



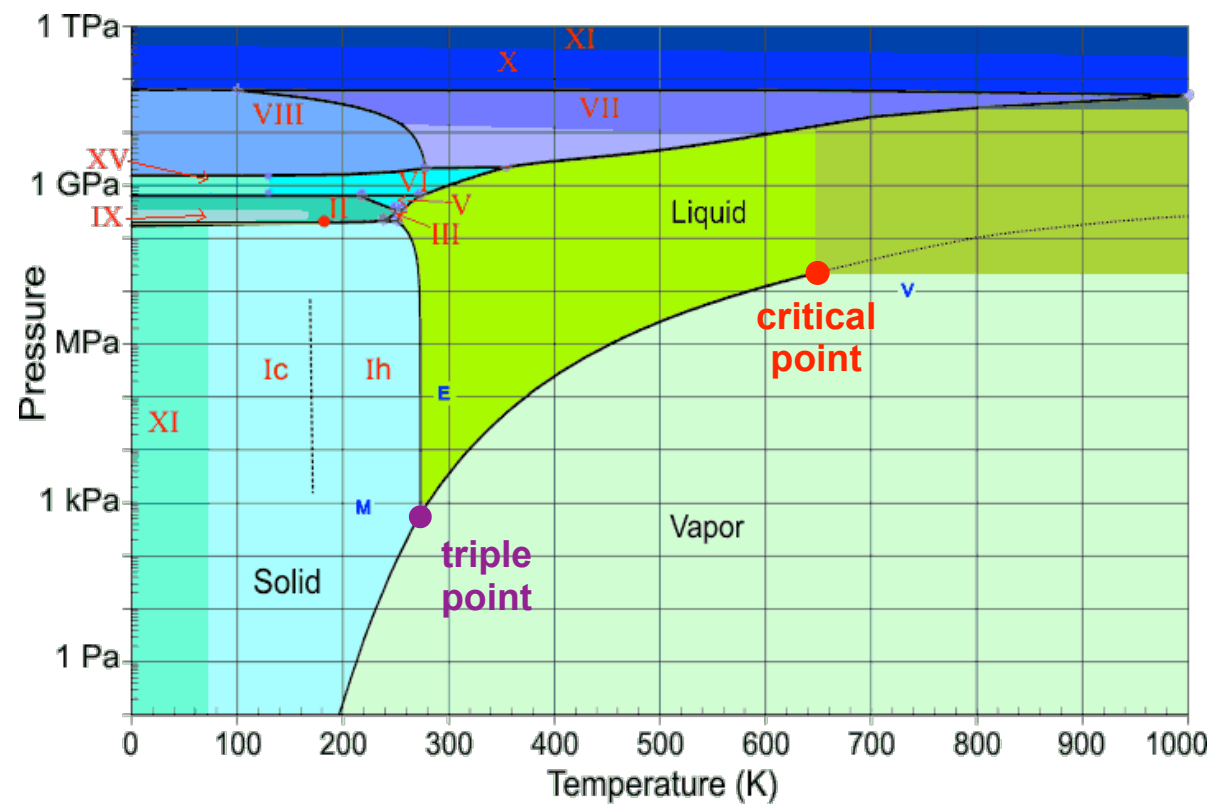
Critical dynamics in QCD matter

Lorenz von Smekal

with: Mattis Harhoff, Johannes Roth, Sören Schlichting,
Leon Sieke, Yunxin Ye

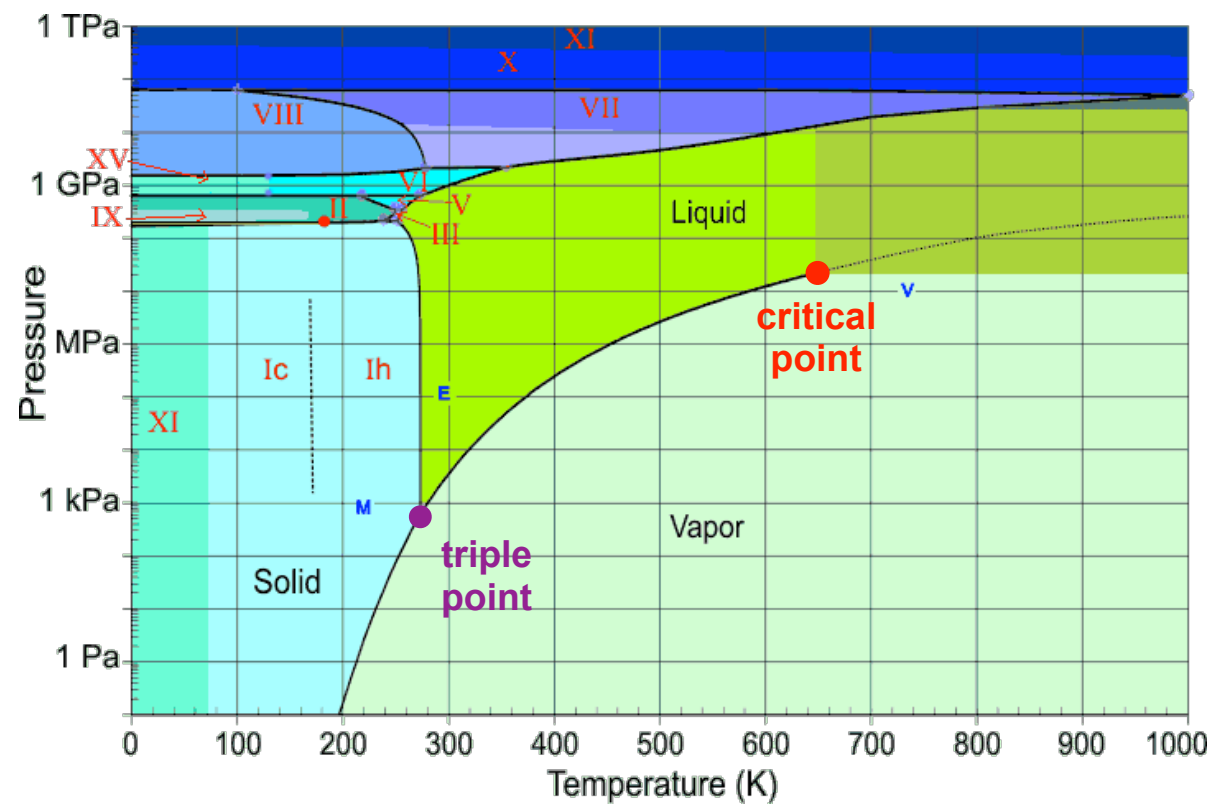
Universality near Critical Point

Water



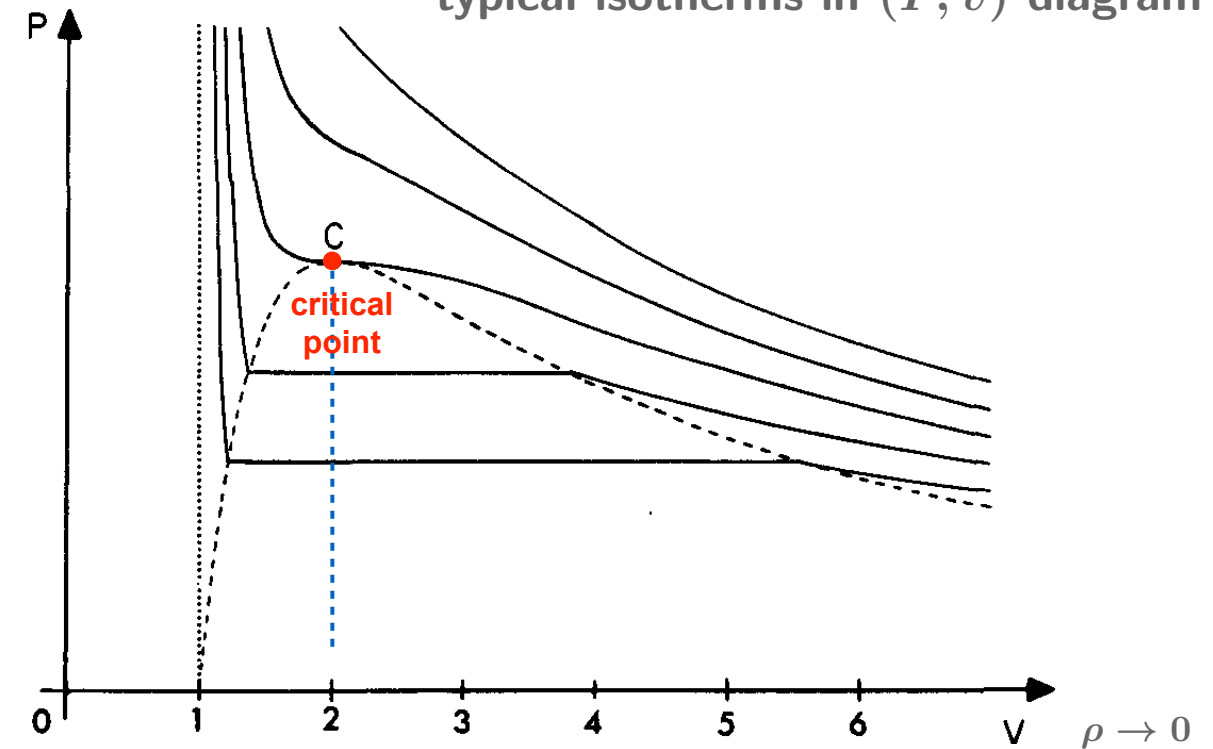
<http://www.lsbu.ac.uk/water/phase.html>

Water

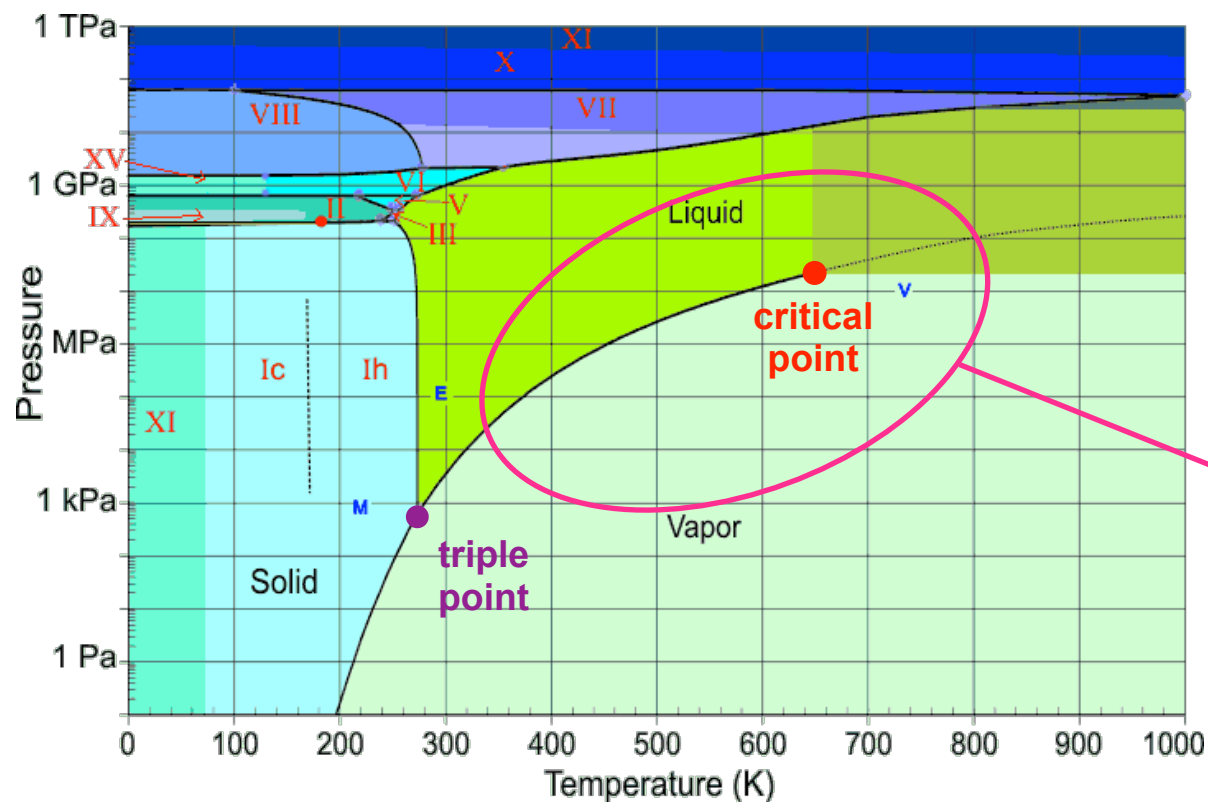


<http://www.lsbu.ac.uk/water/phase.html>

typical isotherms in (P, v) diagram



Water



<http://www.lsbu.ac.uk/water/phase.html>

critical isotherm:

$$T = T_c :$$

$$\rho - \rho_c \sim M ,$$

$$v_c - v \sim \pm |H|^{1/\delta} ,$$

$$P - P_c \sim \pm |v_c - v|^\delta$$

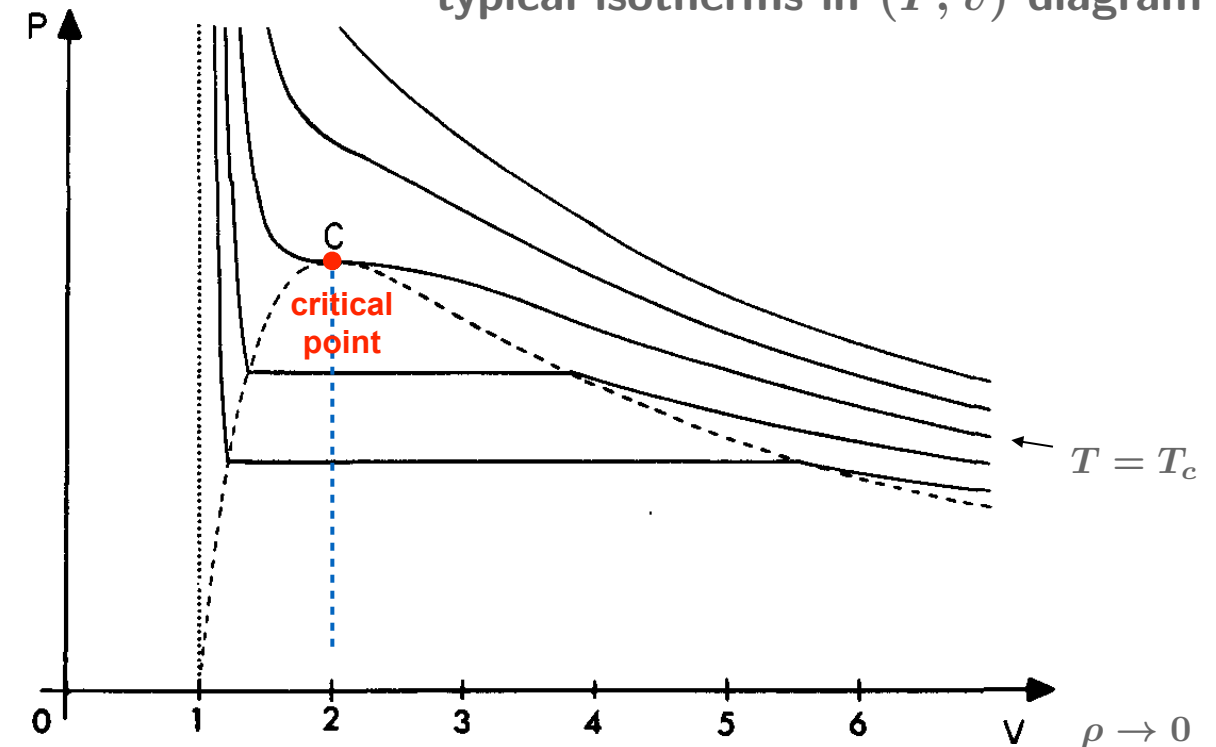
along coexistence curve:

$$(H = 0, T < T_c)$$

$$v_c - v \sim \pm (T_c - T)^\beta$$

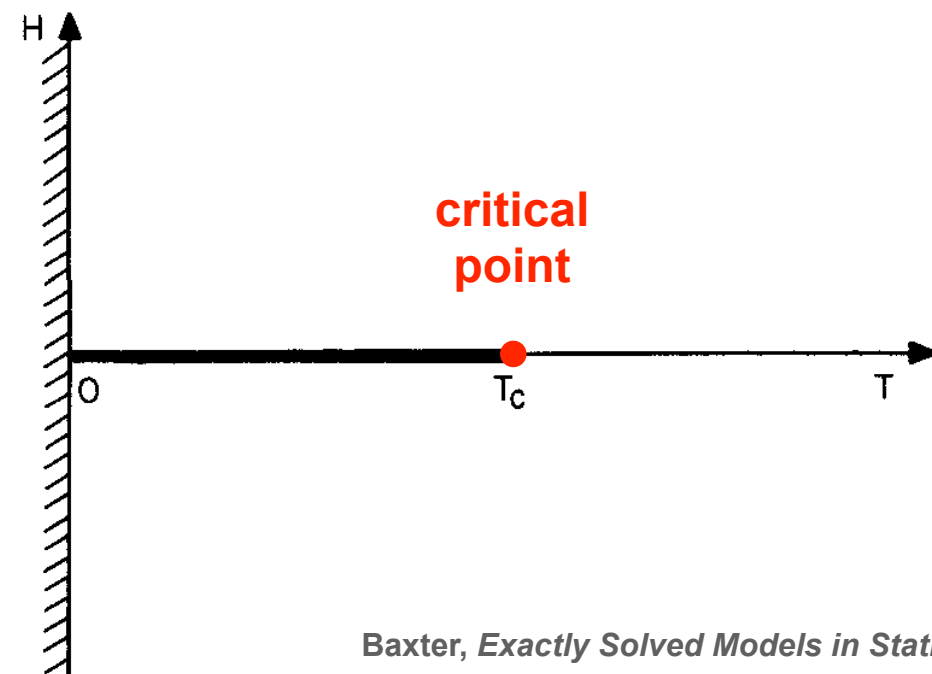
$$\sim \pm (P_c - P)^\beta$$

typical isotherms in (P, v) diagram



Ising Model:

Magnetic
Field



Baxter, *Exactly Solved Models in Statistical Mechanics*

QCD phase diagram

- Worldwide effort: find **critical point**

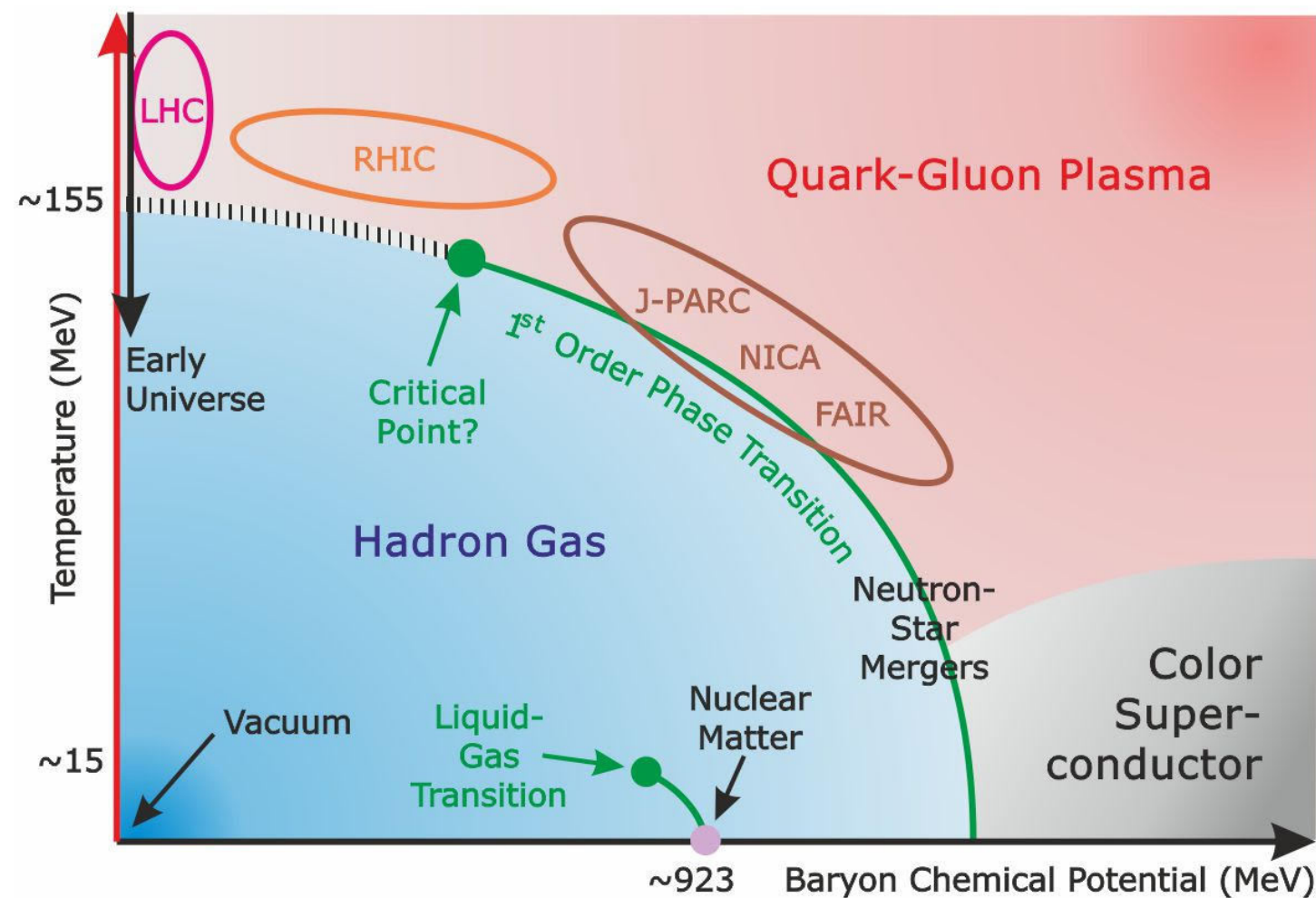


Figure: A. Steidl, CRC-TR 211

QCD phase diagram

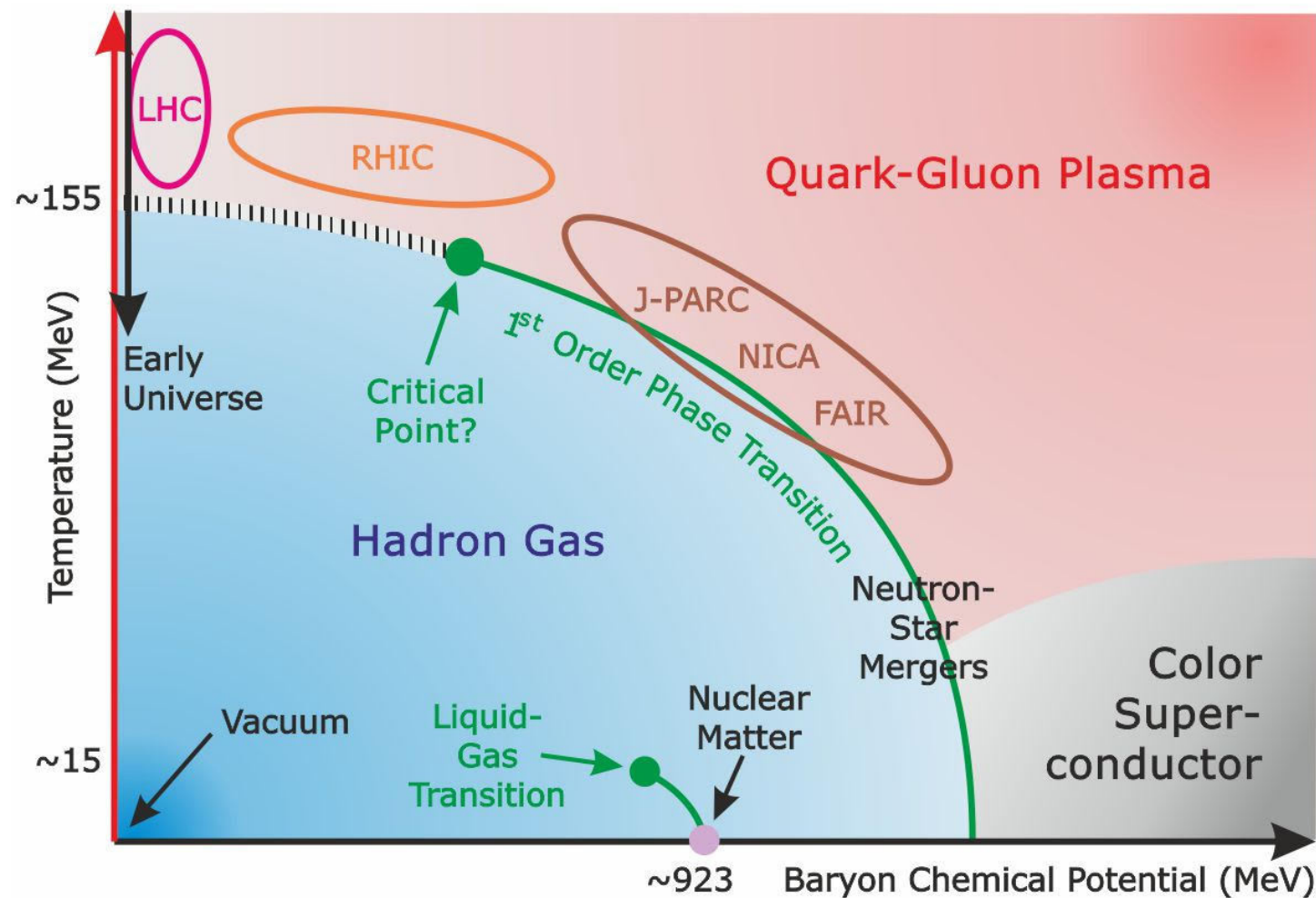
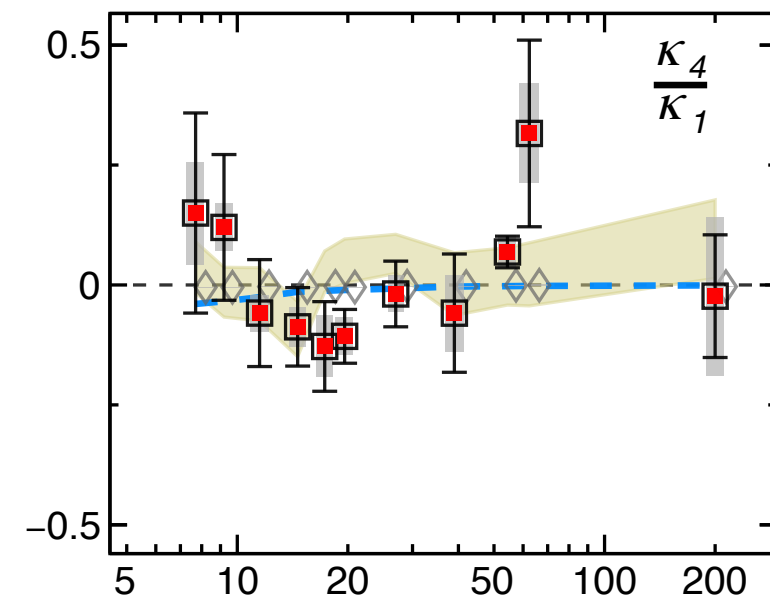


Figure: A. Steidl, CRC-TR 211

- Worldwide effort: find **critical point**
- Possible signatures in heavy-ion collisions: **critical fluctuations**

Stephanov, Rajagopal, Shuryak, PRD **60** (1999) 114028

Stephanov, PRL **107** (2011) 052301



STAR Collaboration, arXiv:2504.00817

QCD phase diagram

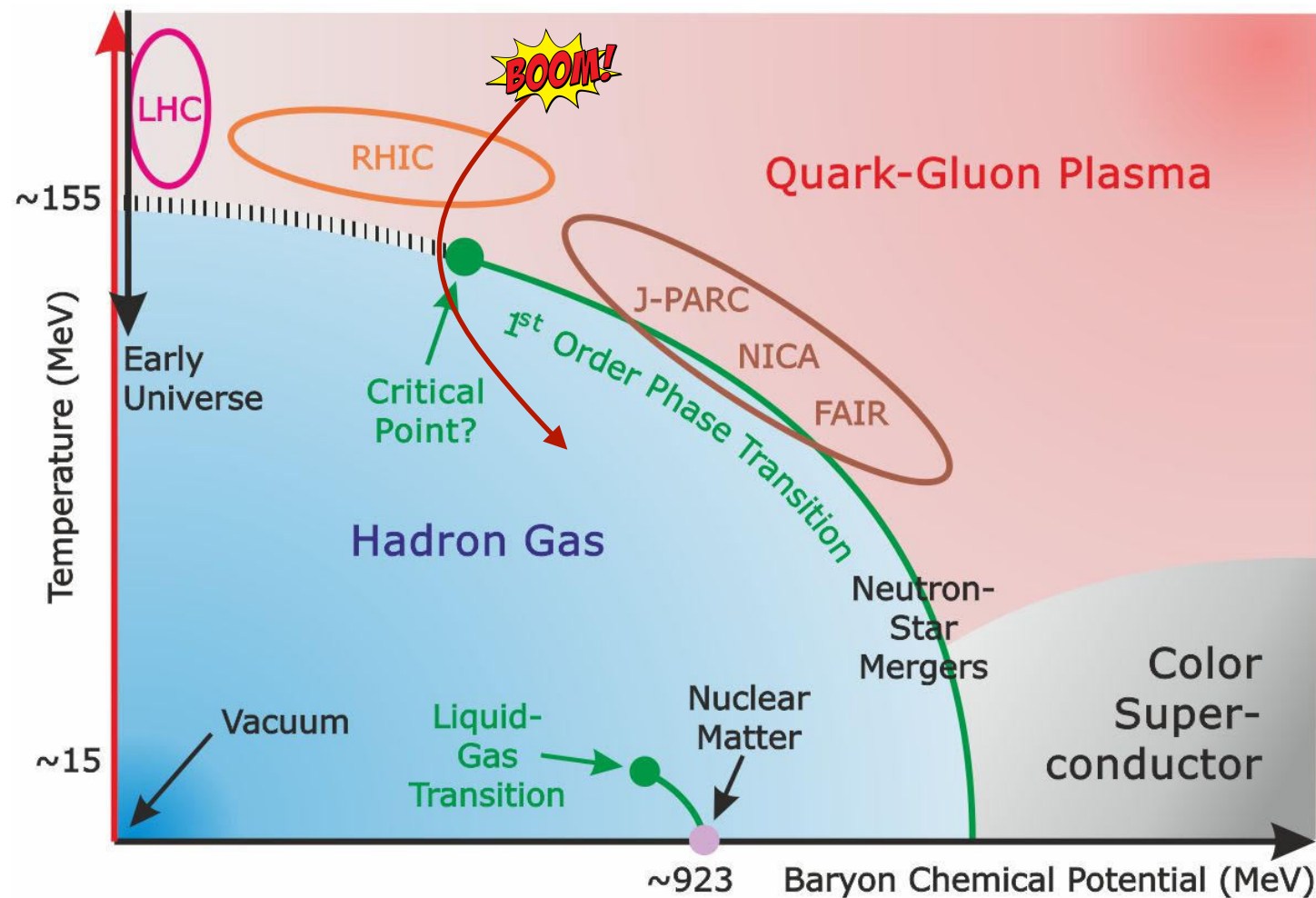
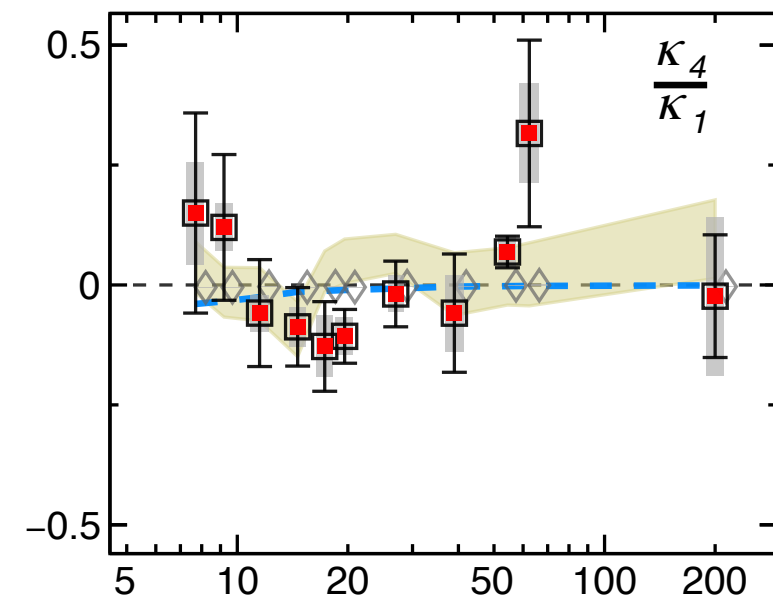


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- Expanding fireball: understand **non-equilibrium transition** and **critical dynamics**

QCD phase diagram

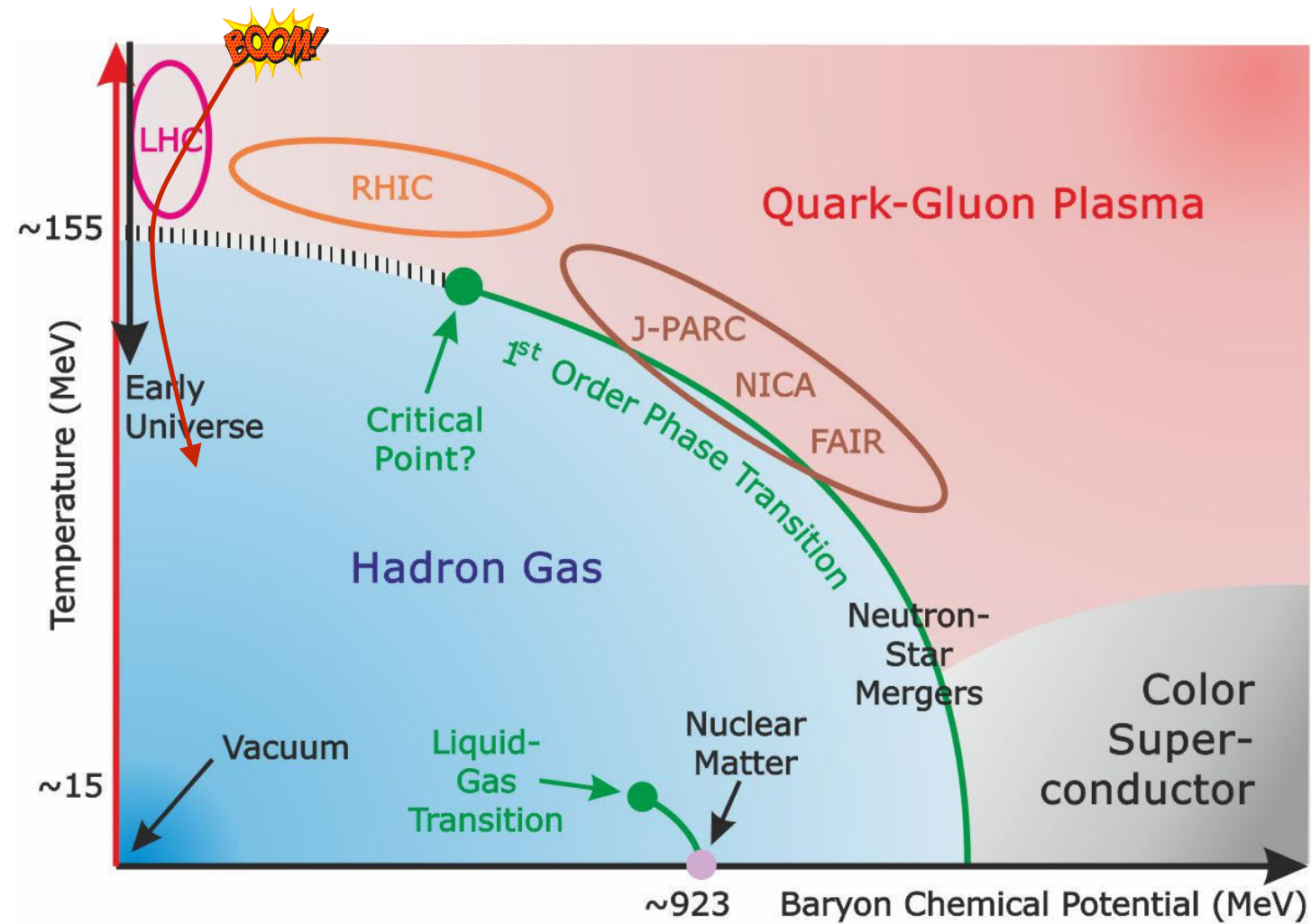


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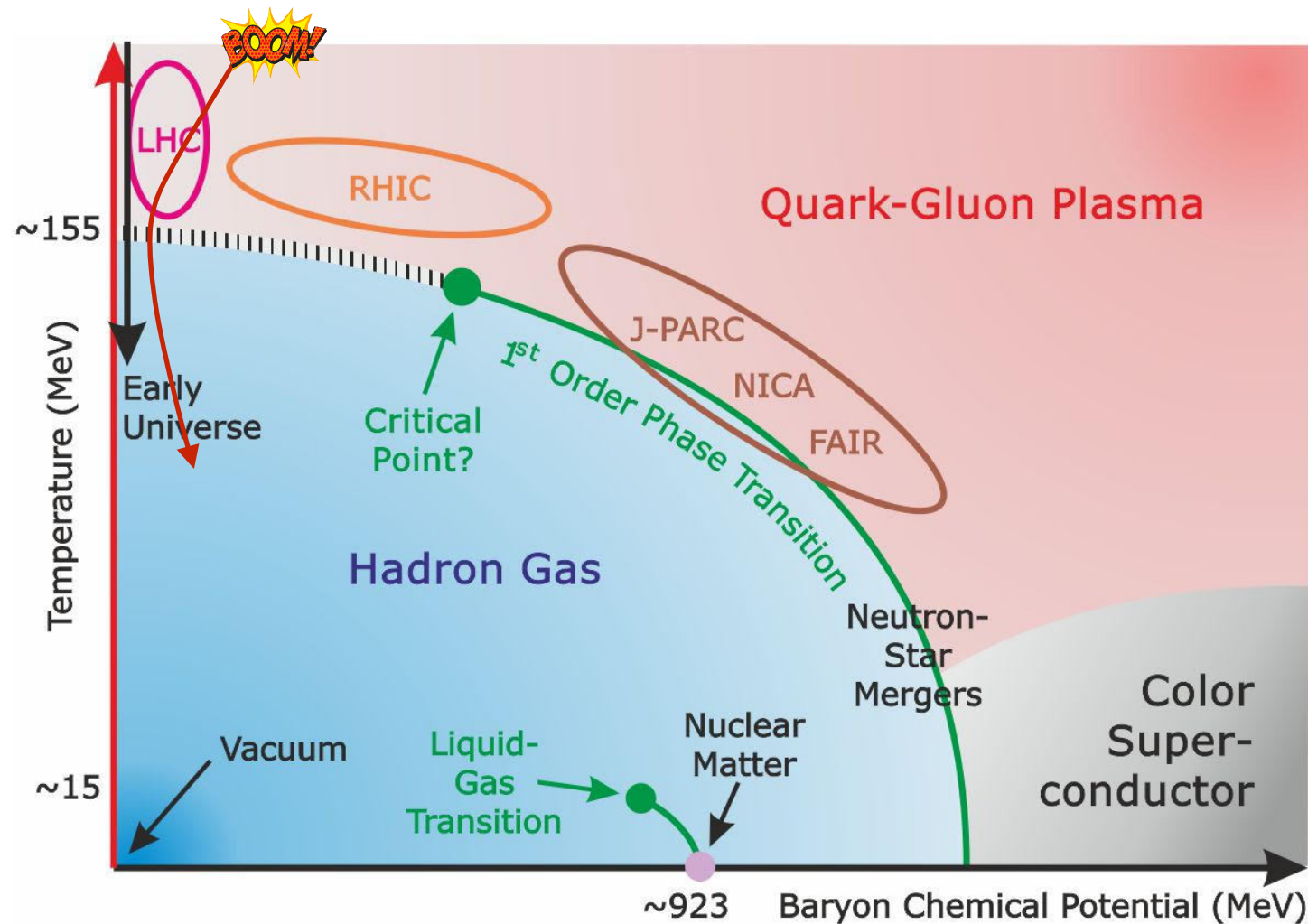
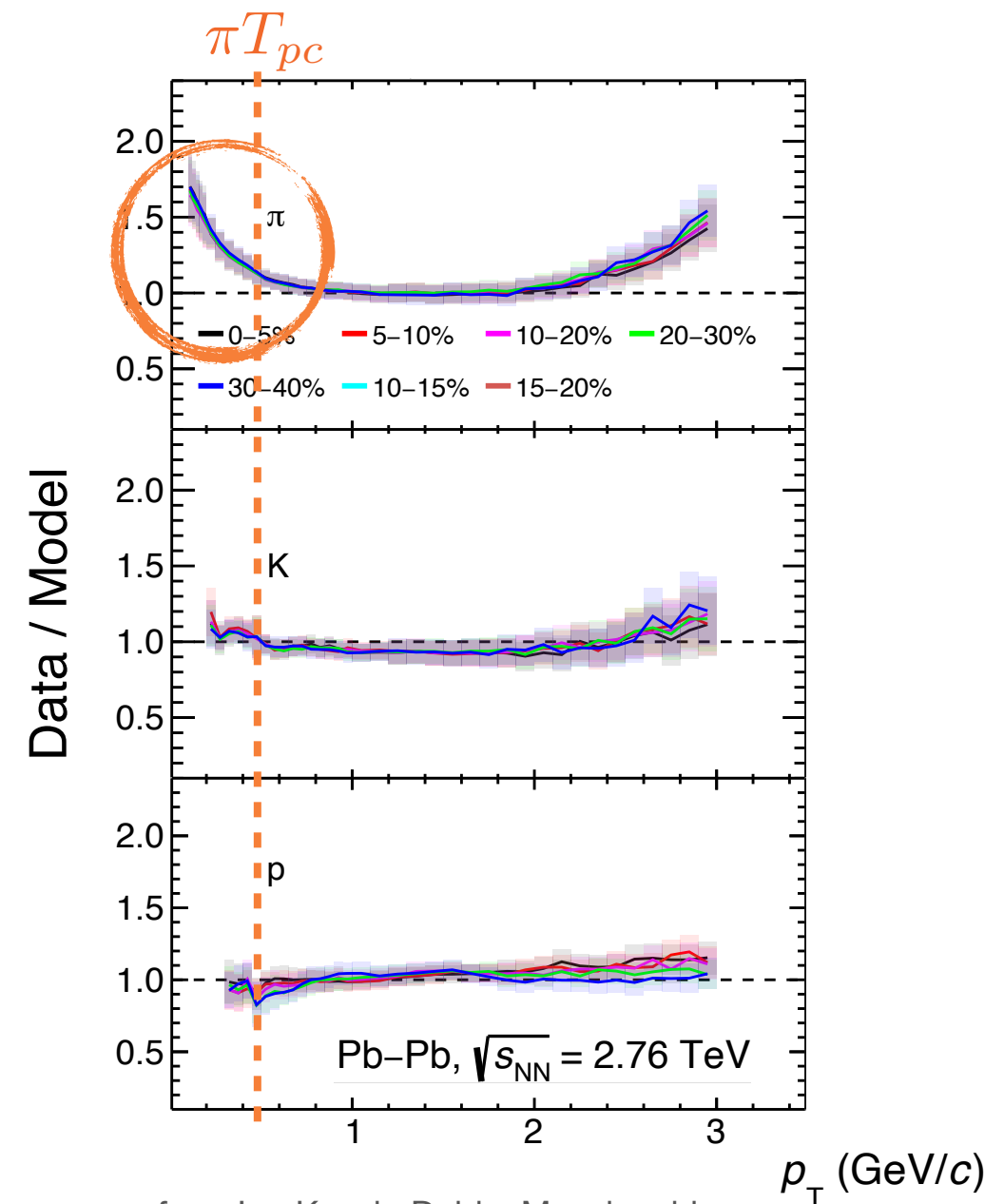


Figure: A. Steidl, CRC-TR 211

- Excess of soft pions: imprint of **chiral crossover**?

Florio, Grossi, Soloviev, Teaney, PRD **104** (2021) 034025

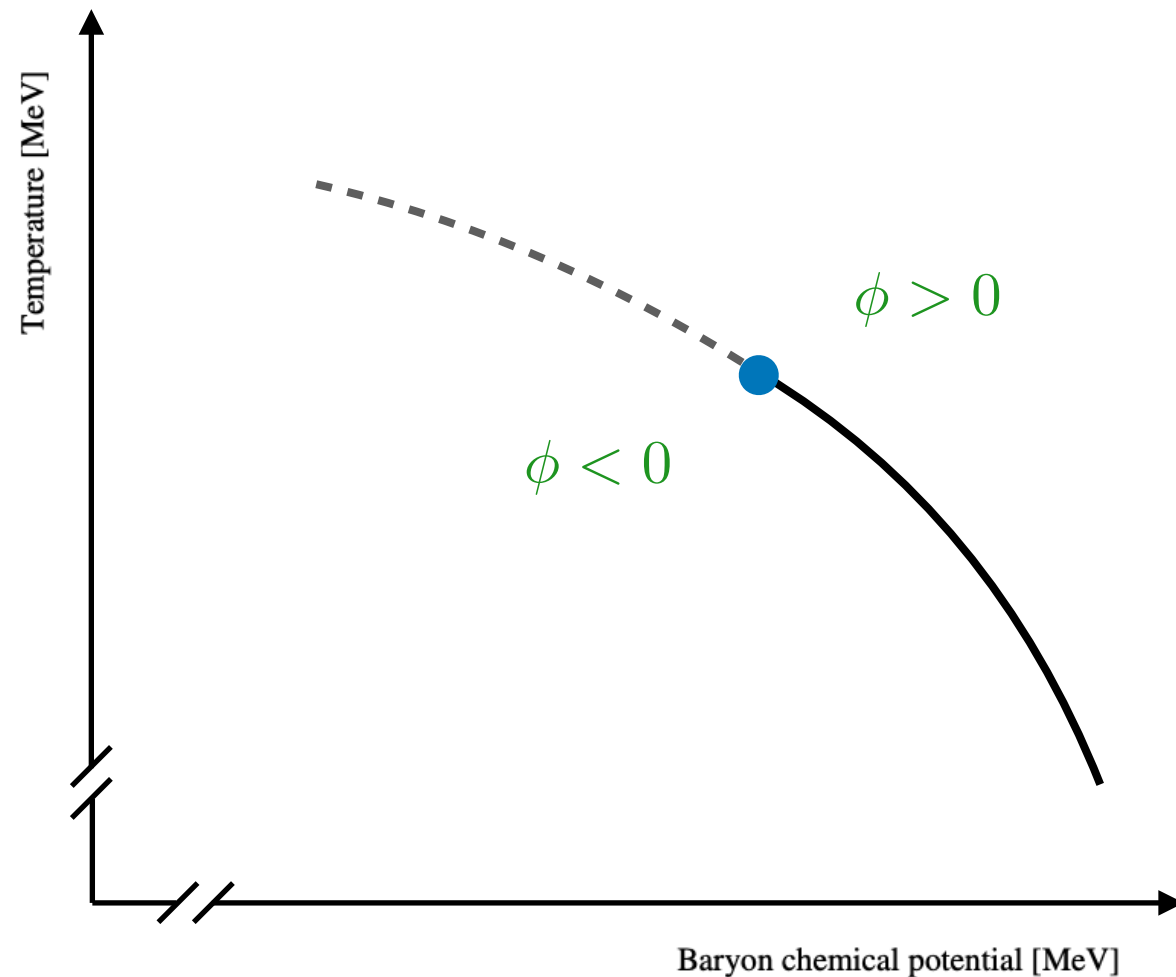


from Lu, Kavak, Dubla, Masciocchi,
Selyuzhenkov, arXiv:2407.09207

Order parameter: ϕ

Critical point: Correlation length ξ diverges

- Microscopic details **irrelevant**
- Critical properties **only** depend on symmetries, number of dimensions, ...
~ Universality (classes)



non-universal mapping to Ising model:

Parotto et al., PRC **101** (2020) 3, 034901

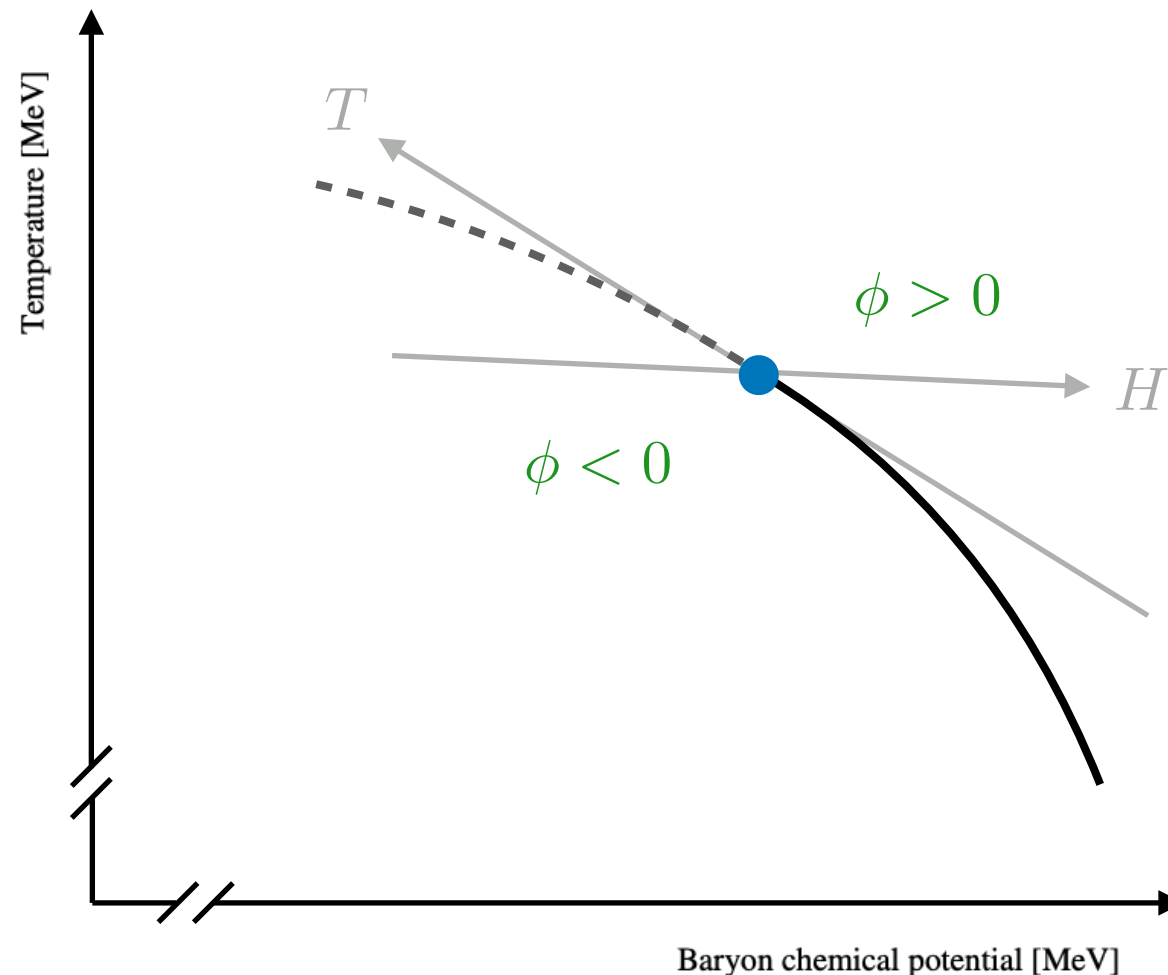
Pradeep, Sogabe, Stephanov, Yee, PRC **109** (2024) 6, 064905

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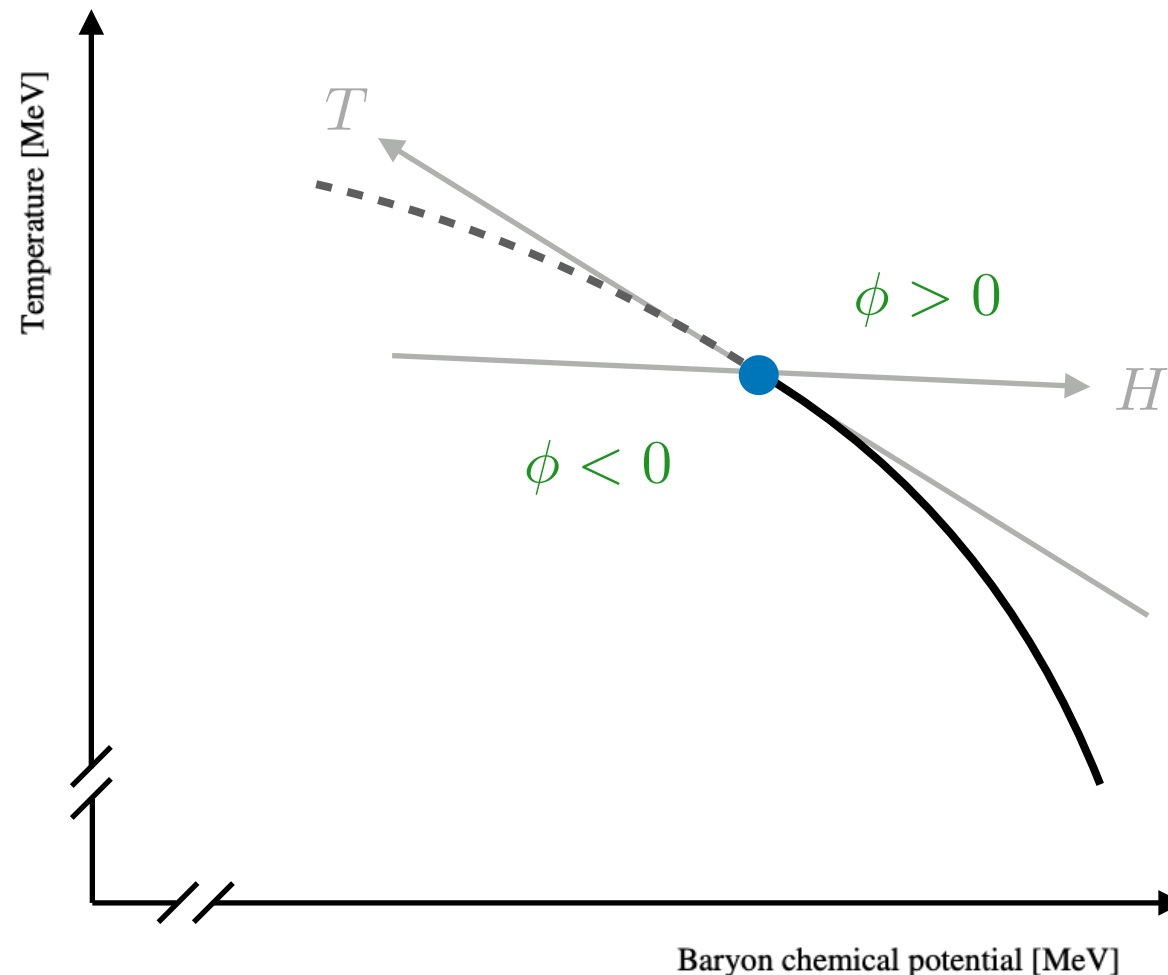


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- For example: $\xi \sim |\tau|^{-\nu}$ $\nu = 0.6301(4)$
Pelissetto, Vicari (2002)
- $\nearrow$
reduced temperature $\tau = (T - T_c)/T_c$

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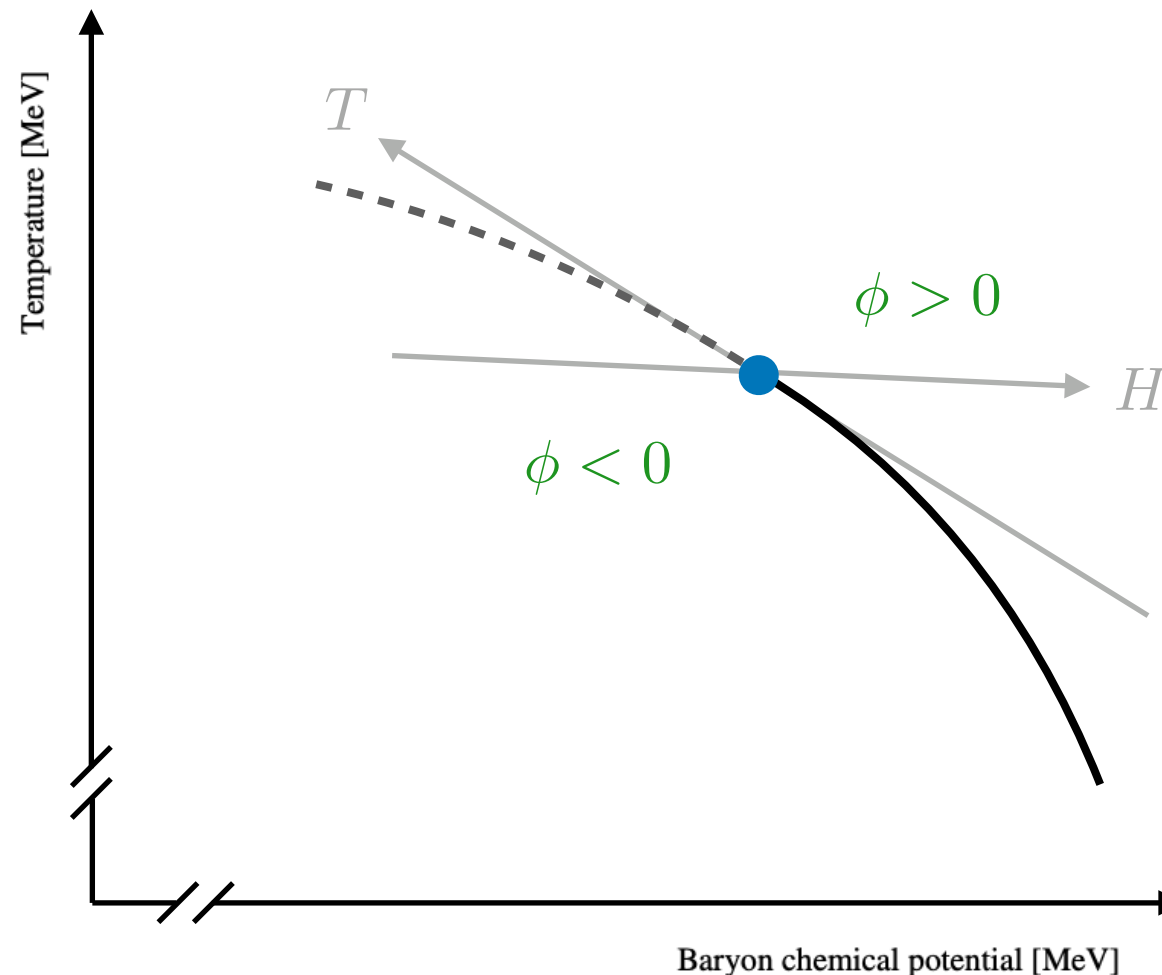
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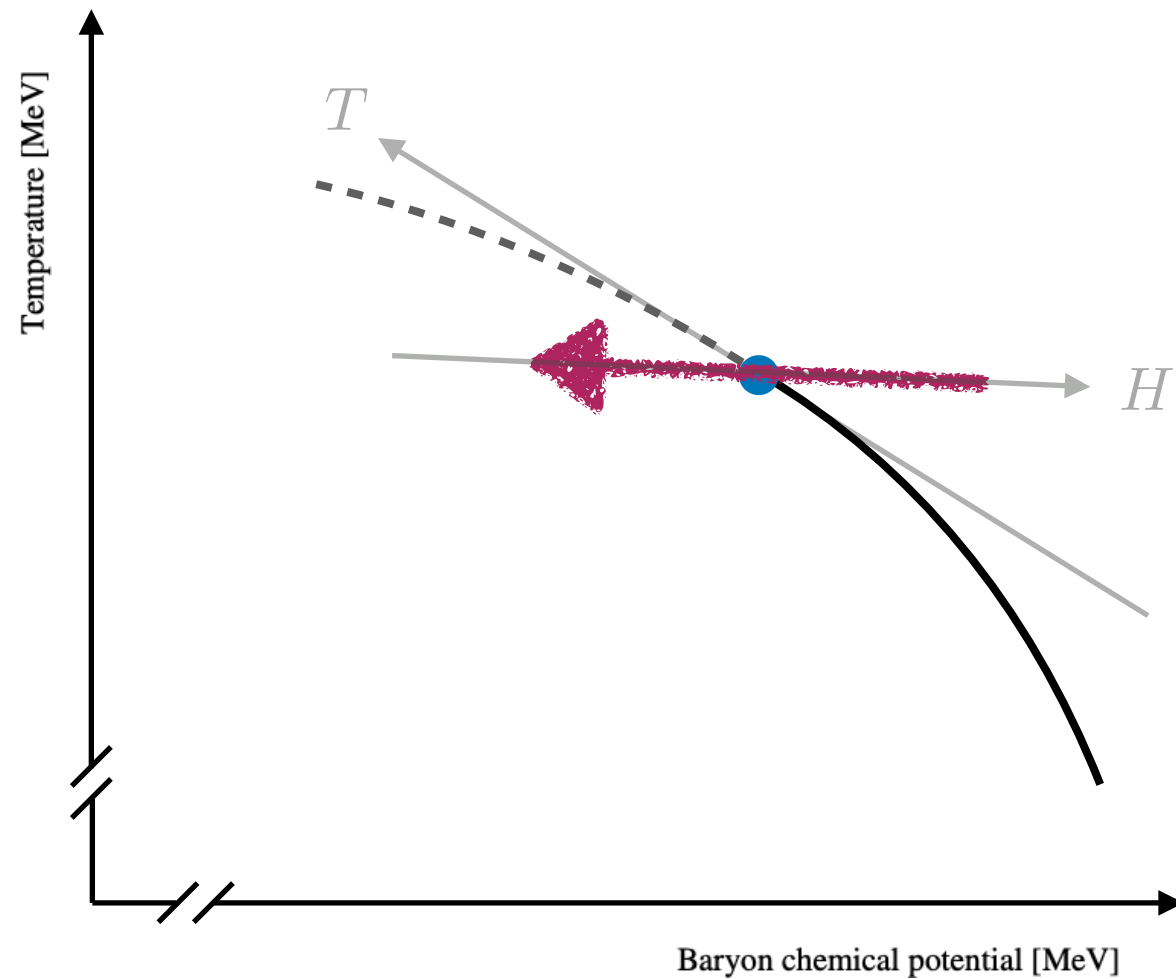


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Ising model $\Rightarrow$ (equilibrium) universal properties of the QCD critical point

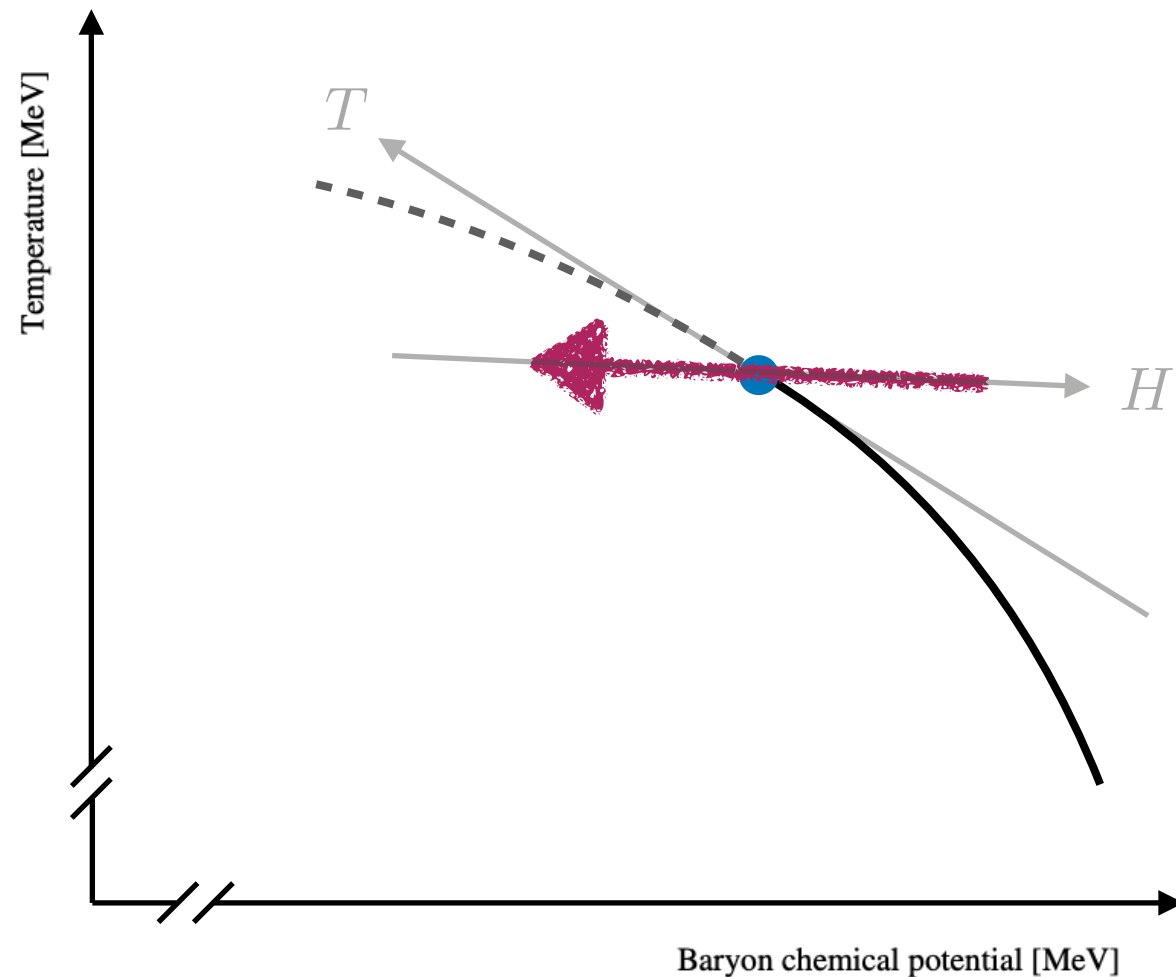
- study trans-critical protocol:

Sieke, Harhoff, Schlichting, von Smekal, NPB **1011**, 116808 (2025)

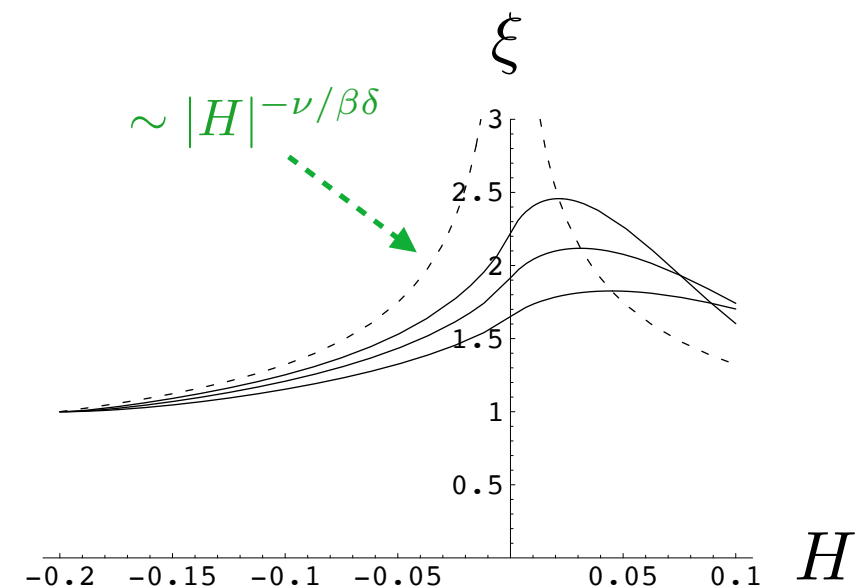


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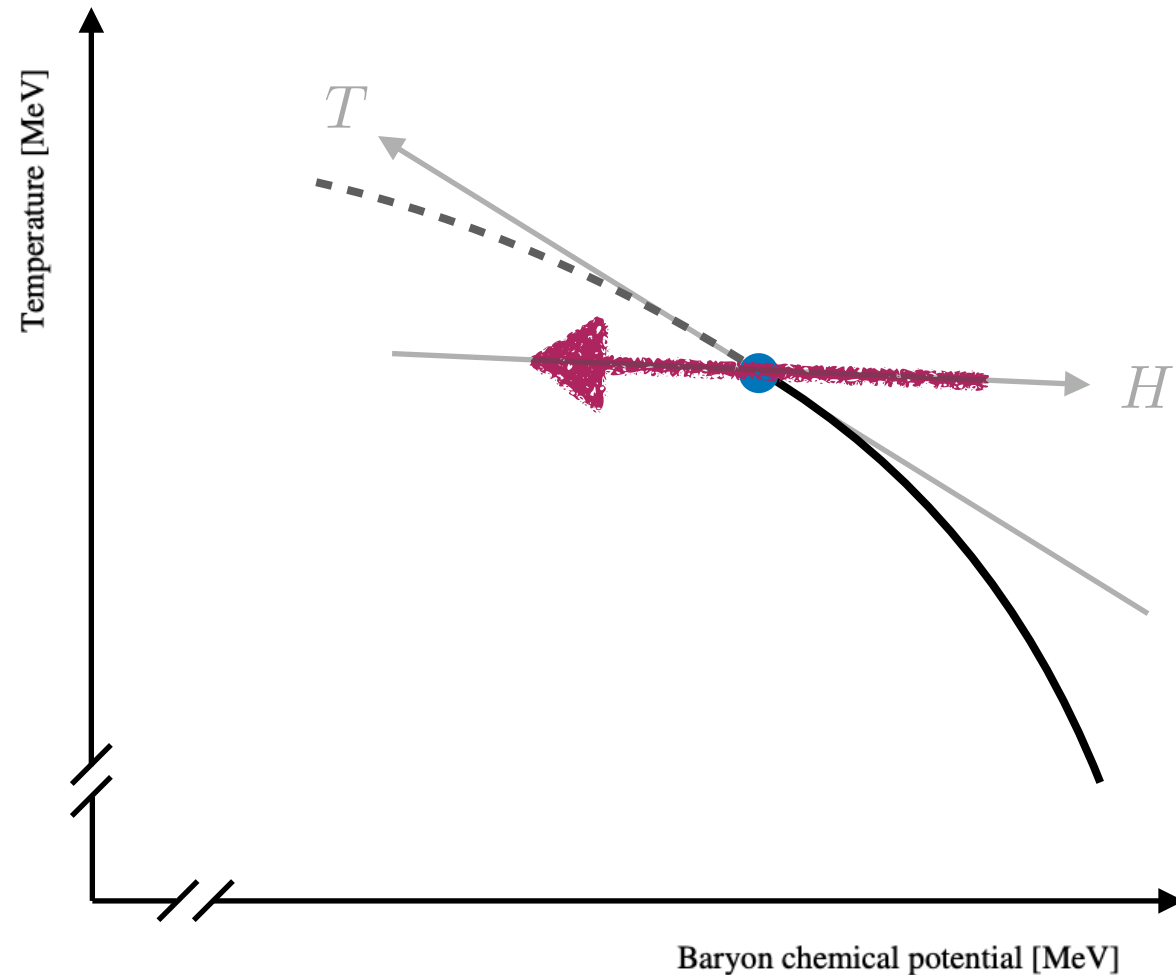
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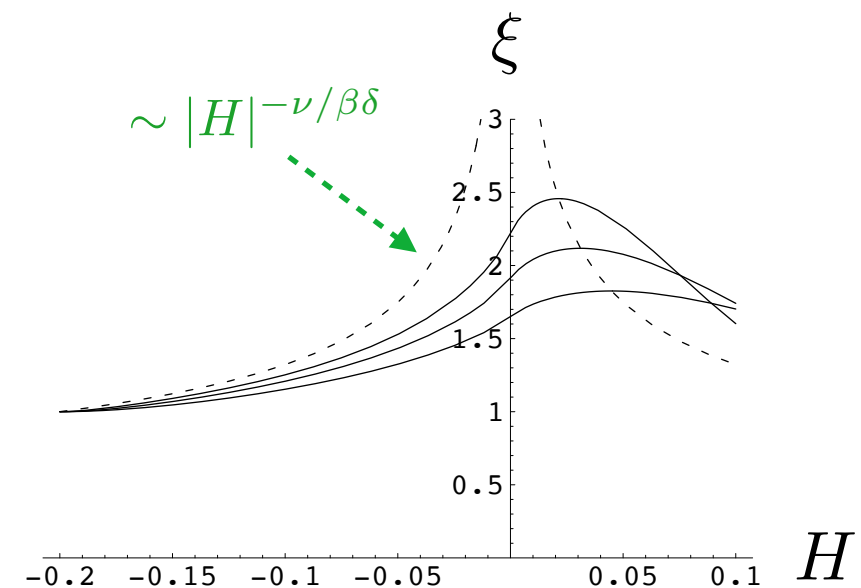
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→ **Critical mode falls out of (local) equilibrium!**

Reason:

critical slowing down

$$\xi_t \sim \xi^z$$

correlation time correlation length

The diagram illustrates the scaling relationship between correlation time and correlation length at a critical point. It features the equation $\xi_t \sim \xi^z$ in the center. Below the equation, two blue arrows point upwards towards the terms ξ_t and ξ^z . The arrow pointing to ξ_t is labeled 'correlation time' in blue text. The arrow pointing to ξ^z is labeled 'correlation length' in blue text.

Reason:

critical slowing down

dynamic critical exponent

$$\xi_t \sim \xi^z$$

correlation time

correlation length

The diagram illustrates the scaling relationship between the correlation time ξ_t and the correlation length ξ at a critical point. The equation $\xi_t \sim \xi^z$ is shown, where ξ_t is labeled 'correlation time' and ξ is labeled 'correlation length'. A red circle highlights the exponent z , which is also labeled 'dynamic critical exponent' with a red arrow. The text 'critical slowing down' is written in blue above the equation.

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correlation time correlation length

- z determined by **dynamic universality class**
- group theories by equations of motion for critical modes
- critical mode arbitrarily slow $\leadsto$ **hydrodynamic theory**
(depends on: order parameter conserved or not, potential other conserved quantities, etc.)

Dynamics near Critical Point (Model H — intuitive)

(see *Phase Transition Dynamics* by Onuki)

- conserved order parameter (specific entropy)

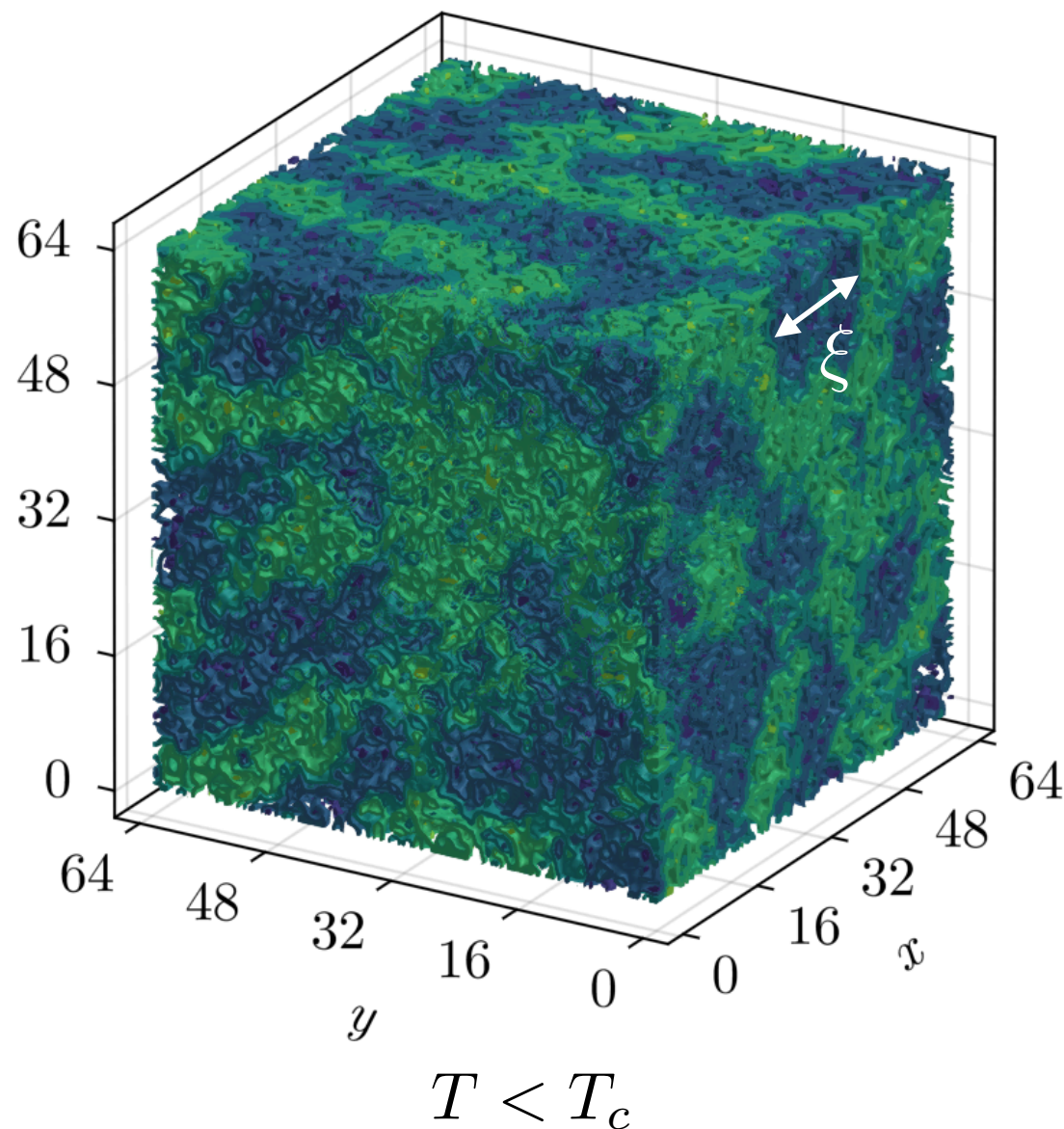
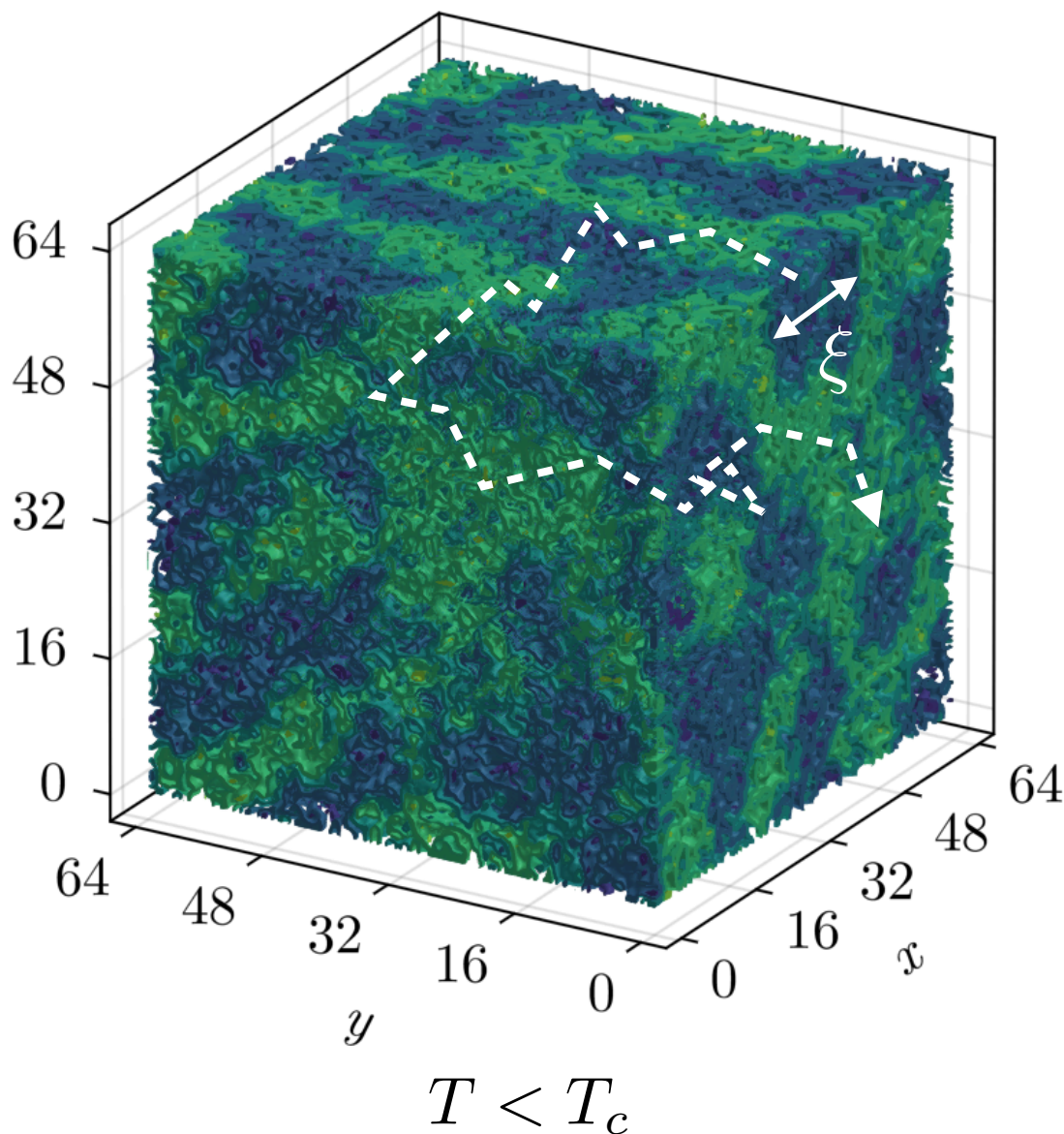


Figure from Thomas Schaefer's talk at Quark Matter 2025

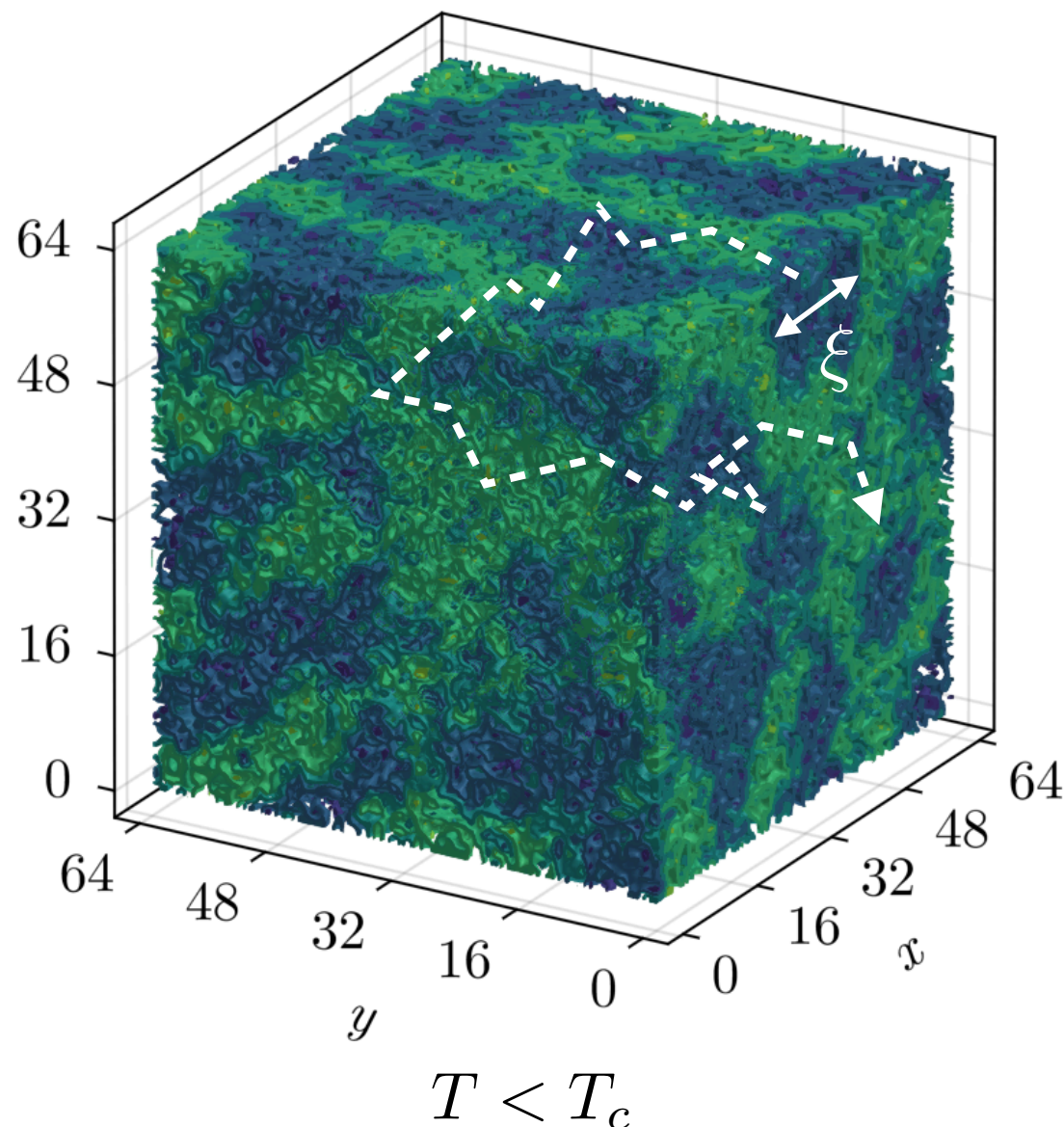
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- conserved order parameter (specific entropy)
- Clusters of linear extent $\sim \xi$ undergo Brownian motion due to advection by thermal velocity fluctuations

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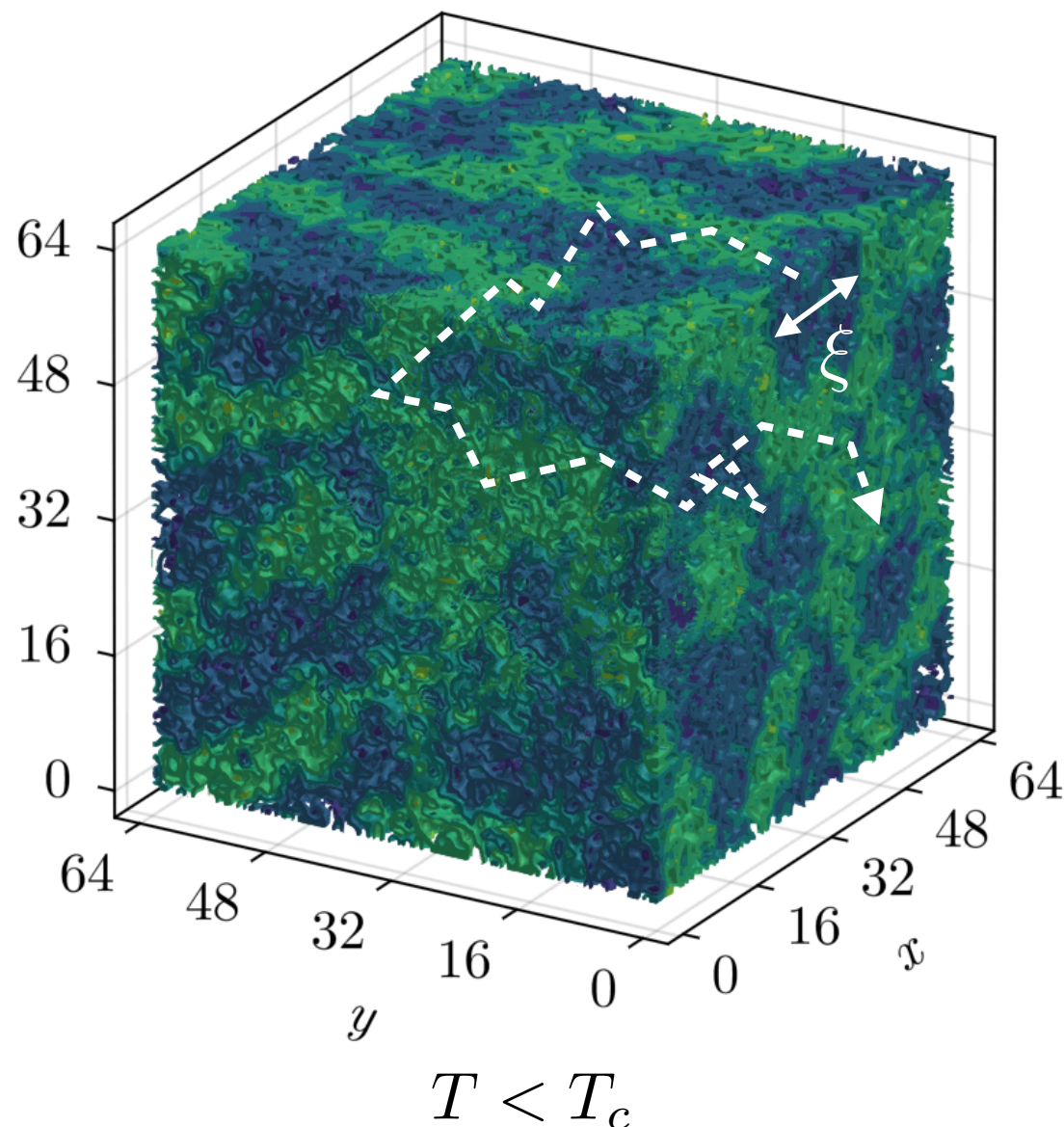


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- Einstein-Stokes relation for diffusion coefficient of spheres with radius $\sim \xi$

$$D_\phi \sim \frac{g^2 T_c}{\eta \xi^{d-2}}$$

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- Relaxation time of modes with wavelength $\ell \ll \xi$:
- $$\xi_t(\ell) \sim \frac{\ell^2}{D_\phi}$$

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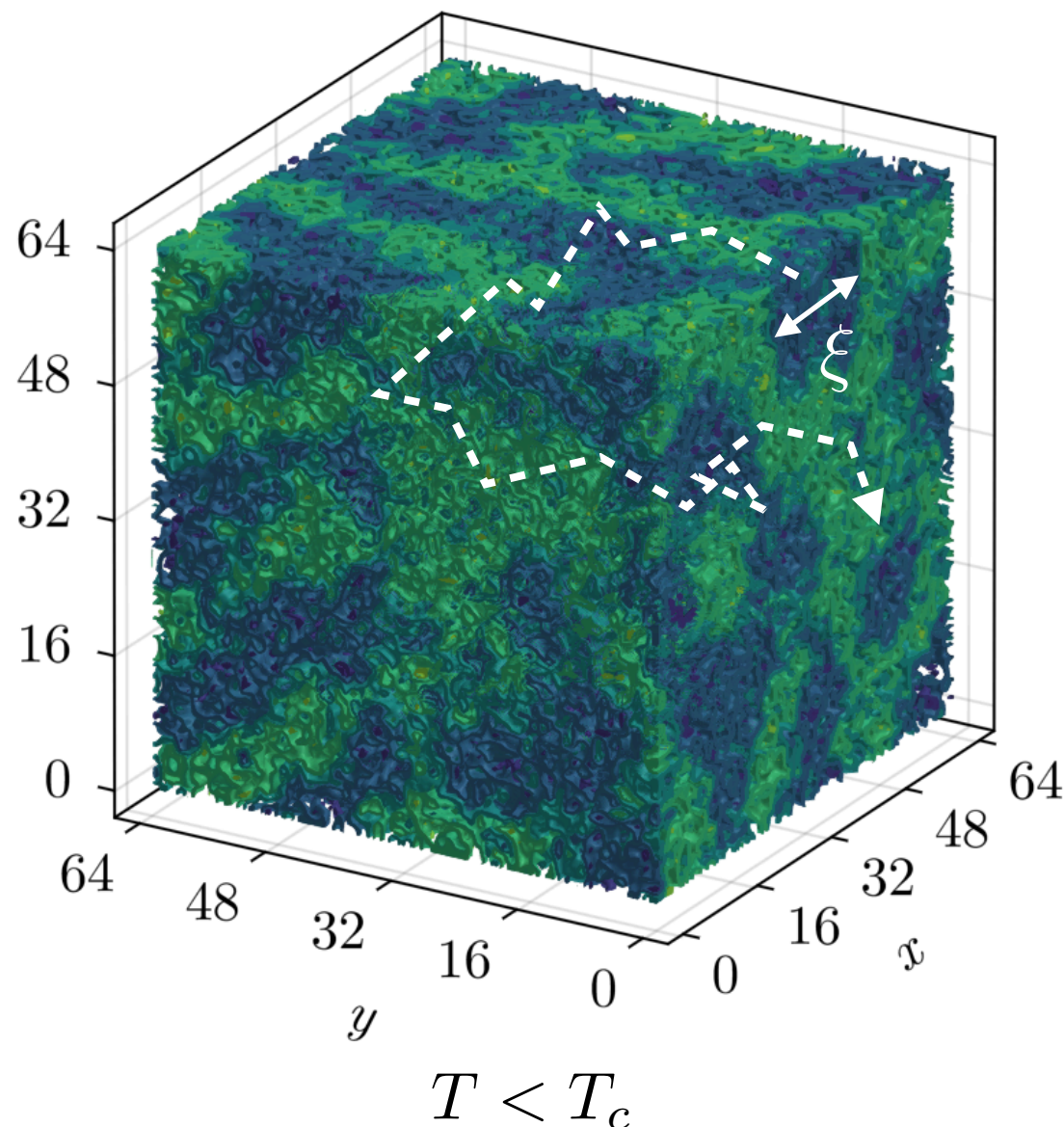


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- Match to critical scaling at $\ell \sim \xi$

$$\xi_t = D_\phi^{-1} \xi^2 \sim \frac{\eta \xi^{d-2}}{g^2 T_c} \xi^2 \sim \eta \xi^d$$

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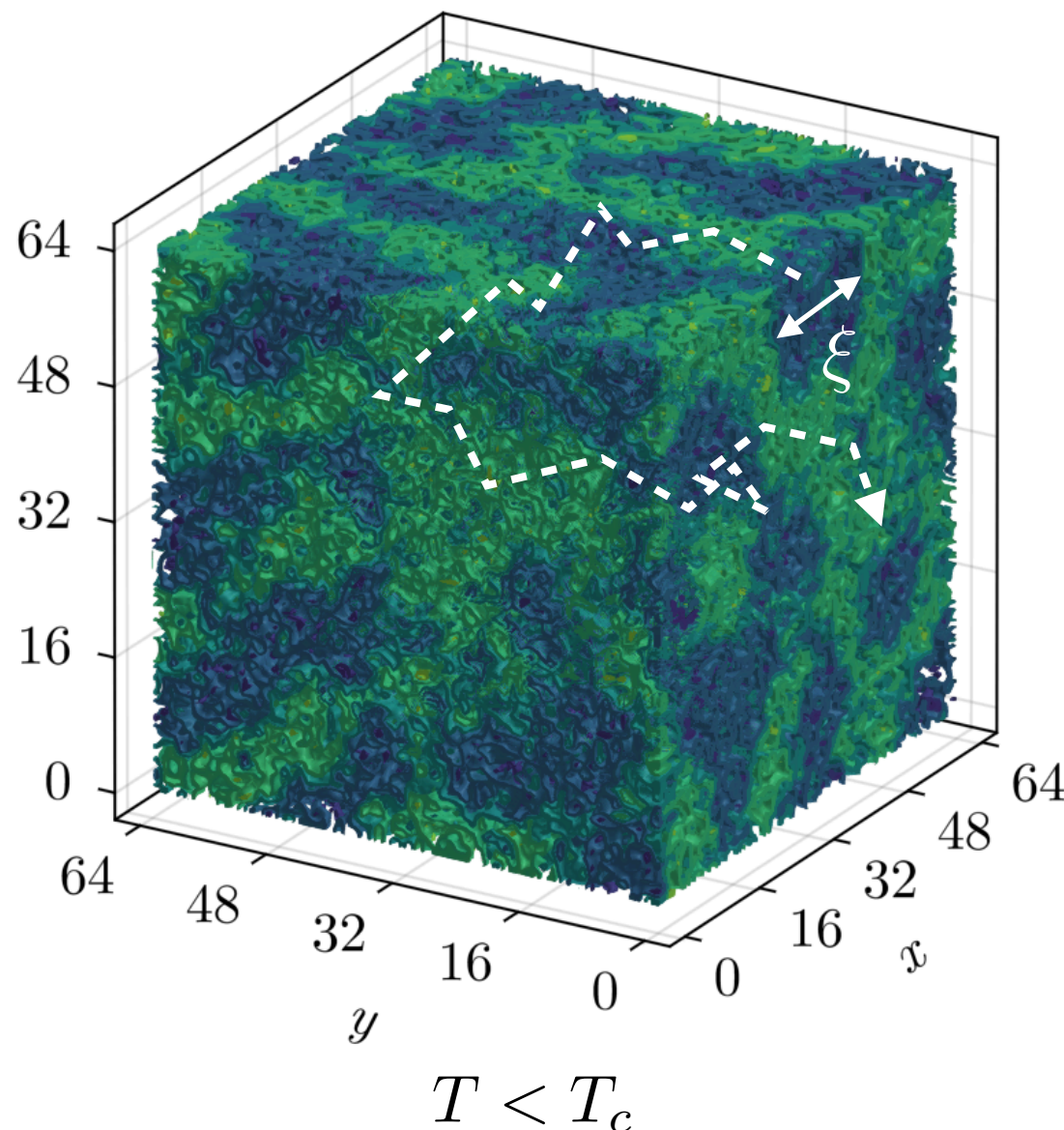


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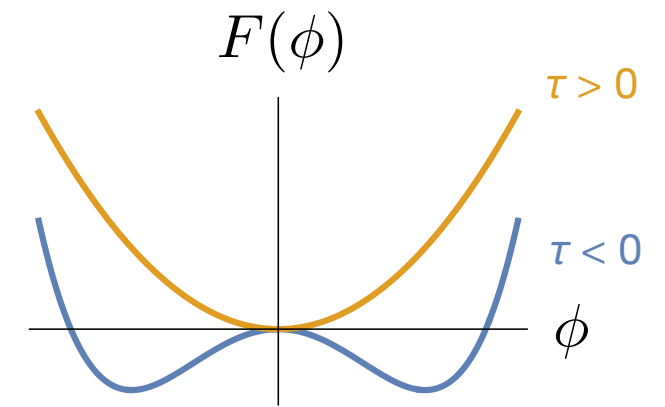
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$z = d + x_\eta$

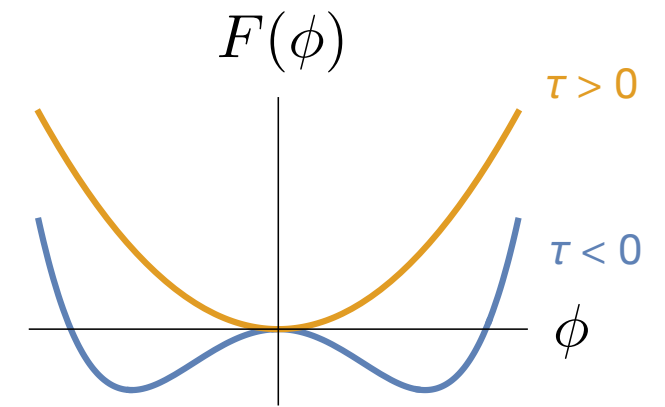
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- Langevin equations of motion

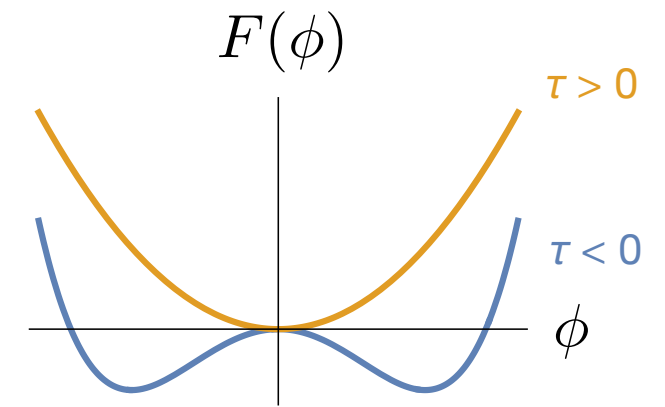
Hohenberg and Halperin, Rev. Mod. Phys. **49**, 435 (1977)

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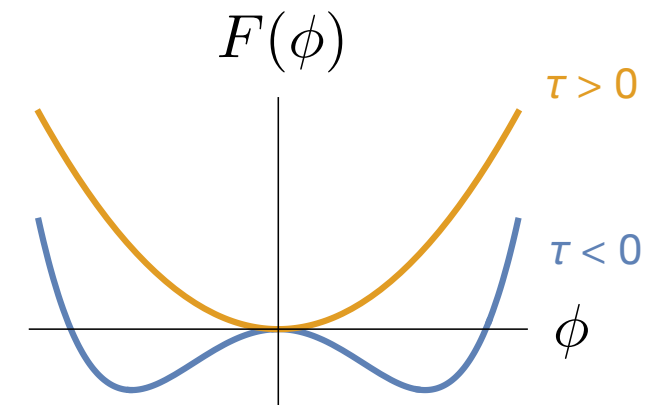
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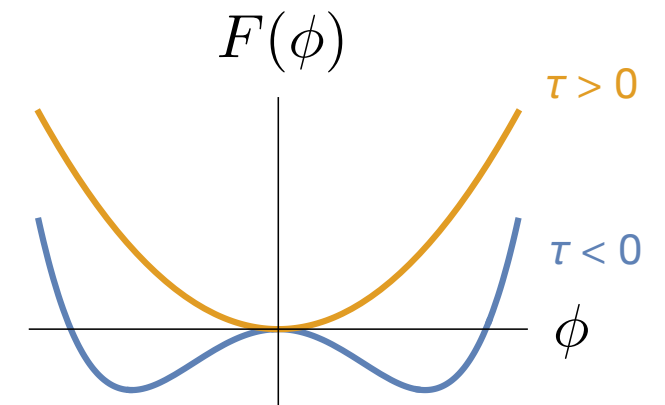
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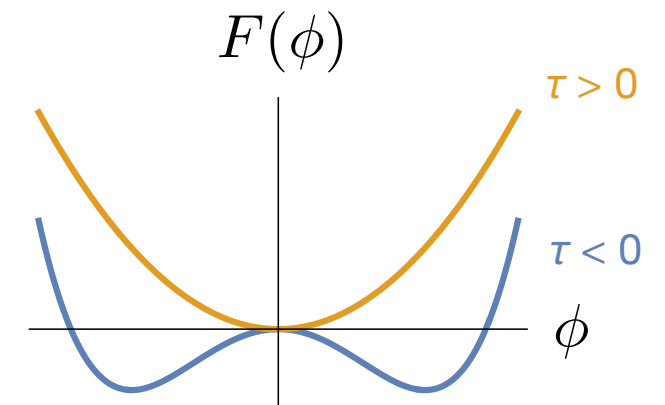
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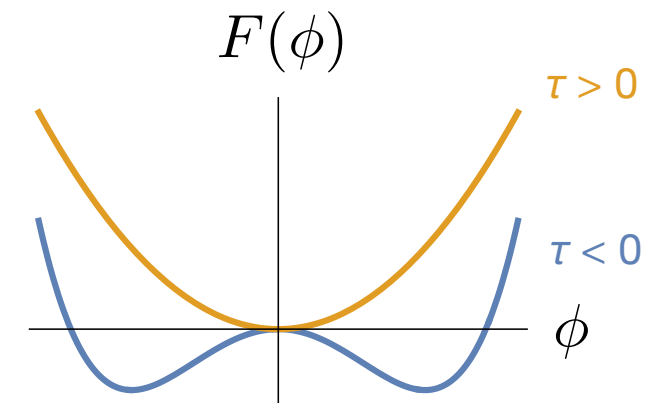
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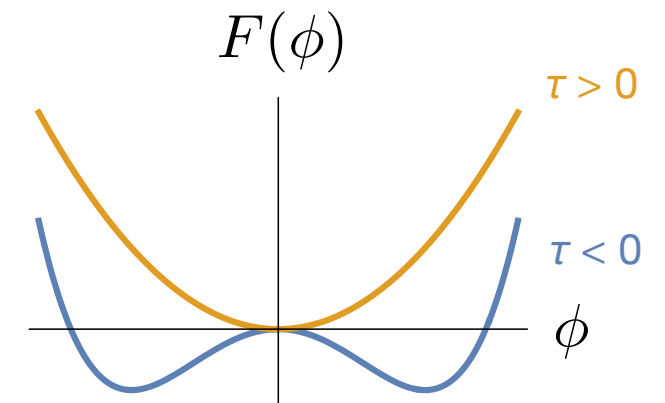
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fluctuation-dissipation relation

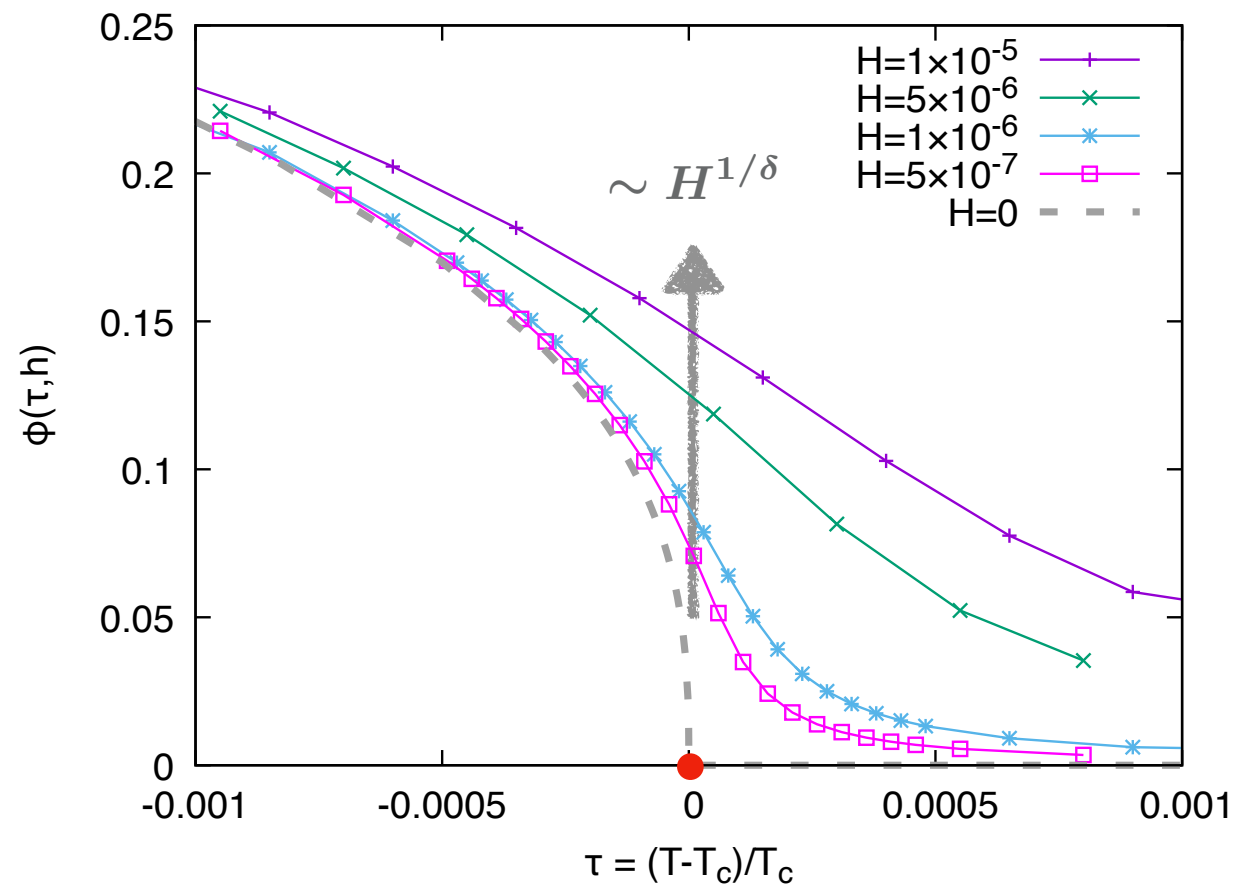
$$\frac{\partial \vec{j}}{\partial t} = \mathcal{T} \left[\underbrace{\eta \vec{\nabla}^2 \frac{\delta F}{\delta \vec{j}}}_{\text{shear viscosity}} + \underbrace{\vec{\xi}}_{\text{noise}} + \underbrace{g \frac{\delta F}{\delta \phi} \vec{\nabla} \phi}_{\text{reversibility of advection}} - \underbrace{g \left(\frac{\delta F}{\delta \vec{j}} \cdot \vec{\nabla} \right) \vec{j}}_{\text{convection}} \right]$$

Son and Stephanov,
PRD **70**, 056001 (2004)

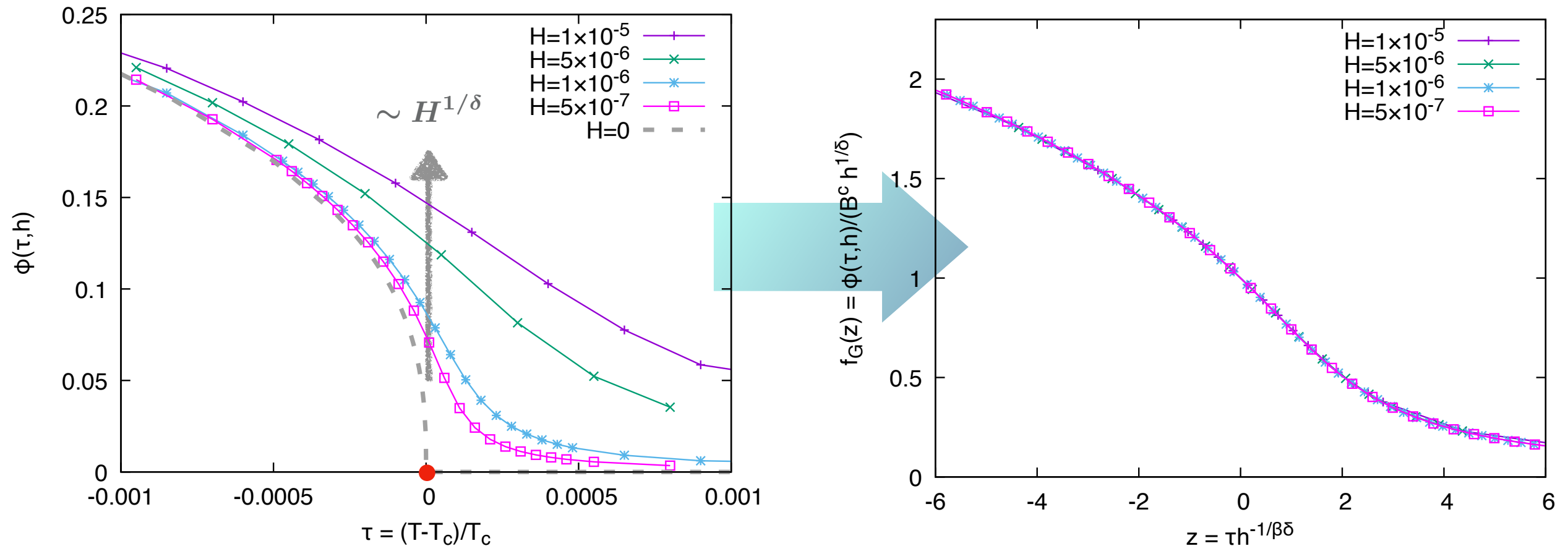
Model H
 $z = d + x_\eta$

Liquid-gas critical point in pure fluid

- Benchmark: Magnetic EoS (3d Ising)



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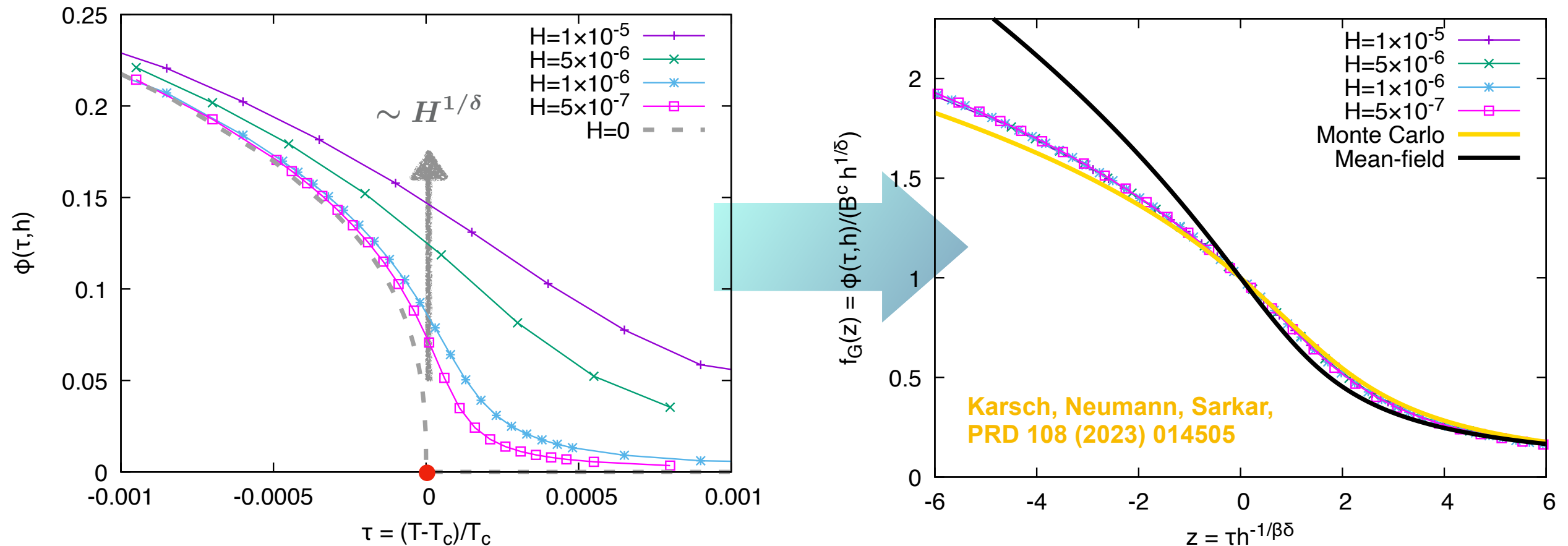


$$\phi(T, H) \propto \pm h^{1/\delta} f_G(\tau h^{-1/\beta\delta})$$

Widom-Griffiths (static) scaling function

Widom (1965), Griffiths (1967)

- Benchmark: Magnetic EoS (3d Ising)

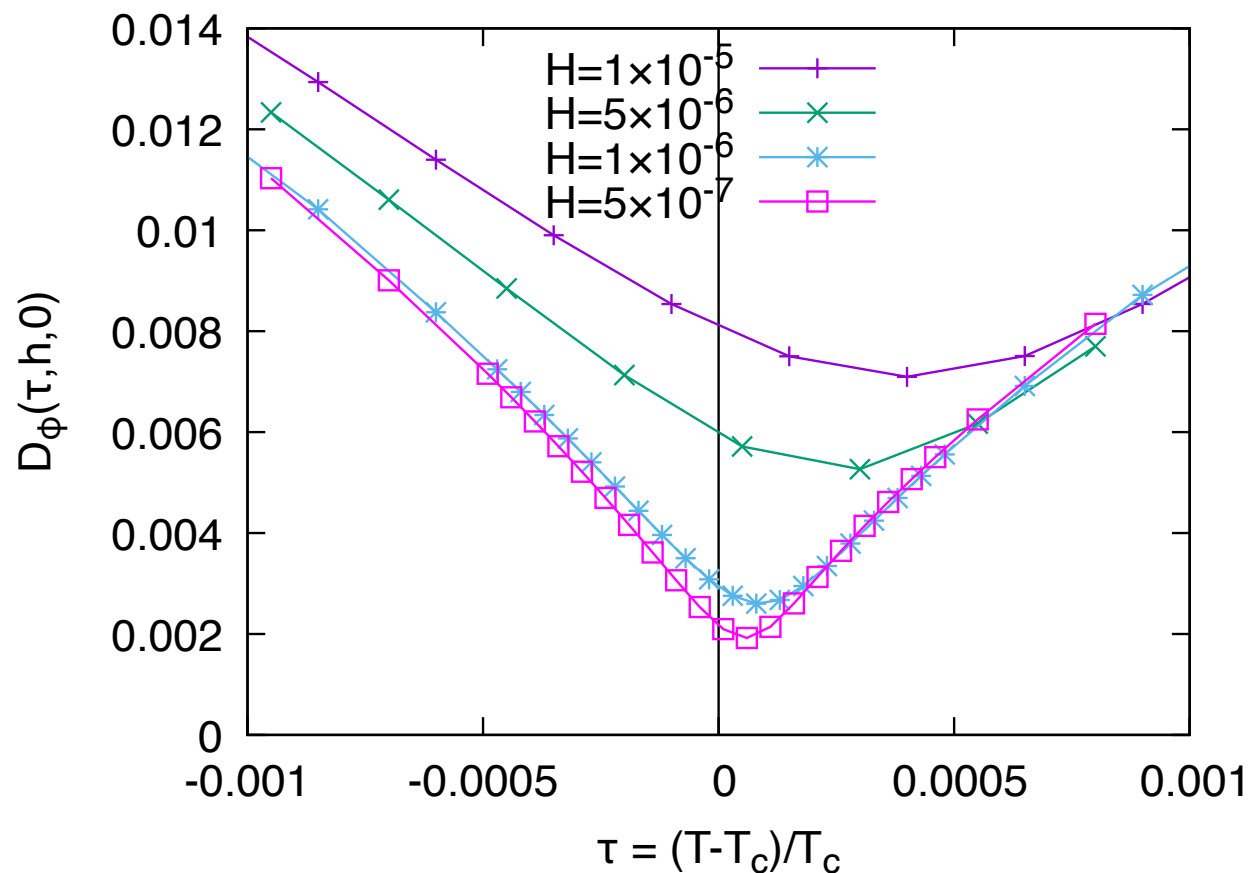


$$\phi(T, H) \propto \pm h^{1/\delta} f_G(\tau h^{-1/\beta\delta})$$

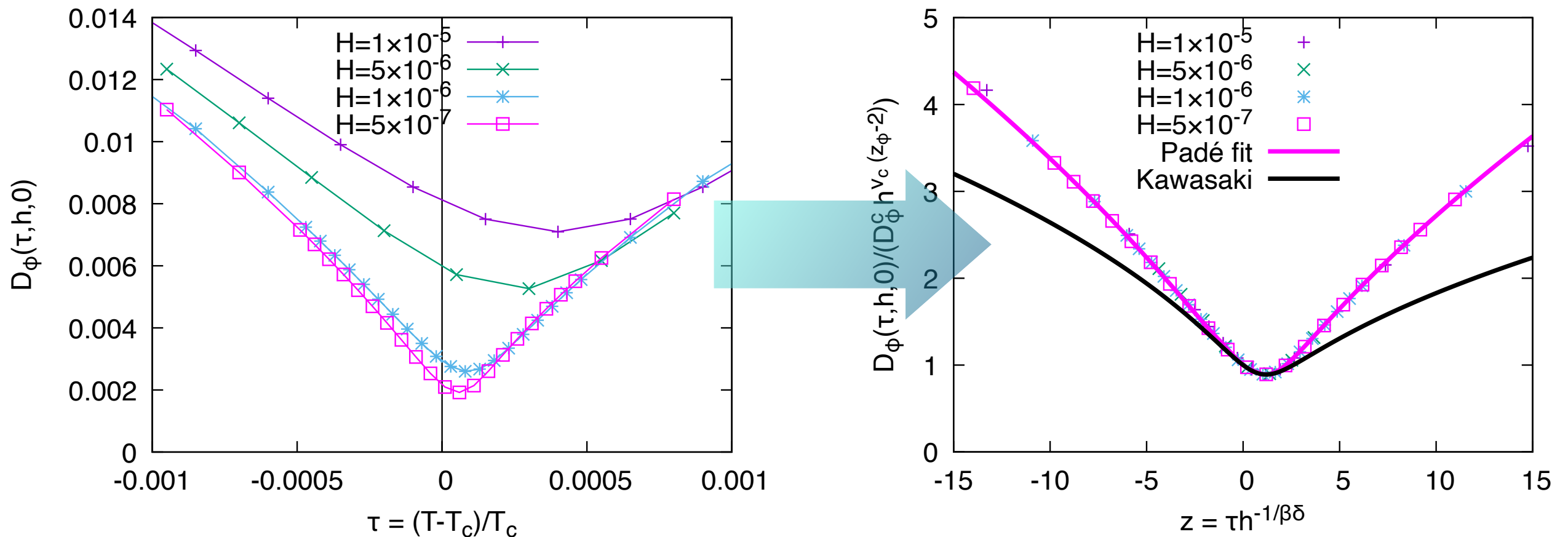
Widom-Griffiths (static) scaling function

Widom (1965), Griffiths (1967)

- Magnetic scaling of heat diffusion $D_\phi = \frac{\sigma}{\chi_\phi}$

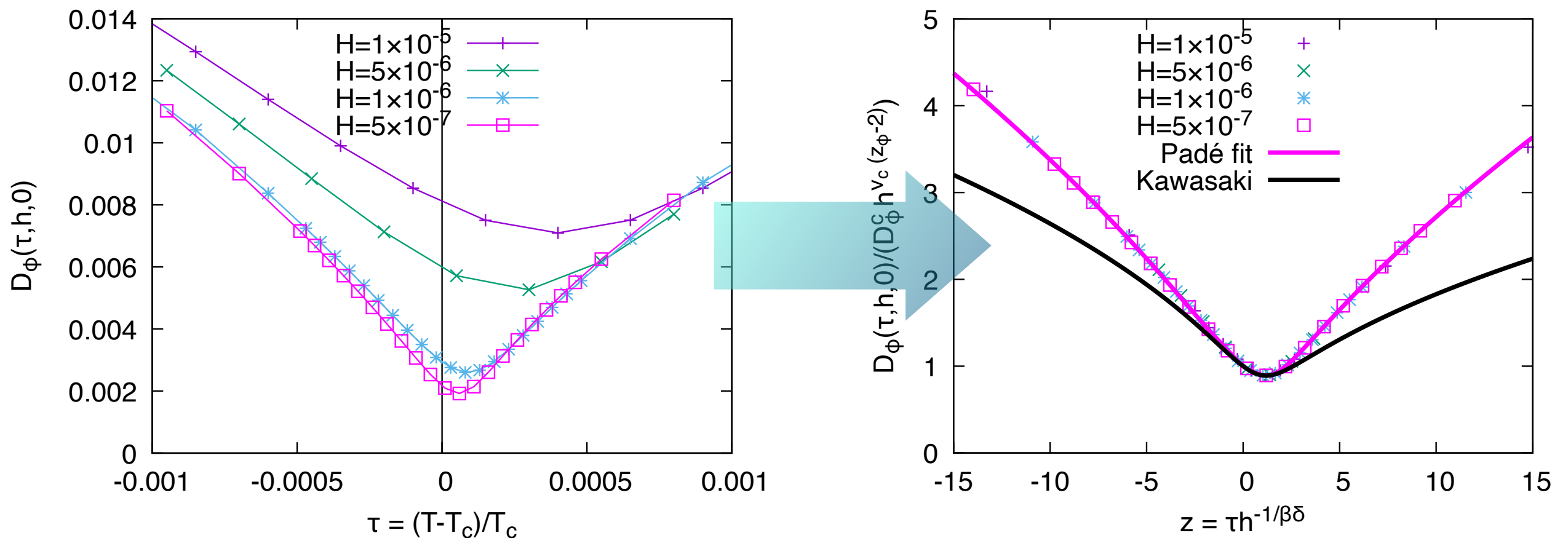


- Magnetic scaling of heat diffusion $D_\phi = \frac{\sigma}{\chi_\phi}$



$$D_\phi(T, H) \propto h^{-\nu} (z - 2)^{\beta\delta} f_D(\tau h^{-1/\beta\delta})$$

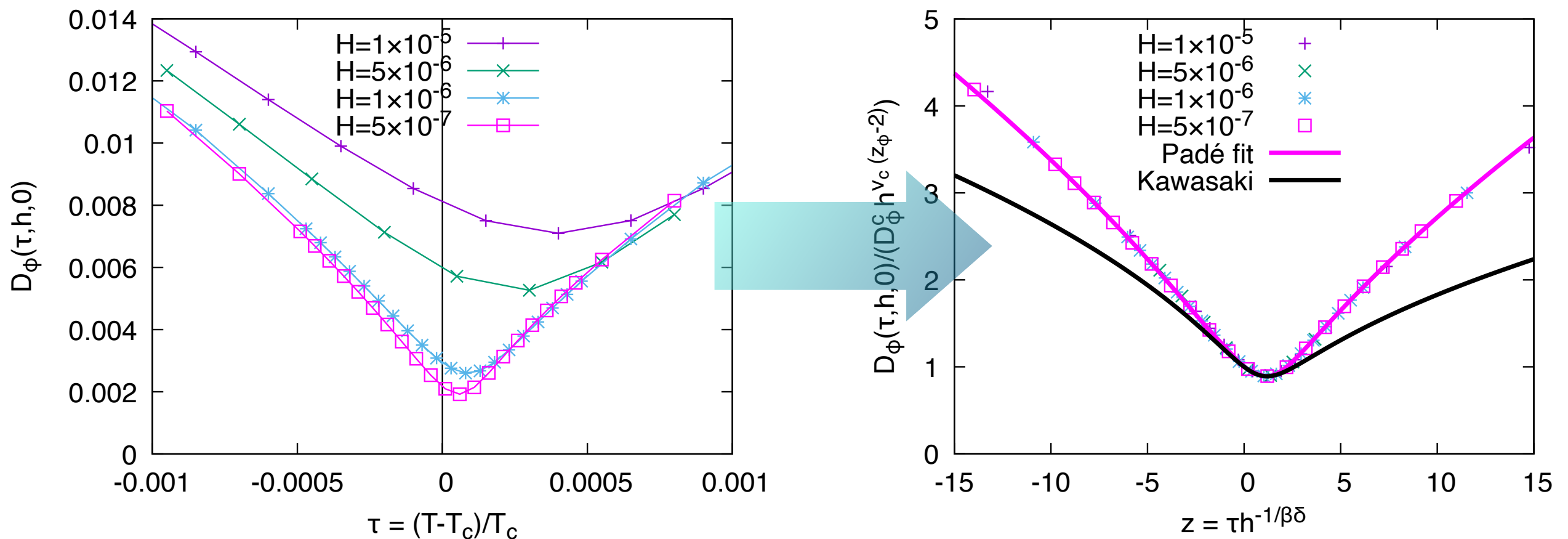
- Magnetic scaling of heat diffusion $D_\phi = \frac{\sigma}{\chi_\phi}$



$$z = d + x_\eta \approx 3.06$$

$$D_\phi(T, H) \propto h^{-\nu(z-2)/\beta\delta} f_D(\tau h^{-1/\beta\delta})$$

- Magnetic scaling of heat diffusion $D_\phi = \frac{\sigma}{\chi_\phi}$



$$D_\phi(T, H) \propto h^{-\nu(z-2)/\beta\delta} f_D(\tau h^{-1/\beta\delta})$$

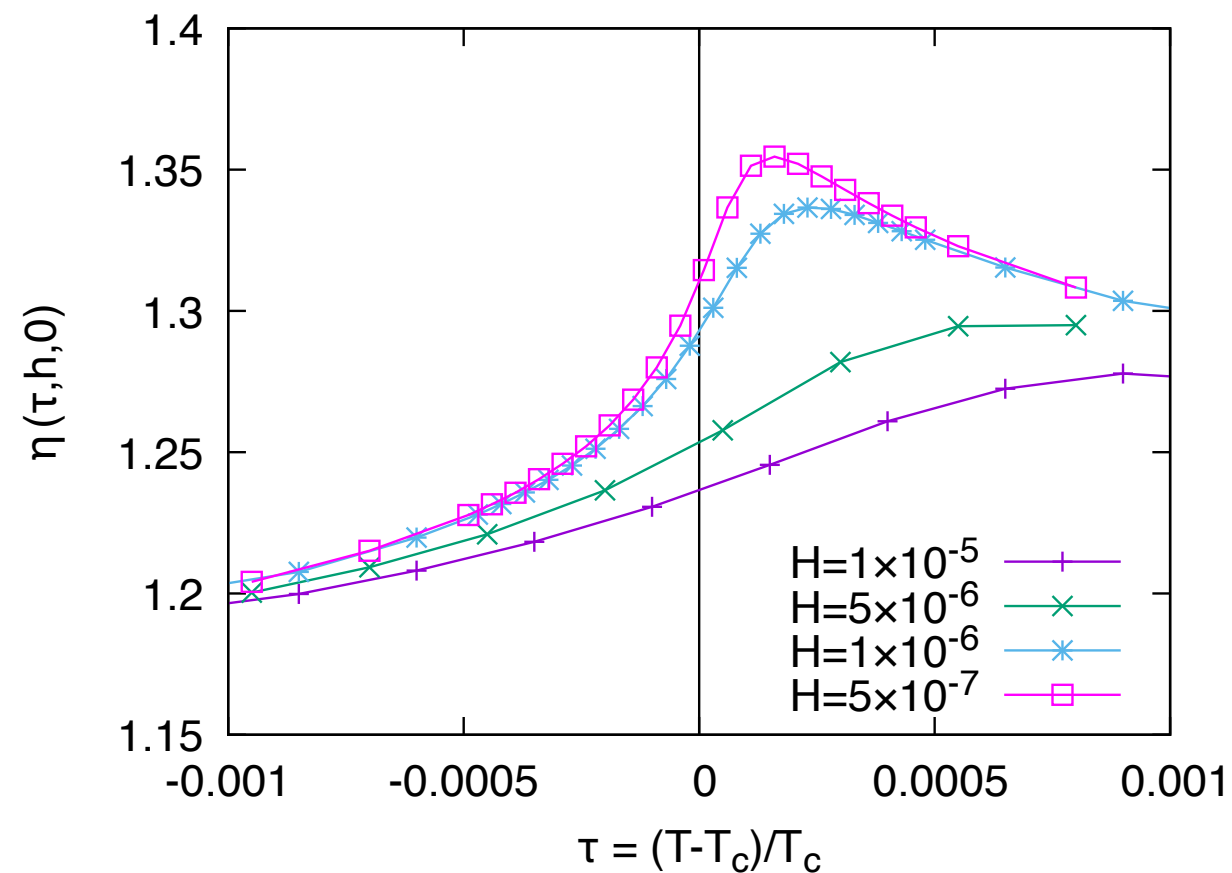
- Critical Viscosity of Xenon (CVX) experiment: $x_\eta = 0.0690(6)$

Berg et al., Phys. Rev. E **60** (1999) 4079 — Discovery

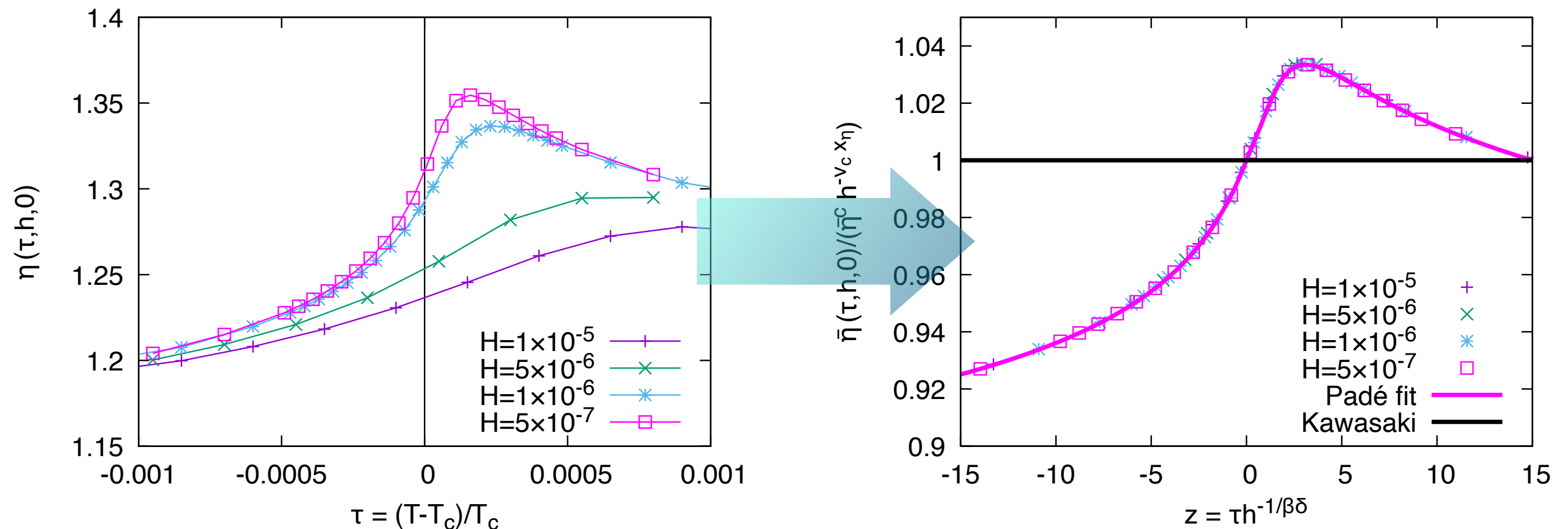
Berg et al., Phys. Rev. E **77** (2008) 041116 — Columbia



- Magnetic scaling of shear viscosity

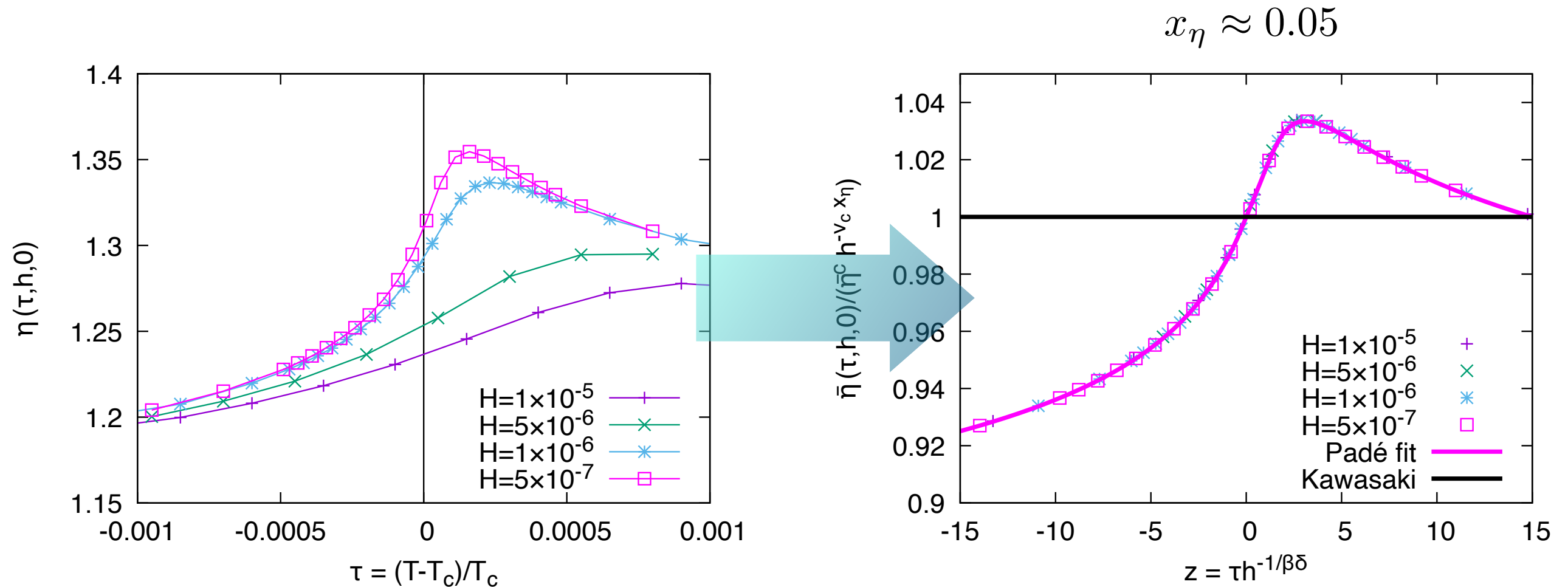


- Magnetic scaling of shear viscosity



$$\eta(T, H) \propto h^{-\nu_{\text{red}}/\beta\delta} f_\eta(\tau h^{-1/\beta\delta})$$

- Magnetic scaling of shear viscosity



$$\eta(T, H) \propto h^{-\nu_{\text{red}} x_\eta / \beta\delta} f_\eta(\tau h^{-1/\beta\delta})$$

Roth, Ye, Schlichting, von Smekal, arXiv:2603.17874 (2026)

- Dynamic scaling (wavenumber dependent):

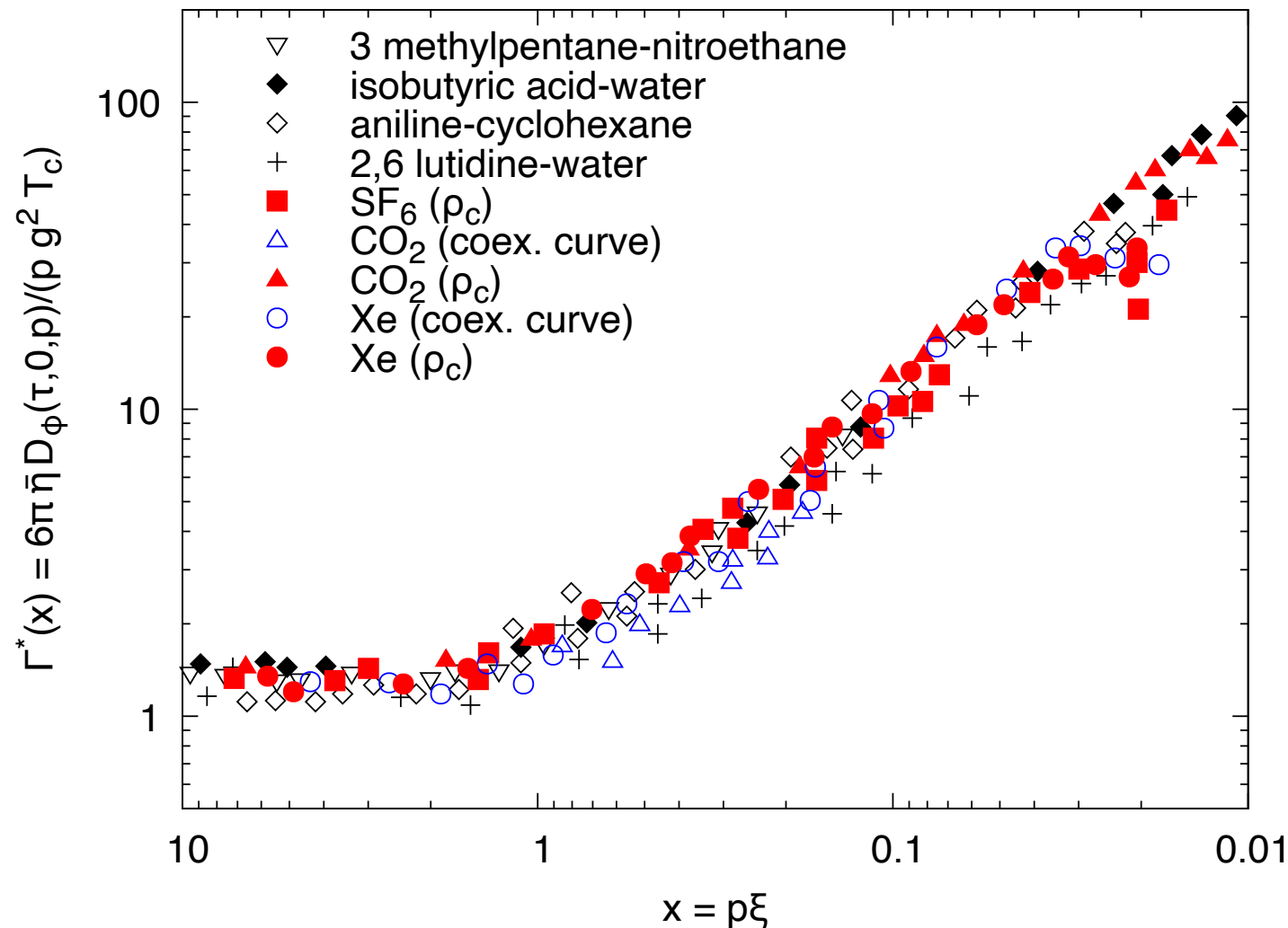
$$D_{\phi}(\tau, p) = \frac{g^2 T_c p}{6\pi\eta(\tau)} \Gamma^*(p\xi(\tau))$$

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$$D_\phi(\tau, p) = \frac{g^2 T_c p}{6\pi\eta(\tau)} \Gamma^*(p\xi(\tau))$$

$$\xrightarrow{p \rightarrow 0} R^\pm \frac{g^2 T_c}{\eta\xi}$$

- Dynamic scaling (wavenumber dependent):



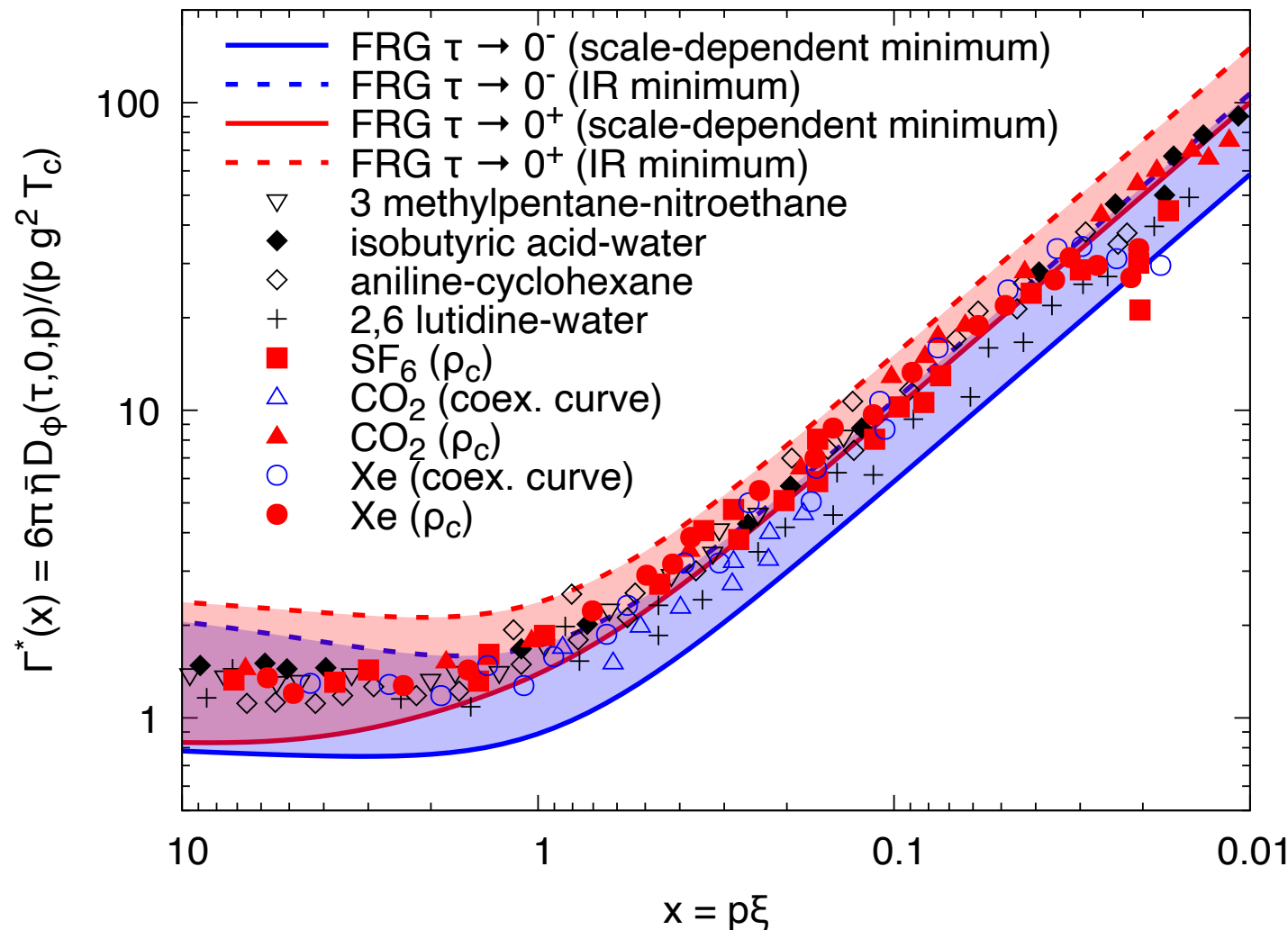
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- Experimental data for $\Gamma^*(x)$

Swinney, Henry, Phys. Rev. A **8** (1973) 2586

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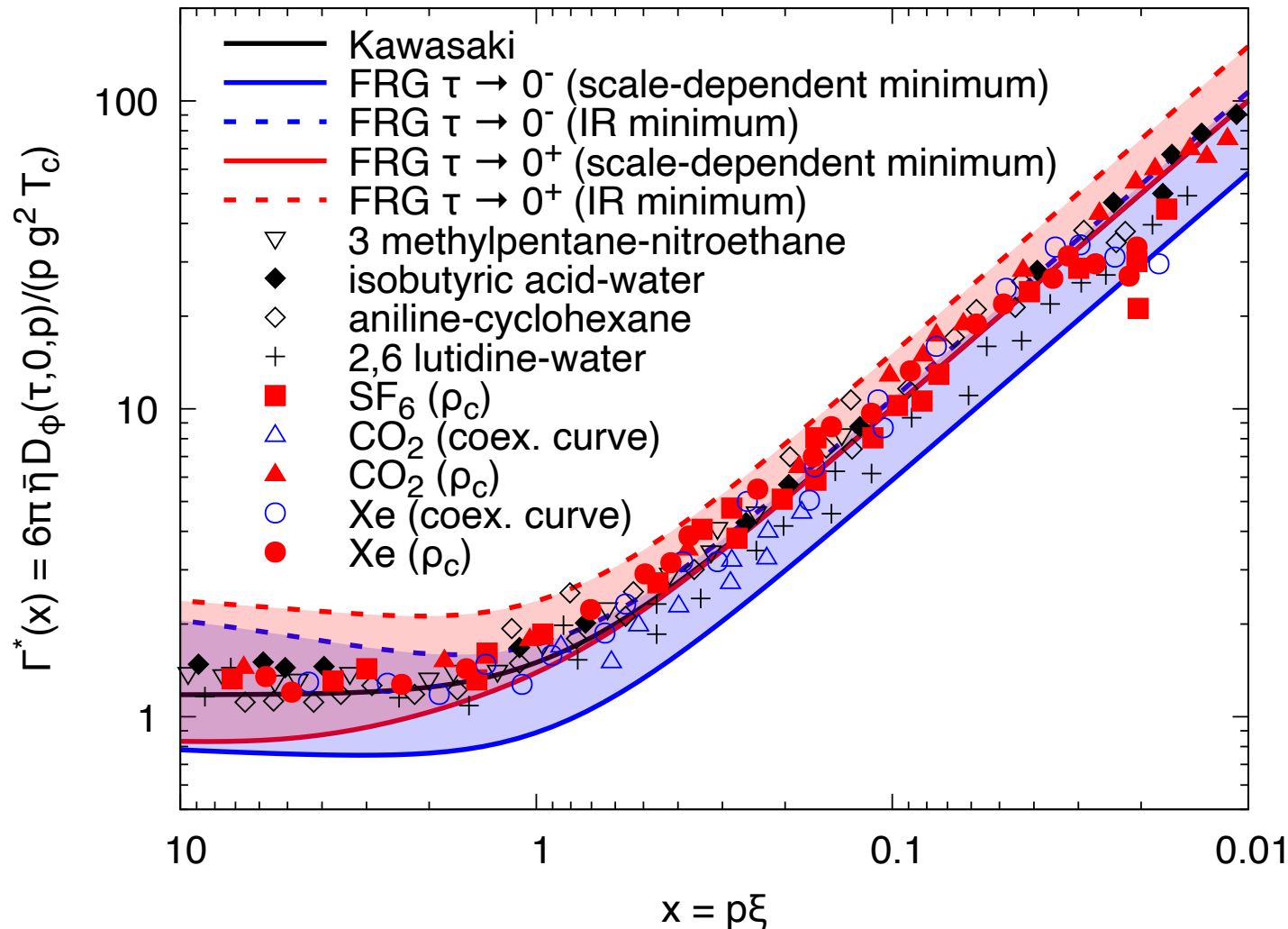
- real-time FRG: different $\Gamma^*(x)$ for **critical isochore**/**coexistence line**

Roth, Ye, Schlichting, von Smekal, arXiv:2603.17874

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- real-time FRG: different $\Gamma^*(x)$ for **critical isochore**/**coexistence line**

Roth, Ye, Schlichting, von Smekal, arXiv:2603.17874

- Kawasaki approximation:

$$\Gamma^*(x) = \frac{3}{4x^3} (1 + x^2 + (x^3 - x^{-1}) \arctan x)$$

$$\rightsquigarrow R^\pm = \frac{1}{6\pi}$$

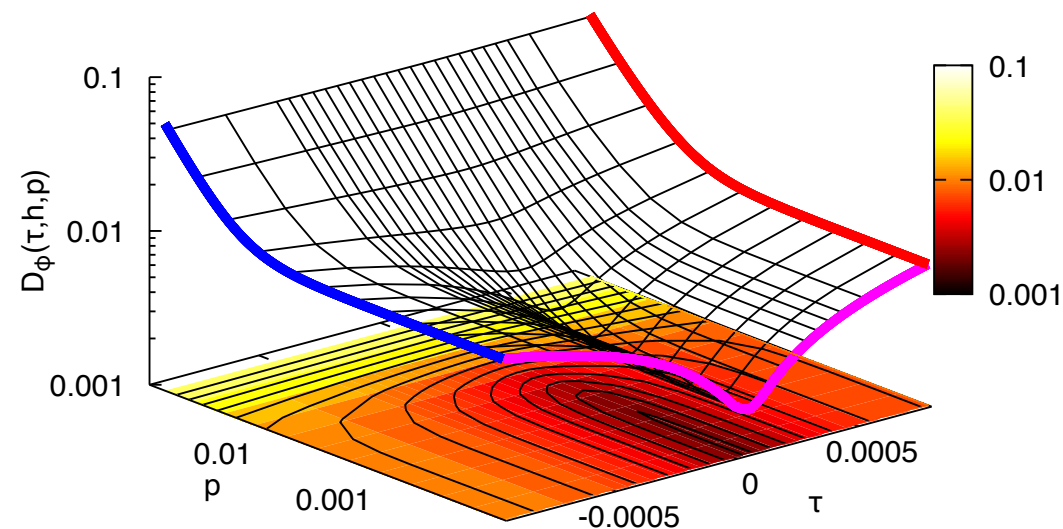
Kawasaki, Ann. Phys. **61** (1970) 1

Universal critical dynamics near:

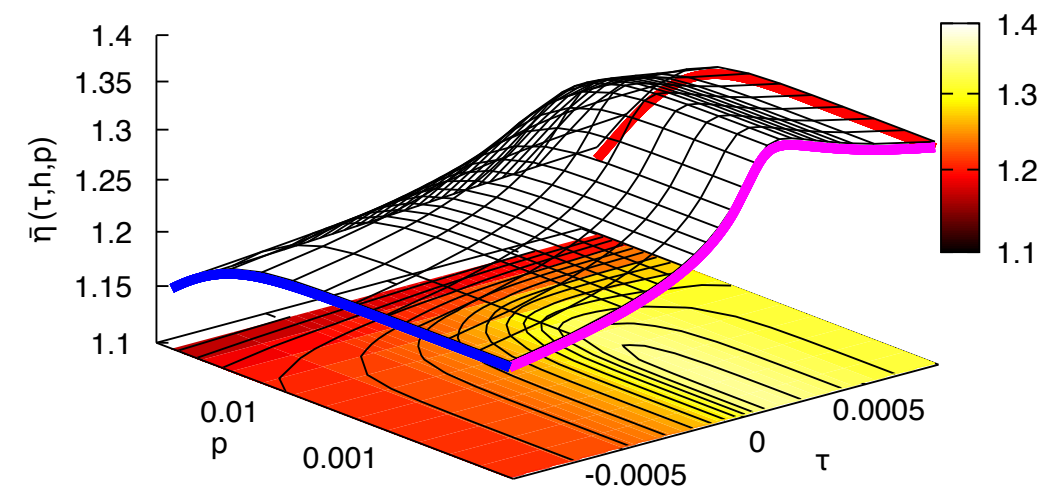
1. QCD critical point

Roth, Ye, Schlichting, von Smekal, Phys. Rev. D **111** (2025) L111901; arXiv:2603.17874

heat diffusion



shear viscosity

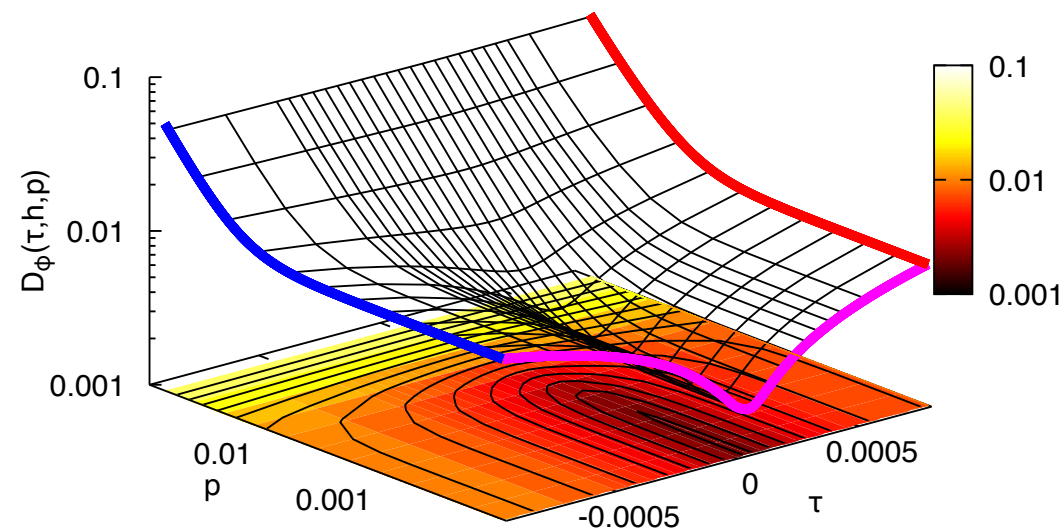


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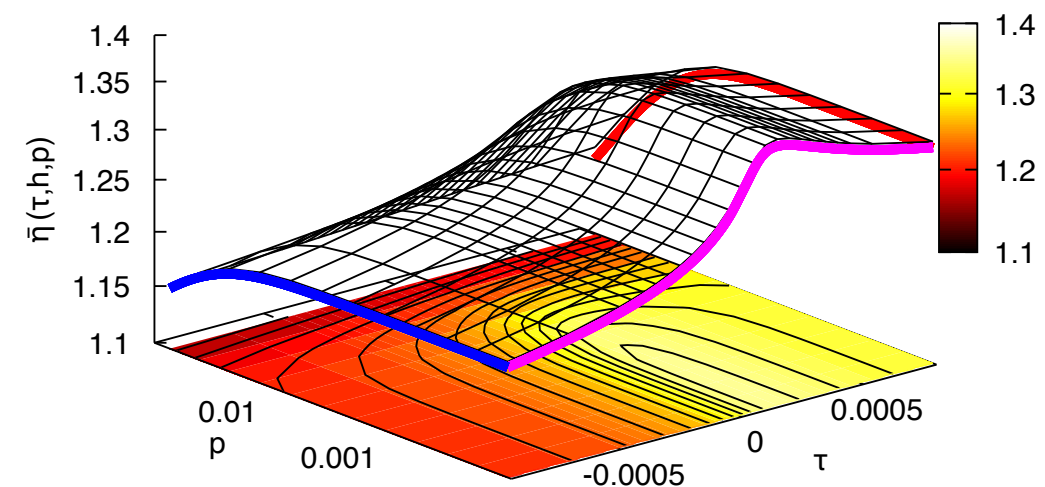
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Roth, Ye, Schlichting, von Smekal, Phys. Rev. D **111** (2025) L111901; arXiv:2603.17874

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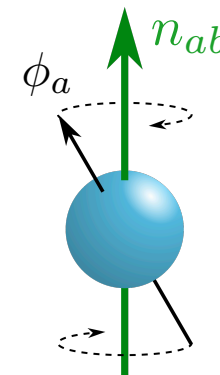
shear viscosity



2. chiral phase transition

Roth, Ye, Schlichting, von Smekal, JHEP **01** (2025) 118

reversible mode couplings $\leftrightarrow$ Larmor precession



Model G

$$z = d/2$$

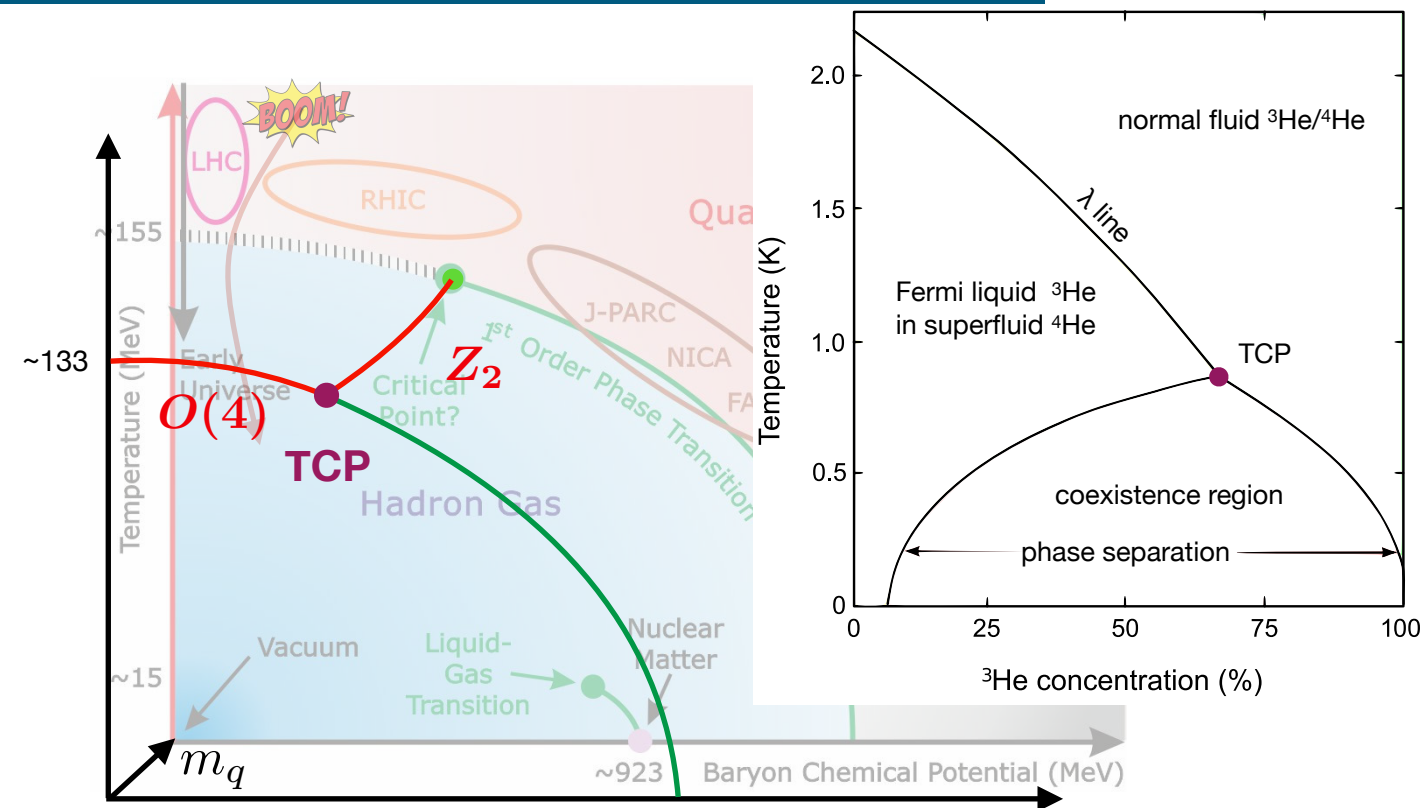
$O(4)$ Heisenberg **antiferromagnet**

What I did not talk about:

Connect the two:

3. tricritical dynamics in QCD and ^3He - ^4He mixtures

Pradeep, Roth, von Smekal, arXiv:2608.11065



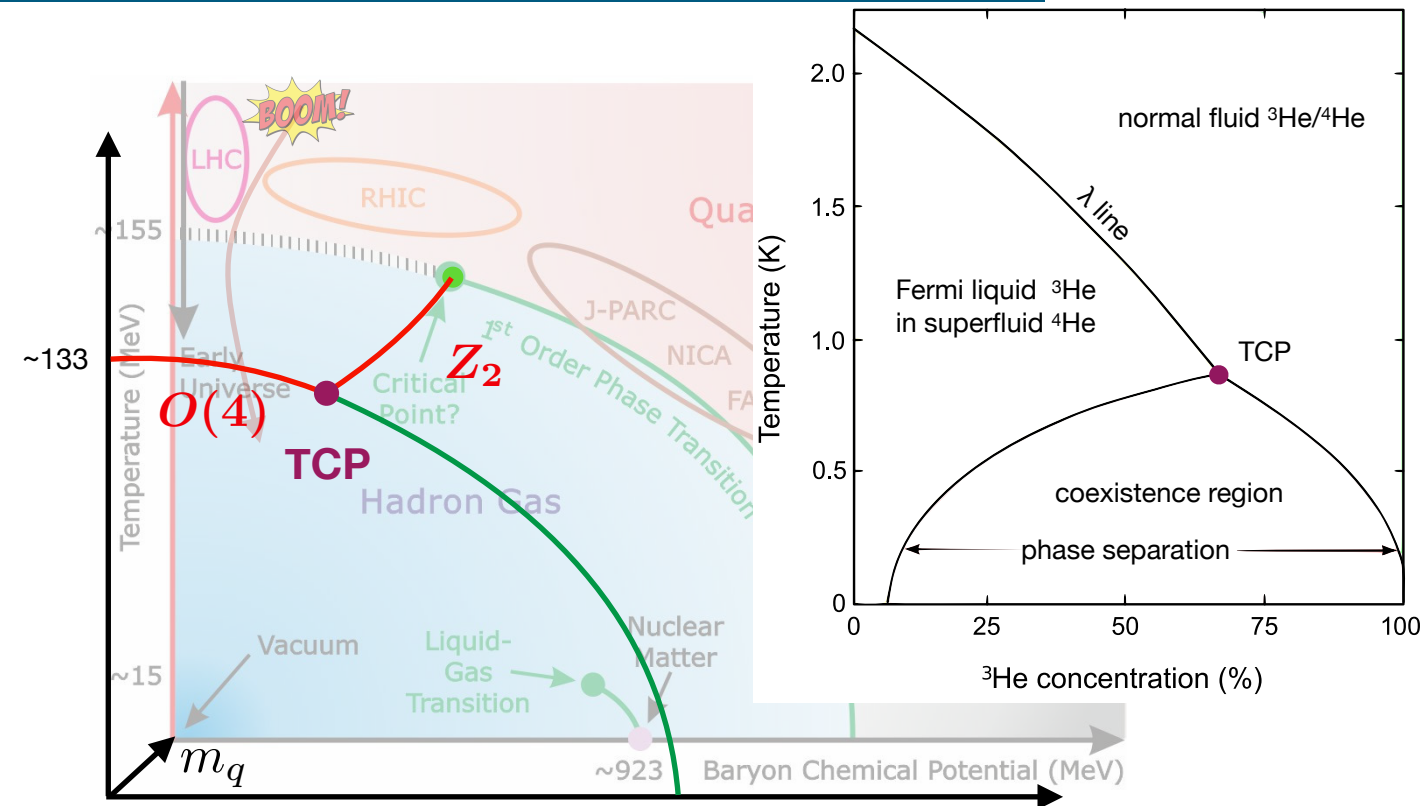
Poster 66: Johannes Roth

What I did not talk about:

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Pradeep, Roth, von Smekal, arXiv:2608.11065



4. Stochastic hydrodynamics simulations for dynamics near the QCD critical point

Poster 66: Johannes Roth

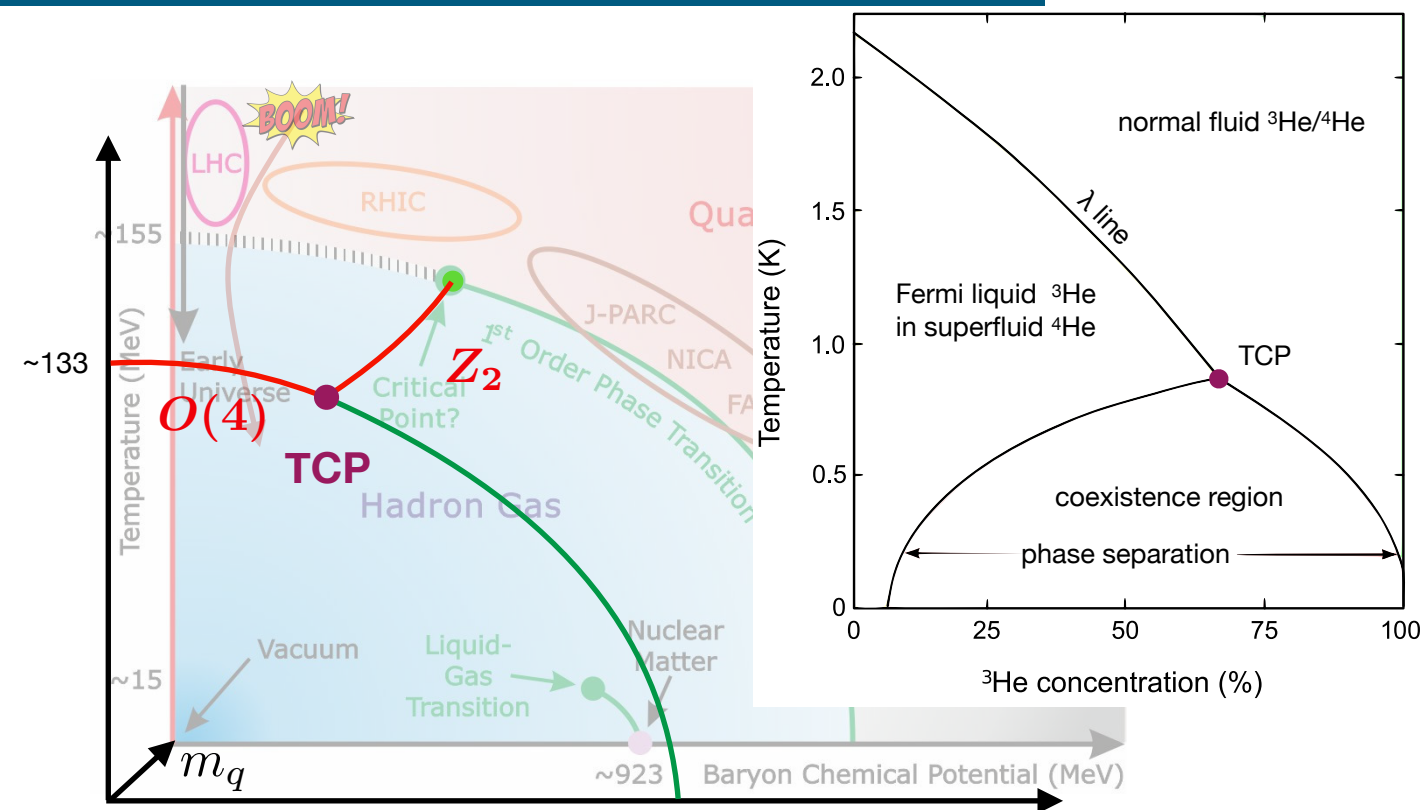
Poster 81: Mattis Harhoff

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Pradeep, Roth, von Smekal, arXiv:2608.11065



Poster 66: Johannes Roth

4. Stochastic hydrodynamics simulations for dynamics near the QCD critical point

Poster 81: Mattis Harhoff

5. Non-equilibrium phase transitions and finite-time scaling

Kibble-Zurek in transcritical quenches:

Sieke, Harhoff, Schlichting, von Smekal, NPB **1011** (2025) 116808

non-equilibrium scaling across first-order transitions:

Sieke, Fuchs, von Smekal, NPB **1029** (2026) 117585

Poster 59: Leon Sieke

Mattis Harhoff

Johannes Roth

Leon Sieke

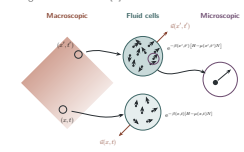
Non-Relativistic Stochastic Hydrodynamics

Mattis Harhoff, Cedric Riedel, Johannes Roth, Sören Schlichting, Lorenz von Smekal

UNIVERSITÄT BIELEFELD

Hydrodynamics as an effective field theory

Hydrodynamics is an effective field theory on large length and time scales [1]



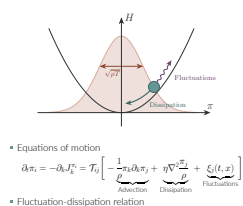
• Separation of scales
Microscopic theory (e.g. QFT) has conservation laws (energy, momentum, etc.)
 $\partial_t T^{\mu\nu}(x) = 0, \quad \partial_i T^{\mu\nu}(x) = 0$
→ Slow variables, since relaxation happens via transport
• Coarse-grain by "integrating out" fast modes
Postulate: On large scales, system locally behaves as thermal ensemble with parameters $T(x), w(x), \mu(x)$
 $T^{\mu\nu}(x) \rightarrow T^{\mu\nu}(T(x), w(x), \mu(x))$
• Conservation laws contain two types of hydrodynamic fluxes
• "Ideal" fluxes advect the fluid without producing entropy, pure equilibrium contribution
• Dissipative fluxes couple to gradients of thermodynamic quantities, produce entropy, relax the fluid to equilibrium

Thermal fluctuations

• Classical hydrodynamics neglects thermal fluctuations
• All correlations beyond $\langle T^{\mu\nu} \rangle$ lost
• Fluctuations needed for thermodynamic consistency (fluctuation-dissipation relations)
• Non-linear interactions of fluctuations drastically change behavior of fluid (renormalization, long time tails, etc.) [1, 2]
Single realization: Hydro + thermal fluctuations
 $\phi(t, x) = \phi(t, x) + \Delta\phi(t, x)$
 $(T^{\mu\nu}[\phi]) = T^{\mu\nu}(\phi) + \frac{1}{2} \frac{\delta^2 T^{\mu\nu}}{\delta\phi^2} (\Delta\phi, \Delta\phi) + \dots$
• Fluctuations are especially relevant to small systems and phase transitions
• Include stochastic fluxes into hydrodynamic e.o.m.
• Conserved quantities stay conserved
• Formal derivation from microscopic QFT possible on Keldysh contour [1]

Example: Incomp. hydrodynamics with thermal fluctuations

• Conserved non-relativistic momentum density π
• Constant density ρ
• Incompressibility $\nabla \cdot \pi = 0$
• Coupling to heat bath of temperature $T = 1/\beta$
• Boltzmann distribution for configurations
 $p_{\text{eq}}[\pi] \propto \exp\left(-\beta \int d^3x \left(\frac{1}{2} \pi^2 + \pi(\nabla \cdot \pi)\right)\right)$
• Thermal fluctuations around equilibrium state



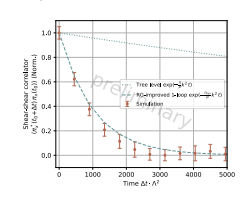
• Equations of motion
 $\partial_t \pi_i = -\partial_i \pi_j = T_0 \left[-\frac{1}{2} \pi_i \partial_i \pi_j + \eta \nabla^2 \pi_i + \zeta_i(t, x) \right]$
• Fluctuation-dissipation relation
 $\langle \zeta_i(t, x) \zeta_j(t', x') \rangle = -2T \eta \delta_{ij} \nabla^2 \delta(x - x') \delta(t - t')$
• Guarantees that system relaxes to correct equilibrium state via dissipation

Challenges

• Analytically
• Theory is not well-defined, correlation functions have UV divergences
• Regularization needed
• Numerically
• Finite difference operators act on noisy configurations (lattice spacing acts as effective UV cutoff)
• Bad performance of numerical PDE solvers
• Numerical violation of additional conservation laws

Option 2: Modify Dissipation

• Suppress UV modes via stronger dissipation
 $\eta \rightarrow \eta(k) = [1 + S(k)]\eta$
• Effectively higher order gradients as dissipative terms
• New $\eta(k)$ enters in dissipative term and fluctuation-dissipation theorem
• Theory becomes UV finite
• Can compare e.g. shear correlators to perturbation theory



References

- [1] M. Crossley, R. Ghorbani and H. Liu, *JHEP* 09, 095 (2017), arXiv:1511.03646 [hep-th].
- [2] G. Basar, *Prog. Part. Nucl. Phys.* 143, 104175 (2025), arXiv:2410.02864 [hep-th].
- [3] I. E. Dzyaloshinski and G. E. Volovick, *Ann. Phys.* 125, No. 1 (1985), 47-97.
- [4] C. Chattopadhyay, J. Ott, T. Schaefer and V. V. Skokov, *Phys. Rev. Lett.* 133, no.3, 3 (2024), arXiv:2403.10608 [nucl-th].
- [5] I. Bhandari, R. Singh and D. Teaney, *Phys. Rev. C* 111, no.6, 064909 (2025), arXiv:2412.10306 [nucl-th].

BGTS HGS-HiRe FAIR

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Tricritical dynamics in QCD and ³He-⁴He mixtures

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²Fakultät für Physik, Universität Bielefeld, D-33615 Bielefeld, Germany
³Institut für Theoretische Physik, Justus-Liebig-Universität, 35392 Gießen, Germany
Strong and Electro-Weak Matter (SEWM), Helsinki, 17-21 August 2026
Contact: johannes.roth@uni-bielefeld.de

Introduction

Review dynamic universality class of chiral phase transition at $\mu_B = 0$:

[Son, PRL 84, 3771 (2000); Son, Stephanov, PRD 68, 076011 (2003)]

• Order parameter: chiral condensate

$$\Phi_{ij} = \langle \bar{\psi}_i \psi_j \rangle \quad SU(2) \times SU(2)$$

• Below T_c : $\Phi = \sqrt{\mu} e^{i\varphi}$

• Spatially constant φ simply describes different vacuum

→ does not dissipate in time → slow, enters hydrodynamics

• Relevant conserved quantities:

iso-axial-vector charges

$$A_i = \frac{1}{2} \epsilon^{ijk} \pi_j \pi_k \quad V_i = \frac{1}{2} \epsilon^{ijk} \pi_j \pi_k$$

• $\vec{\pi}$ and A_i partly-odd & iso-vectors

→ mix in linearized hydrodynamic equations

• Find propagating modes $\omega(p) = v p$ with

$$v \sim \sqrt{\chi} \sqrt{\kappa}$$

• Static critical scaling:

$$f \sim \zeta^{-d/2+1} \quad \chi_{\text{iso}} \sim \zeta^{d/2}$$

• Characteristic frequency of critical modes: $\omega_c \sim \zeta^{-1} \sim \zeta^{-d/2}$

→ $\omega_c \sim d/2$

as in Model G (Helsenberg antiferromagnet)

[Helsenberg, Hohenberg, RMP 49, 435 (1977)]

• Specific heat is expected to have cusp ($\alpha = -0.213(12) < 0$) in $O(4)$, analogous to $O(2)$ lambda transition in ⁴He:

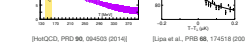


Figure: (Left) Temperature derivative of trace anomaly (same singular behavior as C_V , but regular terms more suppressed). (Right) Characteristic lambda-like shape of specific heat near lambda transition of ⁴He.

Tricritical point

• ϕ^4 theory → mean-field exponents in 3d

In particular, specific-heat exponent: $\alpha_k = 1/2 > 0$

→ Specific heat (at constant pressure) diverges:

$$C_p = T \partial_t \ln Z|_p \sim \zeta^{\alpha_k/10}$$

→ Effect on dynamic universality class?

• Linearized hydrodynamic equation for entropy per baryon, after pressure has equilibrated through sound waves:

$$\frac{\partial \delta s}{\partial t} = -\gamma \partial_i \frac{\delta F}{\partial \pi_i} + g(\pi_{ab}, \pi_{ab}) \frac{\delta F}{\partial \pi_{ab}} + \theta_s$$

• Correlation functions from generating functional

$$Z[\pi, \tilde{\pi}] = \int D\pi D\tilde{\pi} e^{iS[\pi, \tilde{\pi}] + \int d^4x \pi_i \tilde{\pi}_i + \int d^4x \tilde{\pi}_i \pi_i}$$

with

$$S[\pi, \tilde{\pi}] = \int d^4x \left(\tilde{\pi}_i \left(\frac{\partial \pi_i}{\partial t} + \gamma \partial_i \frac{\delta F}{\partial \pi_i} + g(\pi_{ab}, \pi_{ab}) \frac{\delta F}{\partial \pi_{ab}} \right) + i \tilde{\pi}_i \gamma \partial_i \pi_i \right)$$

• Thermal diffusivity $D = \mu/C_p$

• How does heat conductivity μ behave near tricritical point?

Model

• Consider $O(4)$ order parameter ϕ coupled to energy-like density ϵ

Landau-Ginzburg-Wilson (LGW) functional of the form

$$F = \int d^4x \left(\frac{1}{2} (\nabla \phi)^2 + \frac{\lambda}{4N} \phi^2 + \frac{\kappa}{6N^2} \phi^2 \right)$$

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1-particle irreducible (1PI) correlation functions:

$$F[\phi] \rightarrow F[\phi] + \frac{1}{2} \int d^4x \phi_i \phi_i$$

$$S[\phi, \tilde{\phi}] \rightarrow S[\phi, \tilde{\phi}] + \Delta S[\phi, \tilde{\phi}]$$

$$\Delta S[\phi, \tilde{\phi}] = \int d^4x \left(\tilde{\phi}_a \gamma_{ab} \phi_b - g \phi_a \phi_a \right)$$

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