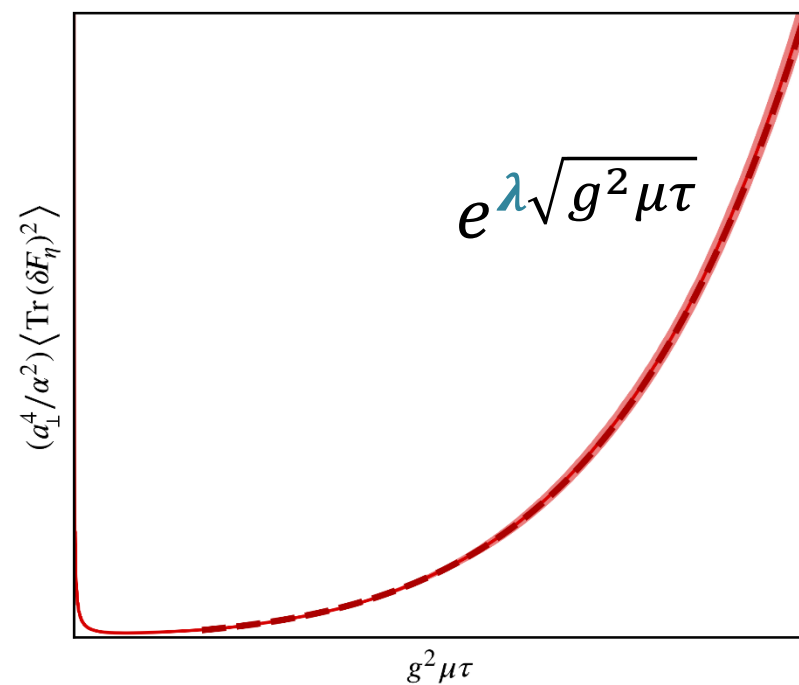
$$\left\langle \text{Tr} \left(\delta F_{\eta}(\tau) \right)^2 \right\rangle \propto e^{\lambda \sqrt{g^2 \mu \tau}}$$



Impact of longitudinal expansion of Glasma

13

Longitudinal expansion dictates a square-root time dependence.



The tracked perturbation in the longitudinal fields, $F = E, B$

The signature of classical chaos: exponential growth

The leading Lyapunov exponent (the intrinsic chaoticity being extracted)

Square-root time dependency: A direct physical consequence of the 2+1 D boost-invariant Glasma

$$\left\langle \text{Tr} \left(\delta F_\eta(\tau) \right)^2 \right\rangle \propto e^{\lambda \sqrt{g^2 \mu \tau}}$$

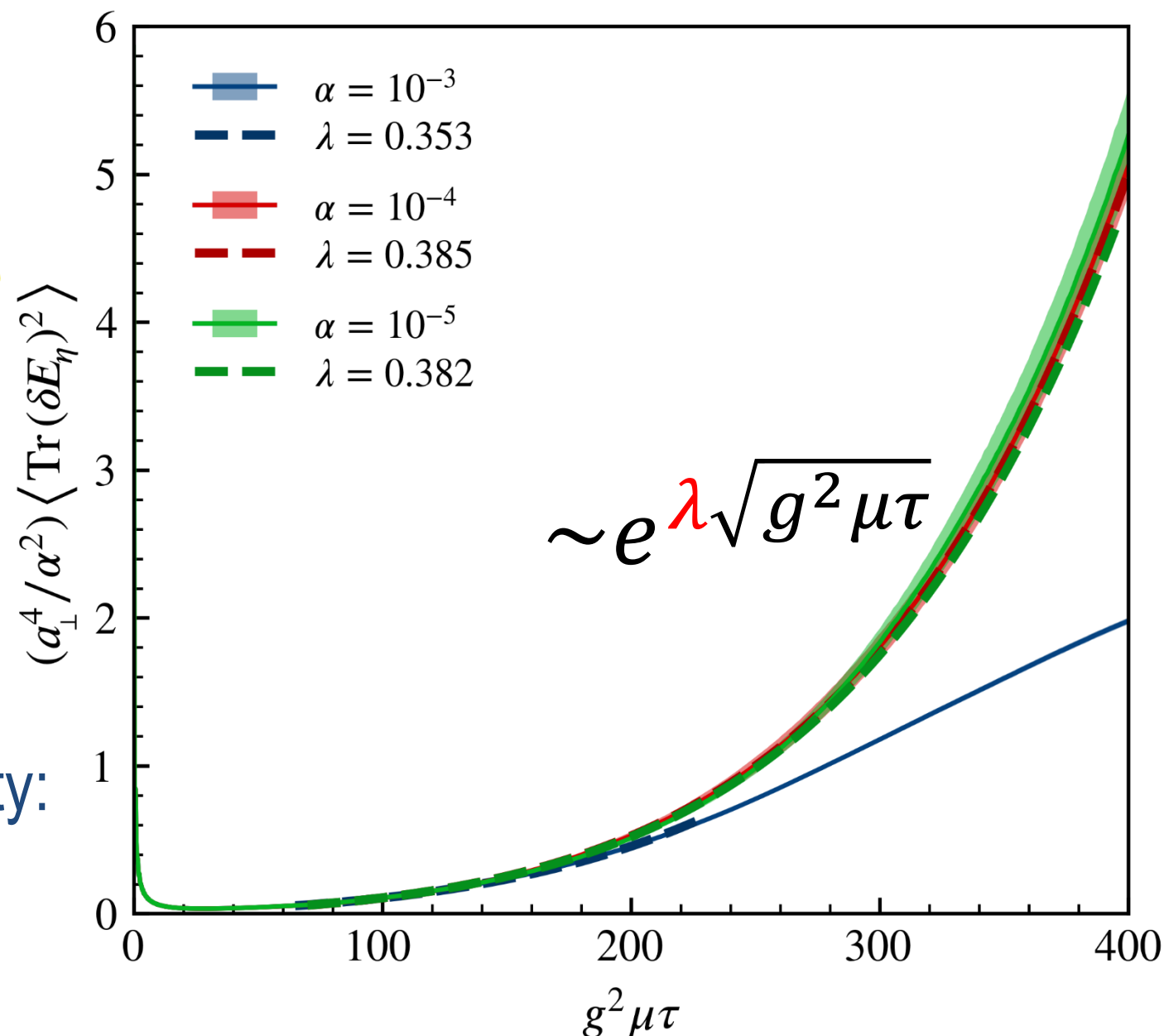
The MV model parameter, $g^2 \mu \sim Q_s$
 Q_s : Saturation scale of Glasma

Step 0: Check linearity

14

Are we in the linear regime?

Linear regime stability:
 $\alpha, \beta \ll 1$



$$\left\langle \text{Tr} \left(\delta F_\eta(\tau) \right)^2 \right\rangle \propto e^{\lambda \sqrt{g^2 \mu \tau}}$$

$F = E, B$ λ : Lyapunov exponent

- Scaling by α^2 confirms that the dynamics remain strictly in linear regime for $\alpha < 10^{-3}$
- Initial transient phase: Other modes die away
- Later, a very clear **exponential growth**



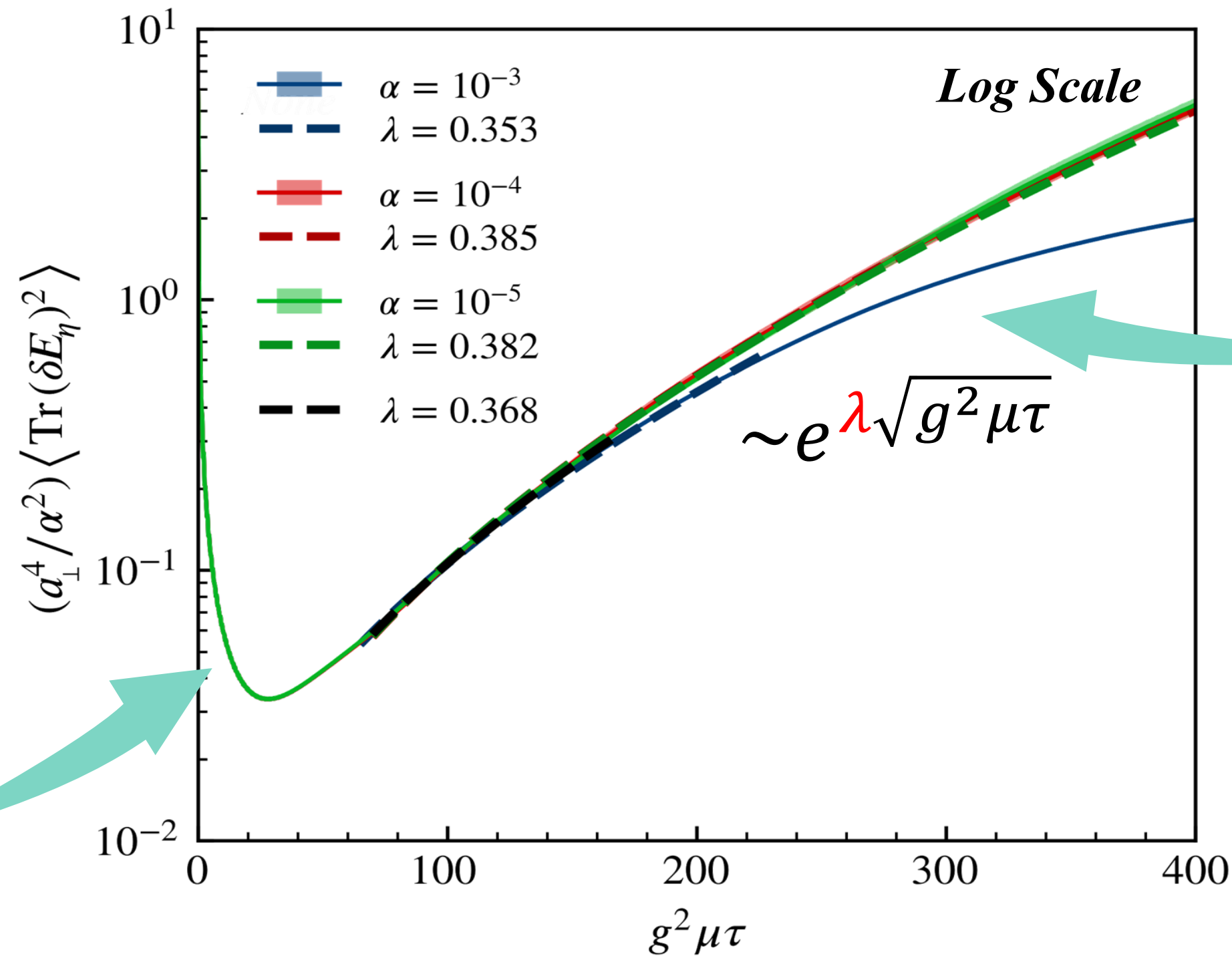
Chaoticity: Perturbations grow exponentially as a **square root of time** due to **boost-invariant expansion** of the Glasma fields.

White-noise perturbation

15

- Perturbations remain in linear regime for $\alpha < 10^{-3}$.
- White noise excites all momentum modes equally.

Transient phase:
System takes time to digest the perturbation



Exponential phase:
Clear linear growth on the logarithm scale

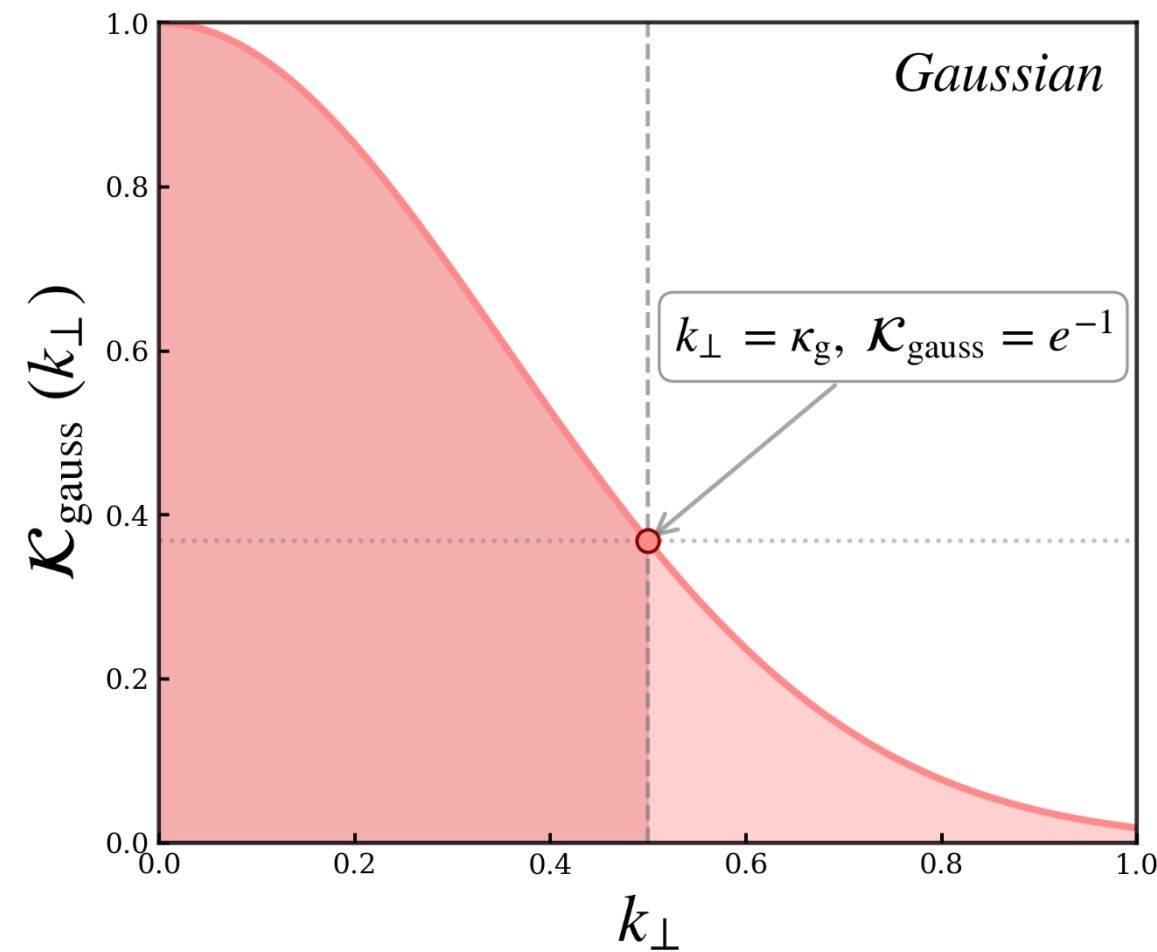
$$\left\langle \text{Tr} \left(\delta E_\eta(\tau) \right)^2 \right\rangle \propto e^{\lambda \sqrt{g^2 \mu \tau}}$$

$\lambda = 0.35, \dots, 0.38$

***Does the chaos in the Glasma
originate from specific instabilities at
specific momentum scales?***

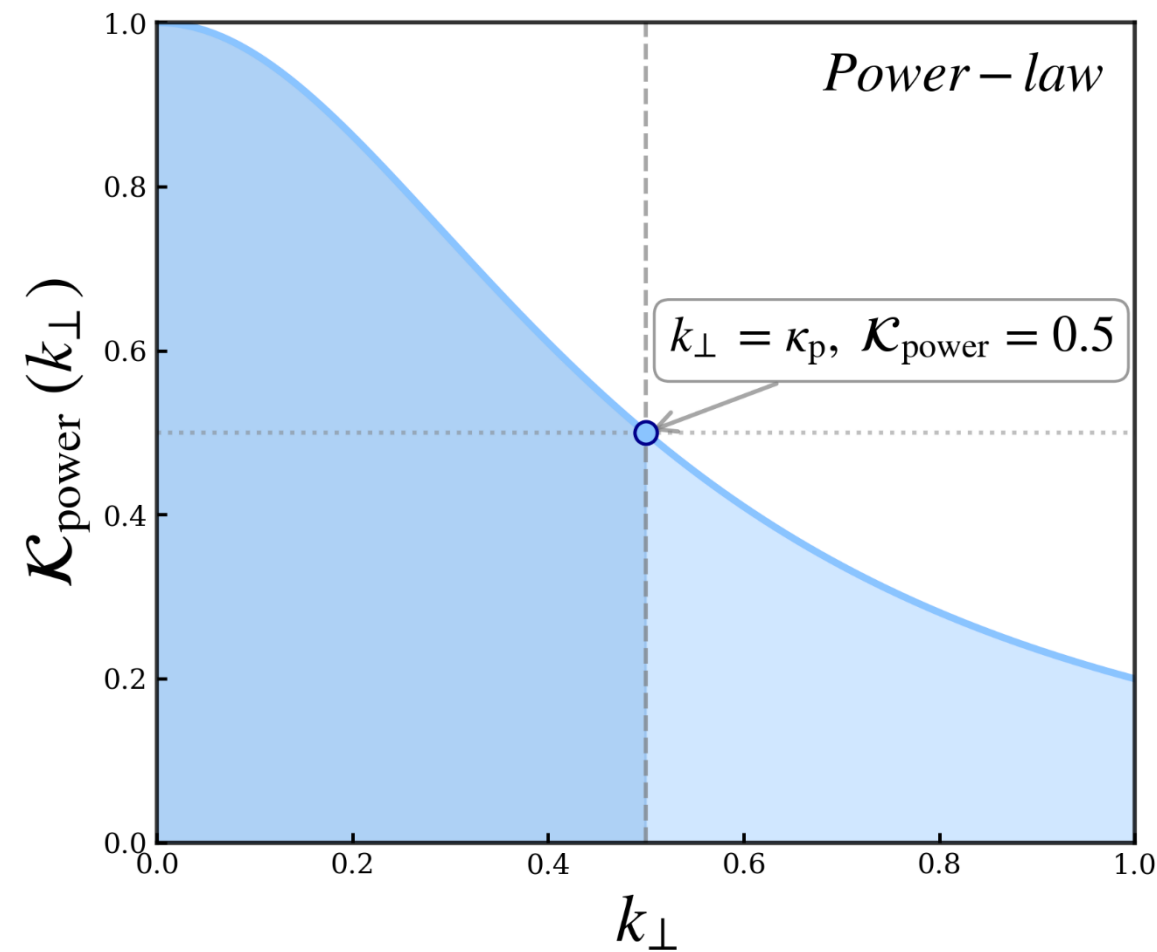
Momentum-space filtering

17



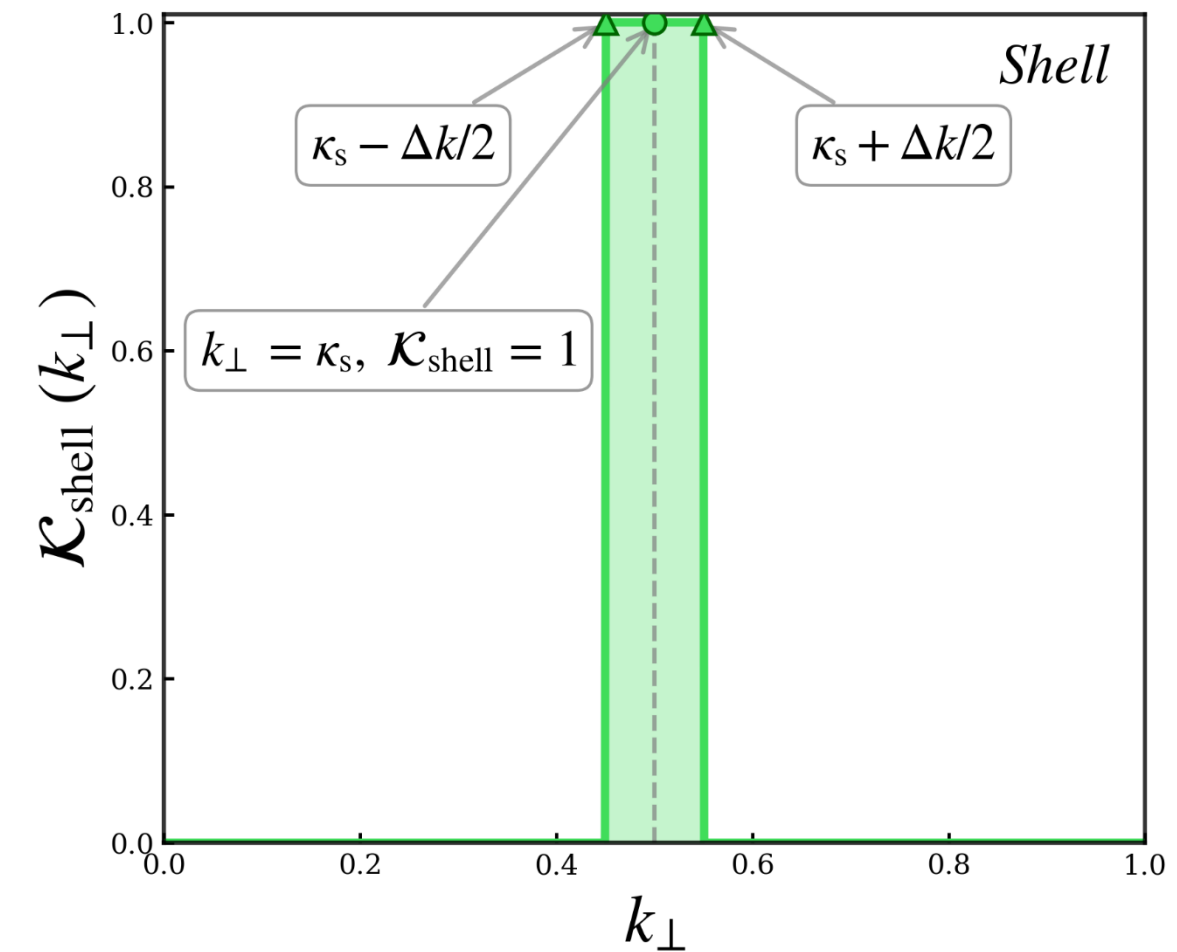
$$\mathcal{K}_{\text{gauss}}(k_{\perp}) = \exp\left(-\frac{\kappa_{\perp}^2}{\kappa_g^2}\right)$$

- Aggressively suppresses high-momentum ultraviolet modes
- Focuses on infrared modes



$$\mathcal{K}_{\text{power}}(k_{\perp}) = \frac{\kappa_p^2}{k_{\perp}^2 + \kappa_p^2}$$

- Milder suppression of high-momentum modes
- Preserves the ultraviolet tail



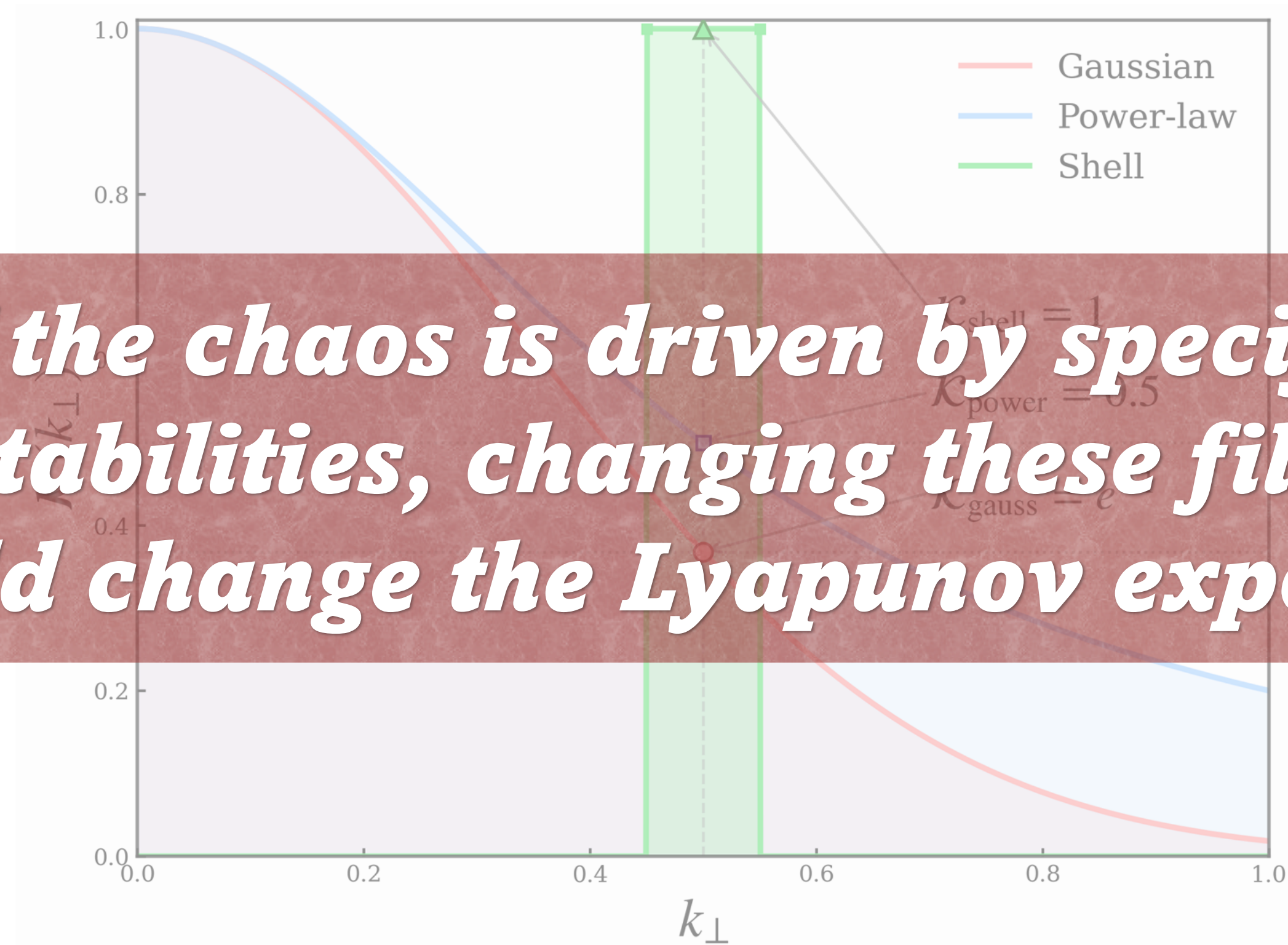
$$\mathcal{K}_{\text{shell}}(k_{\perp}) = \begin{cases} 1, & \kappa_s - \frac{\Delta k}{2} \leq k_{\perp} \leq \kappa_s + \frac{\Delta k}{2} \\ 0, & \text{otherwise} \end{cases}$$

- Isolates a narrow band of momenta
- Tests specific momentum modes independently

Momentum-space filtering

18

If the chaos is driven by specific instabilities, changing these filters should change the Lyapunov exponent.



Gaussian-filtered perturbations

19

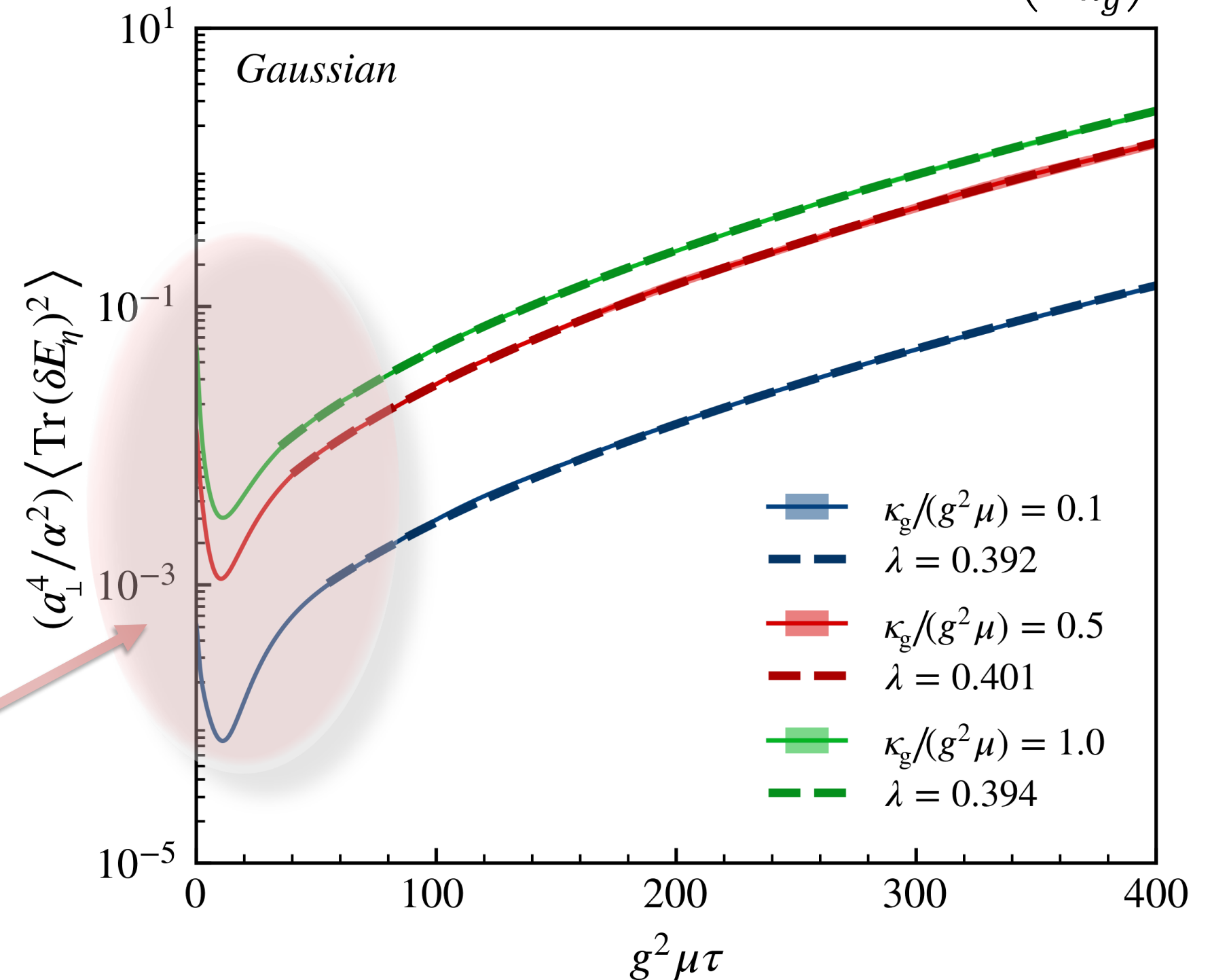
$$\mathcal{K}_{gauss}(k_{\perp}) = \exp\left(-\frac{\kappa_{\perp}^2}{\kappa_g^2}\right)$$

$$a_{\perp}^4 \left\langle \text{Tr} \left(\delta E_{\eta}(\tau = 0) \right)^2 \right\rangle \propto a_{\perp}^2 \kappa_g^2$$

Initial amplitude of the perturbations $\propto \kappa_g$

For smaller values of κ_g , the filter suppresses a larger portion of the momentum spectrum, resulting in fewer modes contributing to the initial fluctuations.

Hence, we get a smaller initial intercept on the y-axis.



Gaussian-filtered perturbations

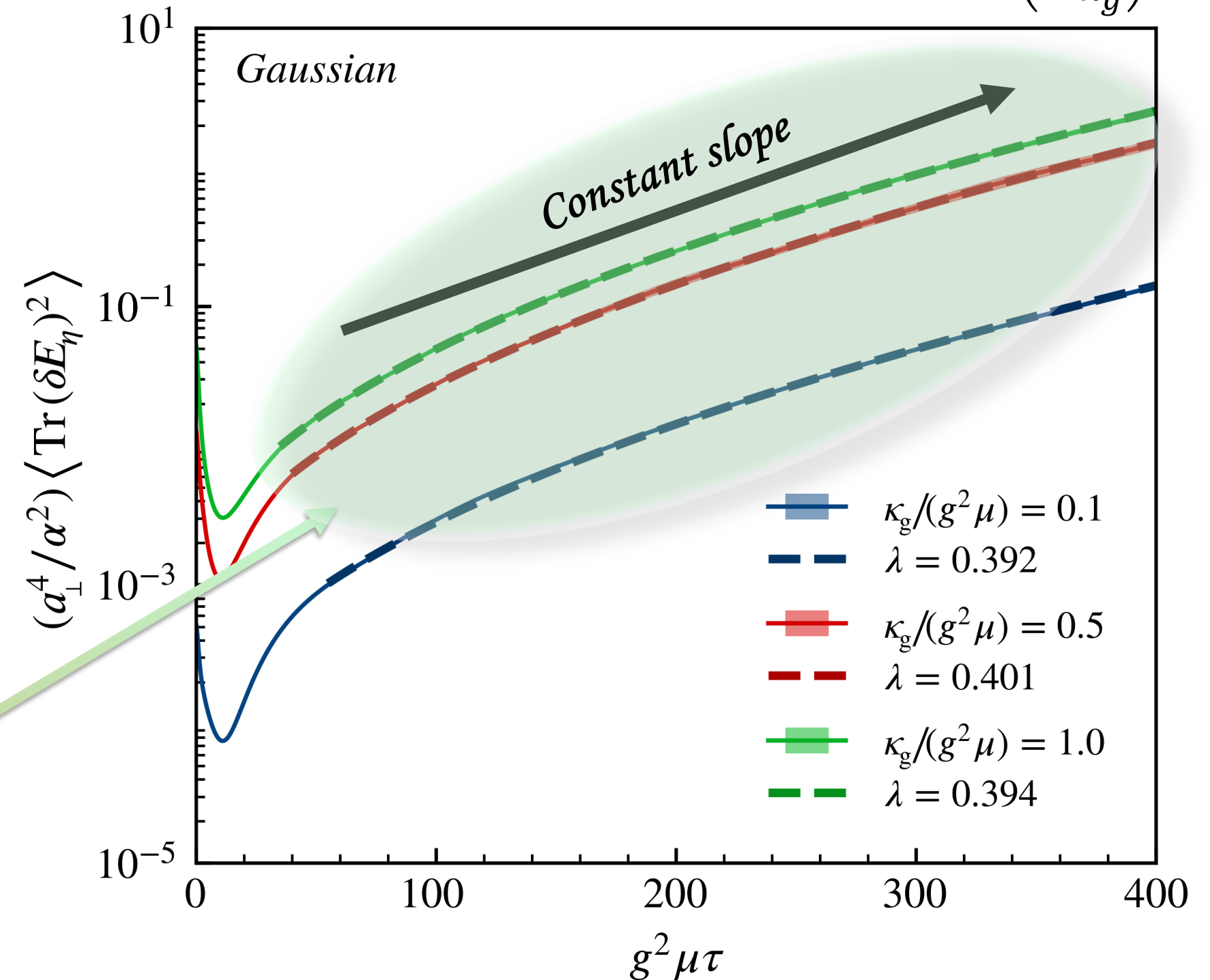
20

$$\mathcal{K}_{gauss}(k_{\perp}) = \exp\left(-\frac{\kappa_{\perp}^2}{\kappa_g^2}\right)$$

$$\left\langle \text{Tr} \left(\delta E_{\eta}(\tau) \right)^2 \right\rangle \propto e^{\lambda \sqrt{g^2 \mu \tau}}$$

The slope is insensitive to the filter scale, κ_g .

Clean exponential growth:
 $\lambda \approx 0.39 - 0.40$



Power-law filtered perturbations

21

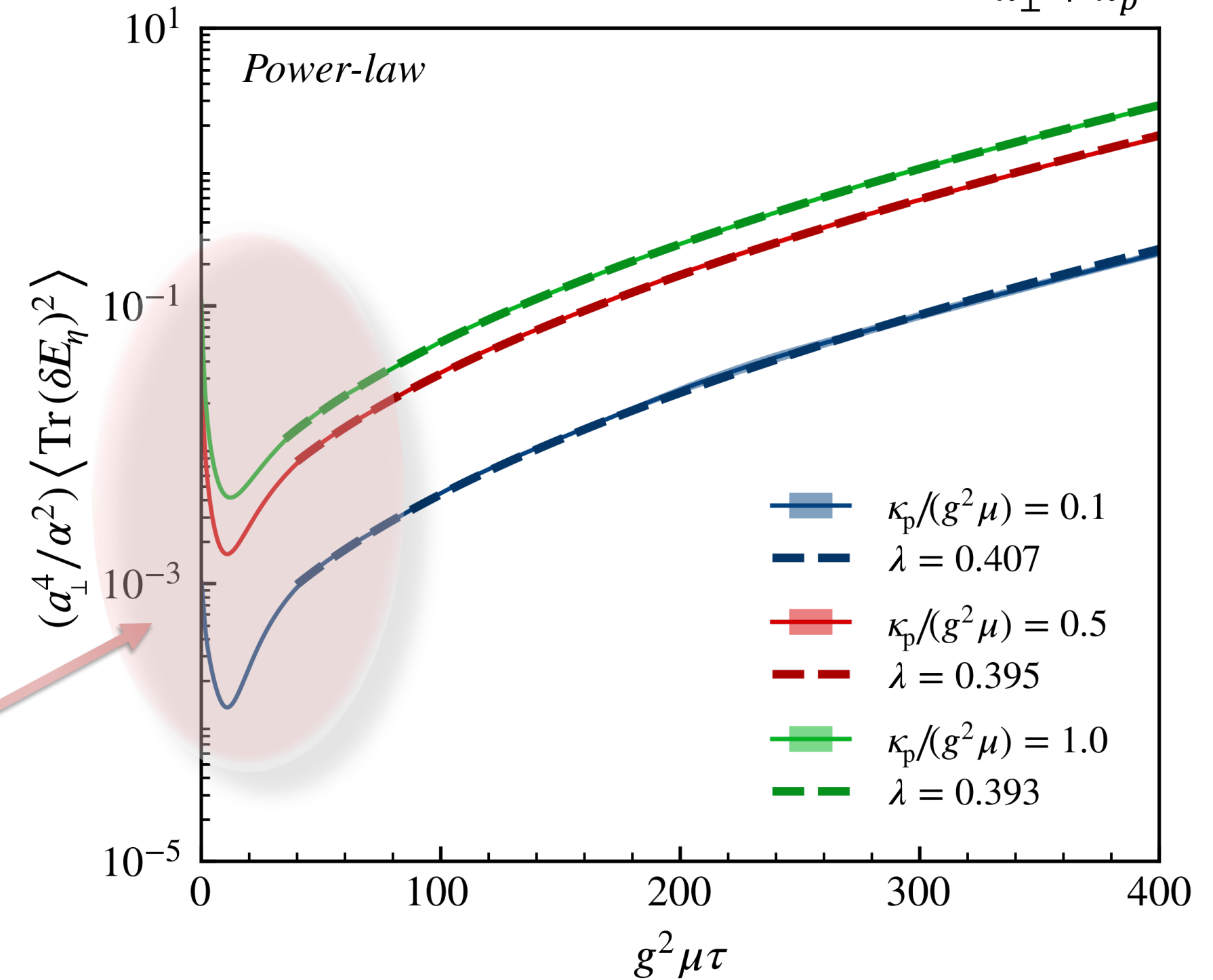
$$\mathcal{K}_{power}(k_{\perp}) = \frac{\kappa_p^2}{k_{\perp}^2 + \kappa_p^2}$$

$$a_{\perp}^4 \left\langle \text{Tr} \left(\delta E_{\eta}(\tau = 0) \right)^2 \right\rangle \propto a_{\perp}^2 \kappa_p^2$$

Initial amplitude of the perturbations $\propto \kappa_p$

For smaller values of κ_p , the filter suppresses a larger portion of the momentum spectrum, resulting in fewer modes contributing to the initial fluctuations.

Hence, we get a smaller initial intercept on the y-axis.



Power-law filtered perturbations

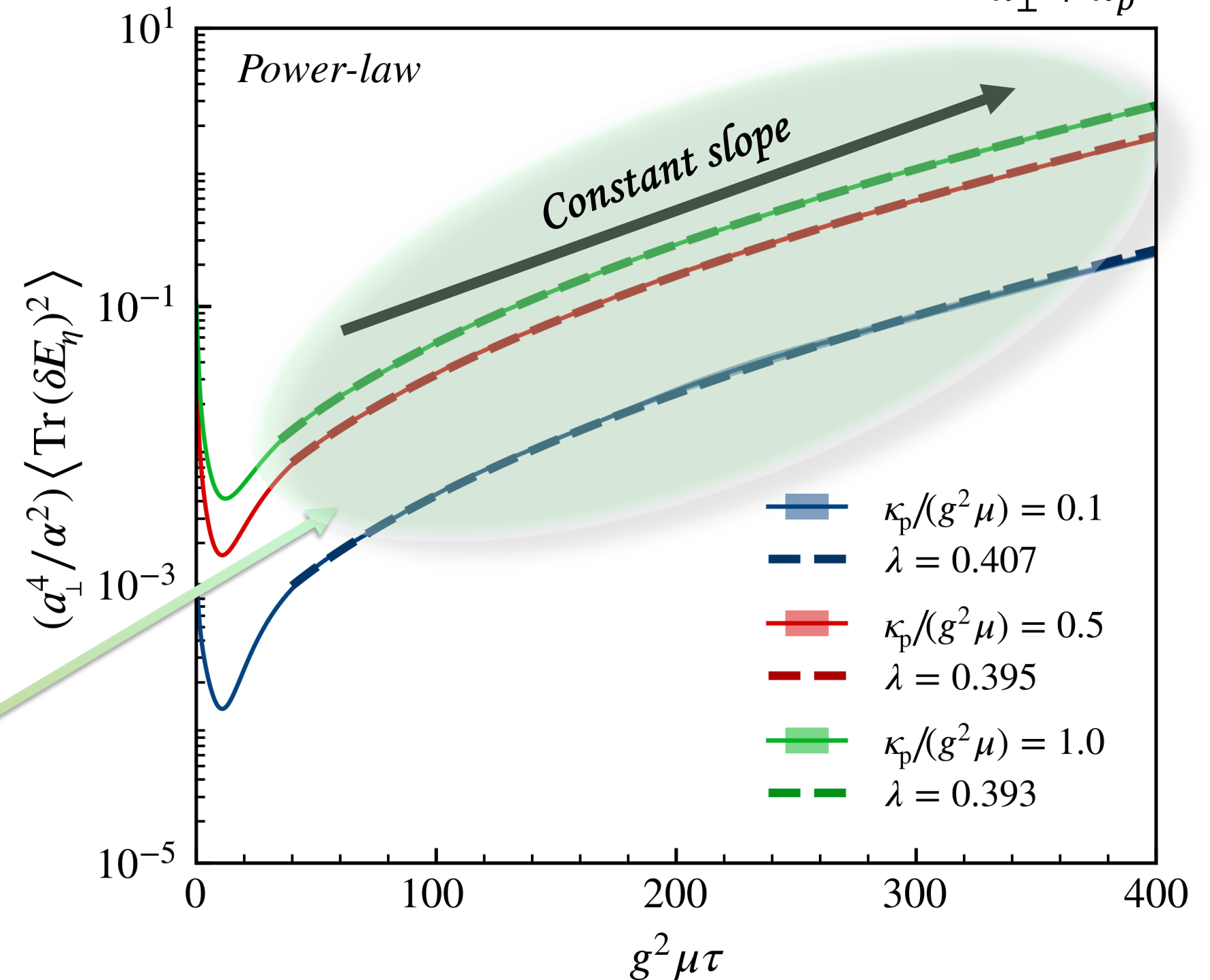
22

$$\left\langle \text{Tr} \left(\delta E_\eta(\tau) \right)^2 \right\rangle \propto e^{\lambda \sqrt{g^2 \mu \tau}}$$

The slope is insensitive to the filter scale, κ_p .

Clean exponential growth:
 $\lambda \approx 0.39 - 0.40$

$$\mathcal{K}_{\text{power}}(k_\perp) = \frac{\kappa_p^2}{k_\perp^2 + \kappa_p^2}$$



Shell-filtered perturbations

23

Clear vertical hierarchy
in the growth curve

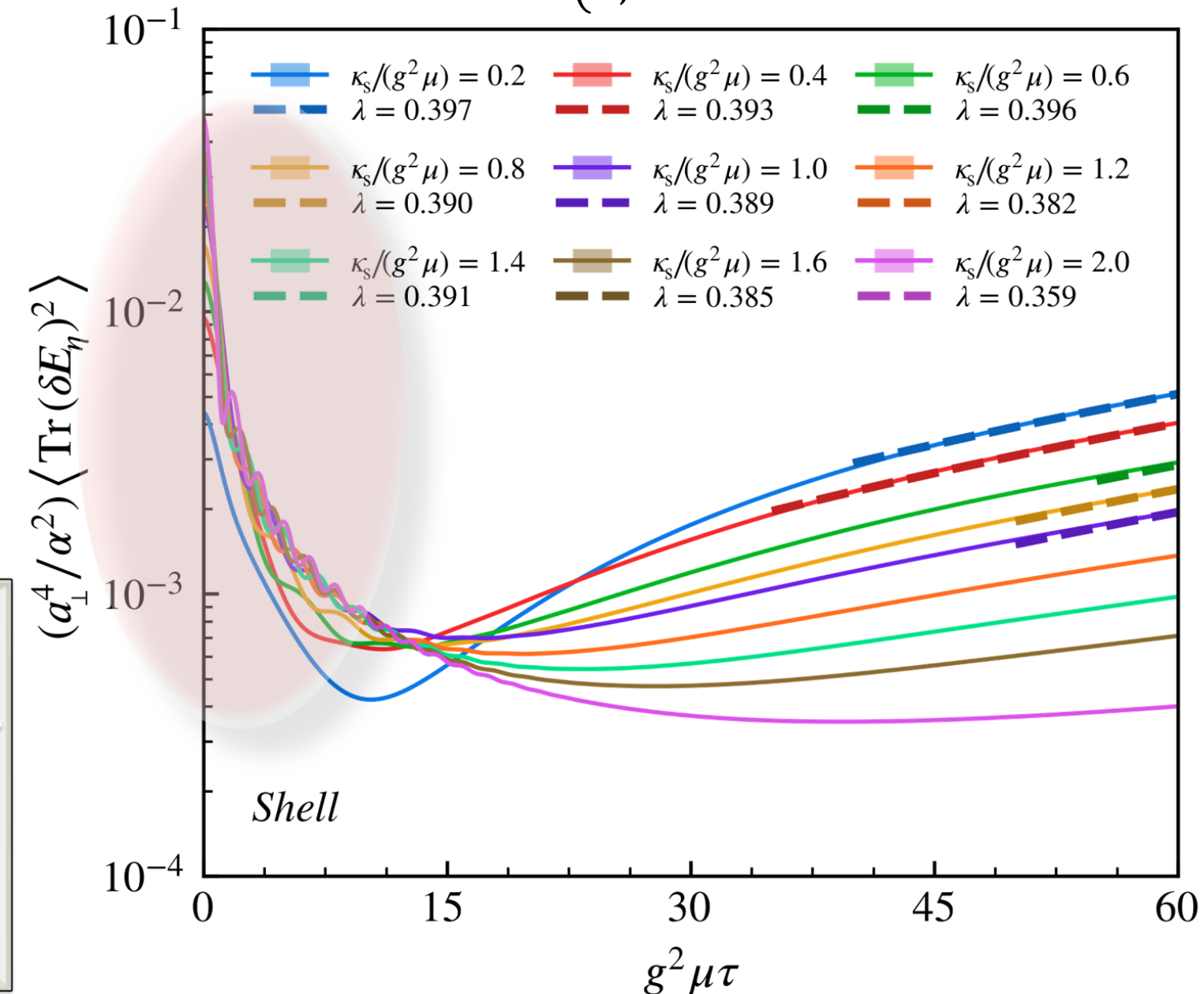
Early time

Higher momentum shells
(larger κ_s) exhibit larger initial
amplitudes due to the larger
phase-space area $A \sim 2\pi\kappa_s\Delta k$.

Late time

Higher- κ_s modes have smaller
overlap with the unstable
modes, leading to lower late
time amplitudes.

$$\mathcal{K}_{shell}(k_{\perp}) = \begin{cases} 1, & \kappa_s - \frac{\Delta k}{2} \leq k_{\perp} \leq \kappa_s + \frac{\Delta k}{2} \\ 0, & \text{otherwise} \end{cases}$$



Shell-filtered perturbations

24

Clear vertical hierarchy
in the growth curve

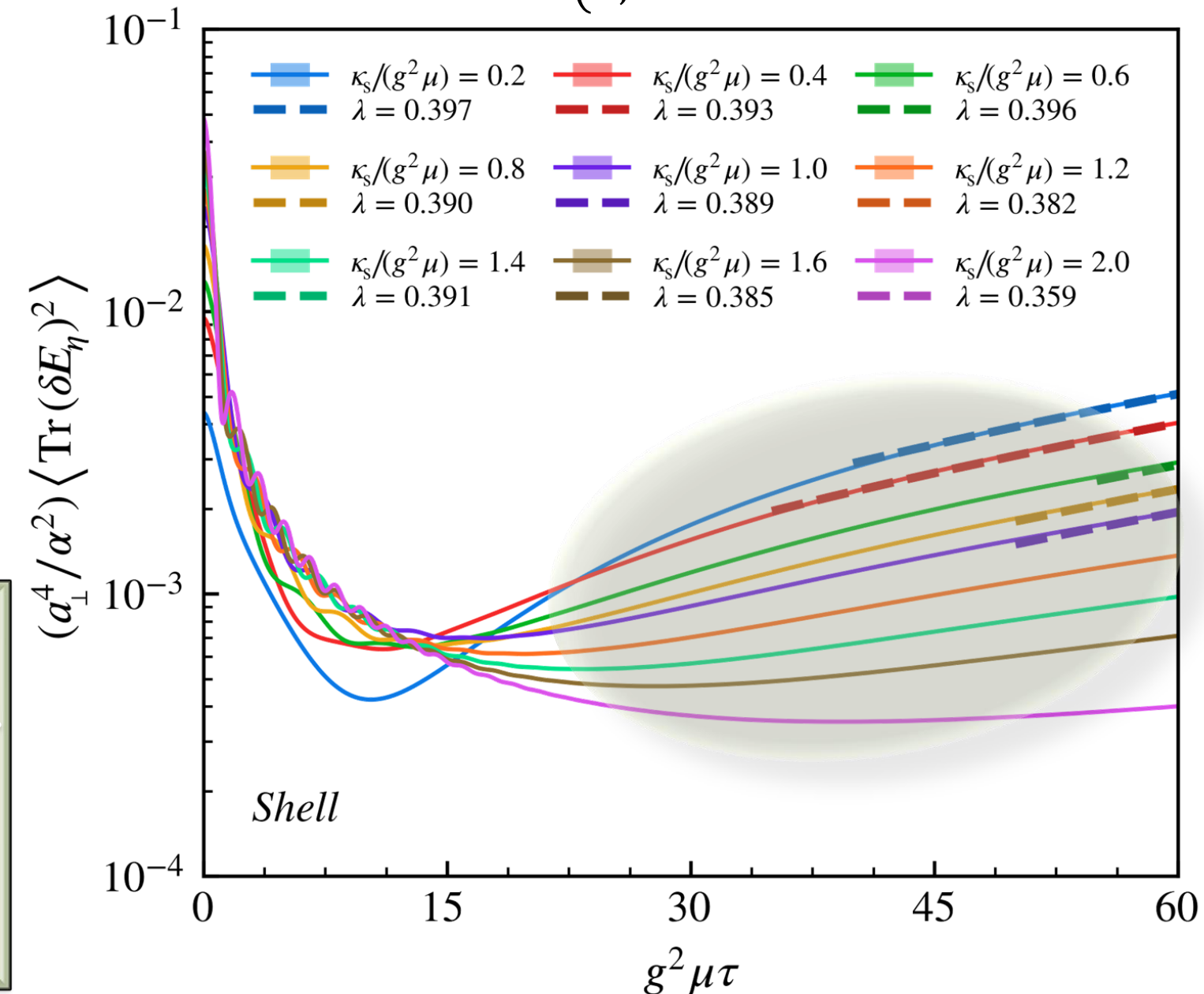
Early time

Higher momentum shells (larger κ_s) exhibit larger initial amplitudes due to the larger phase-space area $A \sim 2\pi\kappa_s\Delta k$.

Late time

Higher- κ_s modes have smaller overlap with the unstable modes, leading to lower late time amplitudes.

$$\mathcal{K}_{shell}(k_{\perp}) = \begin{cases} 1, & \kappa_s - \frac{\Delta k}{2} \leq k_{\perp} \leq \kappa_s + \frac{\Delta k}{2} \\ 0, & \text{otherwise} \end{cases}$$



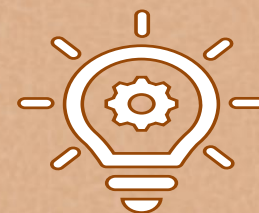
Shell-filtered perturbations

25

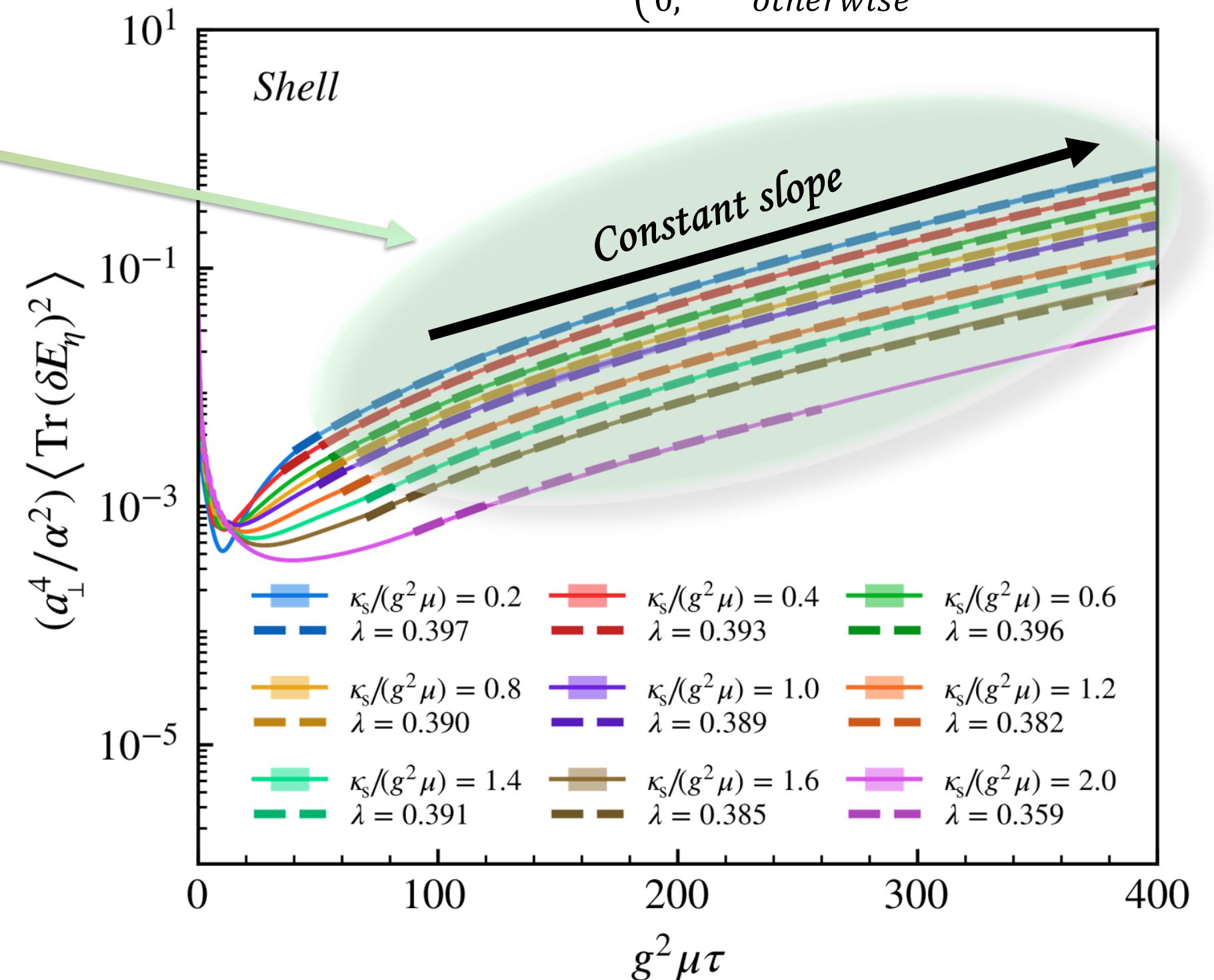
Despite the oscillatory start and shifting amplitudes, every single shell eventually locks into the same slope:

$$\lambda \approx 0.38 - 0.40$$

- The unstable mode has a nontrivial momentum-space structure, rather than being associated with a single momentum scale.
- Need a Fourier-space analysis of the perturbation.

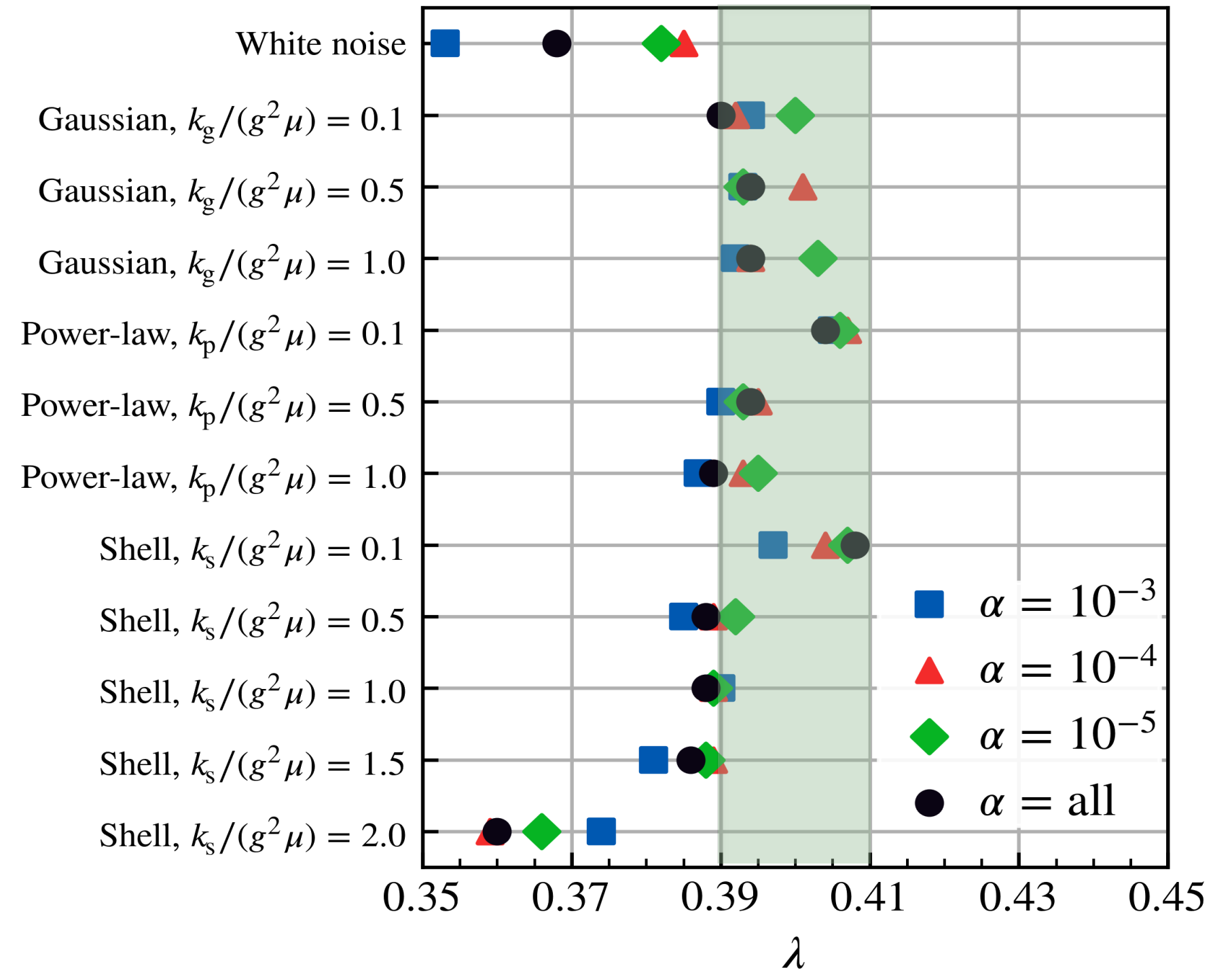
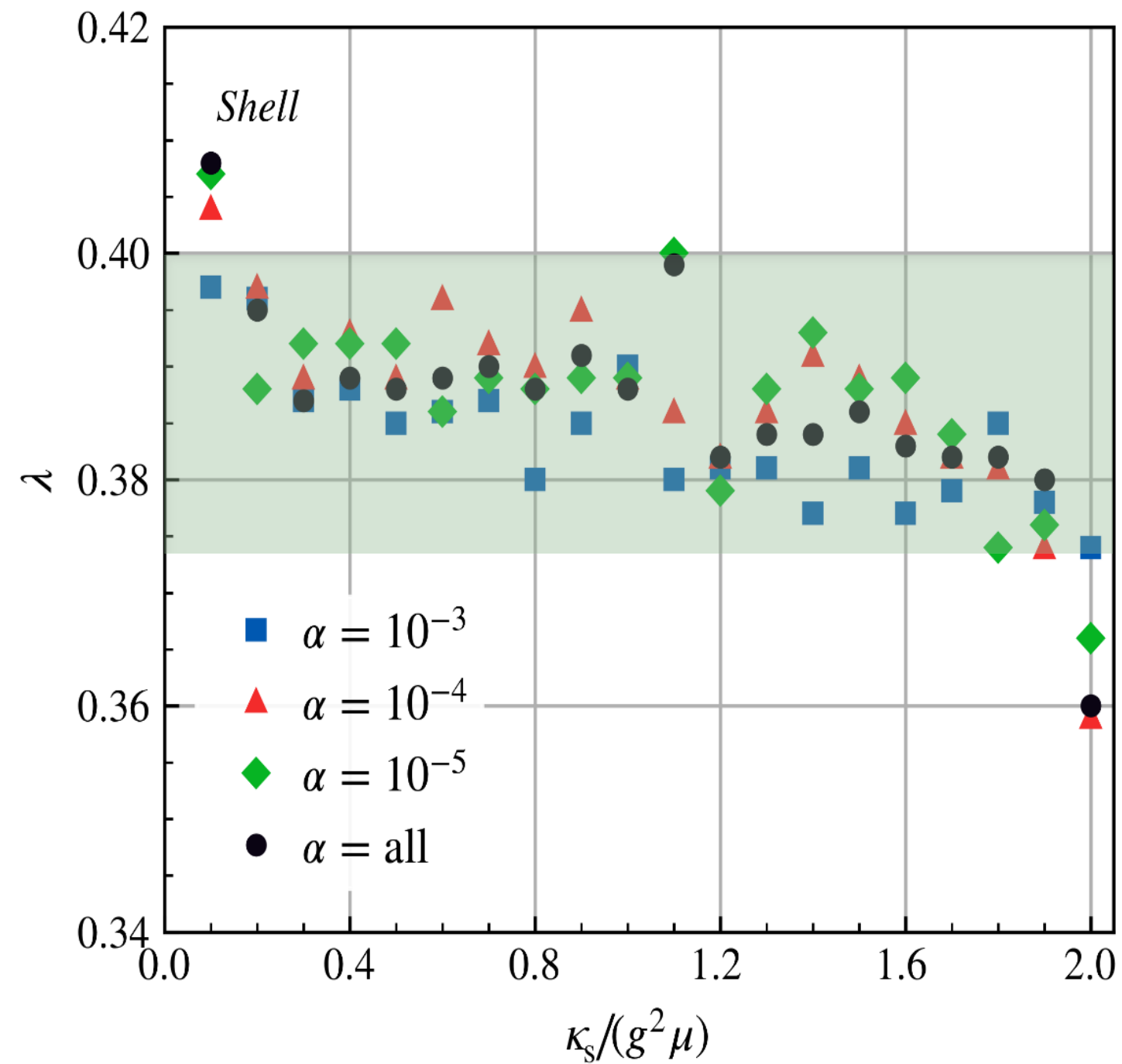


$$\mathcal{K}_{shell}(k_{\perp}) = \begin{cases} 1, & \kappa_s - \frac{\Delta k}{2} \leq k_{\perp} \leq \kappa_s + \frac{\Delta k}{2} \\ 0, & \text{otherwise} \end{cases}$$



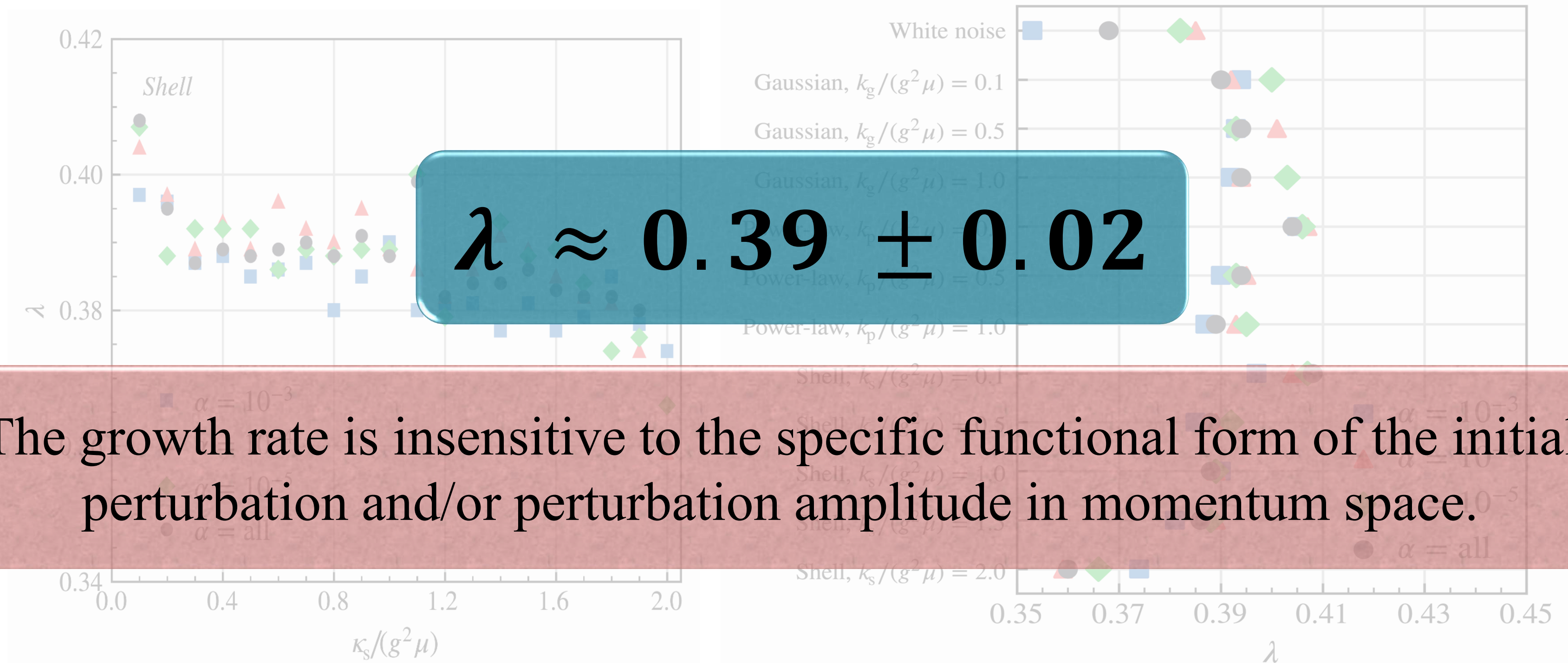
Robustness of the Lyapunov exponent

26



Robustness of the Lyapunov exponent

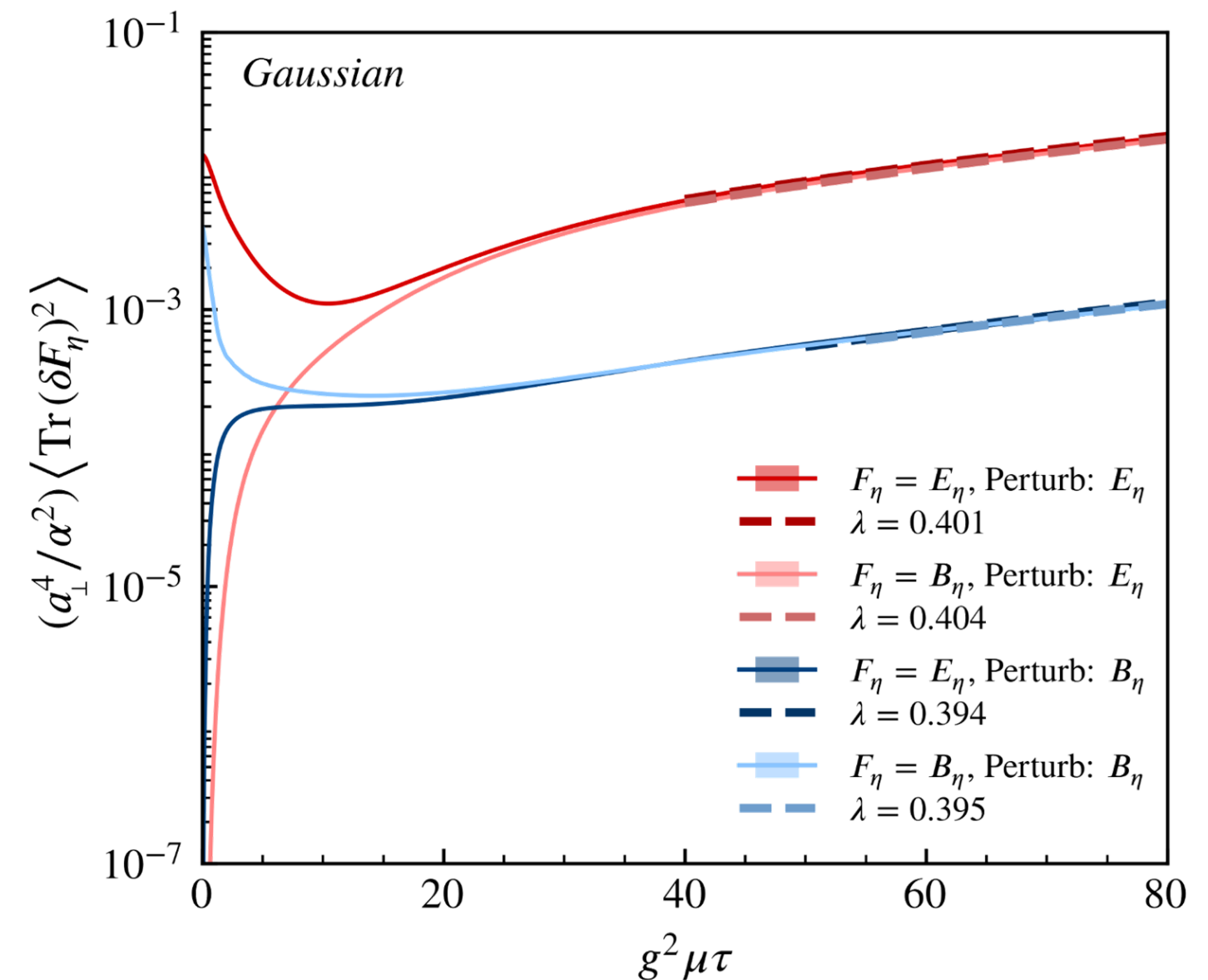
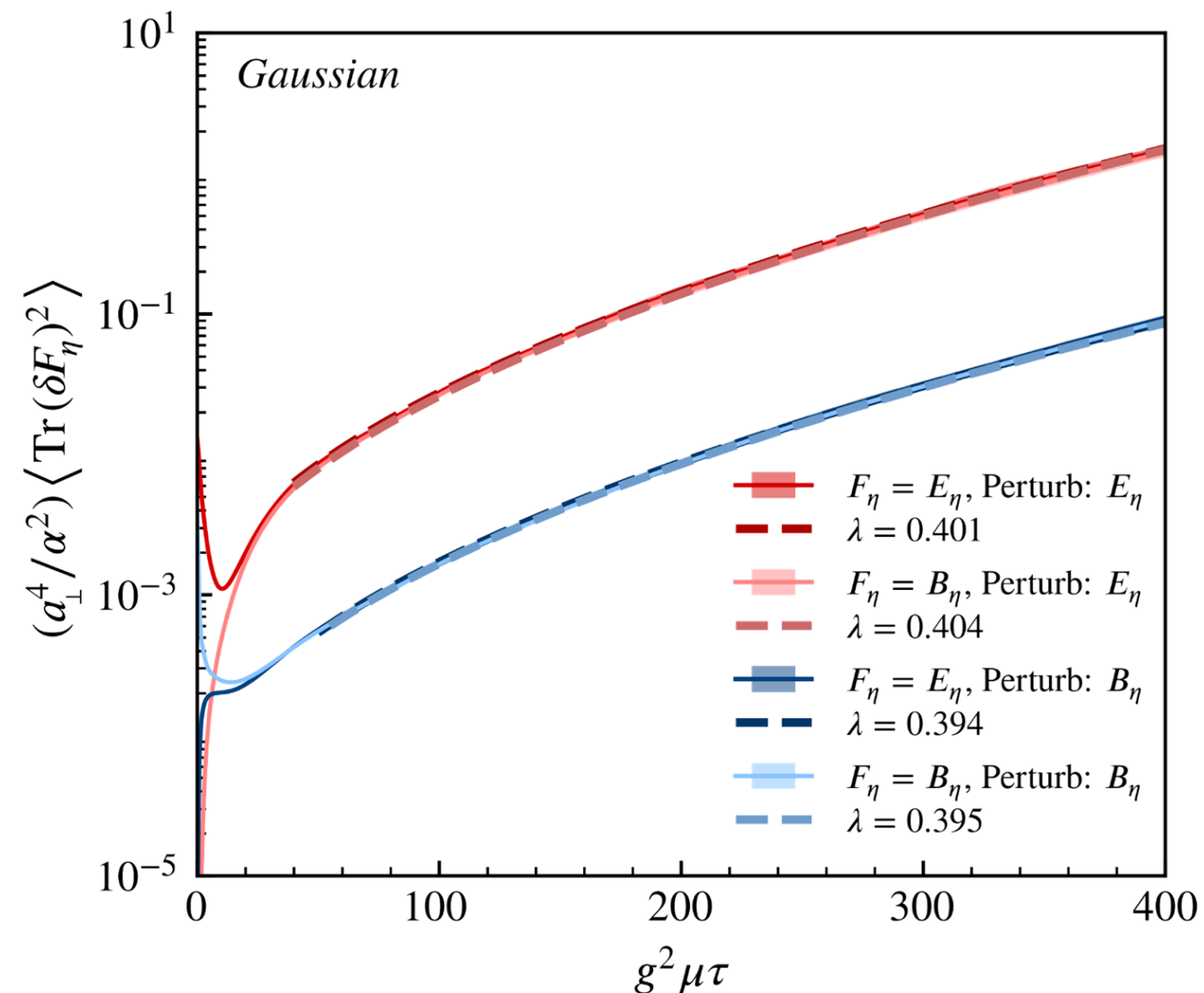
27



The growth rate is insensitive to the specific functional form of the initial perturbation and/or perturbation amplitude in momentum space.

Electric vs magnetic field perturbations

28



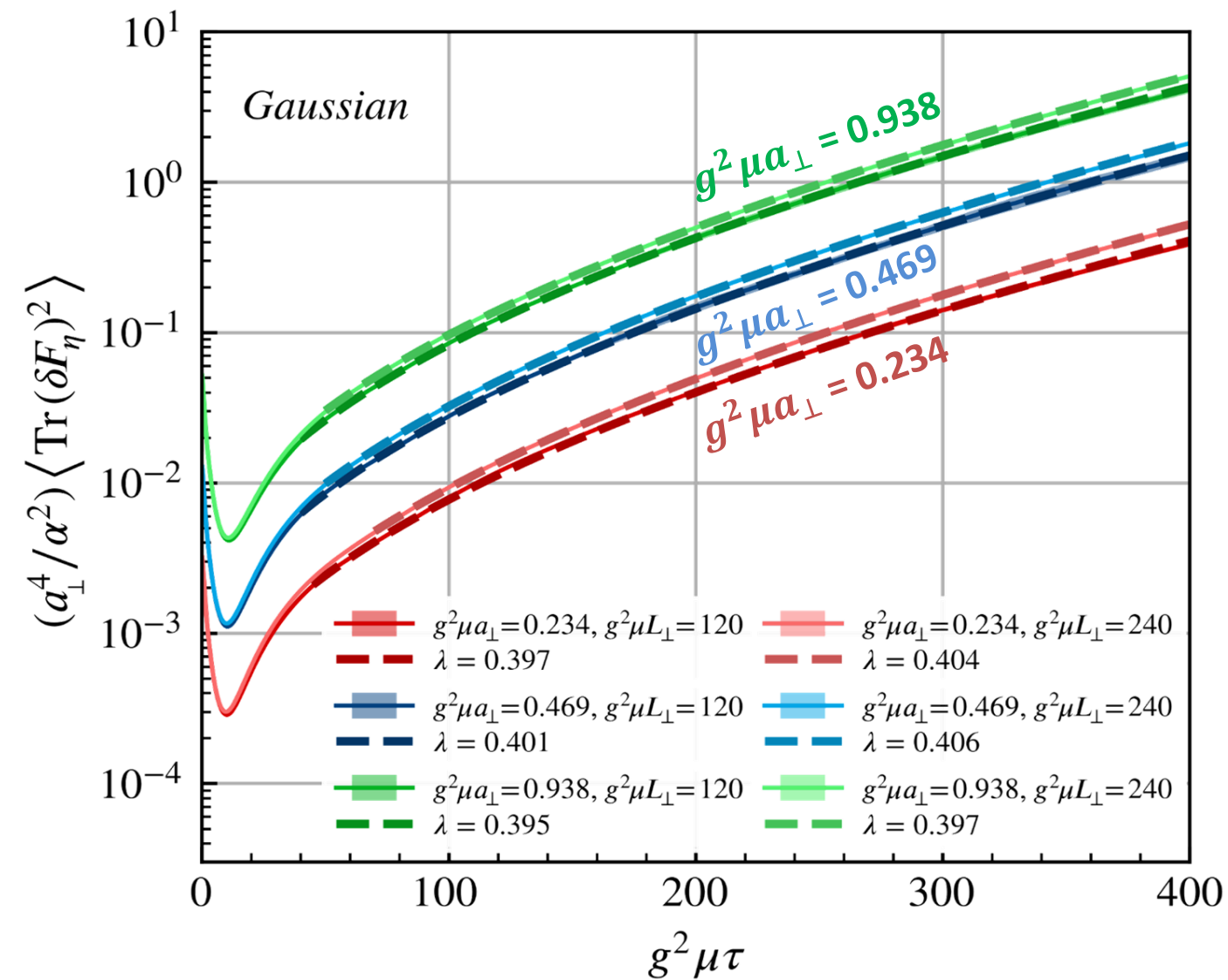
- ✓ Initialize one, measure the other
- ✓ Both couple to same unstable mode

Lyapunov exponent is insensitive to:

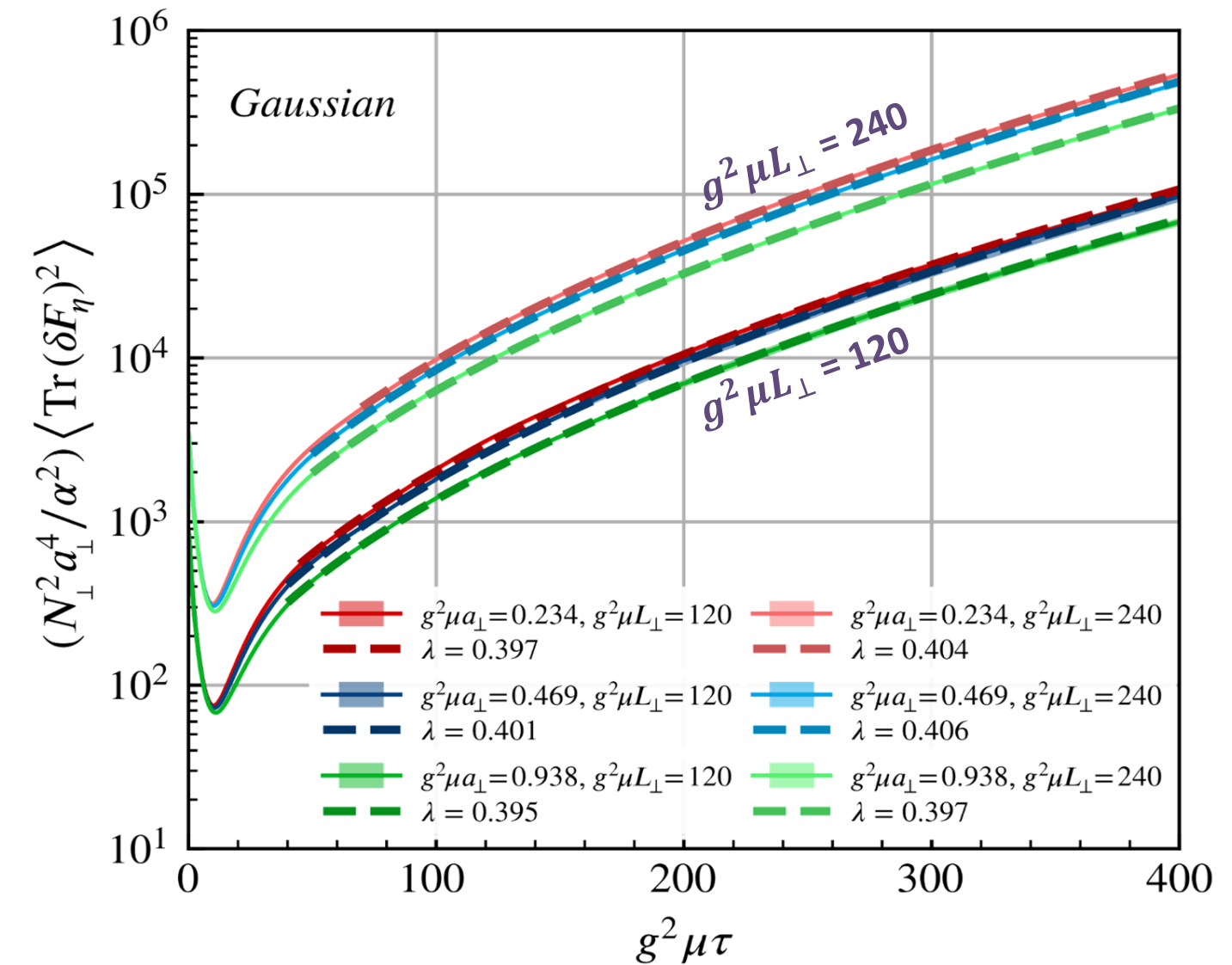
- Component of field perturbed
- Component of field measured

Lattice parameters dependence ??

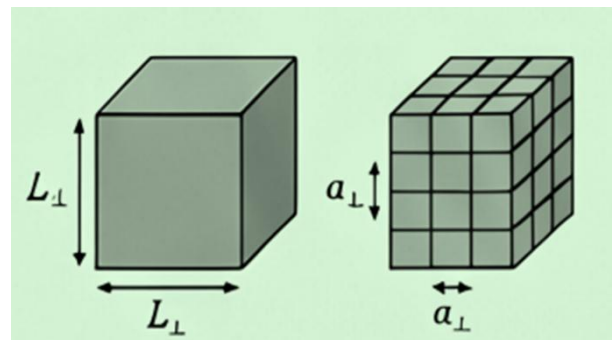
29



Dependence on lattice volume



Dependence on lattice spacing



The Lyapunov exponent ($\lambda \approx 0.38 - 0.40$) of the Glasma is independent of the numerical discretization details.

Conclusions

30

$$\left(\delta F_\eta(\tau)\right)^2 \sim \exp\left[(0.39 \pm 0.02)\sqrt{g^2\mu\tau}\right]$$

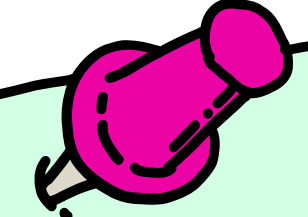
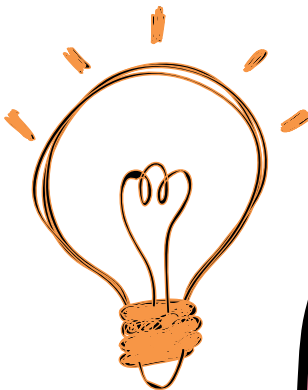
leads to
growth rate, $\Upsilon(\tau)$

$$\frac{d}{d\tau}\delta F_\eta(\tau) = \Upsilon(\tau)F_\eta(\tau)$$

where

$$\Upsilon(\tau) = (0.1 \pm 0.005)\sqrt{\frac{g^2\mu}{\tau}}$$

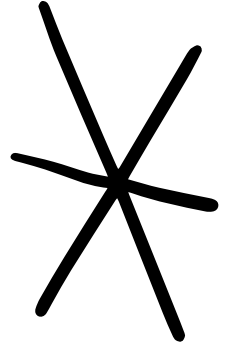
- ❑ We discovered an **unstable leading mode** in the non-linear Glasma fields.
- ❑ Chaos in the Glasma is a scale-invariant, collective phenomenon.
- ❑ The leading mode couples to all momenta scales in the background fields efficiently.



Summary

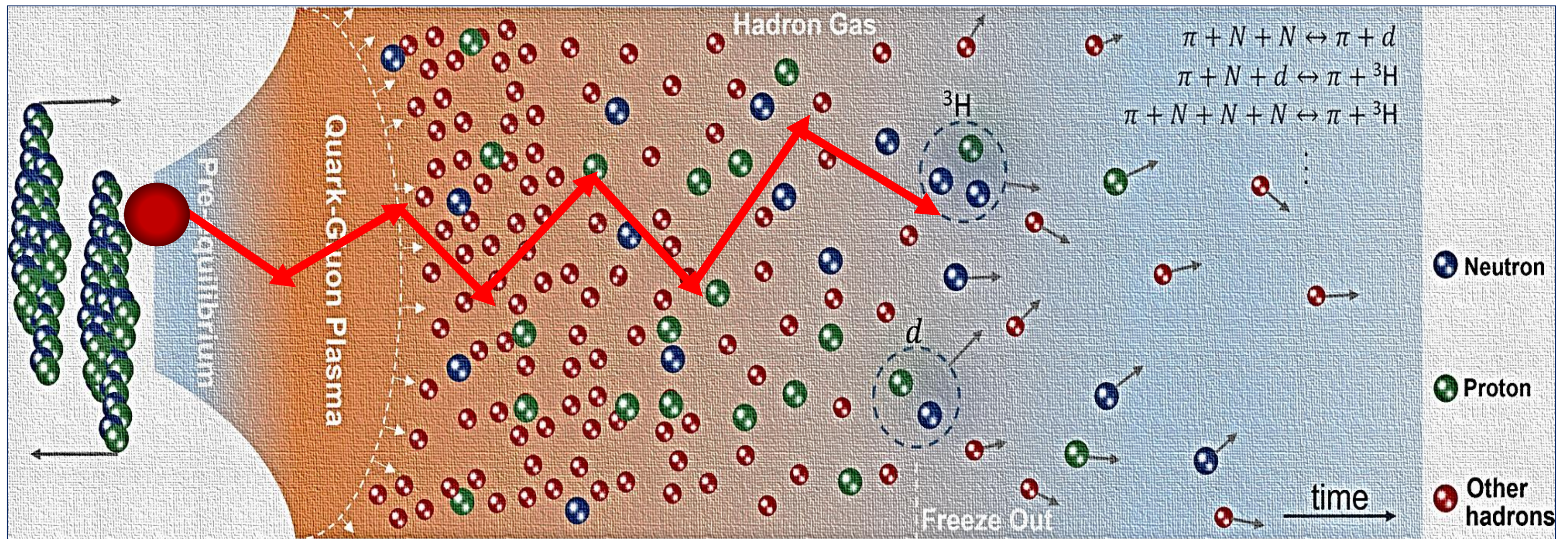
$$\left\langle \text{Tr} \left(\delta F_{\eta}(\tau) \right)^2 \right\rangle \propto e^{\lambda \sqrt{g^2 \mu \tau}}$$

In linear regime of the perturbation, the Lyapunov exponent for the Glasma is $\lambda \sim 0.4$.



Work in progress: Heavy quarks' trajectories

32



K.J. Sun, R. Wang, C.M. Ko, Y.G. Ma, C. Shen, *Nature Communications*, 15 (2024) 1, 1074 [↗](#)

HQs can probe the entire evolution of the medium, from the **early stage** up to hadronization.

Work in progress: Heavy quarks' trajectories

Equations of motion of heavy quarks (Wong Equations)

$$\frac{dx^i}{dt} = \frac{p^i}{E}, \quad E = \sqrt{\mathbf{p}^2 + m^2}$$

$$\frac{dp^i}{dt} = gQ_a F_a^{i\mu} \frac{p_\mu}{E}$$

$$\frac{dQ_a}{dt} = g f_{abc} A_b^\mu \frac{p_\mu}{E} Q_c$$

- Moving beyond field fluctuations to track actual particles (heavy quarks)
- Solving Wong equations for heavy quarks moving in the background CYM glasma system
- Goal: Measure chaotic signatures directly from the diffusion of heavy quarks

Seeding the chaos in HQs

34

$$x_0^{i'} = x_0^i + \chi^i a_\perp$$

$$\begin{aligned}\langle \chi^i(\mathbf{x}_\perp) \rangle &= 0 \\ \langle \chi^i(\mathbf{x}_\perp) \chi^j(\mathbf{y}_\perp) \rangle &= \alpha_x^2 \delta^{ij} \delta_{\mathbf{x}_\perp, \mathbf{y}_\perp}\end{aligned}$$

$$p_0^{i'} = p_0^i + \frac{\xi^i}{a_\perp}$$

$$\begin{aligned}\langle \xi^i(\mathbf{x}_\perp) \rangle &= 0 \\ \langle \xi^i(\mathbf{x}_\perp) \xi^j(\mathbf{y}_\perp) \rangle &= \alpha_p^2 \delta^{ij} \delta_{\mathbf{x}_\perp, \mathbf{y}_\perp}\end{aligned}$$

$$Q_0^{a'} = (Q_0^a + \zeta^a) \frac{|Q_0^a|}{|(Q_0^a + \zeta^a)|}$$

$$\begin{aligned}\langle \zeta^a(\mathbf{x}_\perp) \rangle &= 0 \\ \langle \zeta^a(\mathbf{x}_\perp) \zeta^b(\mathbf{y}_\perp) \rangle &= \alpha_Q^2 \delta^{ab} \delta_{\mathbf{x}_\perp, \mathbf{y}_\perp}\end{aligned}$$

$\alpha_x, \alpha_p, \alpha_Q$:

Perturbation amplitudes

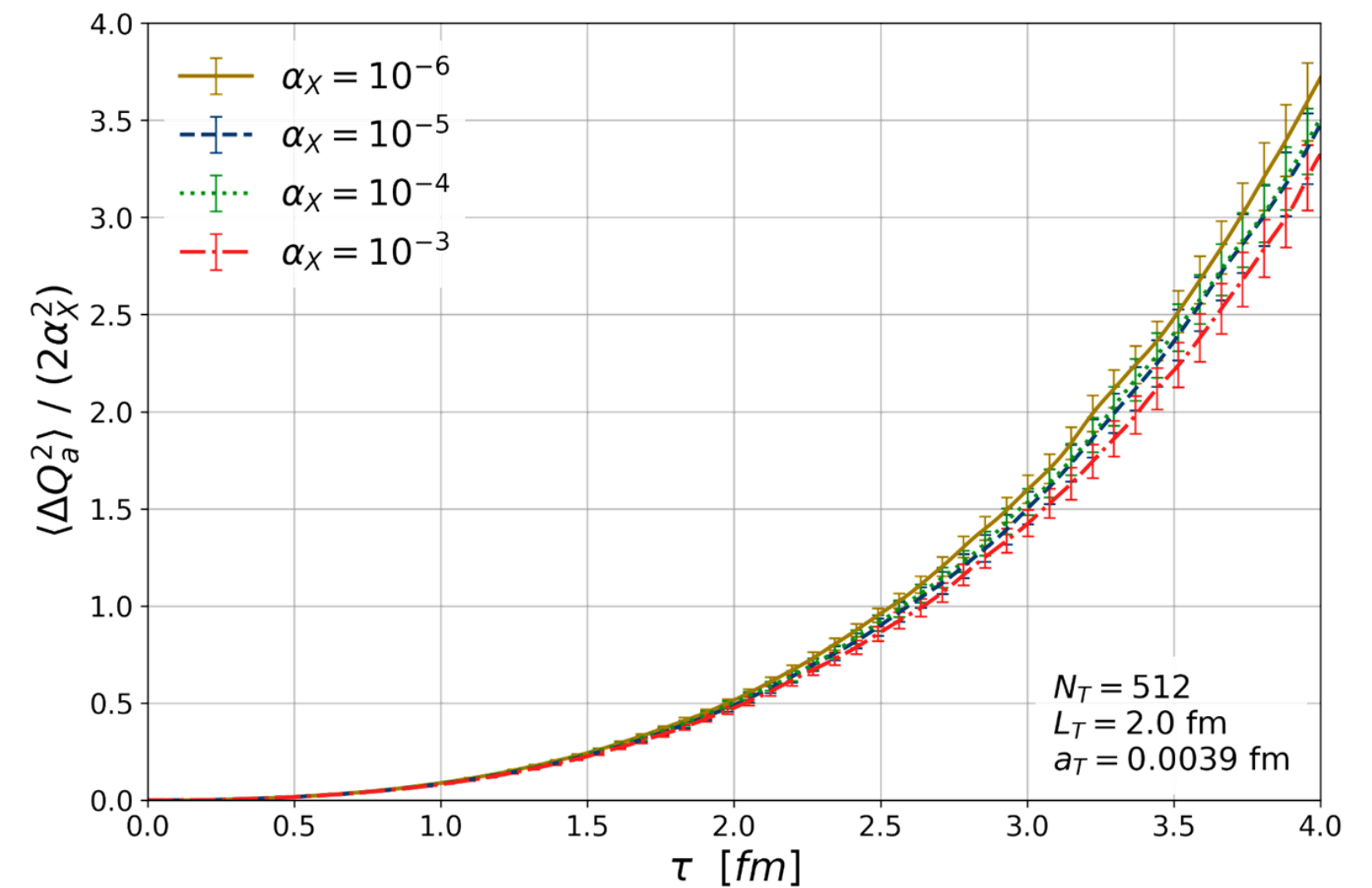
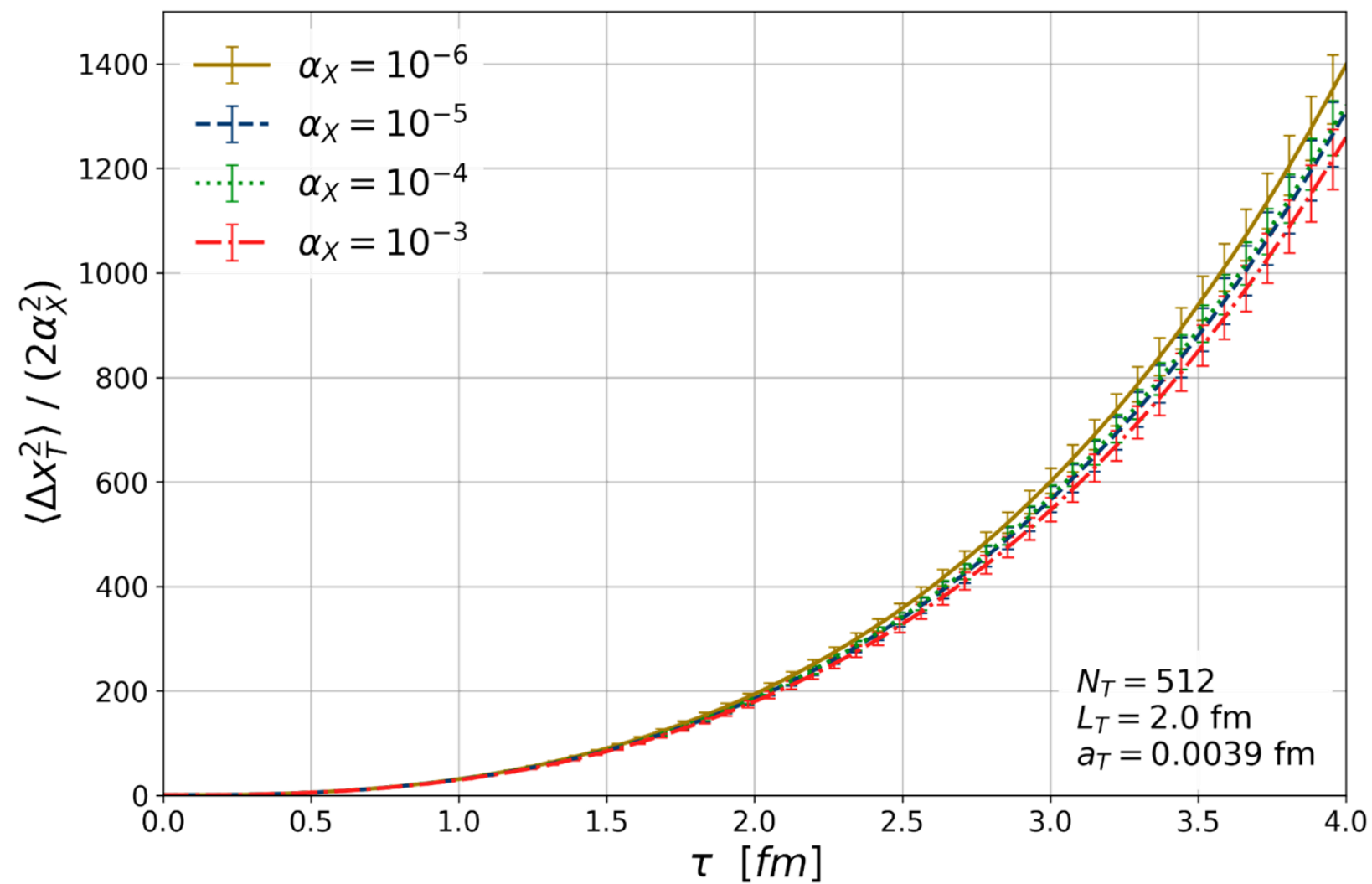
Perturbation in position

Preliminary results

35

$$x_0^{i'} = x_0^i + \chi^i a_\perp$$

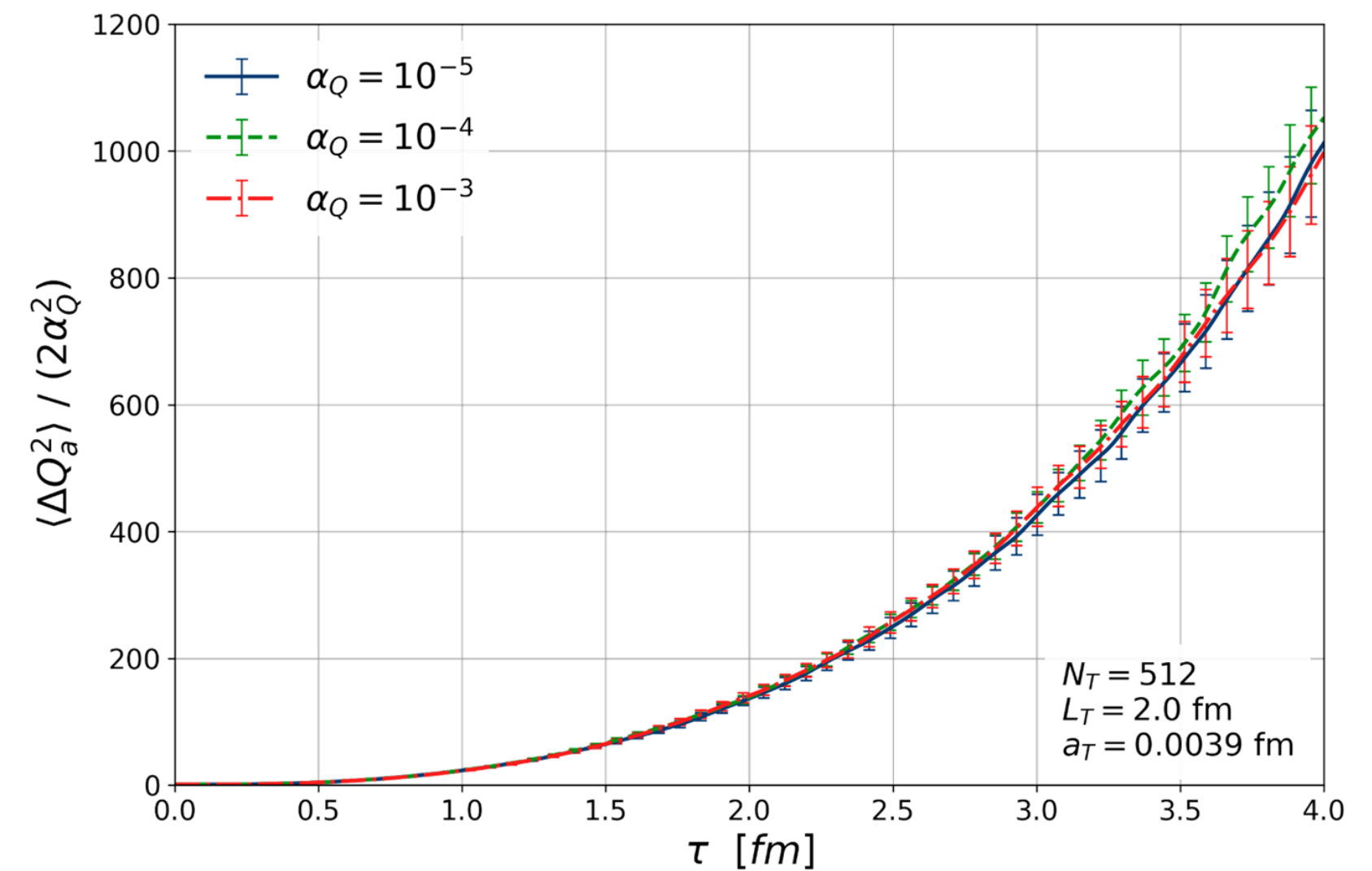
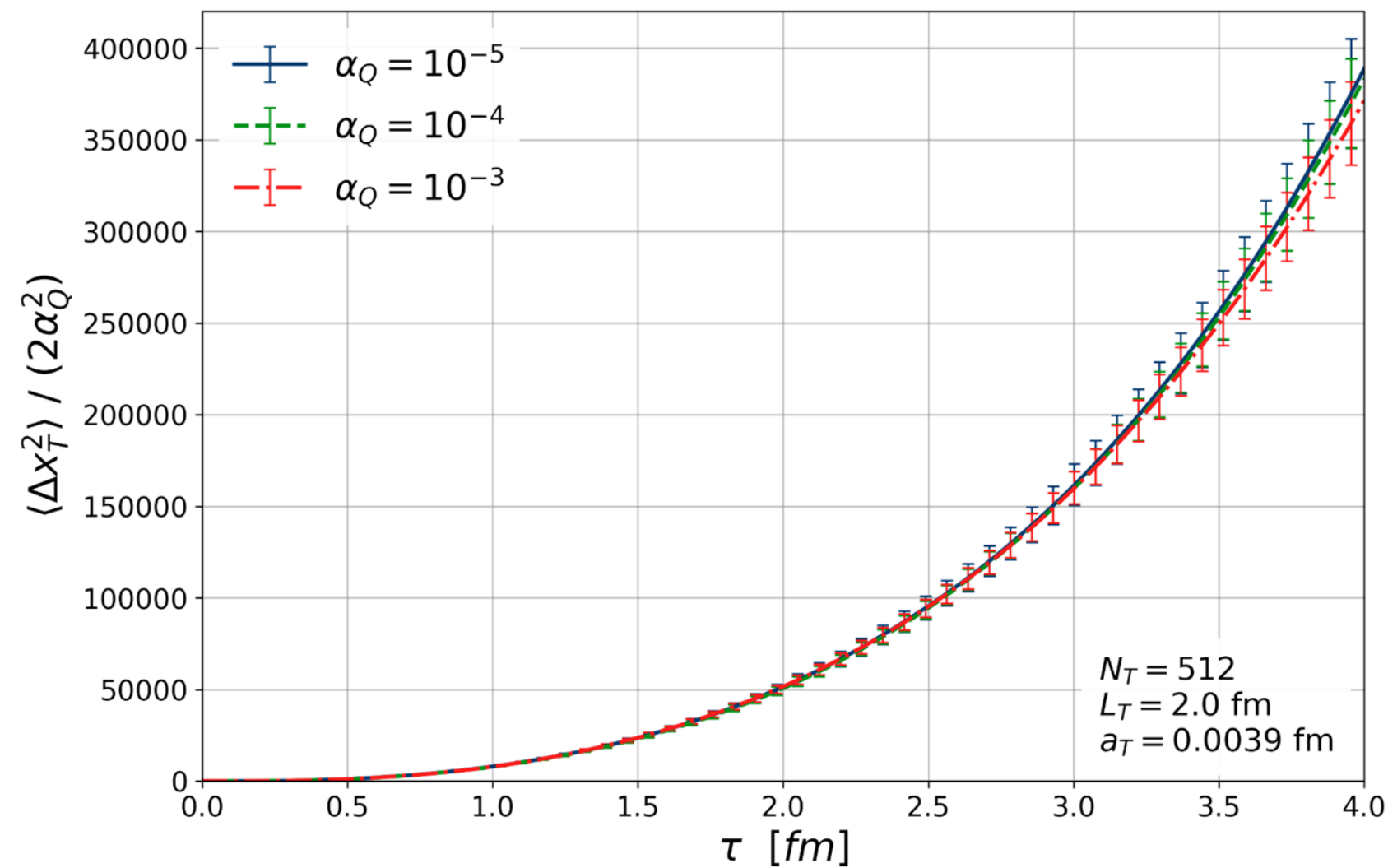
$$\begin{aligned}\langle \chi^i(x_\perp) \rangle &= 0 \\ \langle \chi^i(x_\perp) \chi^j(y_\perp) \rangle &= \alpha_x^2 \delta^{ij} \delta_{x_\perp, y_\perp}\end{aligned}$$



Perturbation in color charge Preliminary results 36

$$Q_0^{a'} = (Q_0^a + \zeta^a) \frac{|Q_0^a|}{|(Q_0^a + \zeta^a)|}$$

$$\begin{aligned}\langle \zeta^a(\mathbf{x}_\perp) \rangle &= 0 \\ \langle \zeta^a(\mathbf{x}_\perp) \zeta^b(\mathbf{y}_\perp) \rangle &= \alpha_Q^2 \delta^{ab} \delta_{\mathbf{x}_\perp, \mathbf{y}_\perp}\end{aligned}$$



The road ahead for Glasma chaos

37

Analyzing leading mode

Extract momentum dependence of the leading mode in Coulomb gauge.

Full spectrum of modes

Extract the complete matrix of Lyapunov exponents, rather than just the leading mode.

Broken boost-invariance

Study of exponentially growing modes related to the Weibel instability.

Thank You.



Pooja
University of Jyväskylä

Strong and Electro-Weak Matter (SEWM) 2026
Helsinki, Finland

Monday,
August 17, 2026

Appendix



Pooja
University of Jyväskylä

Strong and Electro-Weak Matter (SEWM) 2026
Helsinki, Finland

Monday,
August 17, 2026

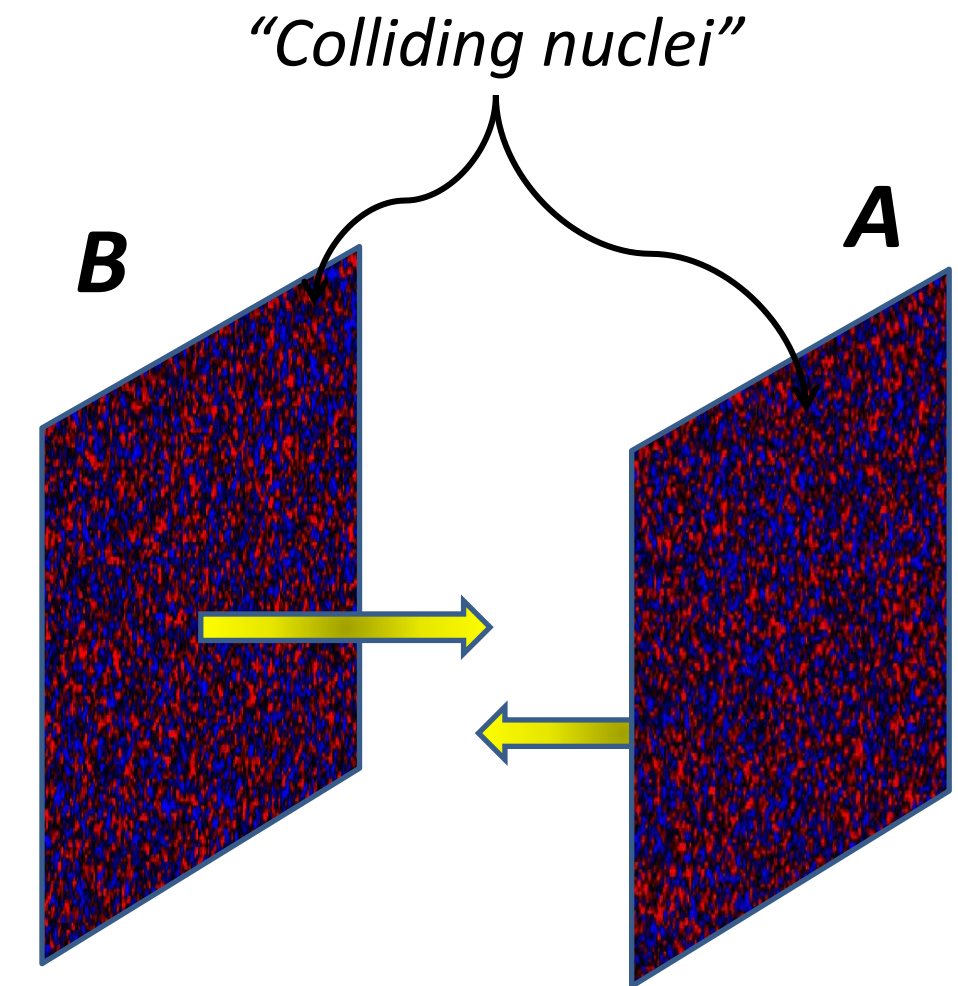
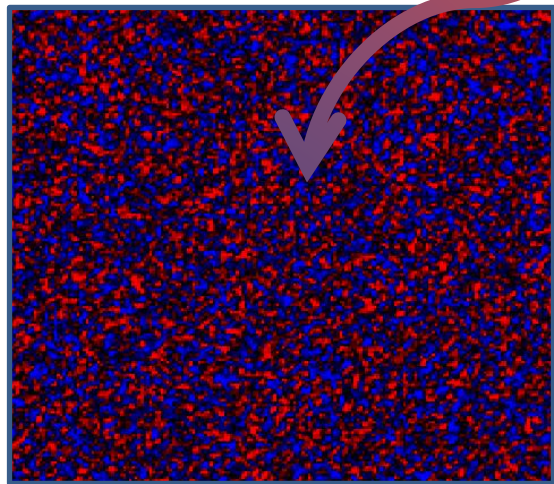
McLerran-Venugopalan (MV) model

40

$$\begin{aligned}\langle \rho^a(x_\perp) \rangle &= 0 \\ \langle \rho^a(x_\perp) \rho^b(y_\perp) \rangle &= (g\mu)^2 \delta^{ab} \delta^{(2)}(x_\perp - y_\perp)\end{aligned}$$

L. McLerran, R. Venugopalan, Phys. Rev. D 49 (1994) 2233-2241

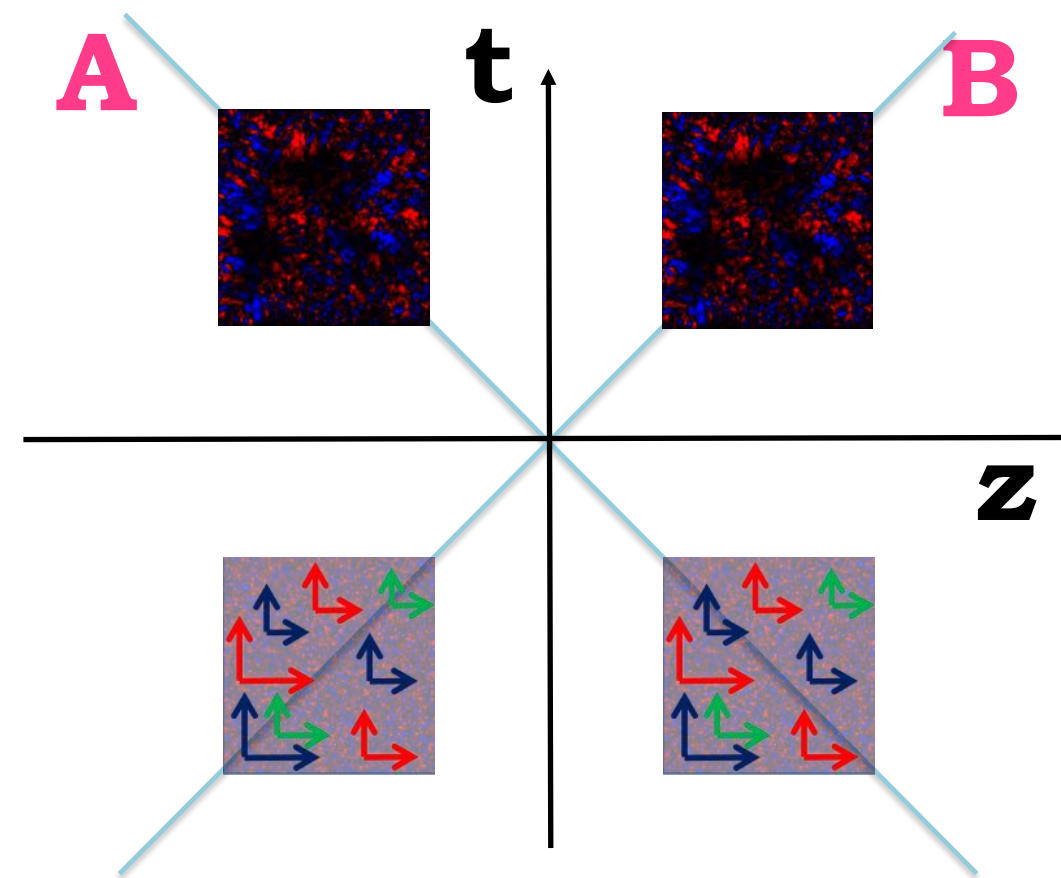
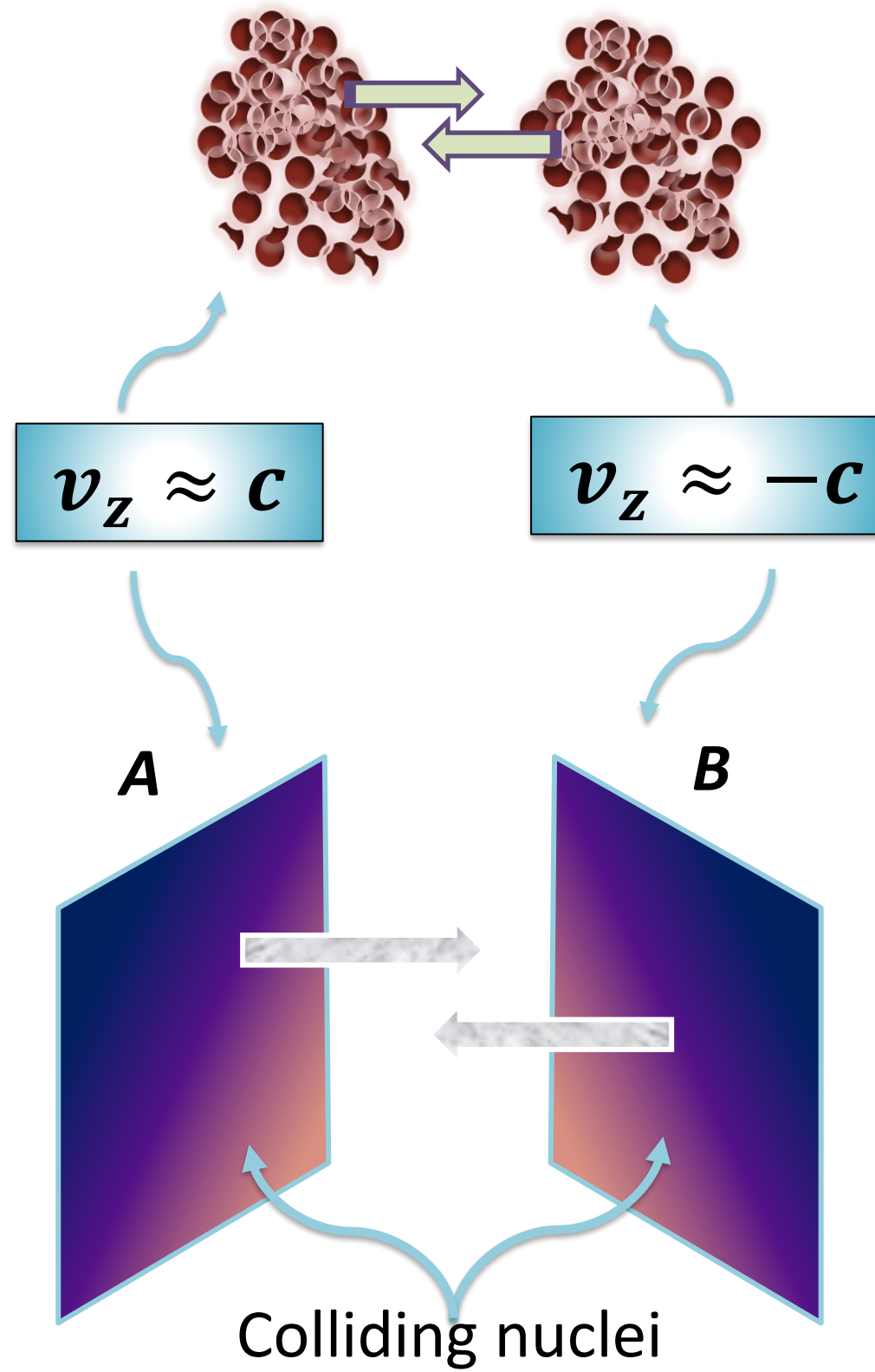
MV model parameter, $g^2\mu \sim Q_s$
 Q_s : Saturation scale of Glasma



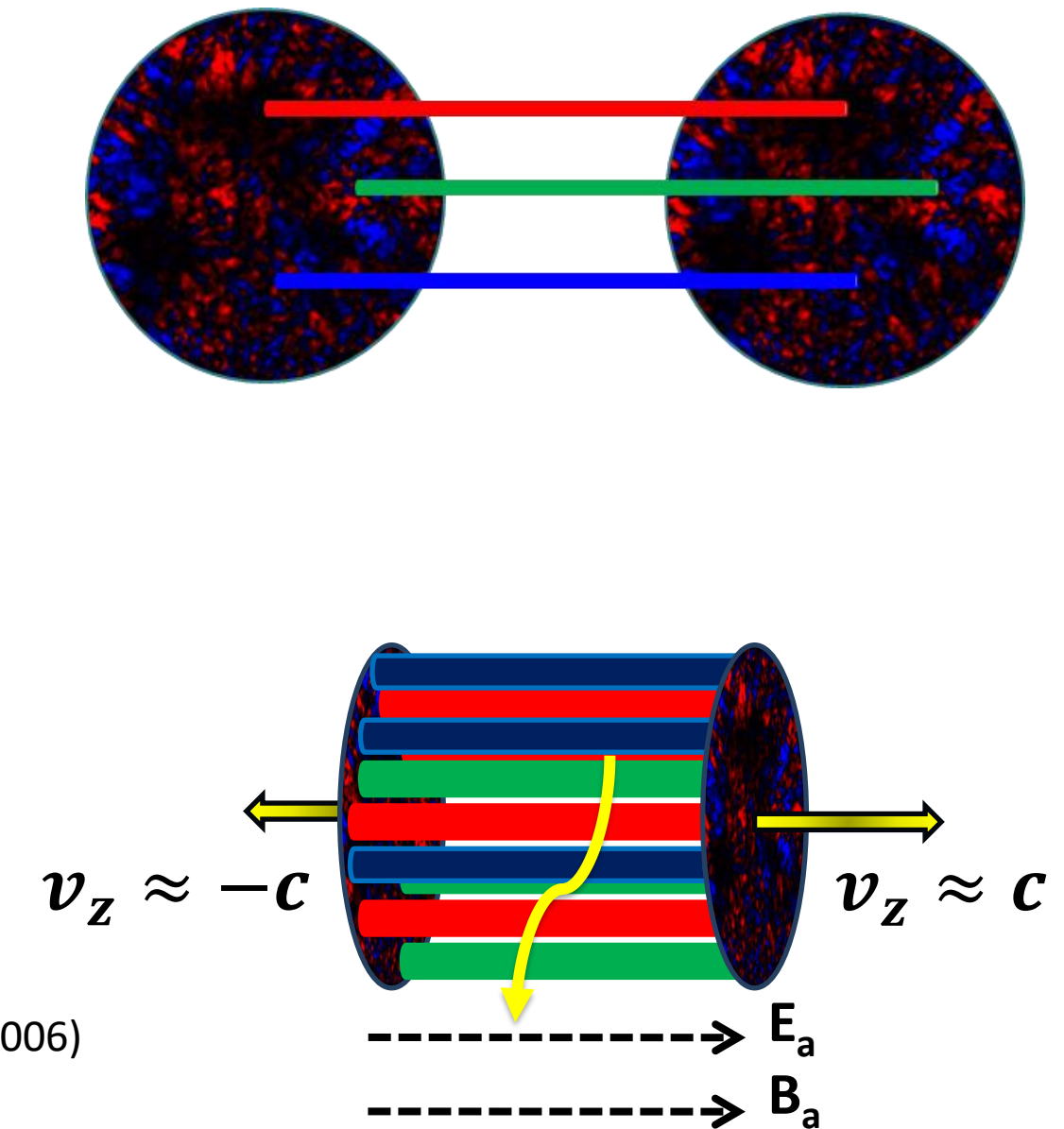
Provides initial condition where large- x partons are modeled as random color charges on a $2D$ plane, sourcing soft gluon fields.

Formation of Glasma

41



Lappi and McLerran (2006)



Glasma: The initial condition

42

$$x = \frac{\text{Parton momentum}}{\text{Nucleon momentum}}$$

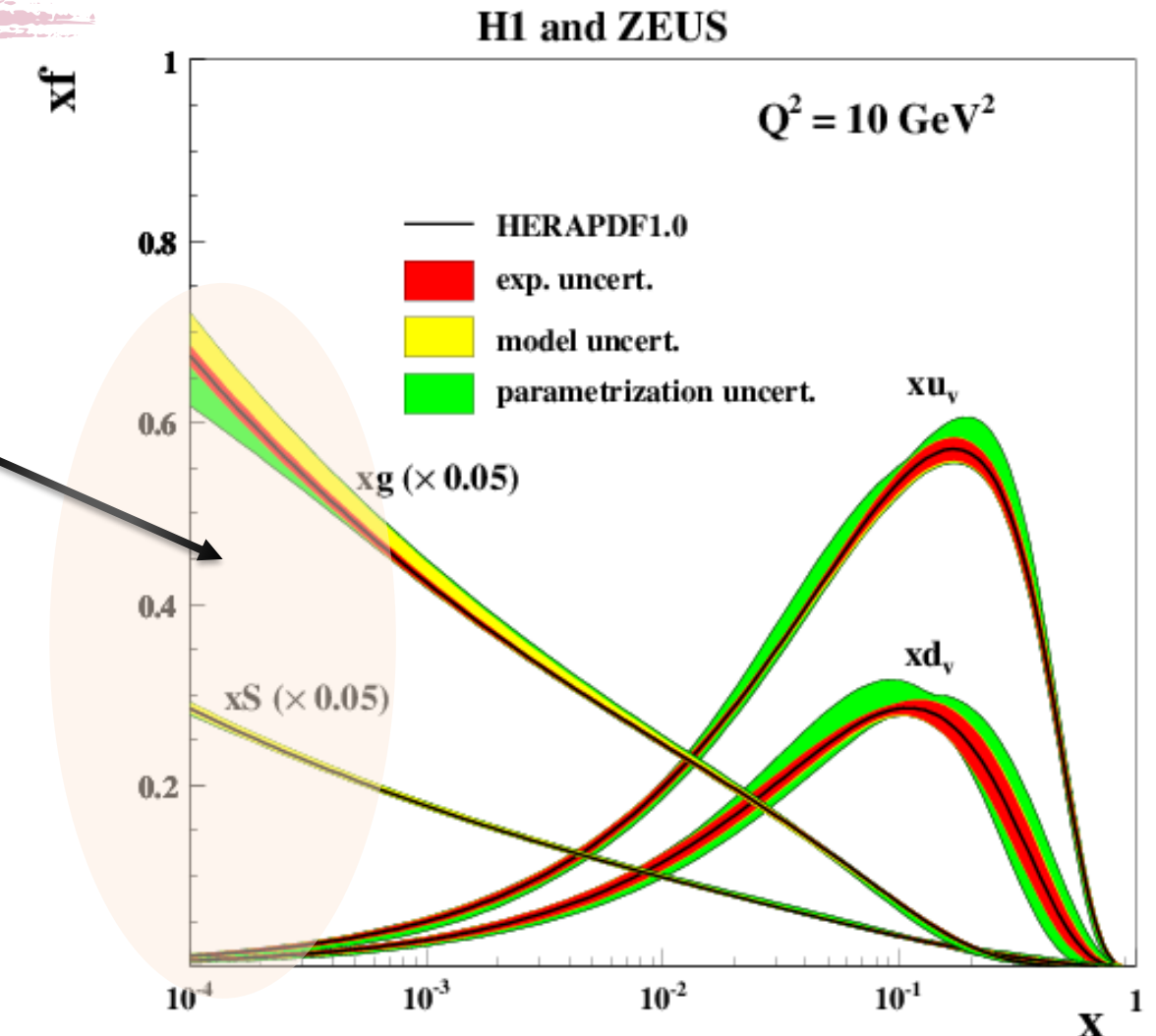
$$x \ll 1$$

Sea quarks + antiquarks
Sea gluons

$$x \approx 1$$

Valence quarks

The small- x
region is
dominated by
the gluons



R. Placakyte (on behalf of the H1 and ZEUS Collaborations),
Proceedings, 31st International Conference on Physics in collisions (PIC 2011)

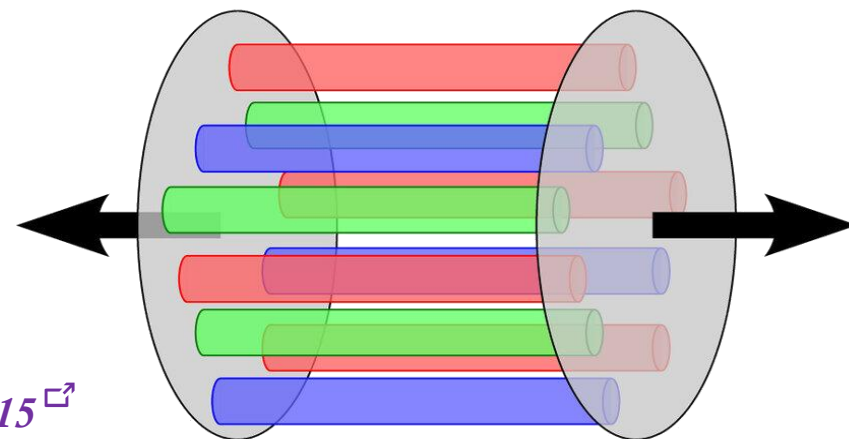
- **Slow partons:** *Gluons* dominate the nucleonic wave function.
- **Fast partons:** *Random sources* of the classical gluon fields.

Color-Glass Condensate (CGC)

43

- ❖ **Color**: Gluons have “colors”.
- ❖ **Glass**: Partons (quarks and gluons) with $x \approx 1$ are very fast ($v \approx c$). So, they appear frozen in lab, like molecules in glasses.
- ❖ **Condensate**: Many small- x gluons: Gluon density is very high and saturated.

The dynamical evolution of the CGC:
Classical Yang-Mills (CYM) equations



K. Fukushima,
Acta Phys. Polon. B 42 (2011) 2697-2715 

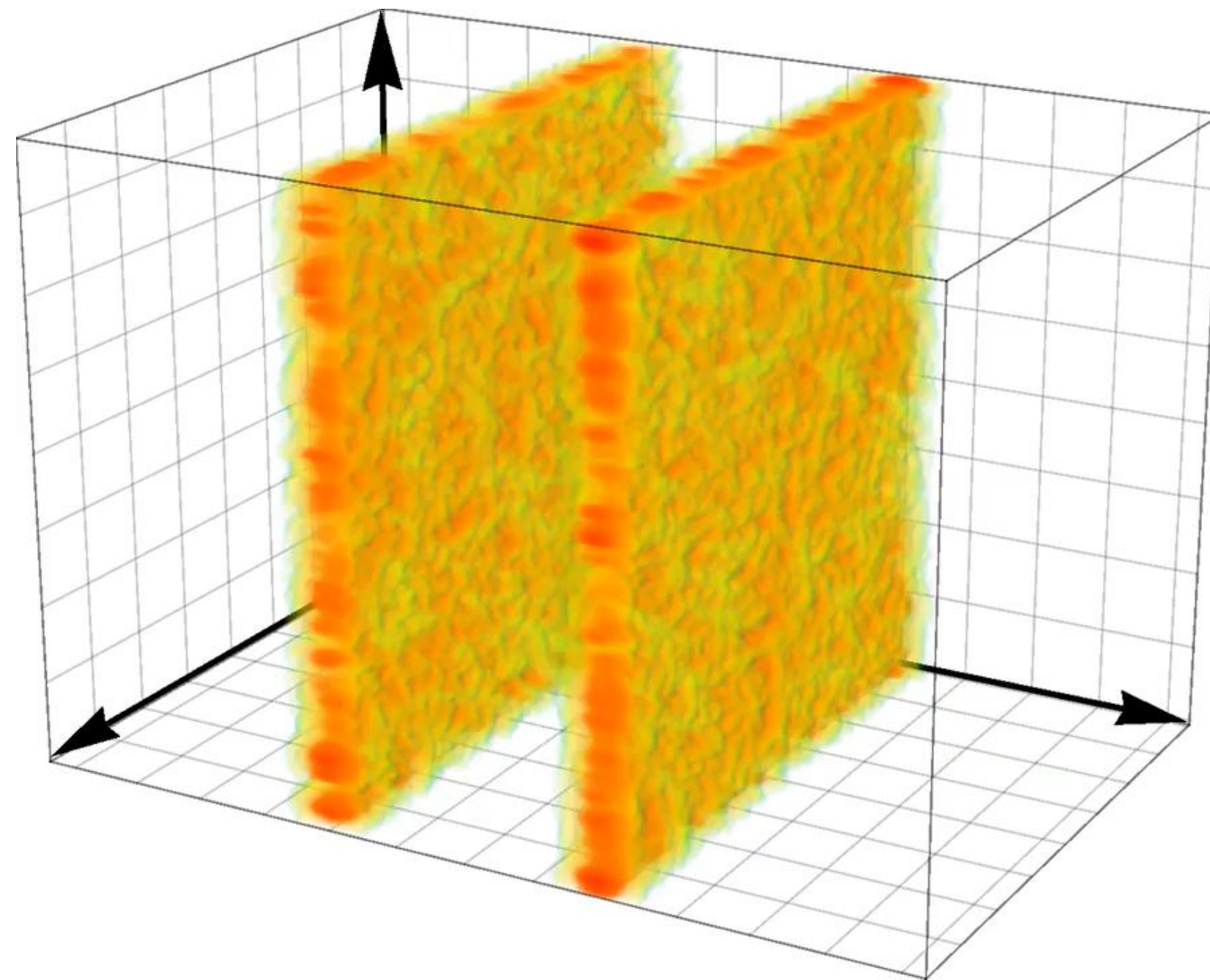
$$\mathcal{D}_\mu F^{\mu\nu} = J^\nu$$

Condensate
($x \approx 0$)
Gluon fields

Glass
($x \approx 1$)
Source

How nuclei collide: A visual sequence

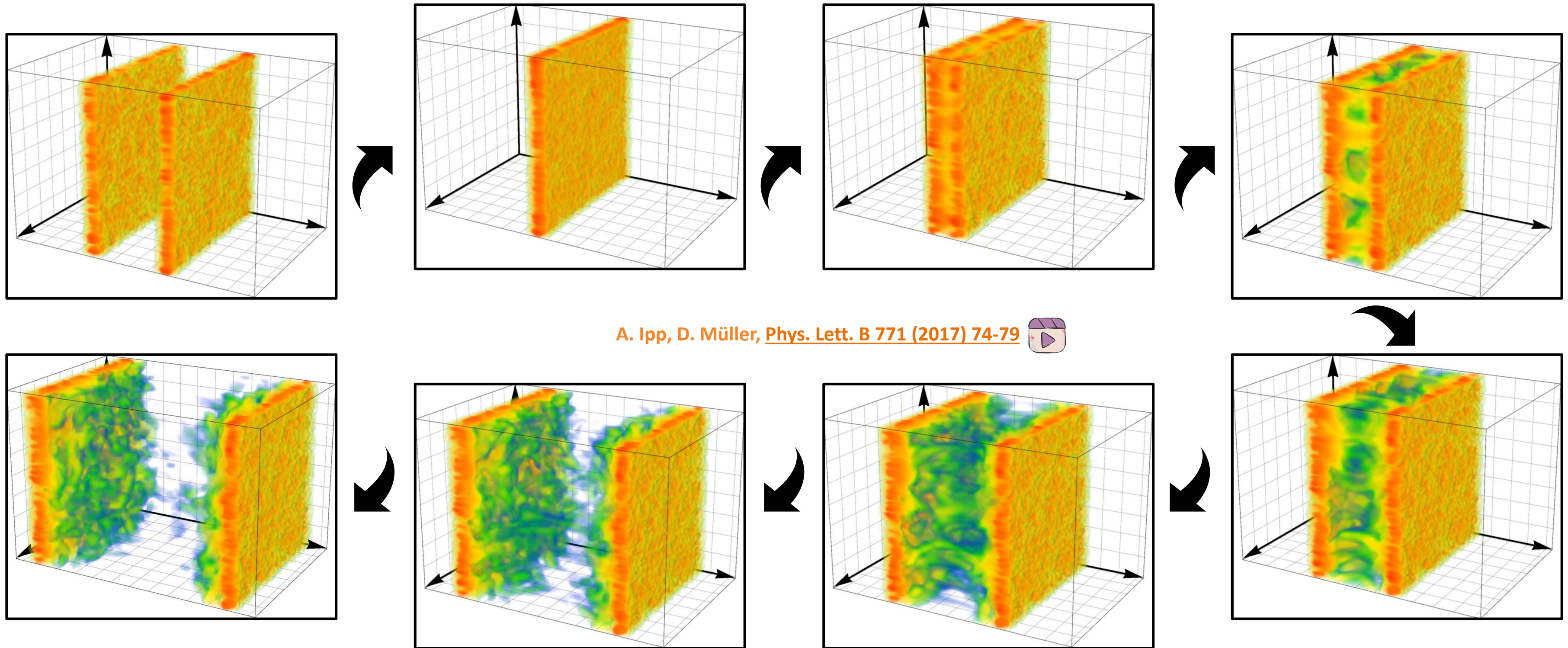
44



A. Ipp, D. Müller, [Phys. Lett. B 771 \(2017\) 74-79](#)

Animatic sketch of collision of two nuclei

45



Evolution of static Background Glasma: CYM Equations

46

Due to the large density,
the gluon field behaves
like a classical field.



Dynamics is governed
by classical EoMs,
namely the **classical
Yang-Mills (CYM)**
equations.

CYM Equations

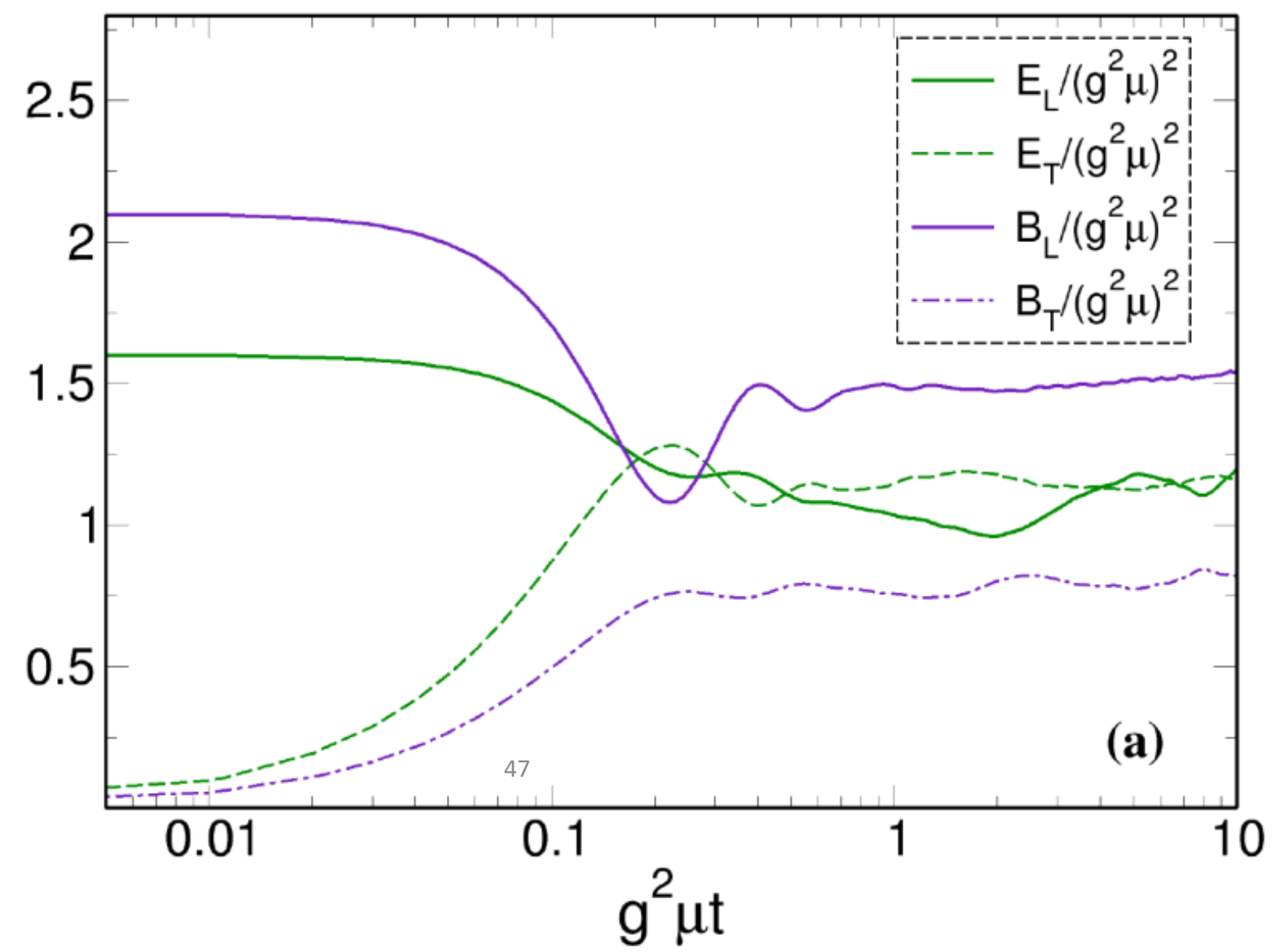
$$\frac{dA_i^a(x)}{dt} = E_i^a(x)$$

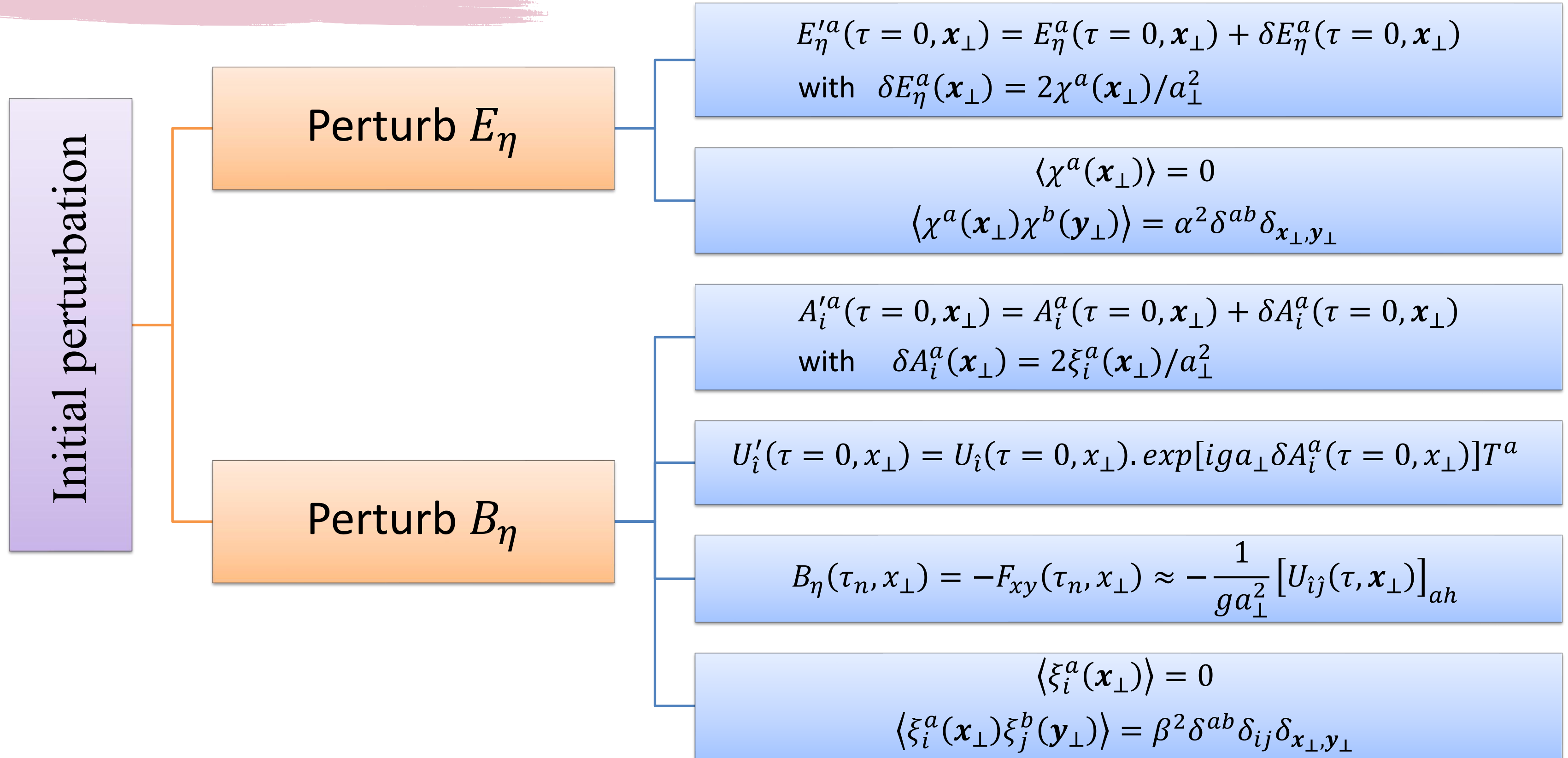
$$\frac{dE_i^a(x)}{dt} = \partial_j F_{ji}^a(x) + gf^{abc} A_j^b(x) F_{ji}^c(x)$$

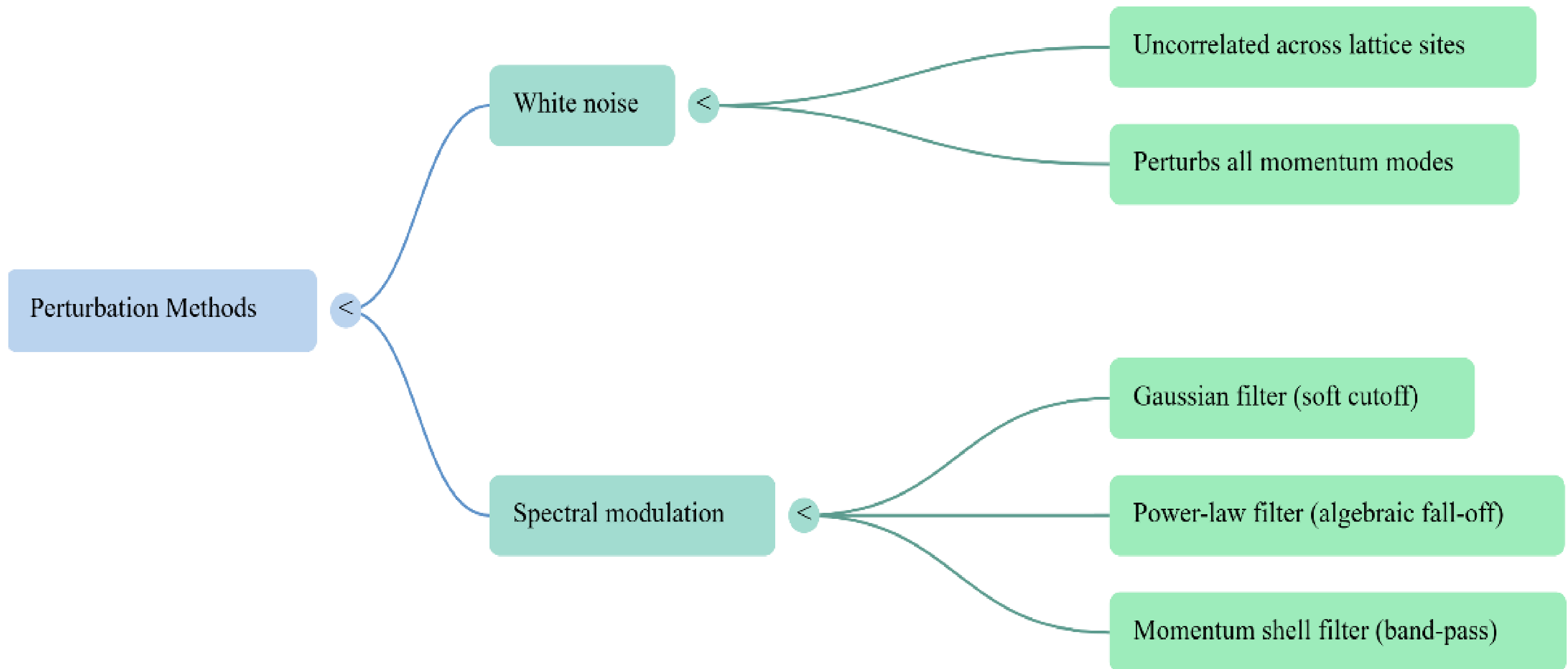
Gluon Field
Strength Tensor

$$\bullet F_{ij}^a(x) = \partial_i A_j^a(x) - \partial_j A_i^a(x) + gf^{abc} A_i^b(x) A_j^c(x)$$

*Evolution of the system is studied assuming Glasma as the initial condition,
and evolving this condition by virtue of the **CYM** equations.*

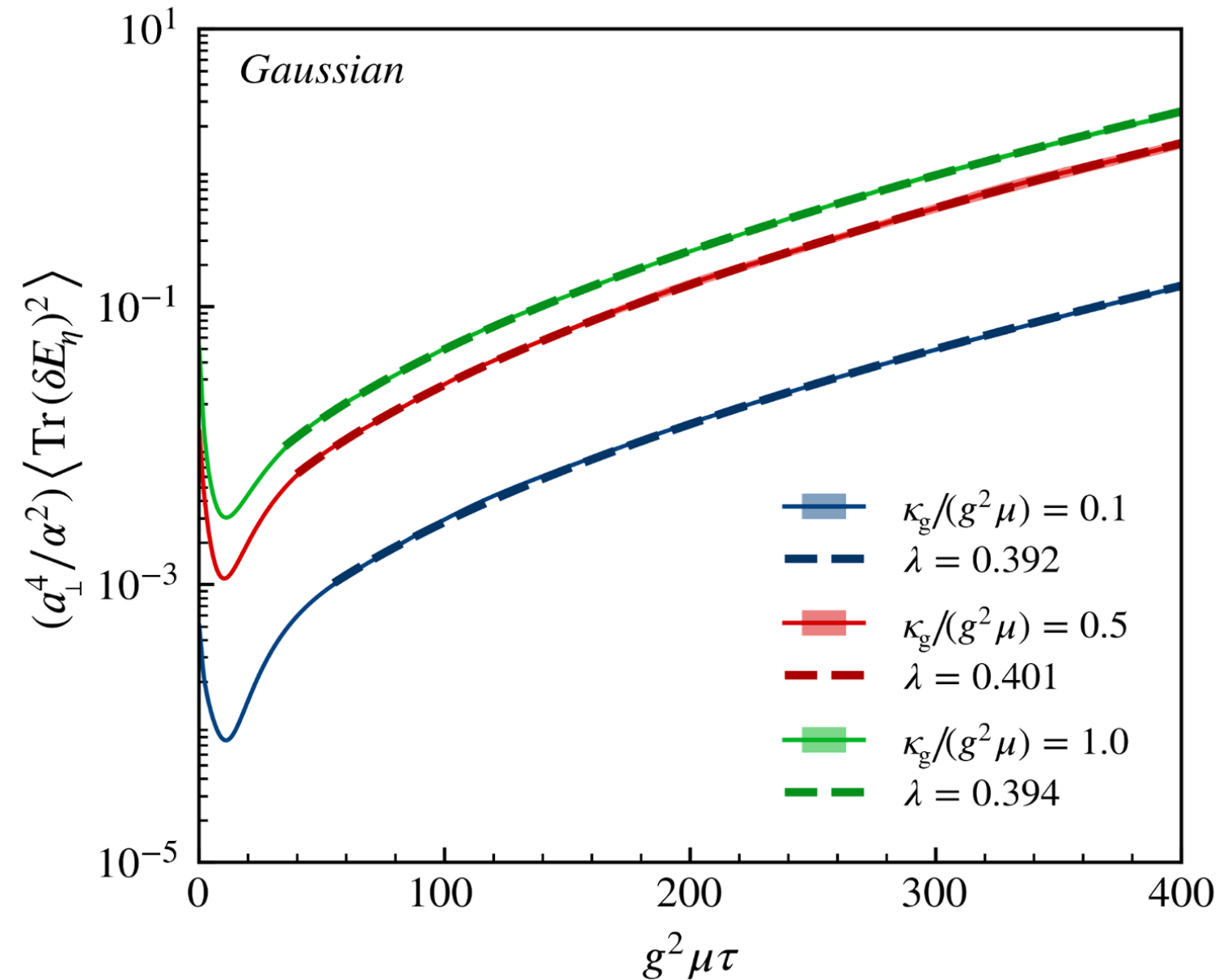
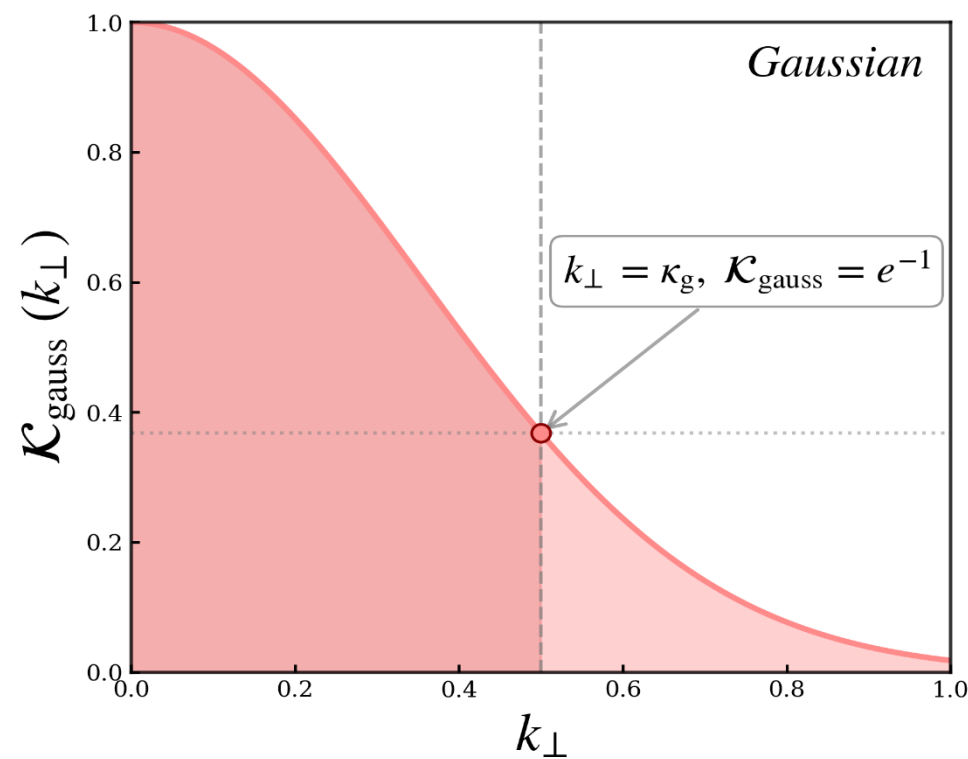
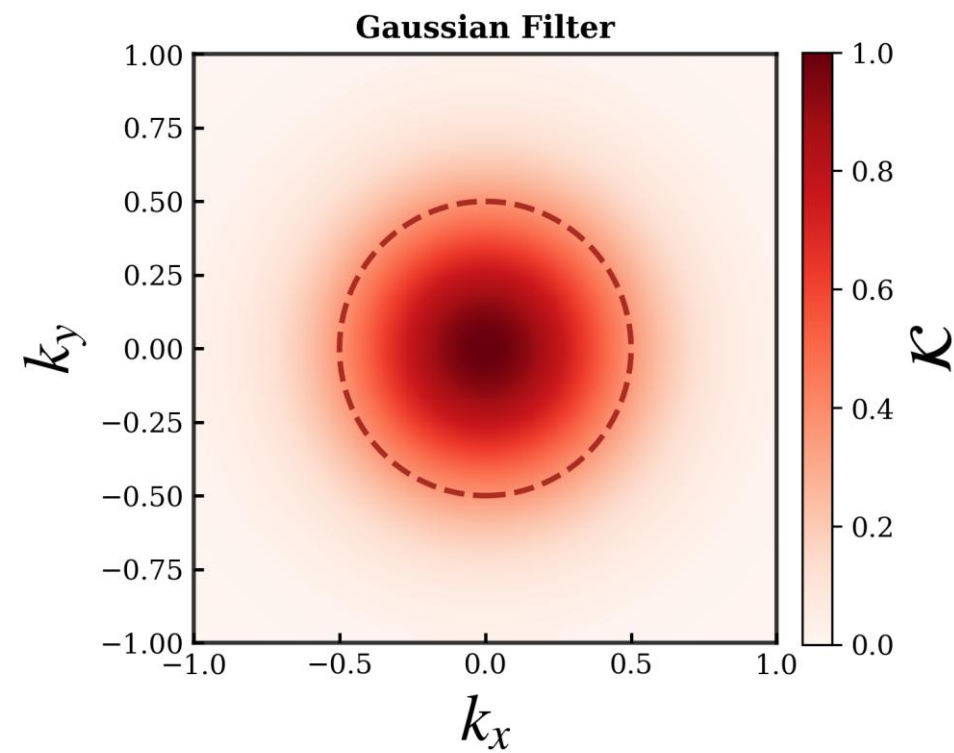






Gaussian-filtered perturbations

50

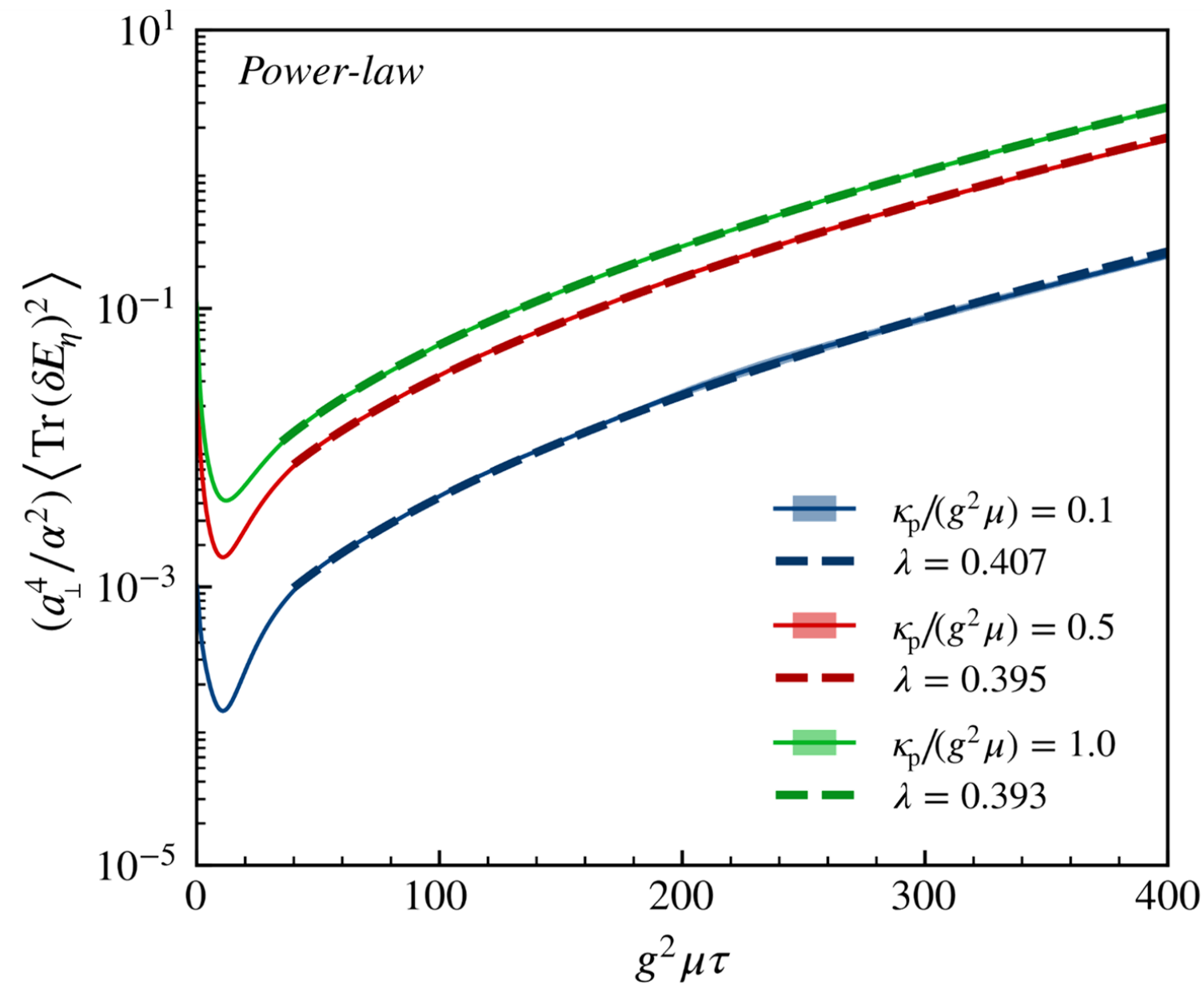
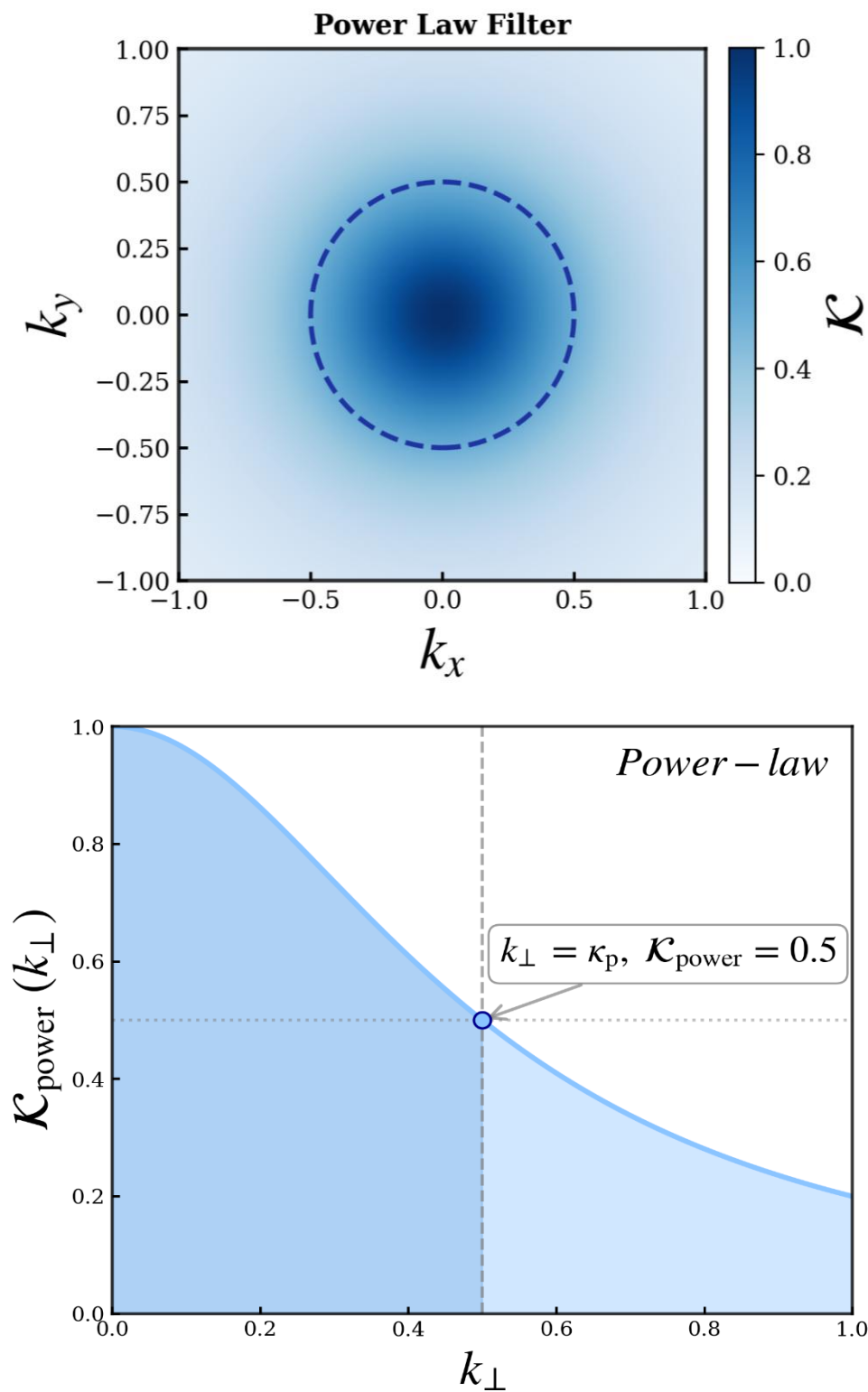


$$\mathcal{K}_{\text{gauss}}(k_{\perp}) = \exp\left(-\frac{\kappa_{\perp}^2}{\kappa_g^2}\right)$$

- Aggressively suppresses high-momentum ultraviolet modes
- Focuses on infrared modes

Power-law filtered perturbations

51

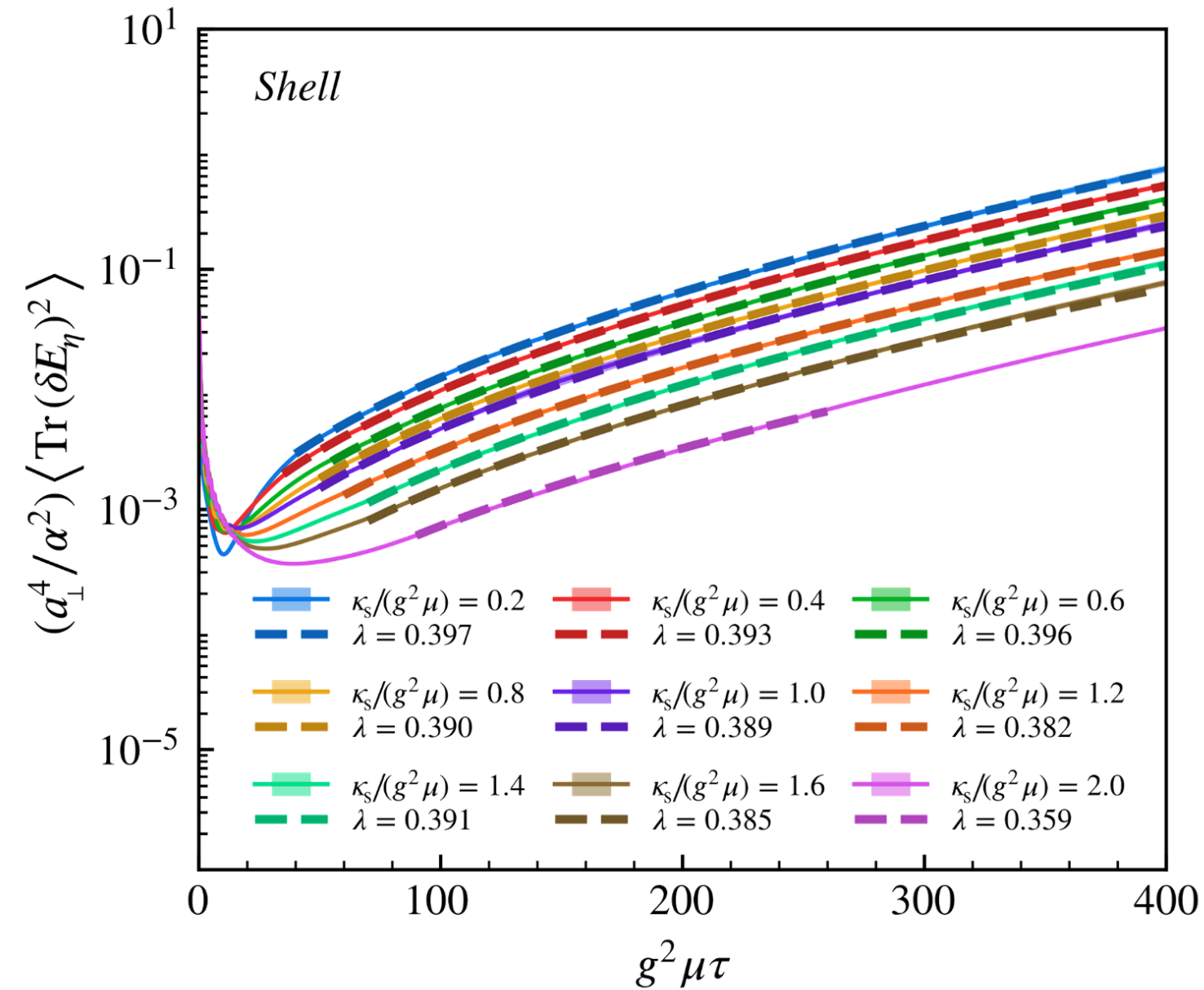
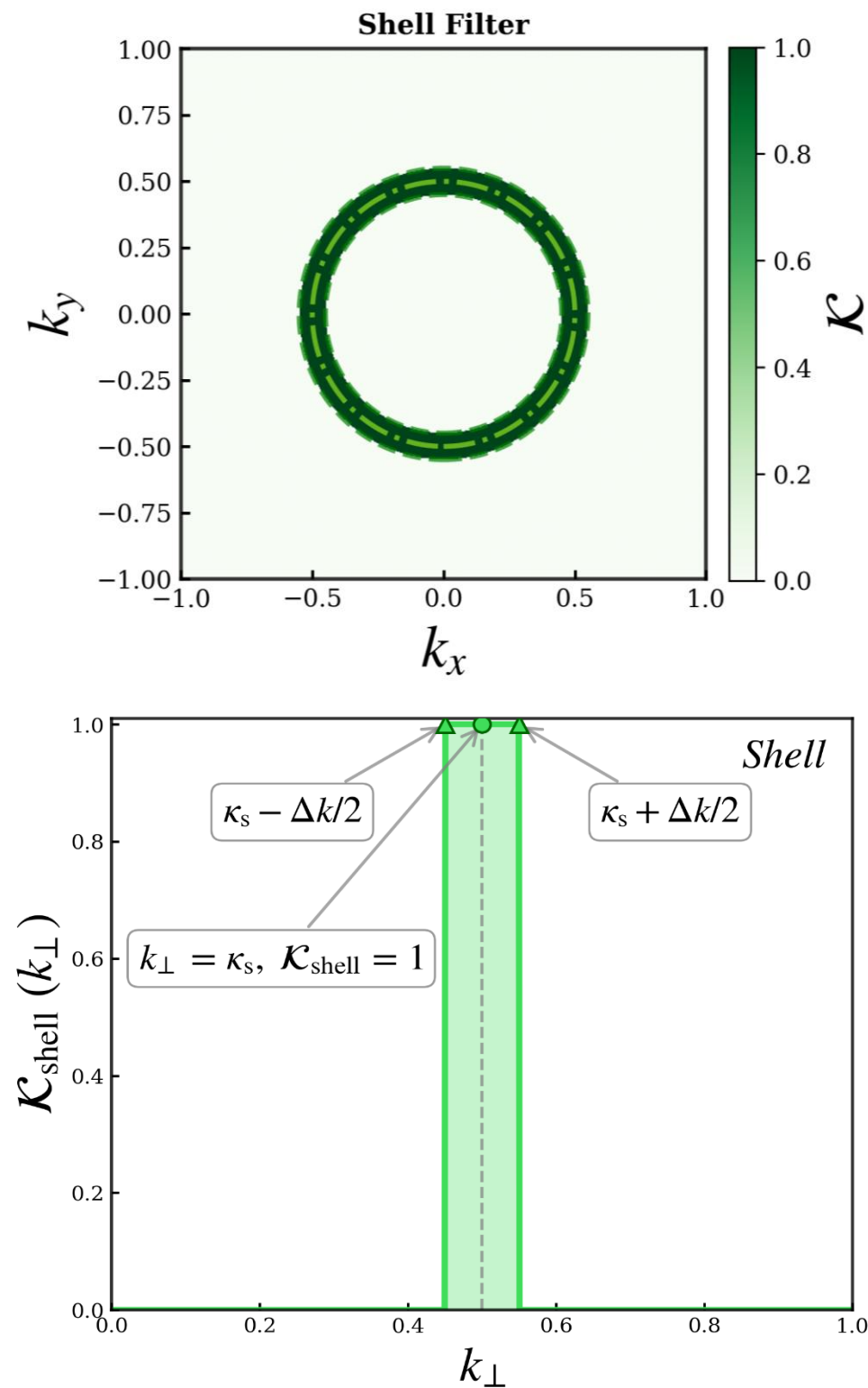


$$\mathcal{K}_{\text{power}}(k_{\perp}) = \frac{\kappa_p^2}{k_{\perp}^2 + \kappa_p^2}$$

- Milder suppression of high-momentum modes (algebraic fall-off)
- Preserves the ultra-violet tail

Shell-filtered perturbations

52



$$\mathcal{K}_{\text{shell}}(k_{\perp}) = \begin{cases} 1, & \kappa_s - \frac{\Delta k}{2} \leq k_{\perp} \leq \kappa_s + \frac{\Delta k}{2} \\ 0, & \text{otherwise} \end{cases}$$

- Isolates a narrow band of momenta
- Tests specific momentum modes independently

Shell-Filtered Perturbations

53

$$\mathcal{K}_{shell}(k_{\perp}) = \begin{cases} 1, & \kappa_s - \frac{\Delta k}{2} \leq k_{\perp} \leq \kappa_s + \frac{\Delta k}{2} \\ 0, & \text{otherwise} \end{cases}$$

- **Clean exponential growth: $\lambda \approx 0.38 - 0.40$**
- Late-time amplitude of the perturbations $\propto \frac{1}{\kappa_s}$
- Non-trivial momentum space structure: **Need a detailed study in Fourier-space analysis of the perturbation**

