Equation of state of hot and dense QCD using the complex Langevin equation

Michael Mandl

with Dénes Sexty and Daniel Unterhuber

University of Graz, Austria

(based on arXiv:2604.19649 [hep-lat])

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FWF Austrian Science Fund

UNI GRAZ

Motivation

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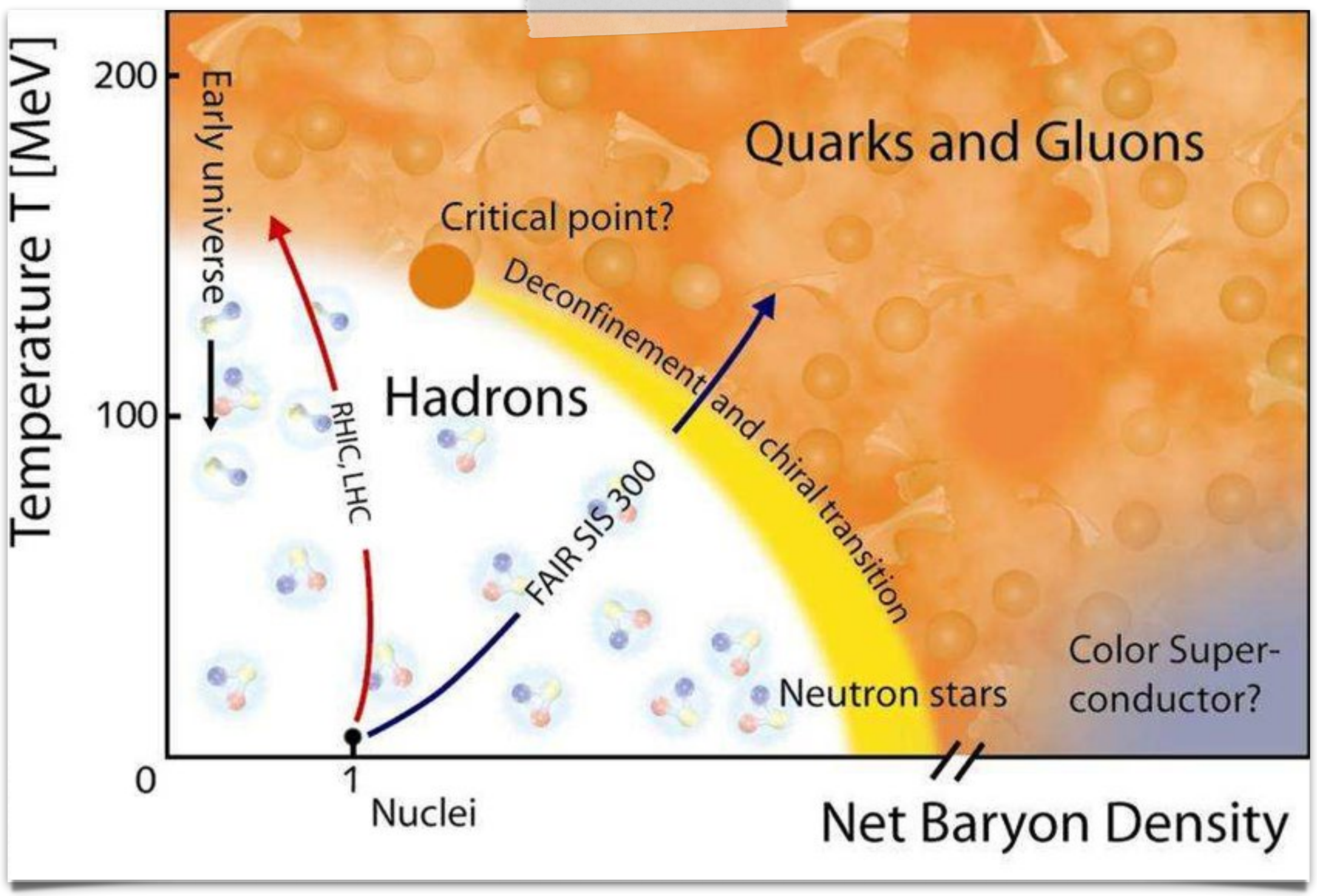
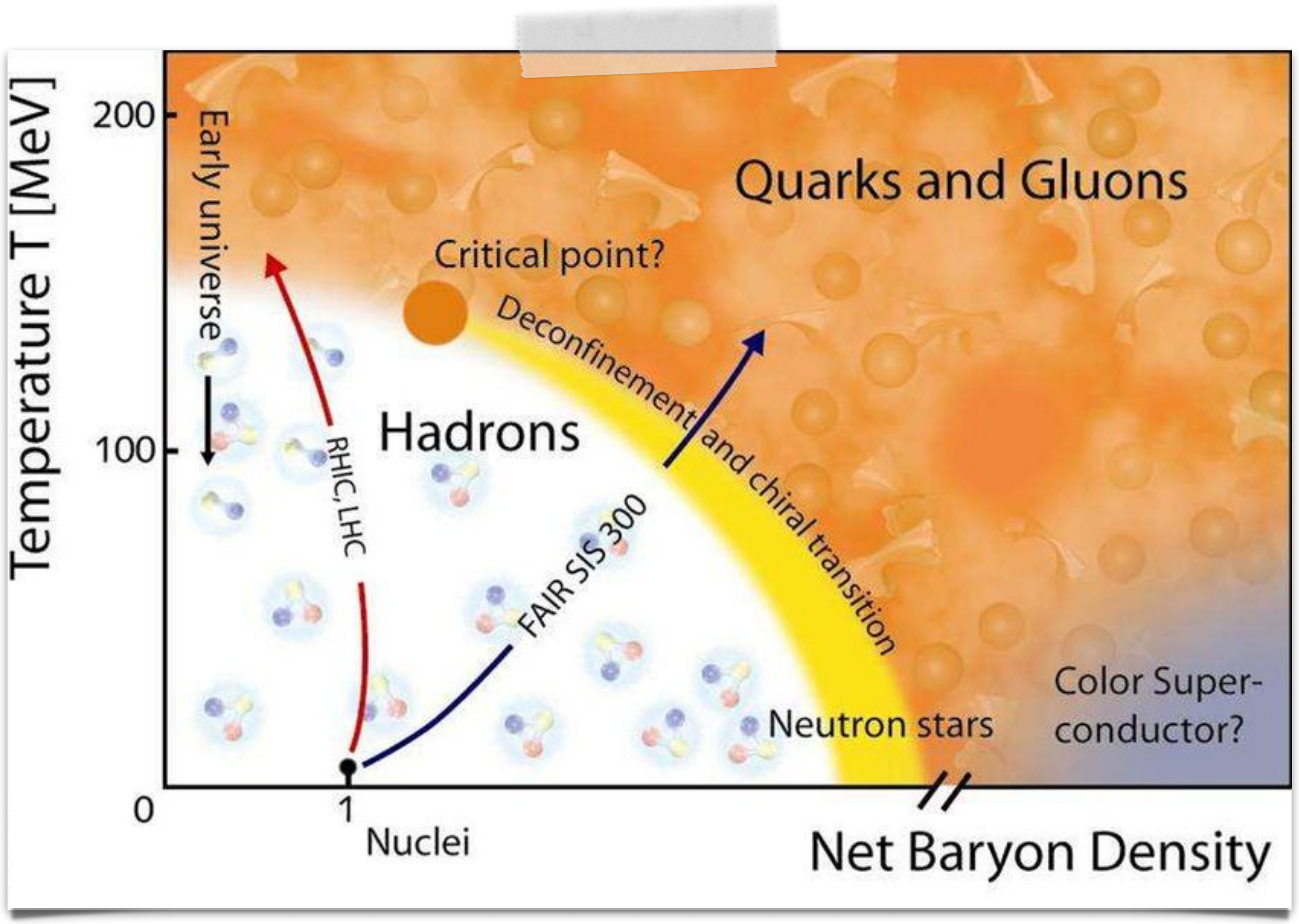


Temperature T [MeV]
200
100
0
1
Nuclei
Net Baryon Density
Early universe
RHIC, LHC
FAIR SIS 300
Hadrons
Quarks and Gluons
Critical point?
Deconfinement and chiral transition
Neutron stars
Color Super-
conductor?

Motivation

- QCD at non-zero density:



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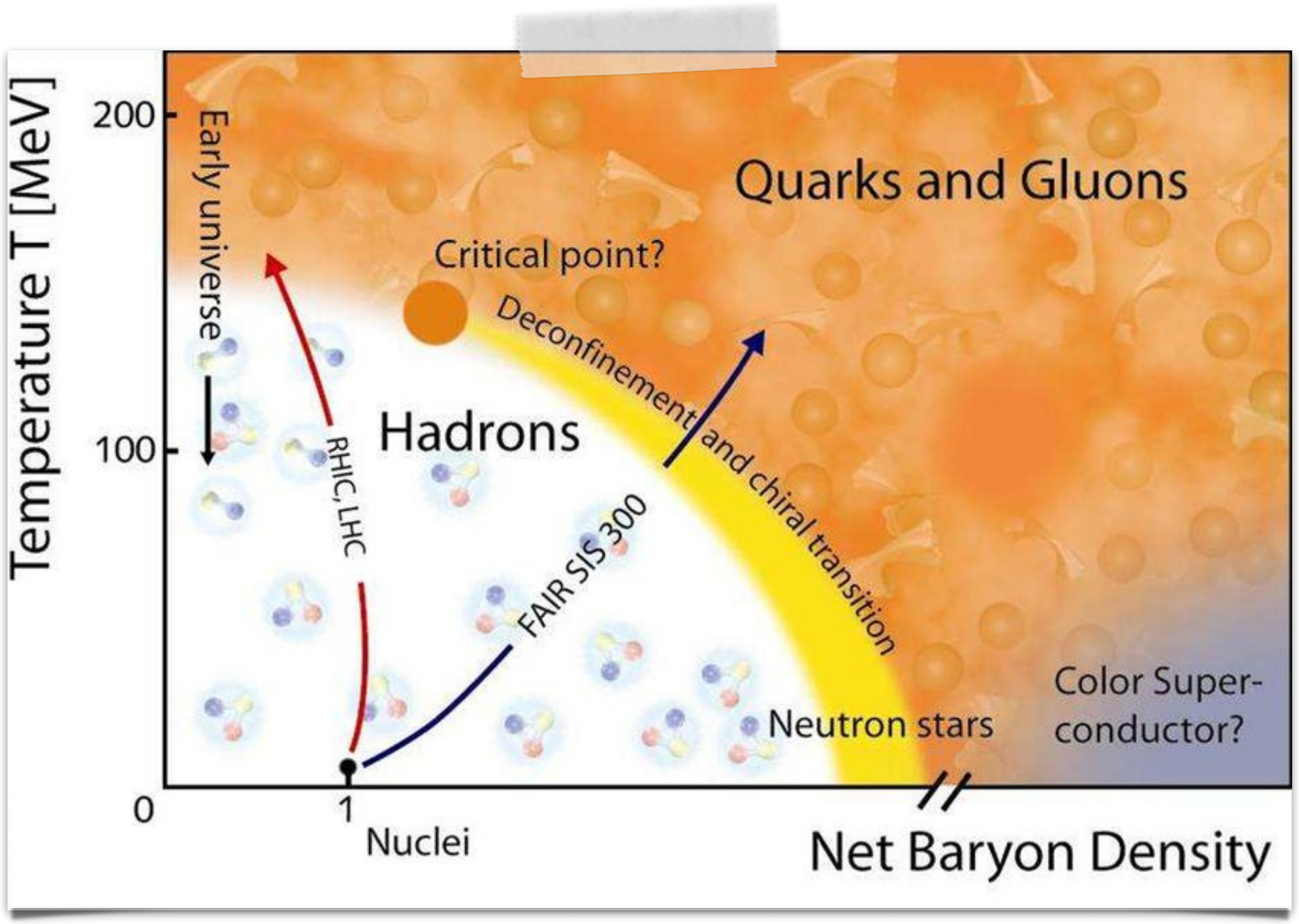

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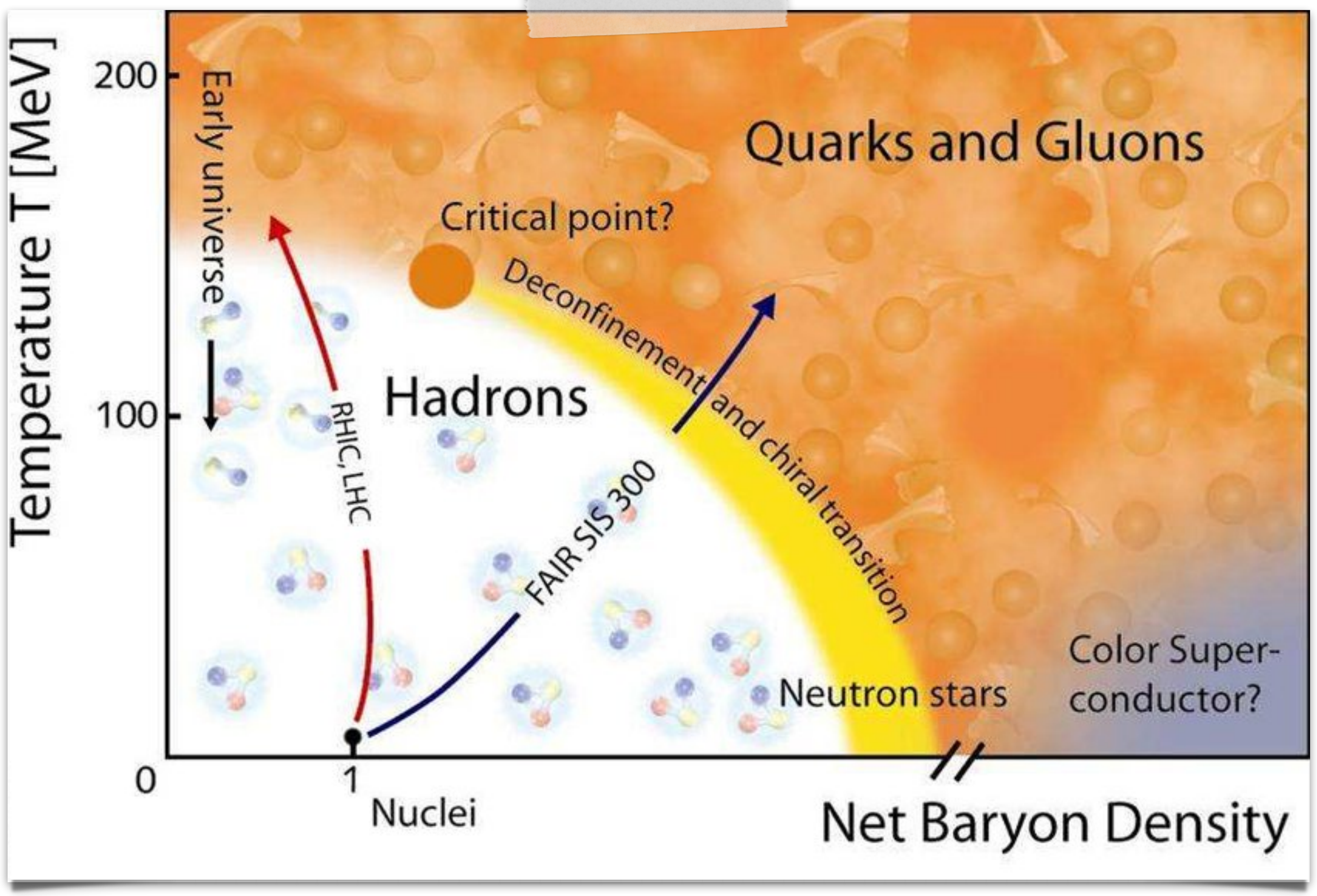

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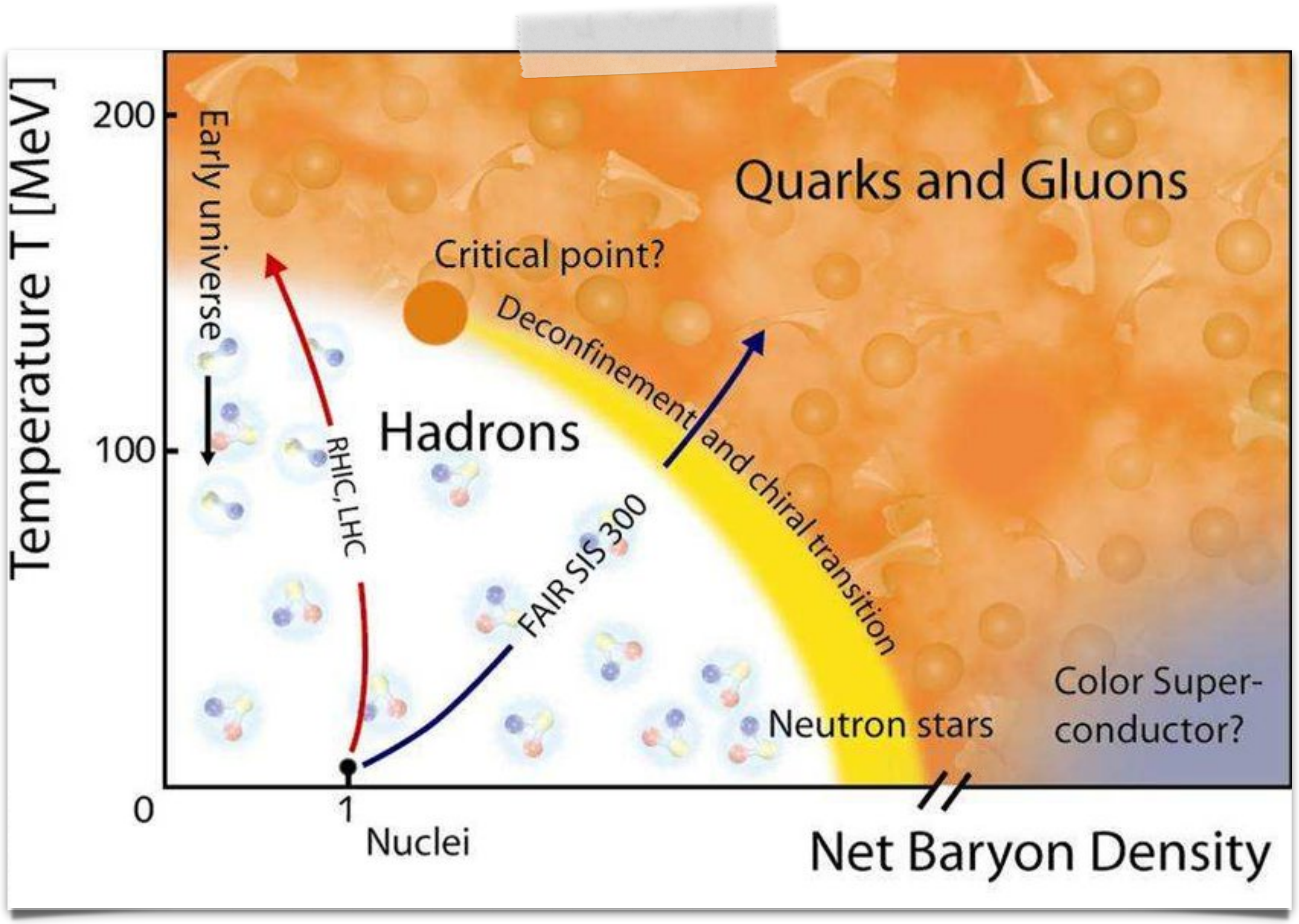

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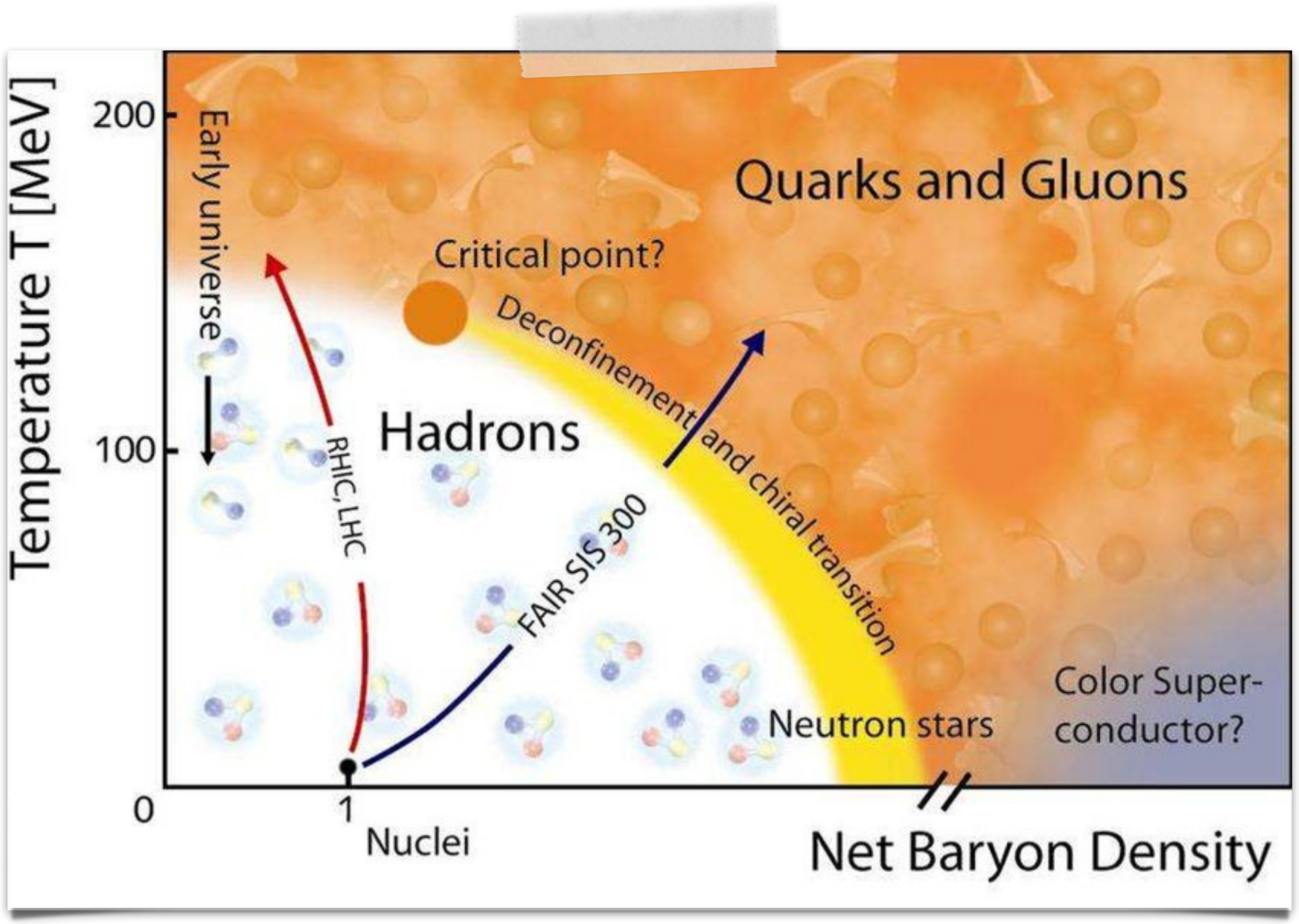
  - $e^{-S_E} \notin \mathbb{R}_0^+ \implies$ importance sampling fails



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- QCD at non-zero density:
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  - Heavy-ion collisions
- Lattice
  - $e^{-S_E} \notin$ … tails

Possible solution: Complex Langevin

Complex Langevin dynamics

Complex Langevin dynamics

Toy example

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- One-variable toy model $S(x) = \dfrac{\lambda}{4}x^4,\ \lambda \in \mathbb{C},\ x \in \mathbb{R}$

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Complex Langevin equation

$$\frac{dz}{d\tau} = -\frac{\partial S(z)}{\partial z} + \eta(\tau)$$

[Parisi '83]
[Klauder '83]

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$$S[U] = S_g[U] - \ln\det D[U]$$

Simulation details

[**M.M.**, Sexty, Unterhuber '26]

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- First ever complex Langevin simulations with physical pion mass

QCD equation of state

[**M.M.**, Sexty, Unterhuber '26]

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Baryon number density

$$n_B(T, \mu_B) = \frac{T}{V} \frac{\partial}{\partial \mu_B} \ln Z(T, \mu_B)$$

QCD equation of state

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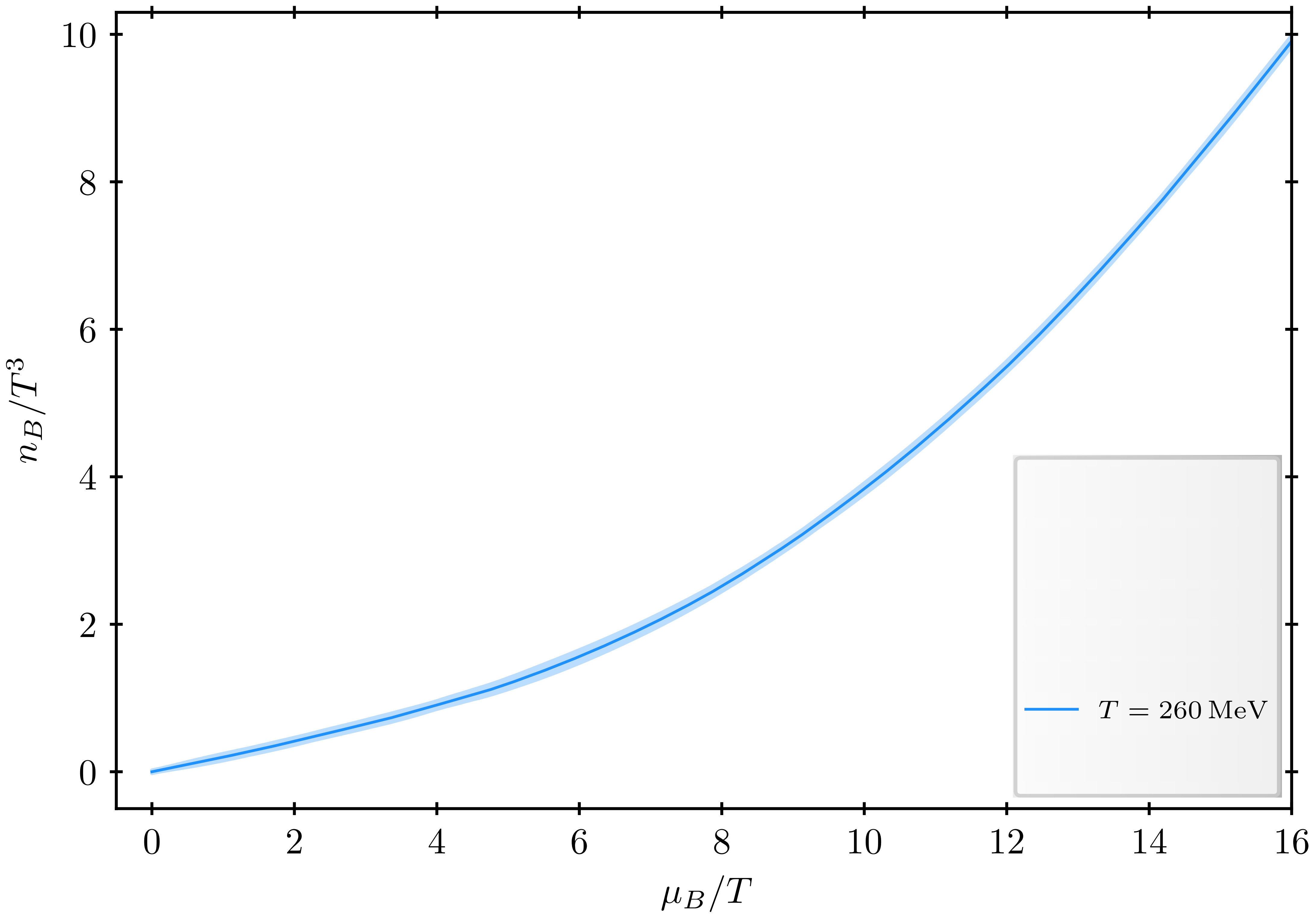

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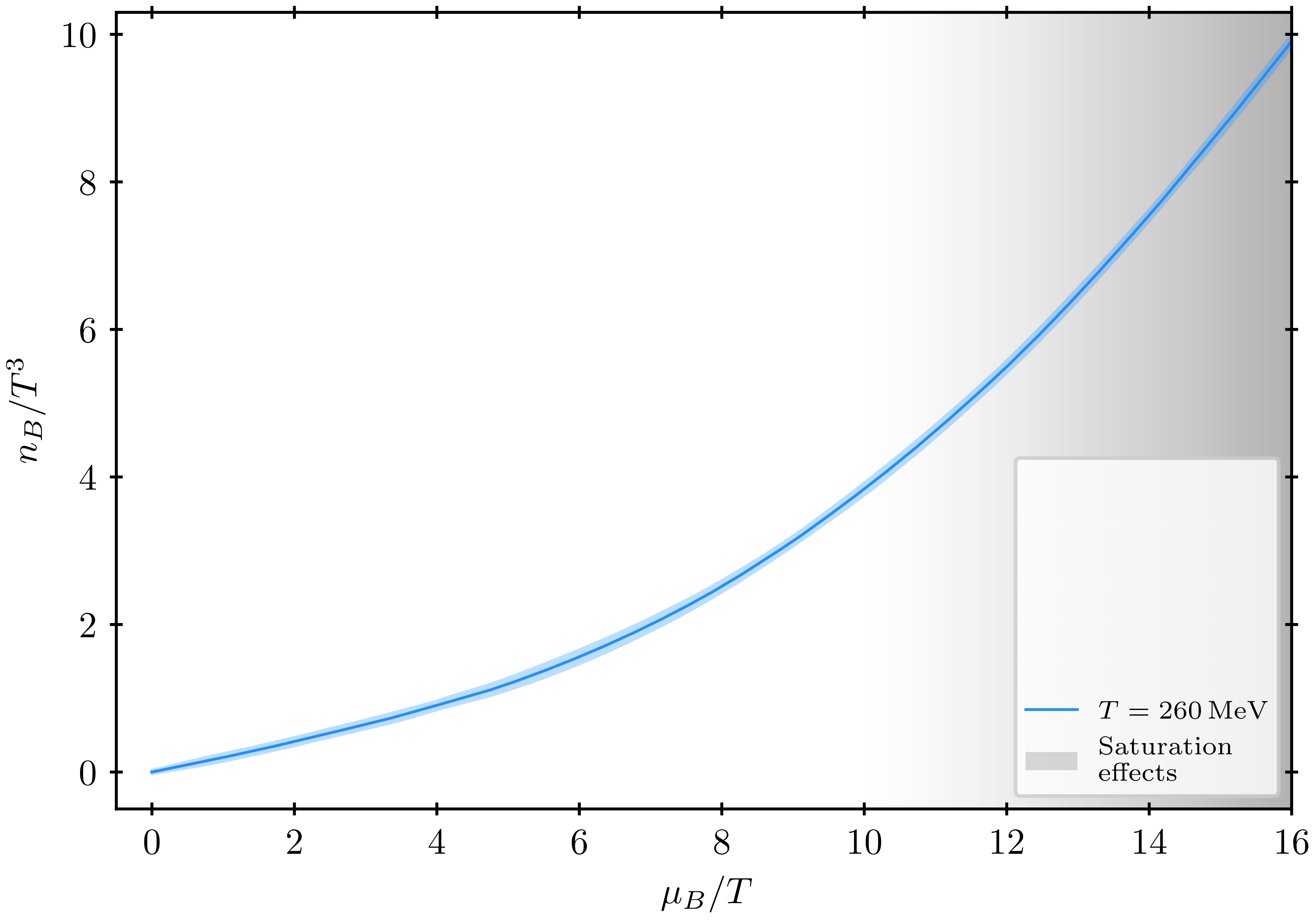

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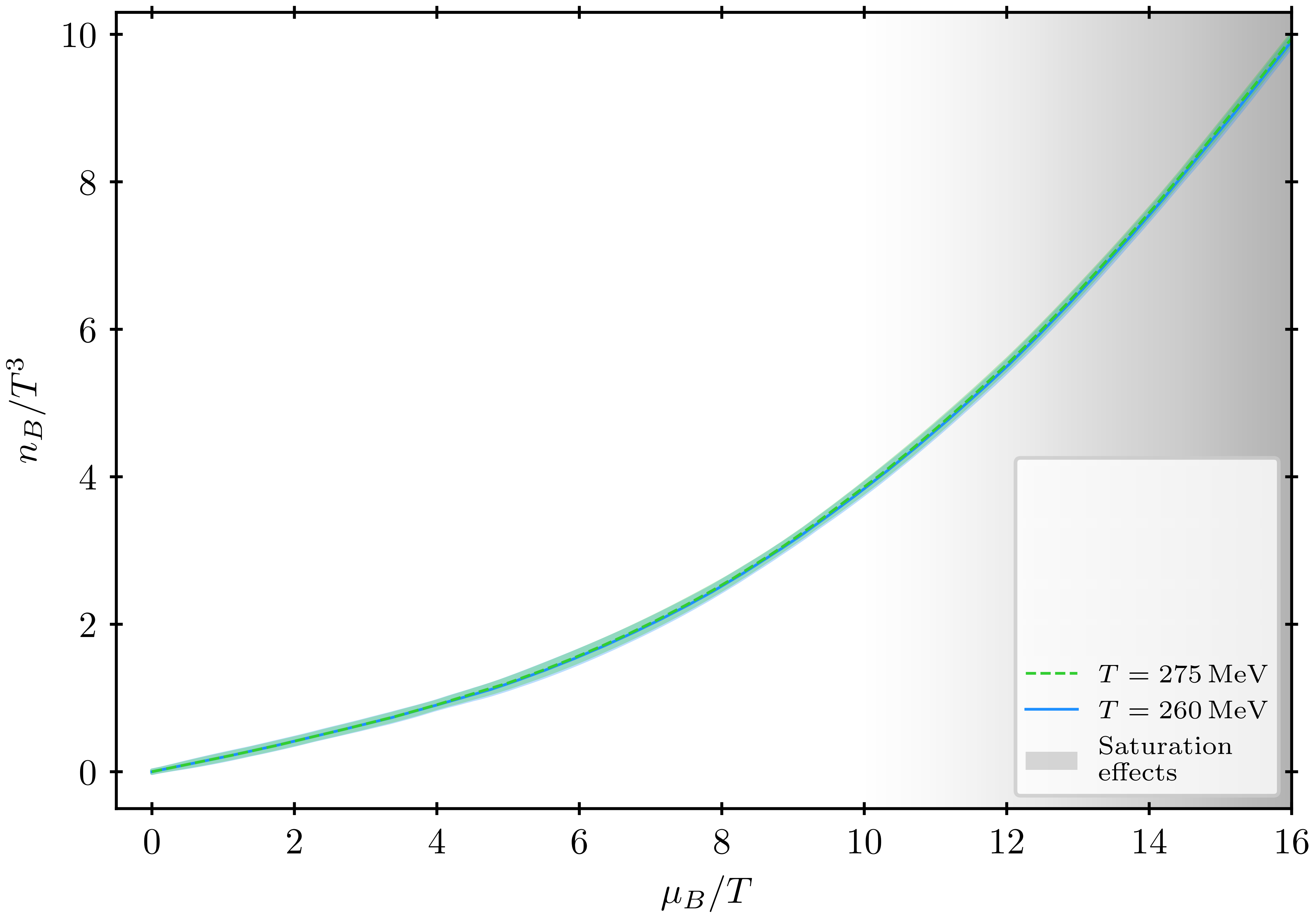

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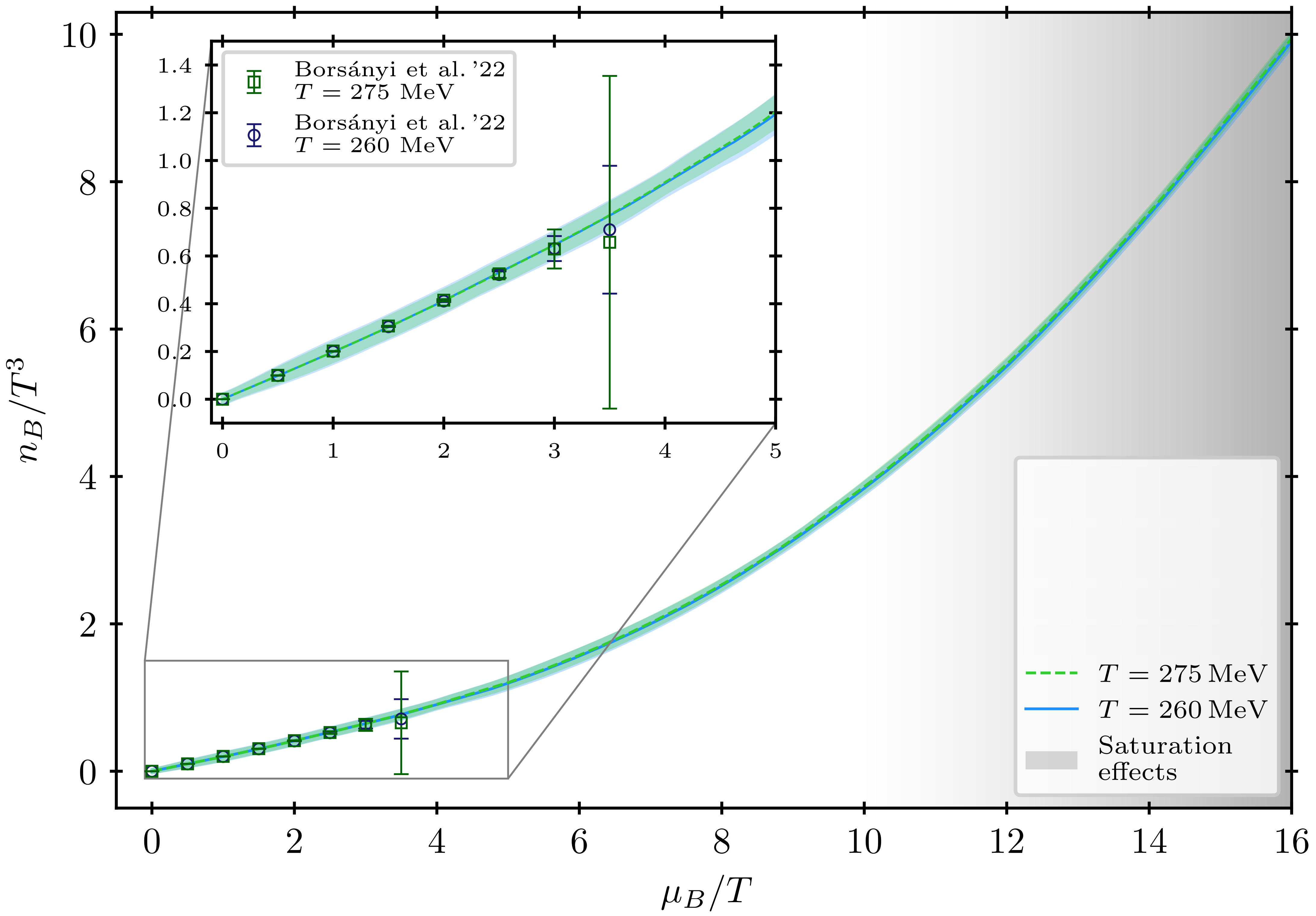

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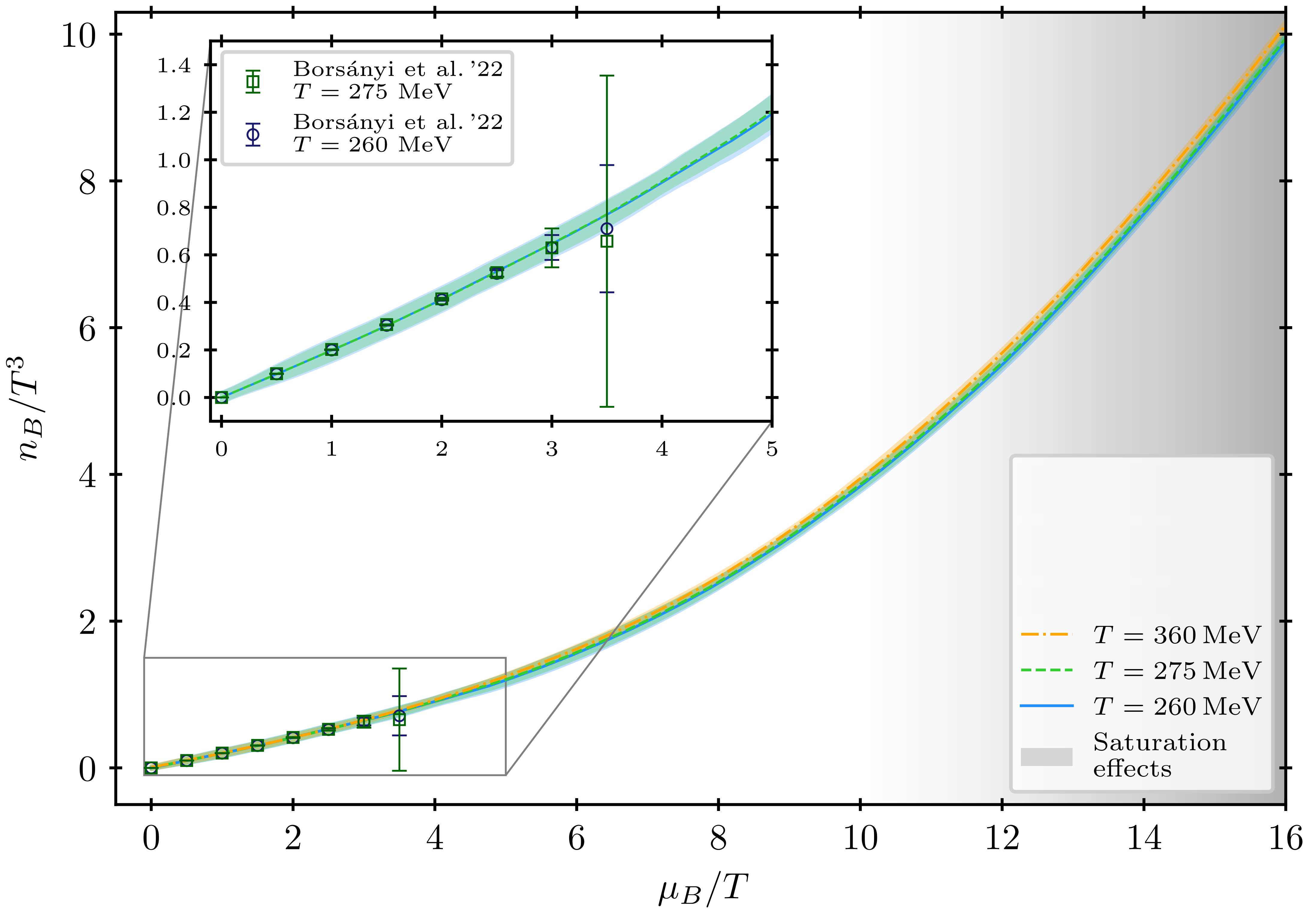

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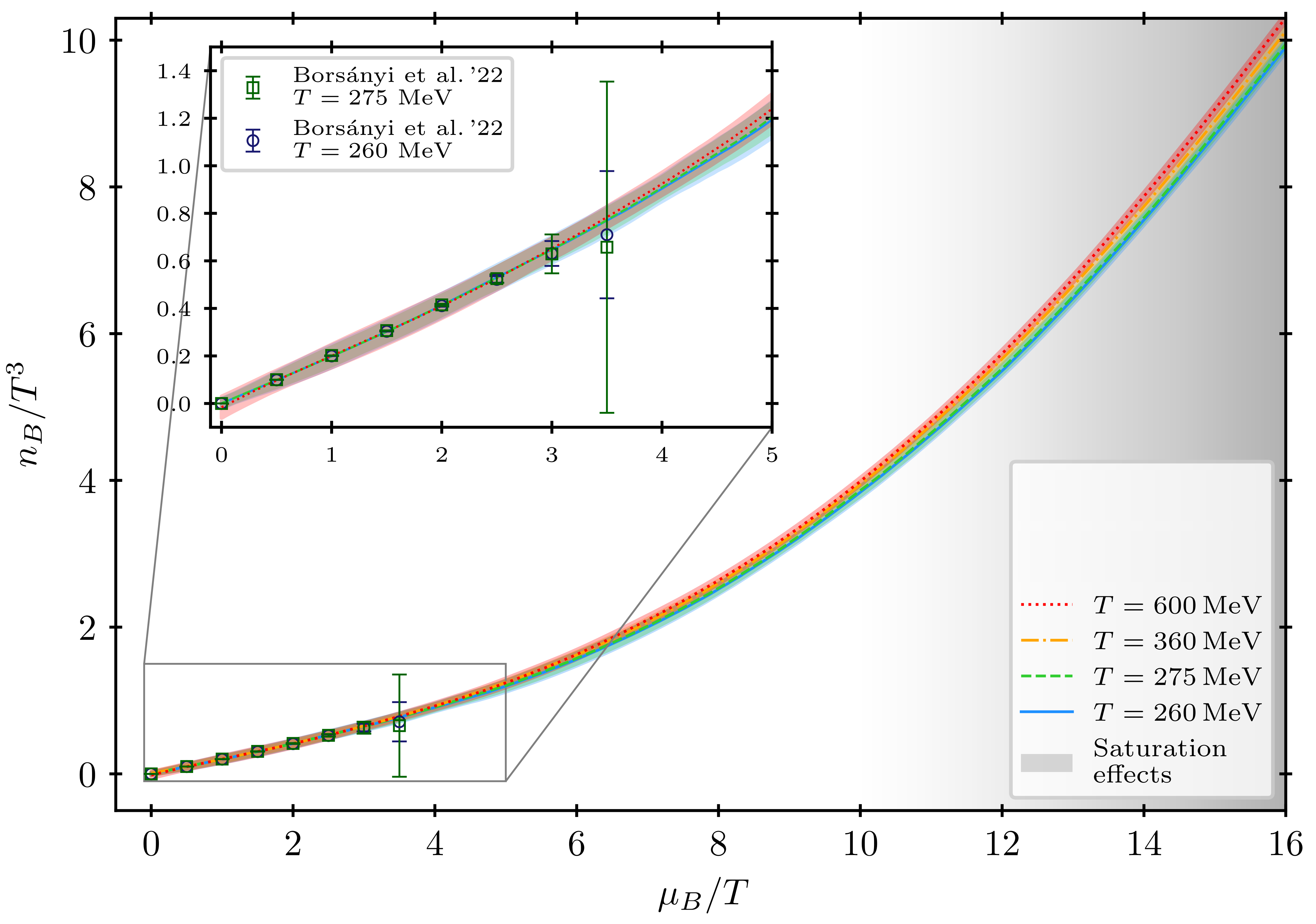

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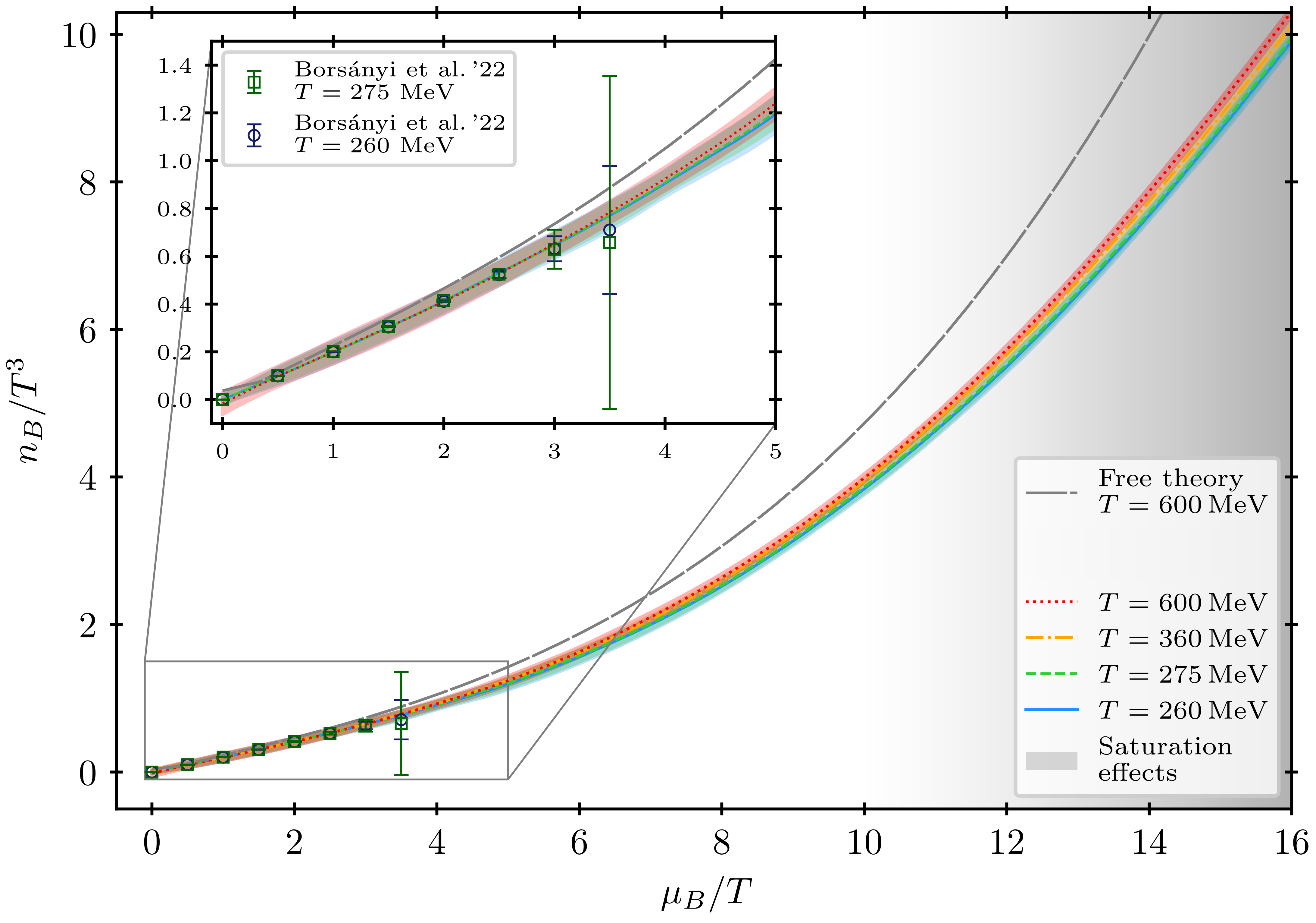

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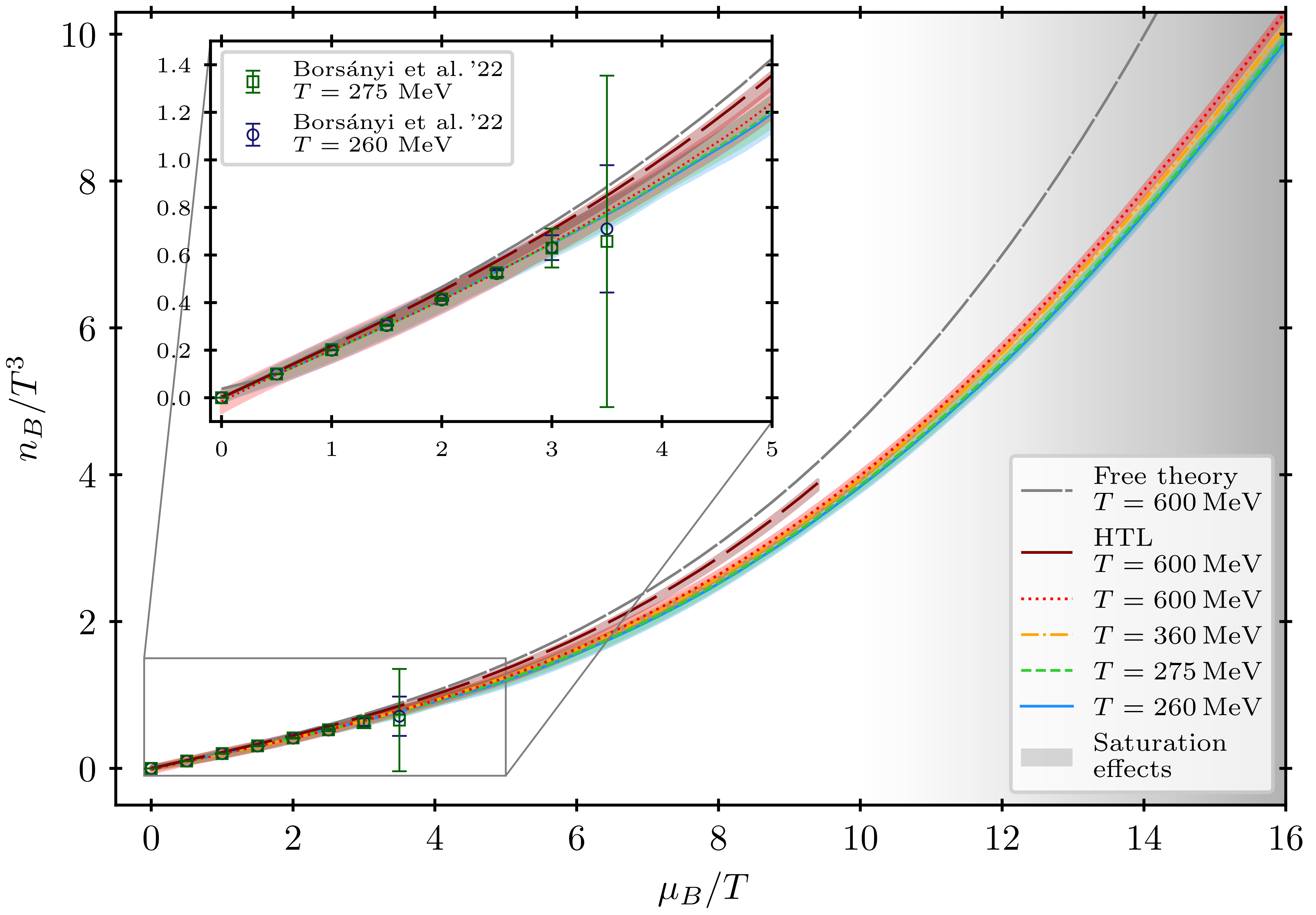

HTL: [Haque et al. ’14]

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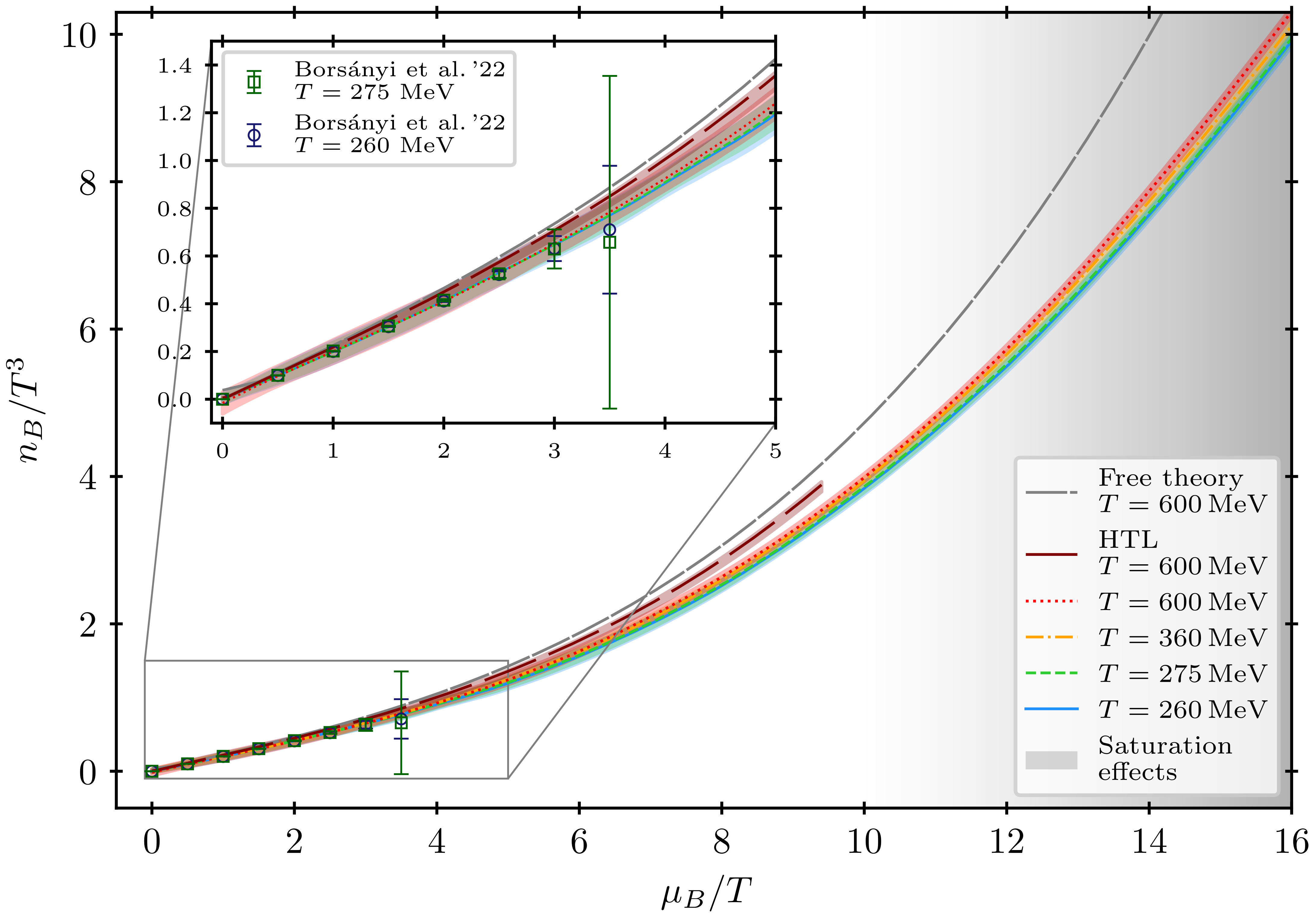

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Pressure difference

$$\Delta p(T, \mu_B) = \int_0^{\mu_B} d\mu_B' \, n_B(T, \mu_B')$$

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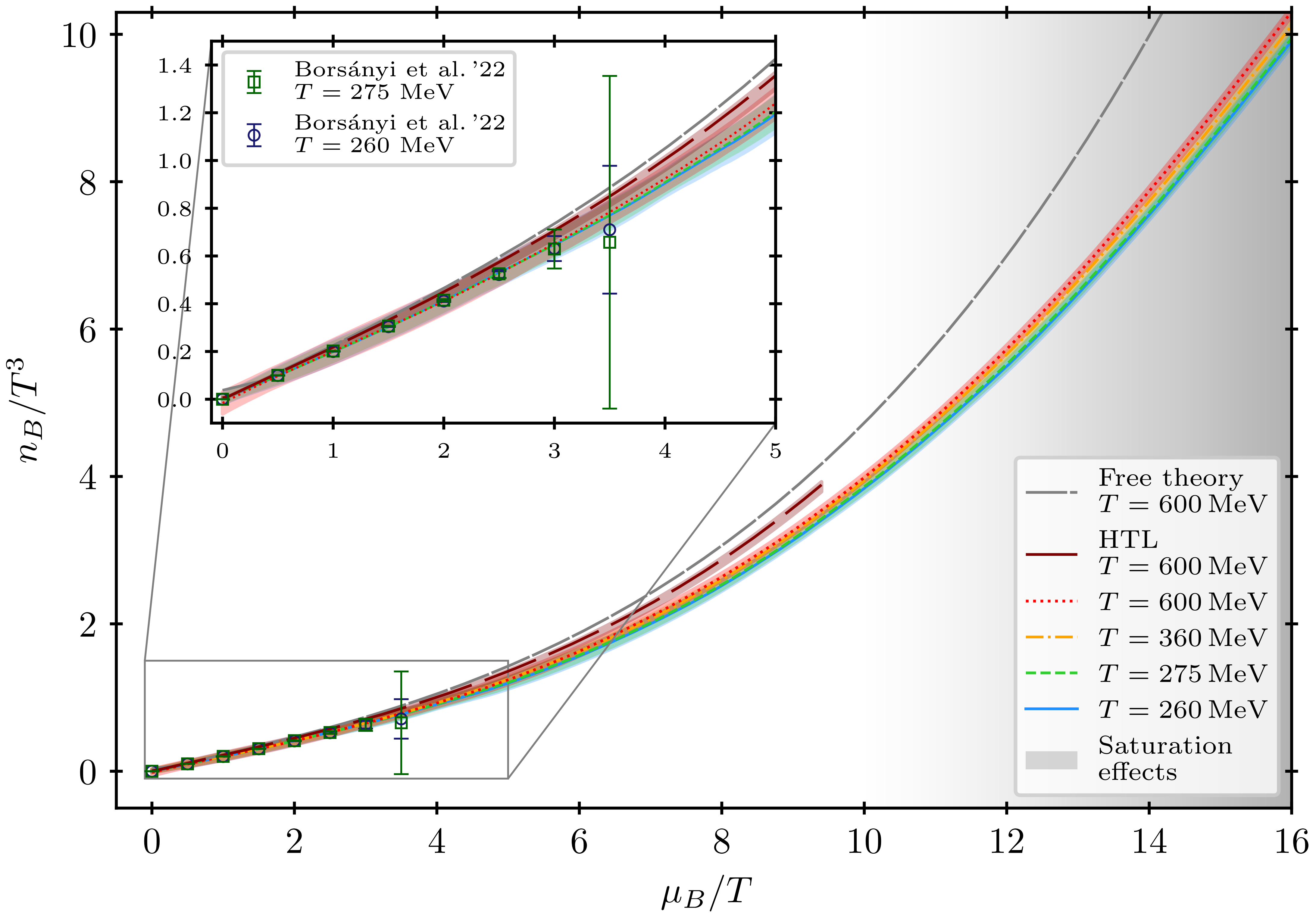

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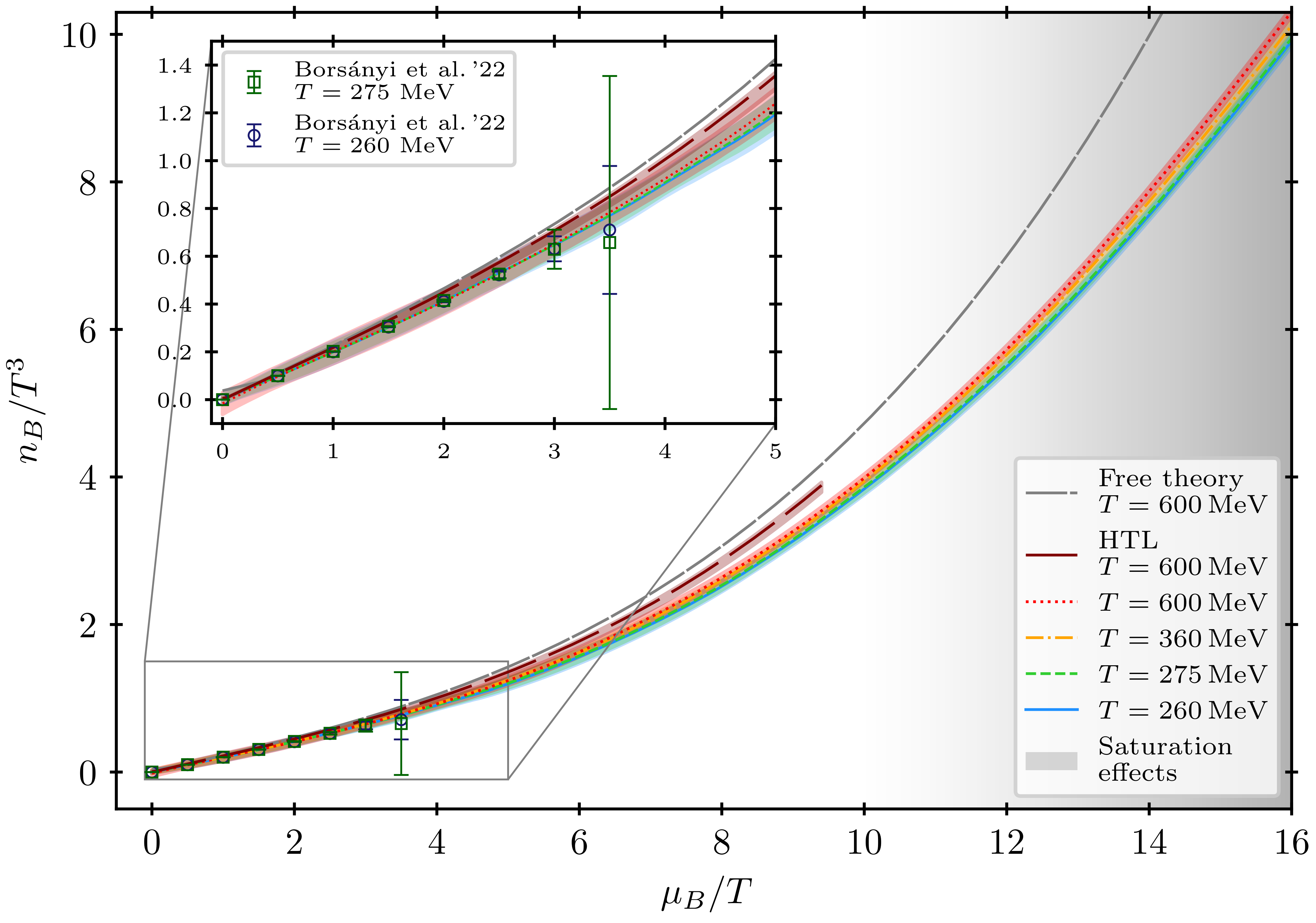

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taken from [Borsanyi et al. '16]

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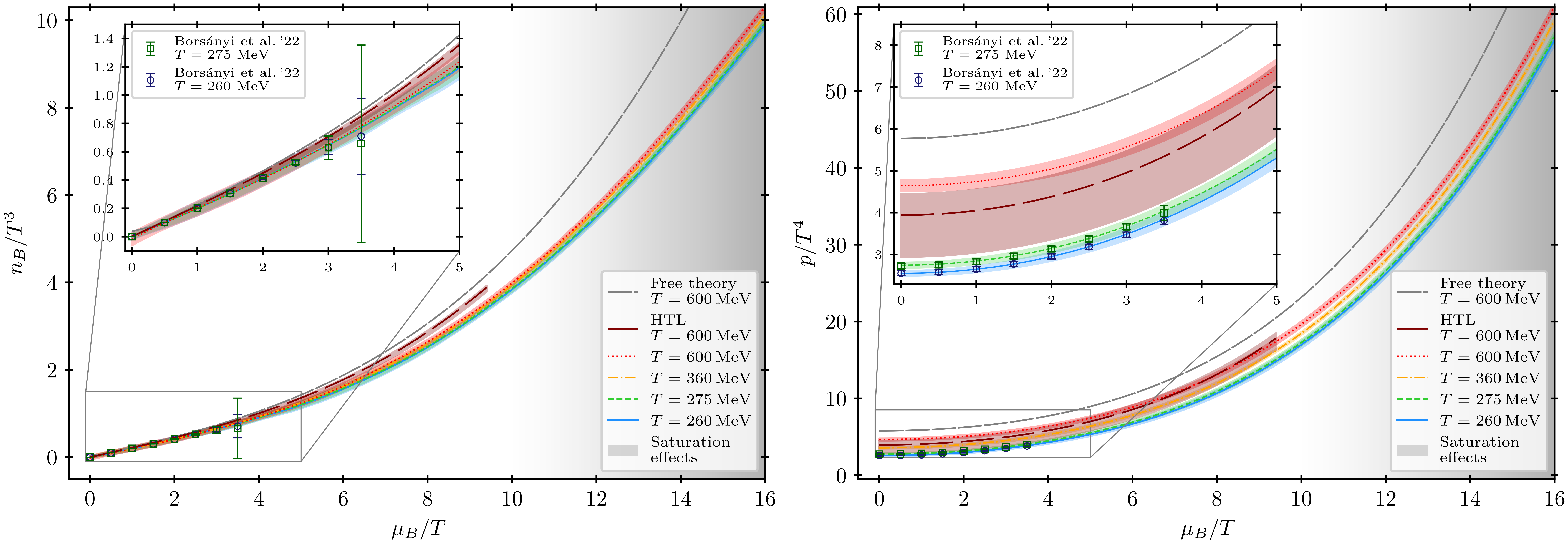

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What is possible with complex Langevin

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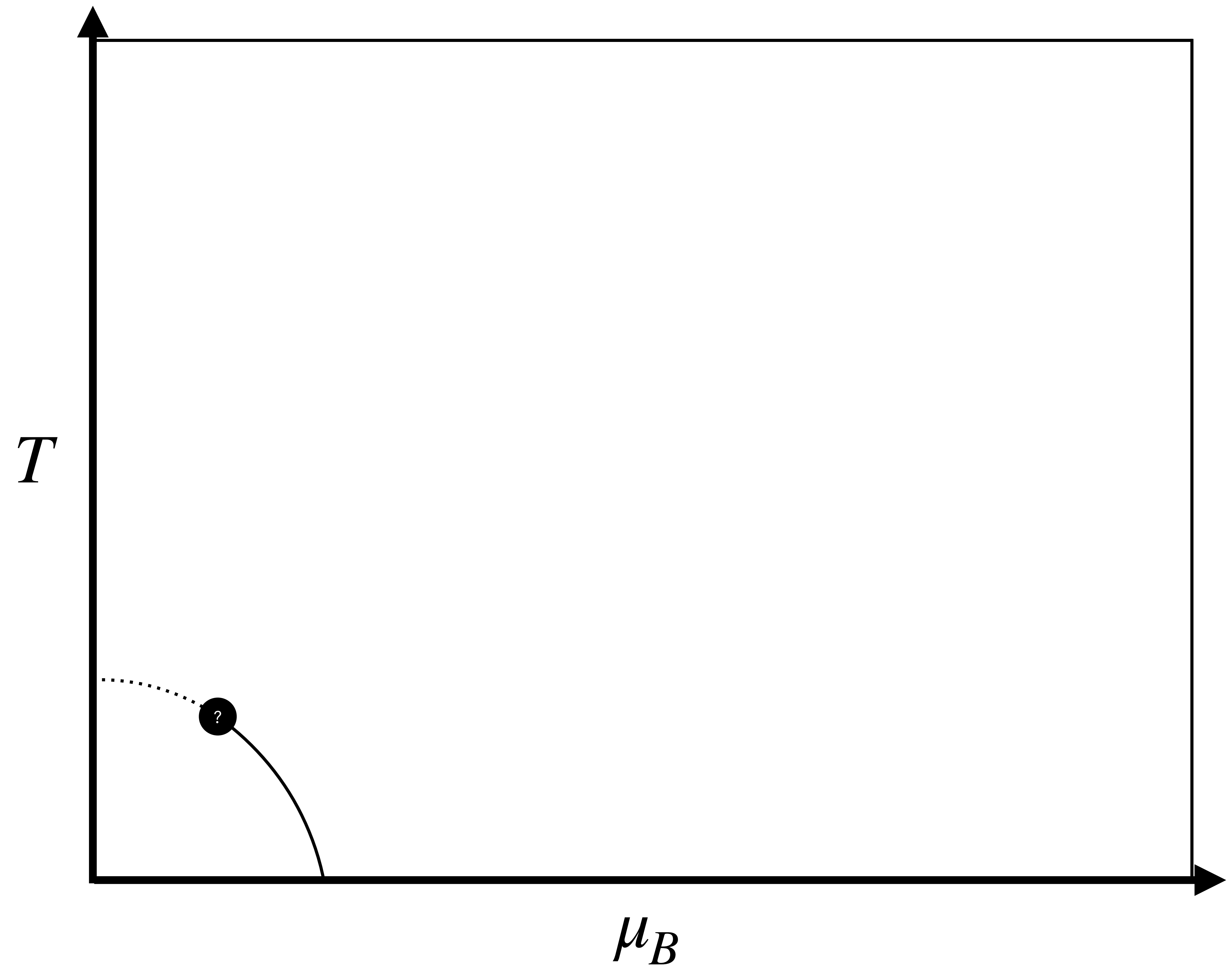

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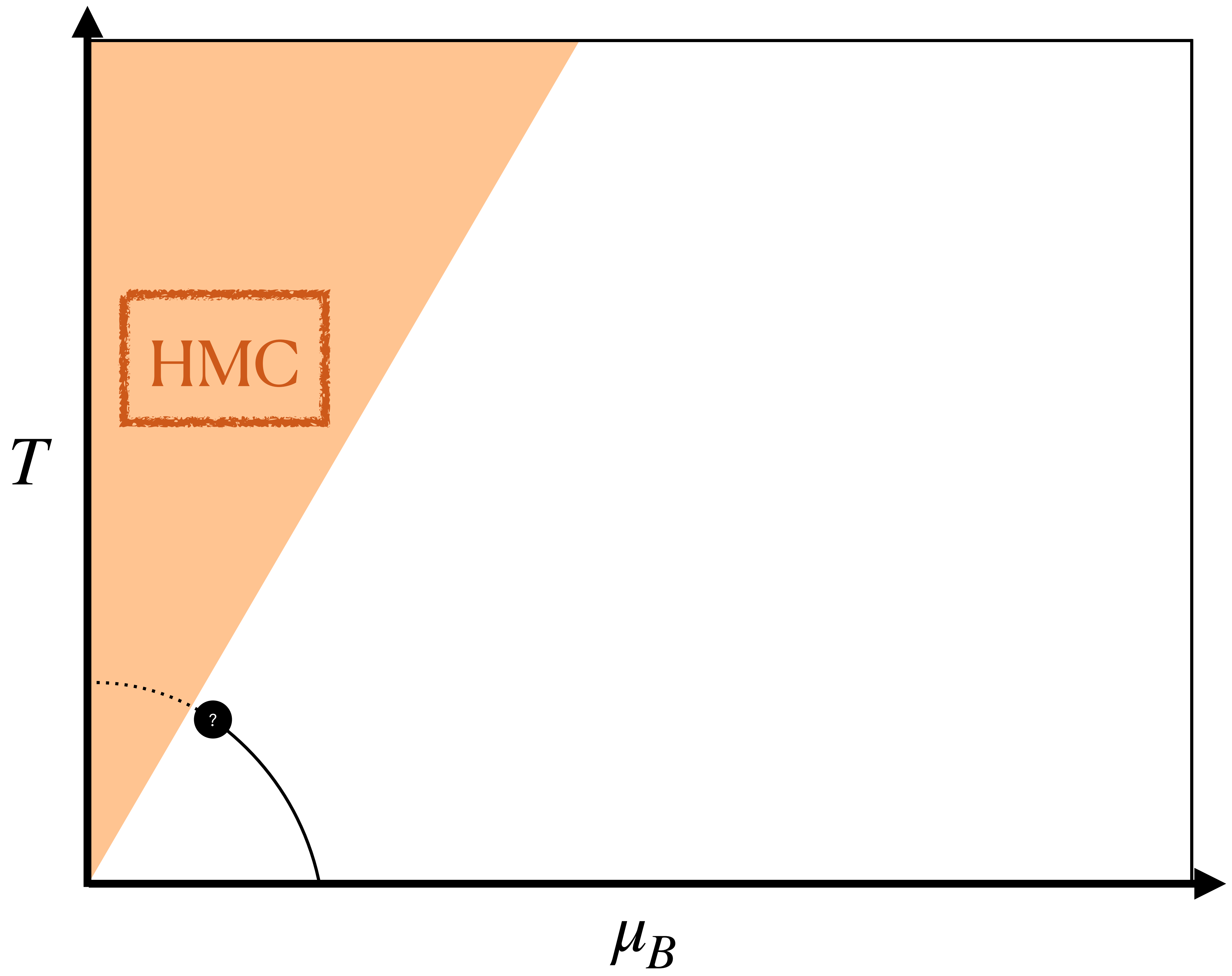

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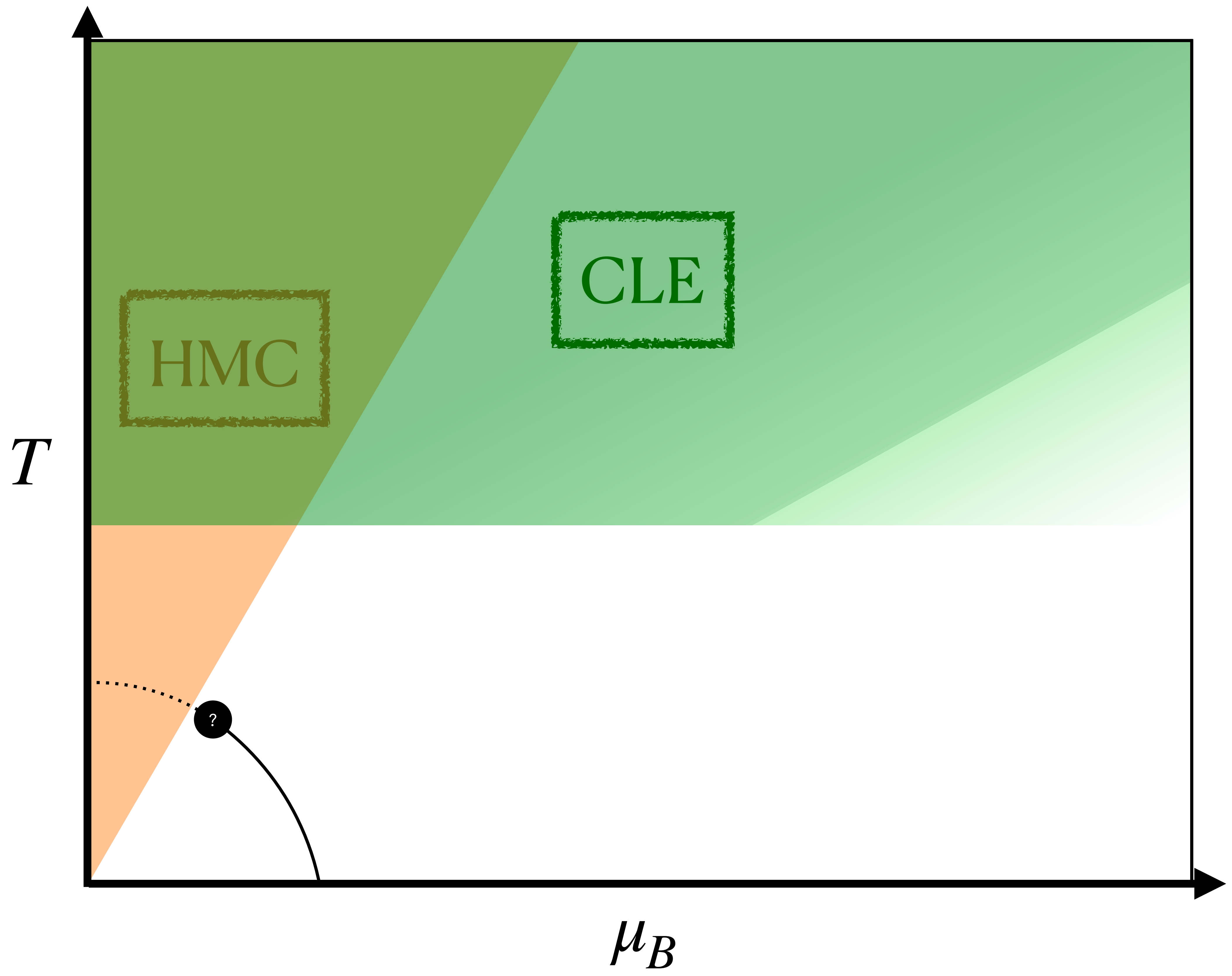

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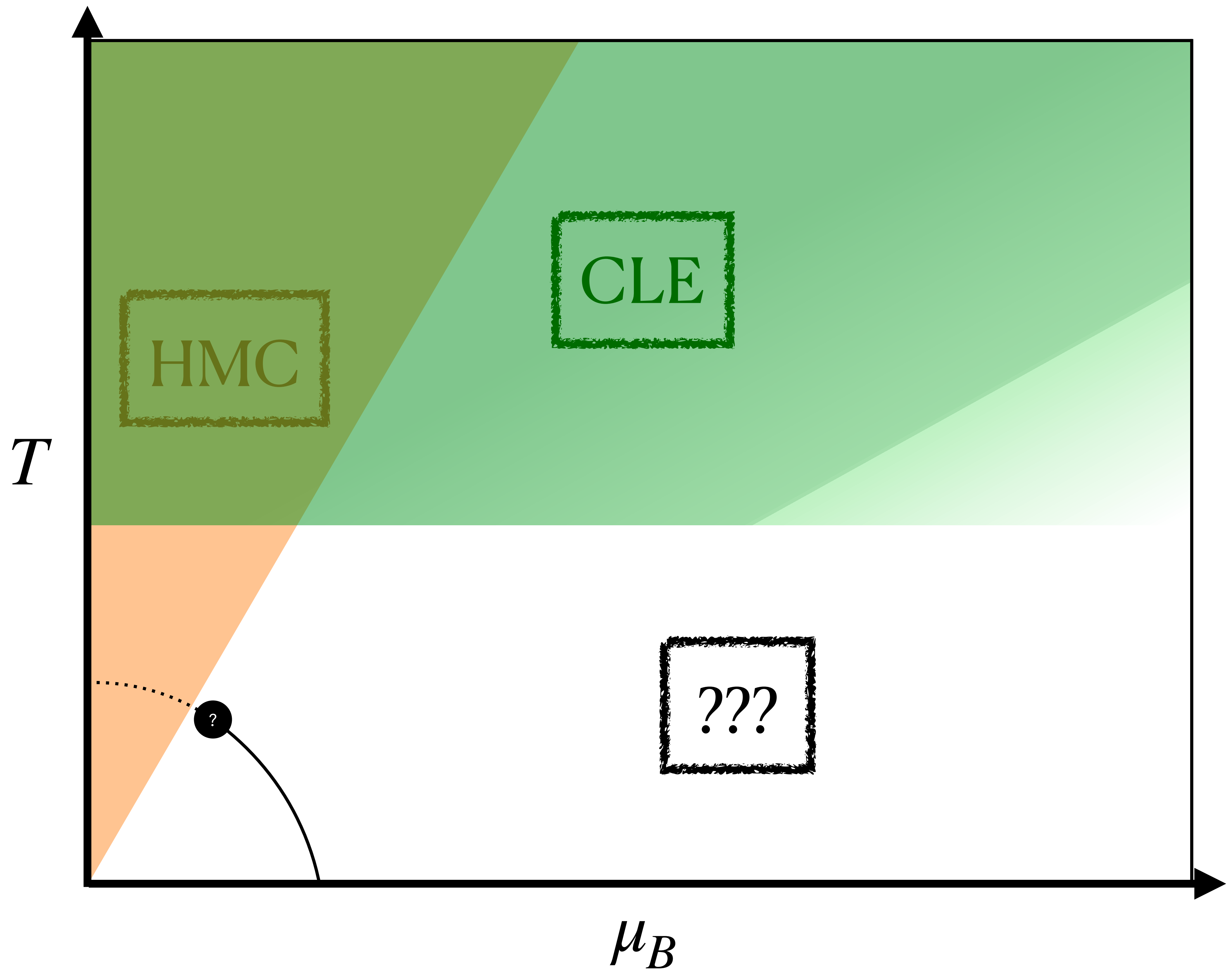

Towards lower temperatures

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Machine-learning kernels

see also [Alvestad, Larsen, Rothkopf ‘23]
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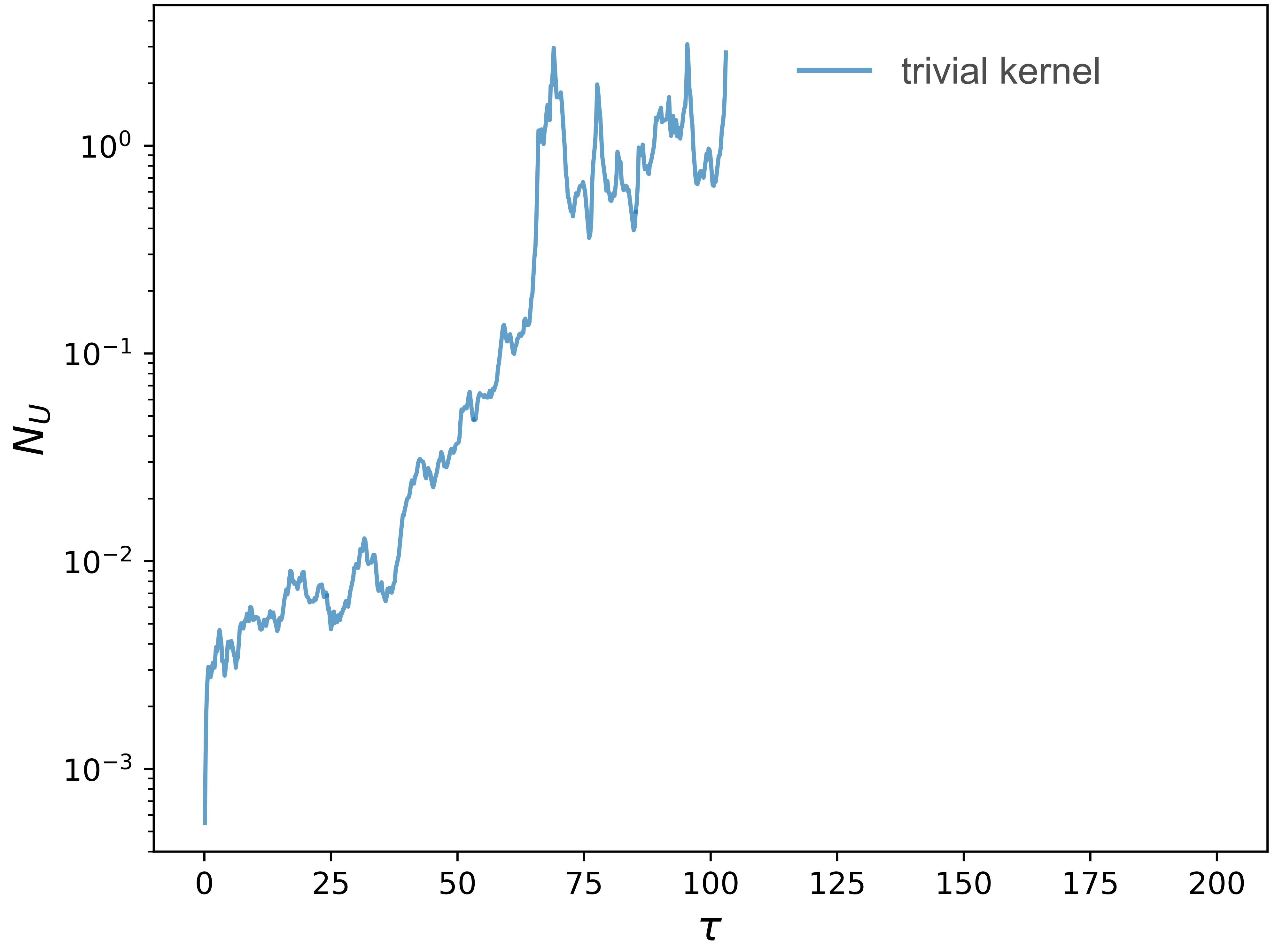

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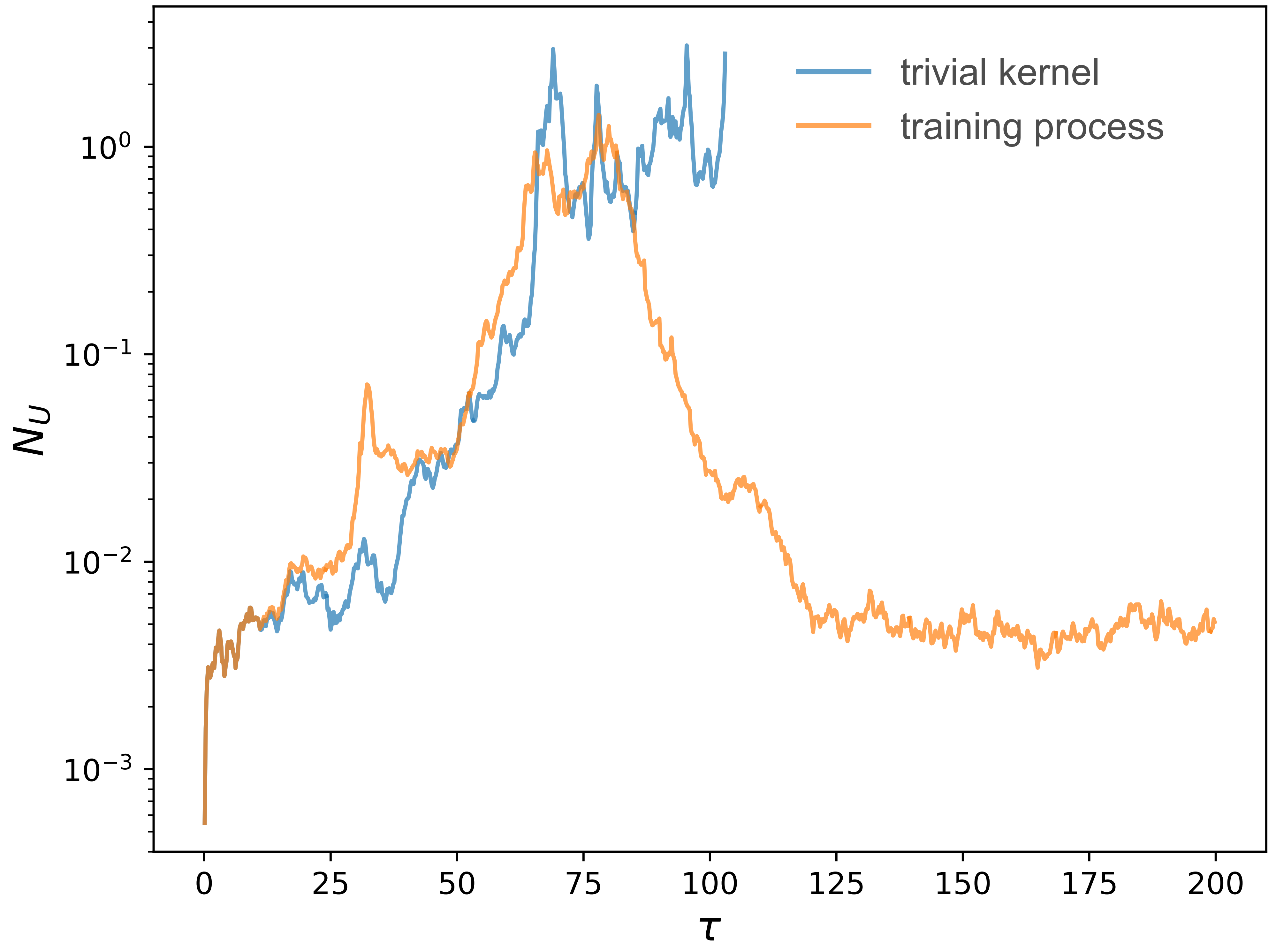

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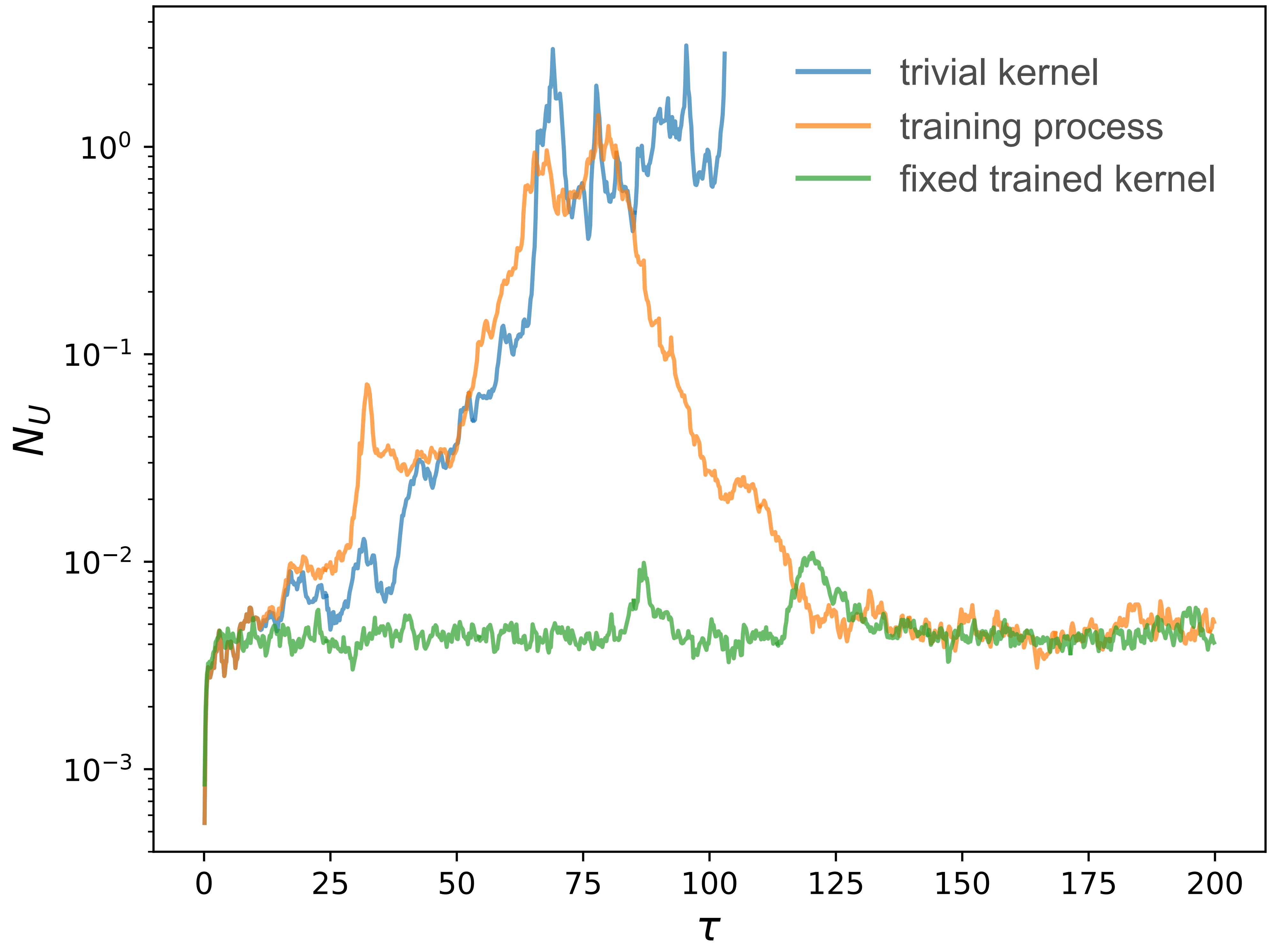

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Summary & Outlook

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Outlook: Lower temperatures (e.g., via kernels, maybe use machine-learning).

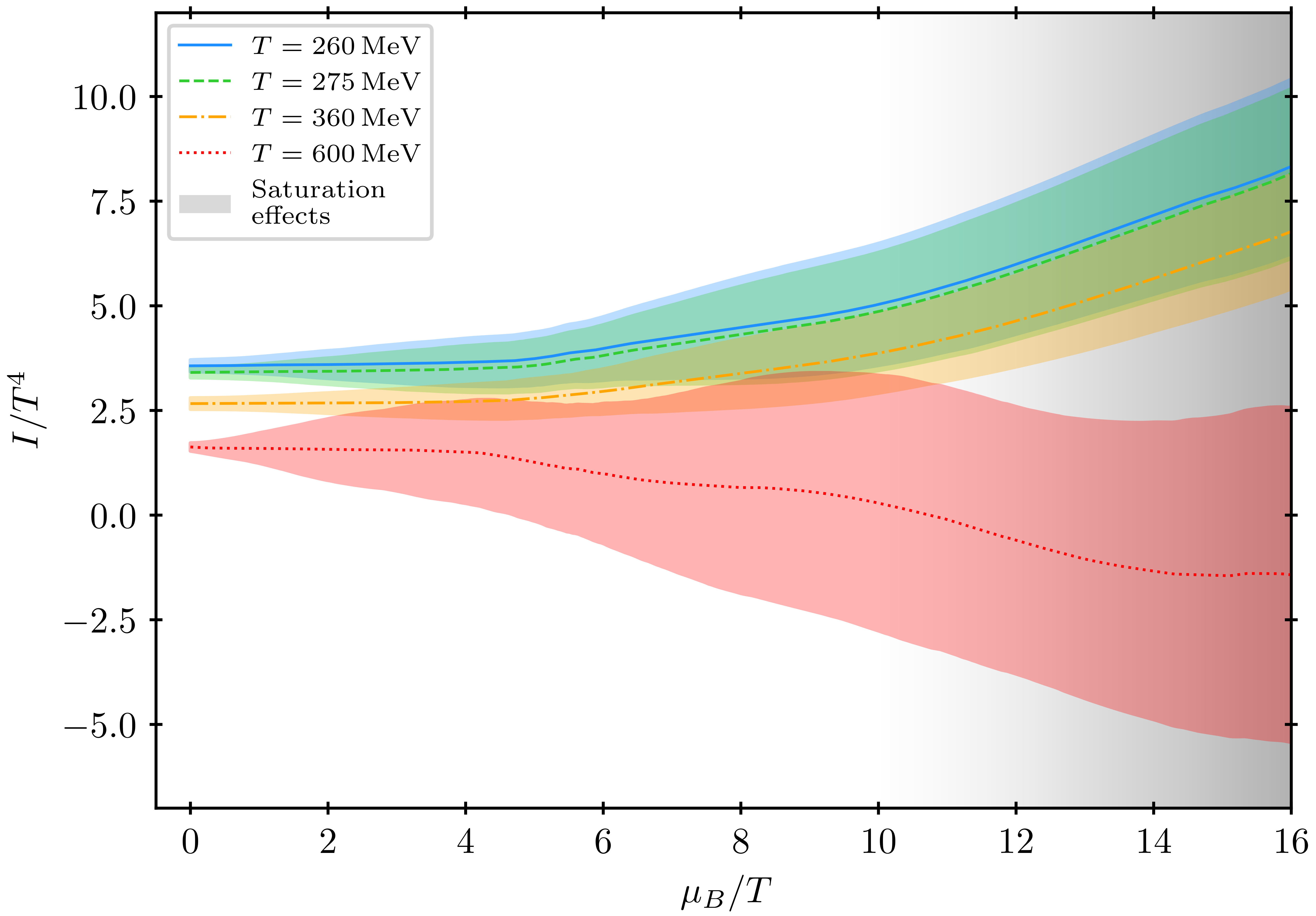
Backup

In other news …

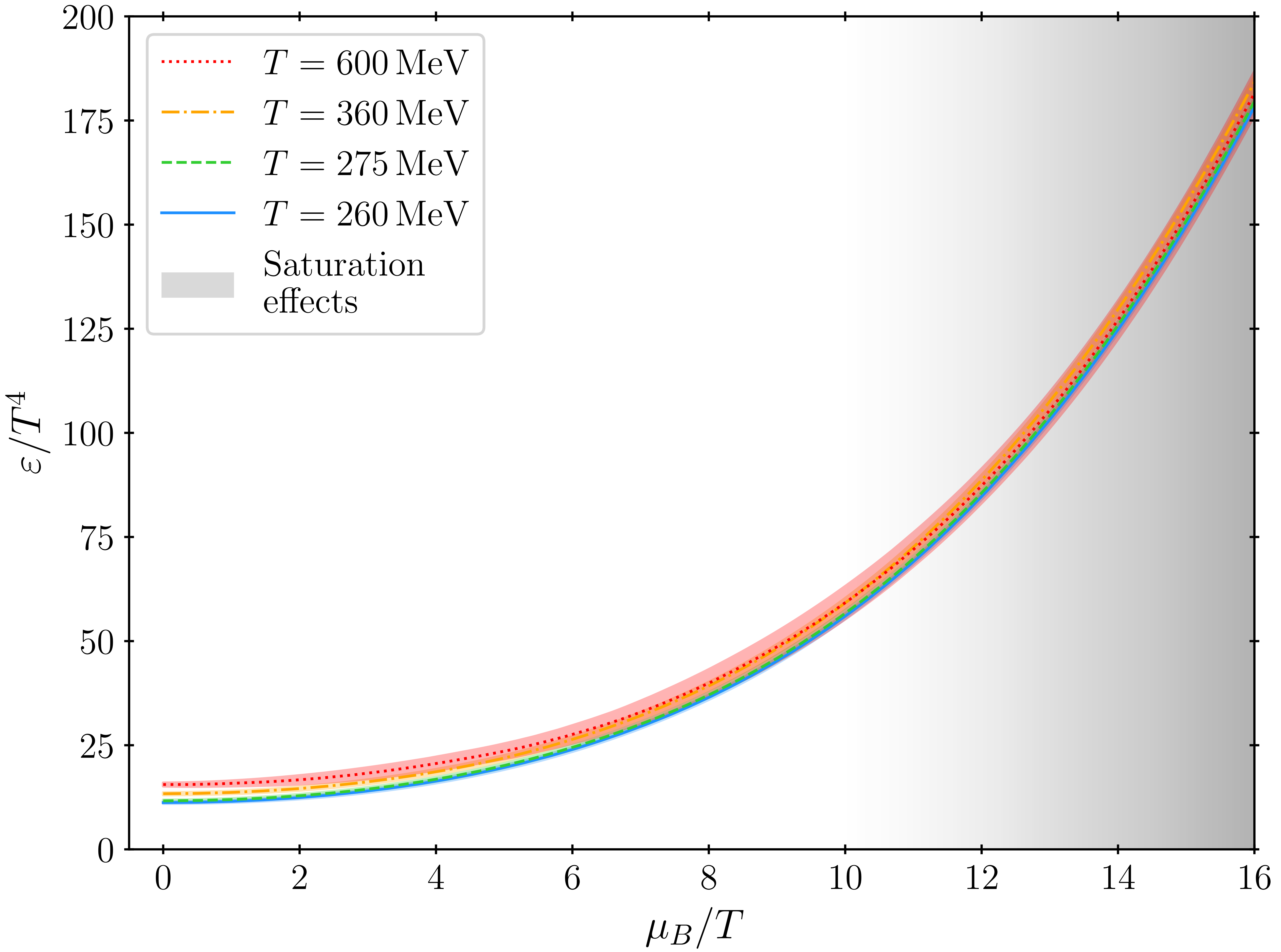
Progress in theoretical understanding

- Relation between kernels and integration cycles [Hansen, **M.M.**, Seiler, Sexty ’25]
- Necessary and sufficient criterion for correctness [**M.M.**, Seiler, Sexty ’25]
- Systematic comparison of diagnostic tools [**M.M.** ’26]

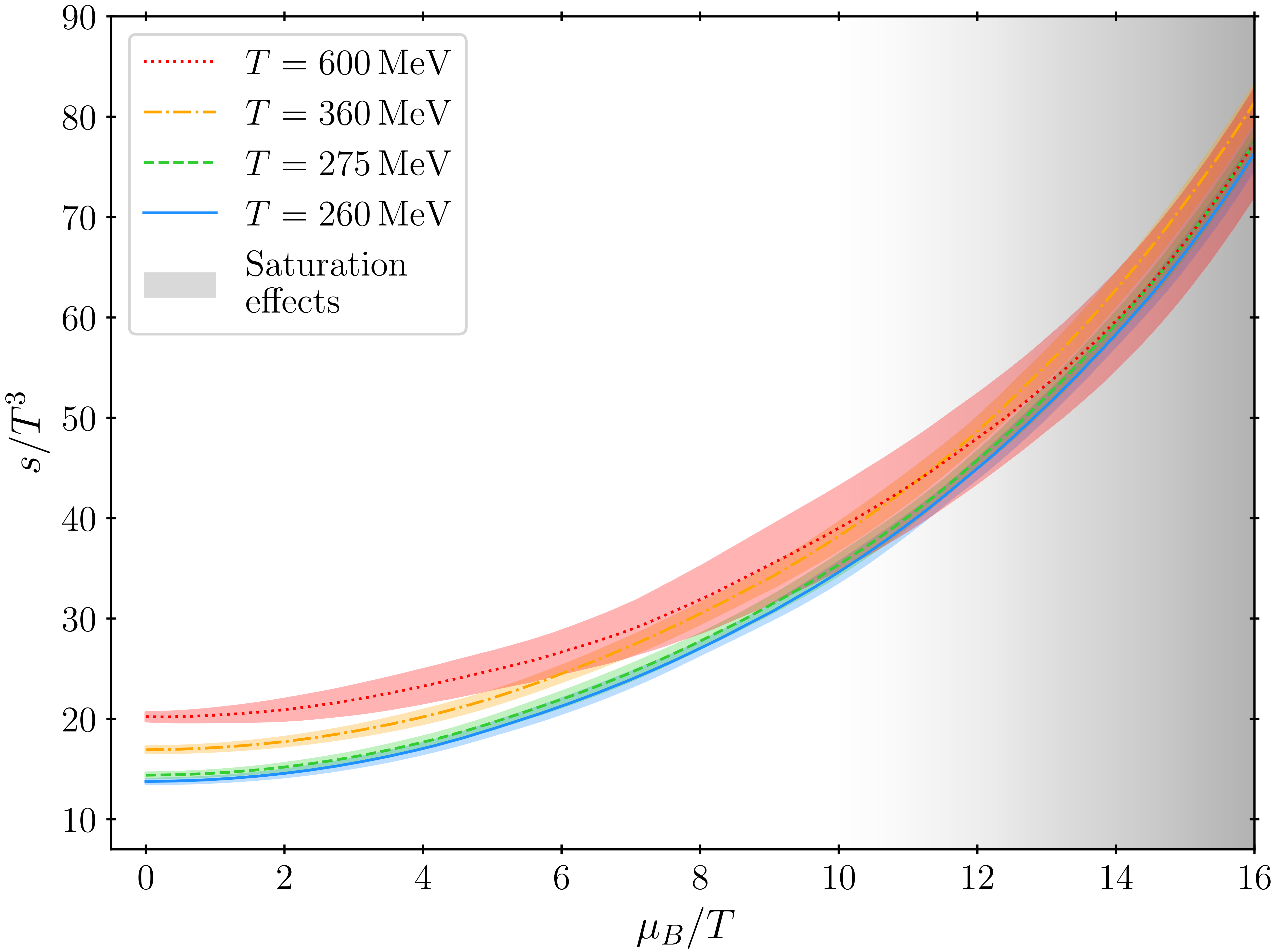
Trace anomaly



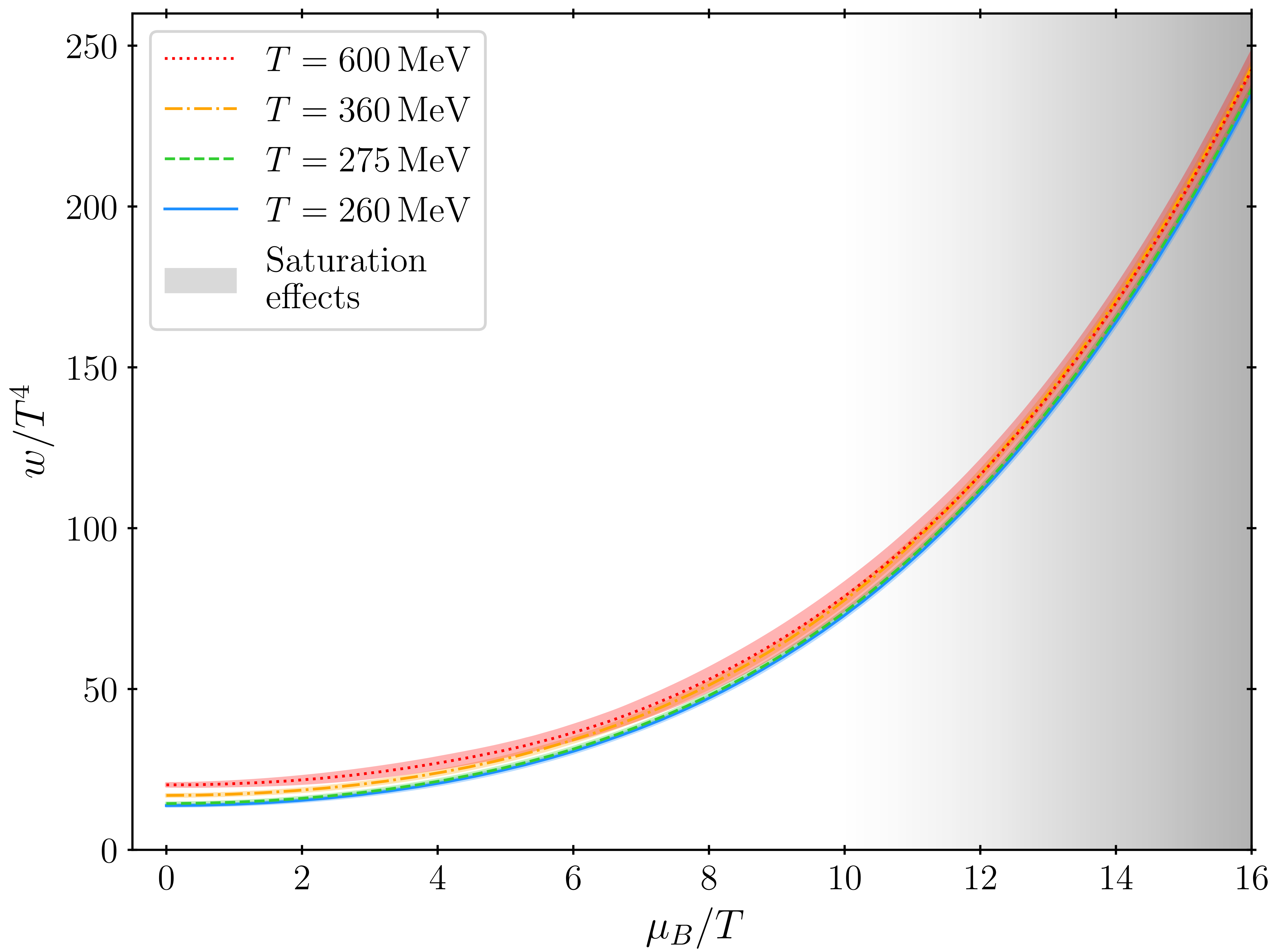
Energy density



Entropy density



Enthalpy density



Specific heat capacity