Tricritical dynamics in QCD and ^{3}He-^{4}He mixtures

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Strong and Electro-Weak Matter (SEWM), Helsinki, 17-21 August 2026
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Introduction

Review dynamic universality class of chiral phase transition at $\mu_B = 0$:

[Son, PRL **84**, 3771 (2000); Son, Stephanov, PRD **66**, 076011 (2002)]

- Order parameter: chiral condensate

$$\Phi_{ij} = \langle \bar{q}_{Li} q_{Rj} \rangle \qquad SU(2)\times SU(2)$$

- Below T_c: $\Phi \equiv \sqrt{\rho}\, e^{i\vec{\varphi}\cdot\vec{\tau}}$

- Spatially constant $\vec{\varphi}$ simply describes different vacuum
  ⇝ does not dissipate in time ⇝ **slow, enters hydrodynamics**

- Relevant conserved quantities:
  Iso-(axial-)vector charges

$$A_i = \langle \tfrac{1}{2}\bar{q}\gamma^0\gamma^5\tau_i q\rangle \qquad V_i = \langle \tfrac{1}{2}\bar{q}\gamma^0\tau_i q\rangle$$

- $\vec{\varphi}$ and A_i parity-odd & iso-vectors
  ⇝ mix in linearized hydrodynamic equations
- Find propagating modes $\omega(p) = vp$ with

$$v = f/\sqrt{\chi_n}$$

pion decay constant — iso-axial-vector charge susceptibility

- Static critical scaling:

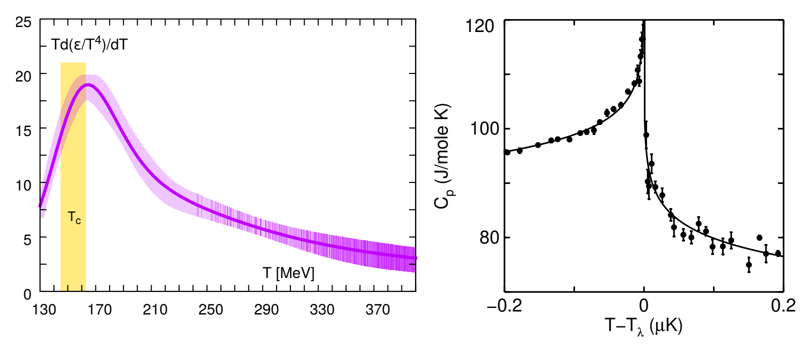
$$f \sim \xi^{-d/2+1} \qquad \chi_n \ \text{finite}$$

- Characteristic frequency of critical modes: $\omega \sim v\xi^{-1} \sim \xi^{-d/2}$

$$\Longrightarrow\ z = d/2$$

as in Model G (Heisenberg antiferromagnet)
[Halperin, Hohenberg, RMP **49**, 435 (1977)]

- Specific heat is expected to have cusp ($\alpha = -0.213(12) < 0$ in $O(4)$), analogous to $O(2)$ lambda transition in ^{4}He:



[HotQCD, PRD **90**, 094503 (2014)] [Lipa et al., PRB **68**, 174518 (2003)]

Figure: (Left) Temperature derivative of trace anomaly (same singular behavior as C_V, but regular terms more suppressed). (Right) Characteristic lambda-like shape of specific heat near lambda transition of ^{4}He.

Tricritical point

- ϕ^6 theory ⇝ mean-field exponents in 3d
  In particular, **specific-heat exponent:** $\alpha_t = 1/2 > 0$
  ⇝ Specific heat (at constant pressure) **diverges:**

$$C_p = T n_B \frac{\partial \hat{s}}{\partial T}\Big|_p \sim \xi^{\alpha_t/\nu_t} \qquad \hat{s} \equiv \frac{s}{n_B}$$

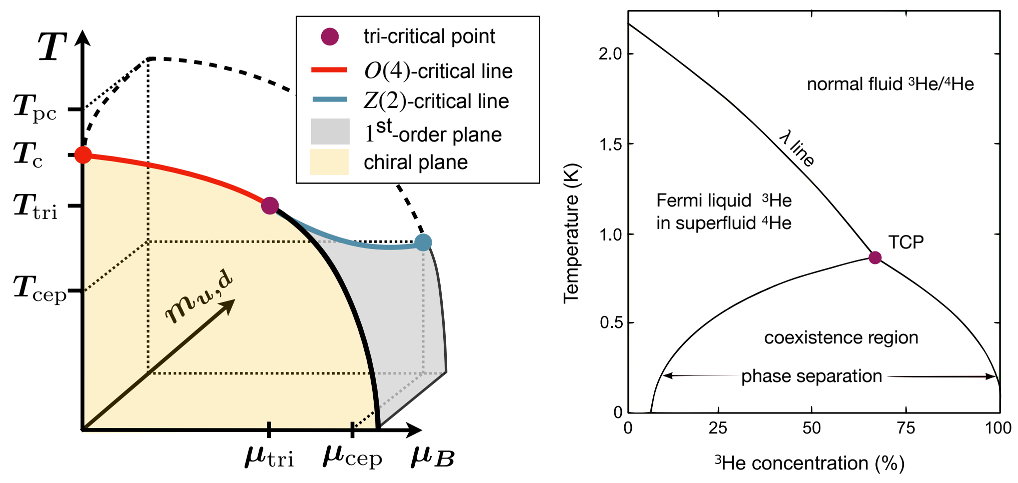
⇝ Effect on dynamic universality class?

- Linearized hydrodynamic equation for entropy per baryon, after pressure has equilibrated through sound waves:

heat conductivity

$$\frac{\partial \delta\hat{s}}{\partial t} = \frac{\mu}{n_B T}\vec{\nabla}^2\delta T \approx \frac{\mu}{C_p}\vec{\nabla}^2\delta\hat{s}$$

- Thermal diffusivity $D = \mu/C_p$
- How does heat conductivity μ behave near tricritical point?



[Ding et al., PRD **109**, 114516 (2024)]

adapted from
[Wikipedia contributors, Helium Phase Diagram, 2014, see F. Pobell, Matter and methods at low temperatures, 3rd ed., Springer, Berlin, 2007]

Figure: (Left) Conjectured phase diagram of QCD in two-flavor chiral limit. (Right) Phase diagram of liquid ^{3}He-^{4}He mixtures at saturated vapor pressure.

Model

- Consider $O(4)$ order parameter ϕ coupled to energy-like density ε
  Landau-Ginzburg-Wilson (LGW) functional of the form
  [Fujii, Ohtani, PRD **70**, 014016 (2004)]

$$F = \int d^dx \left\{ \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!N}(\phi^2)^2 + \frac{\kappa}{6!N^2}(\phi^2)^3 + \frac{\iota}{2\sqrt{N}}\phi^2\varepsilon + \frac{\varepsilon^2}{2C_p} + \frac{n_{ab}n_{ab}}{4\chi_n} \right\}$$

$$n_{0i} = A_i \qquad n_{ij} = \varepsilon_{ijk}V_k$$

- $O(4)$ symmetry forbids linear mixing of ϕ and ε
- Dynamic universality class: Conservation laws, Poisson brackets
- Iso-(axial-)vector charges are generators of $O(4)$ transformations
  Correspondence principle: Poisson brackets from microscopic commutators

$$\{\phi_a, n_{bc}\} = \delta_{ac}\phi_b - \delta_{ab}\phi_c$$
$$\{n_{ab}, n_{cd}\} = \delta_{ac}n_{bd} + \delta_{bd}n_{ac} - \delta_{ad}n_{bc} - \delta_{bc}n_{ad}$$

- Equations of motion:

$$\frac{\partial \phi_a}{\partial t} = -\Gamma^\phi \frac{\delta F}{\delta \phi_a} + \frac{g}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}} + \theta_a\,,$$
$$\frac{\partial n_{ab}}{\partial t} = \gamma\vec{\nabla}^2\frac{\delta F}{\delta n_{ab}} + g\{n_{ab},\phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{g}{2}\{n_{ab},n_{cd}\}\frac{\delta F}{\delta n_{cd}} + \zeta_{ab}\,,$$
$$\frac{\partial \varepsilon}{\partial t} = \mu\vec{\nabla}^2\frac{\delta F}{\delta\varepsilon} + \Xi$$

- Generalize to $O(N)$: Call Model SSS'
  [Sásvari, Schwabl, Szépfalusy, Physica A **81**, 108 (1975)]
  [Folk, Moser, J. Phys. A **39**, R207 (2006)]

- $N=2$: Model F' for tricritical point in ^{3}He-^{4}He mixtures
  [Siggia, Nelson, PRB **15**, 1427 (1977)]
  [Folk, Moser, J. Low. Temp. Phys. **150**, 689 (2008)]

Real-time functional renormalization group (rFRG)

- Equations of motion are of the form

$$\frac{\partial \psi_{\mathbf{a}}}{\partial t} = -\gamma_{\mathbf{ab}}\frac{\delta F}{\delta \psi_{\mathbf{b}}} + g\{\psi_{\mathbf{a}},\psi_{\mathbf{b}}\}\frac{\delta F}{\delta\psi_{\mathbf{b}}} + \theta_{\mathbf{a}}$$
$$\{\psi_{\mathbf{a}},\psi_{\mathbf{b}}\} = f_{\mathbf{abc}}\psi_{\mathbf{c}}$$

- Correlation functions from generating functional

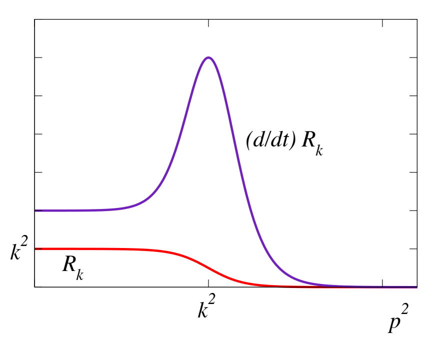
$$Z[j,\tilde{j}] = \int \mathcal{D}\psi\,\mathcal{D}\tilde{\psi}\, e^{iS[\psi,\tilde{\psi}] + i\int \tilde{j}_{\mathbf{a}}\psi_{\mathbf{a}} + i\int\tilde{\psi}_{\mathbf{a}}j_{\mathbf{a}}}$$

with

$$S[\psi,\tilde{\psi}] = \int \left[-\tilde{\psi}_{\mathbf{a}}\left(\frac{\partial\psi_{\mathbf{a}}}{\partial t} + \gamma_{\mathbf{ab}}\frac{\delta F}{\delta\psi_{\mathbf{b}}} - g\{\psi_{\mathbf{a}},\psi_{\mathbf{b}}\}\frac{\delta F}{\delta\psi_{\mathbf{b}}}\right) + iT\tilde{\psi}_{\mathbf{a}}\gamma_{\mathbf{ab}}\tilde{\psi}_{\mathbf{b}}\right]$$

- Introduce infrared (IR) regulator $R_k^{\mathbf{ab}}(\vec{p})$ to LGW functional:

$$F[\psi] \to F[\psi] + \frac{1}{2}\int \psi_{\mathbf{a}}R_k^{\mathbf{ab}}\psi_{\mathbf{b}}$$
$$S[\psi,\tilde{\psi}] \to S[\psi,\tilde{\psi}] + \Delta S_k[\psi,\tilde{\psi}]$$
$$\Delta S_k[\psi,\tilde{\psi}] = -\int \tilde{\psi}_{\mathbf{a}}(\gamma_{\mathbf{ab}} - g f_{\mathbf{abc}}\psi_{\mathbf{c}})R_k^{\mathbf{bd}}\psi_{\mathbf{d}}$$



[Figure by H. Gies]

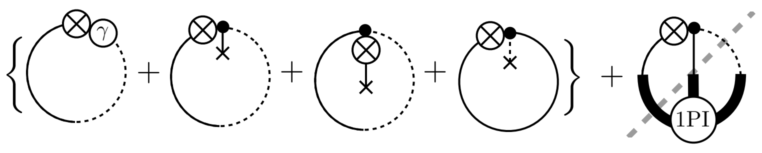
- Connected correlation functions:

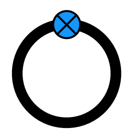
$$W_k[j,\tilde{j}] = -i\ln Z_k[j,\tilde{j}]$$

- 1-particle irreducible (1PI) correlation functions:

$$\Gamma_k[\psi,\tilde{\psi}] = W_k[j,\tilde{j}] - \Delta S_k[\psi,\tilde{\psi}] - \int \tilde{j}_{\mathbf{a}}\psi_{\mathbf{a}} - \int j_{\mathbf{a}}\tilde{\psi}_{\mathbf{a}}$$

- Flow equation:
  [JR, Ye, Schlichting, von Smekal, JHEP **01** (2025) 118]

$\partial_k\Gamma_k = i\{$  $\}$ +

⇝ write: $\partial_k\Gamma_k \approx \frac{i}{2}$  approximate 1-loop form

- Truncation: Promote coupling constants to depend on k
- Find: $k\partial_k\mu_k = 0$ (exact)

Results for critical dynamics

- FRG scale k puts limit on correlation length: $\xi_k \sim 1/k$
- Characteristic frequencies at critical wavenumber $p \sim k$

$$\omega_{\phi,k} \sim -i\Gamma_k^\phi k^2 \qquad \omega_{n,k} \sim -i\gamma_k k^2/\chi_n \qquad \omega_{\varepsilon,k} \sim -i\mu k^2/C_{p,k}$$

- ε:

$$C_{p,k} \sim k^{-\alpha_t/\nu_t} \qquad \mu\ \text{finite} \qquad \Longrightarrow\ \omega_{\varepsilon,k} \sim k^{2+\alpha_t/\nu_t} \sim k^3$$

similar to Model H:
$z \approx 3$

- ϕ_a and n_{ab}: introduce dimensionless couplings

$$w_n \equiv \chi_n\frac{\Gamma_k^\phi}{\gamma_k}\,, \qquad f \equiv \frac{dK_d g^2 T k^{d-4}}{\Gamma_k^\phi\gamma_k}$$

- FRG flow diagram:

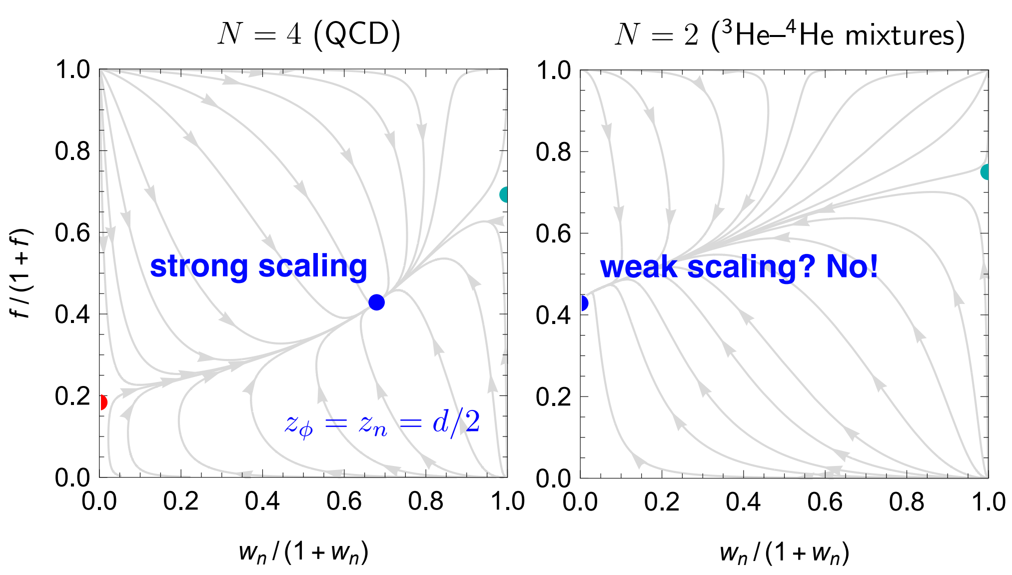
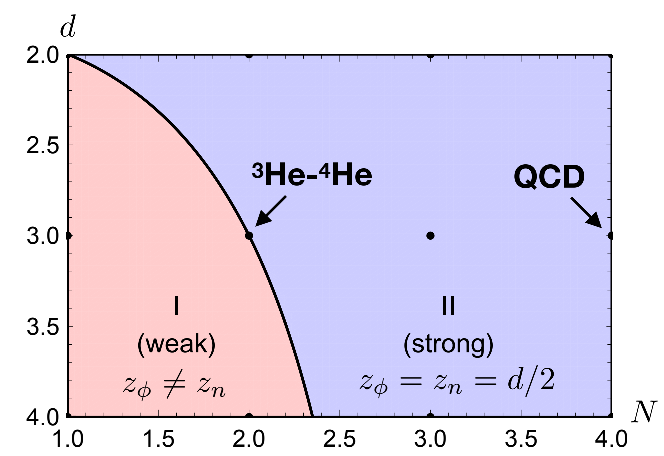


Figure: FRG flow in compactified (w_n,f) plane in 3d at Gaussian fixed point for N = 4 (left) and N = 2 (right).

- Reason:



Take-home message

- Dynamics near QCD tricritical point similar to the one of ^{3}He-^{4}He
- Further exploration of corresponding analogies might be insightful, e.g., for the ongoing search for the QCD critical point

arXiv:2608.11065