Equation of state of hot and dense QCD using the complex Langevin equation

(handout version)

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(based on arXiv:2604.19649 [hep-lat])

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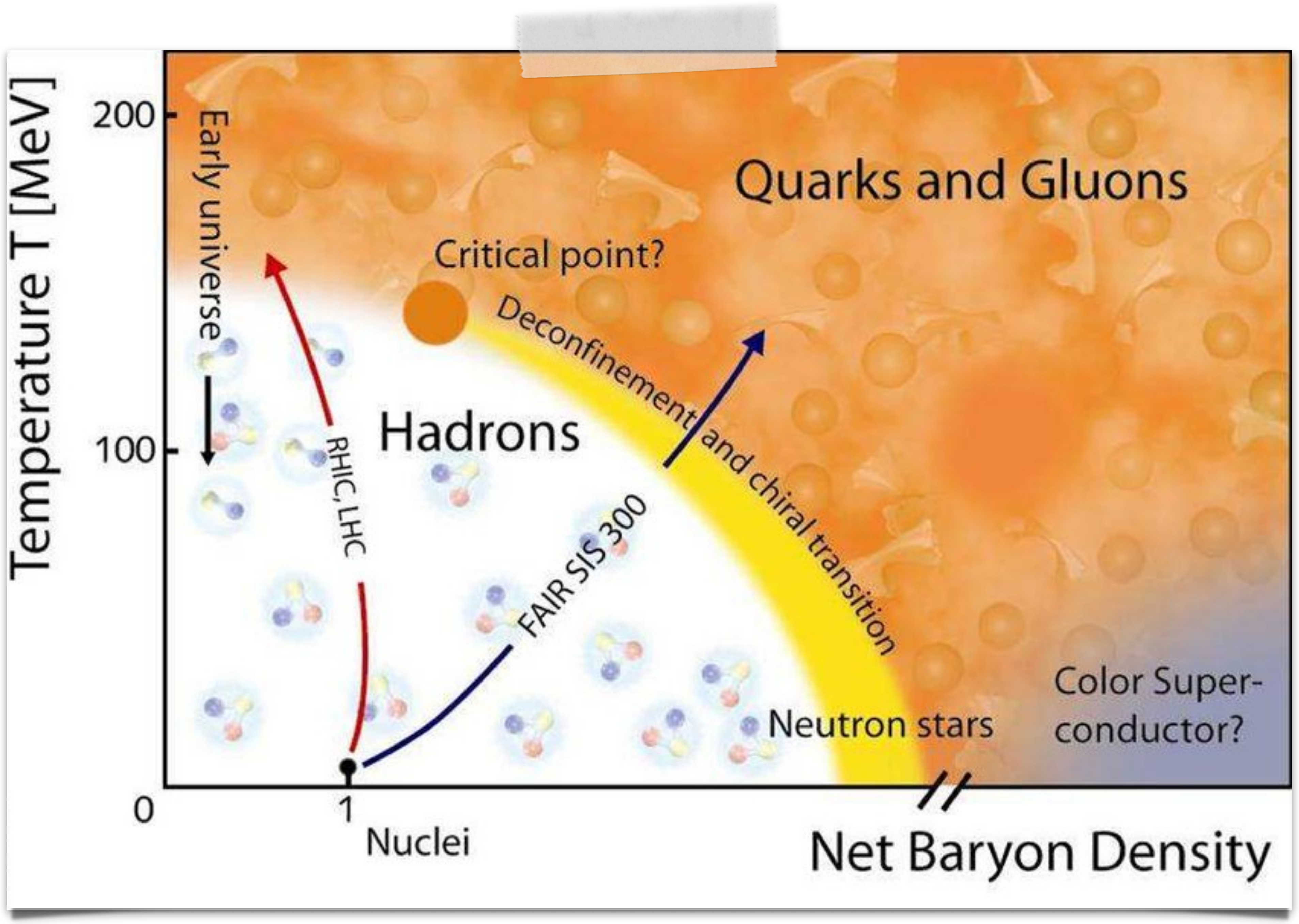
Handout version*

- This handout is a slightly modified version of the talk given at SEWM 2026: some additional comments have been added to give context to the slides.
- Slides with a title marked with an asterisk (*) were not part of the original talk.

Motivation

- QCD at non-zero density:
  - Neutron stars
  - Heavy-ion collisions
- Lattice simulations: sign problem:
  - $e^{-S_E} \notin \mathbb{R}_0^+ \implies$ importance sampling fails



Motivation

- QCD at non-zero density:
  - Neutron stars
  - Heavy-ion collisions
- Lattice
  - $e^{-S_E} \notin$ ... Tails

Possible solution: Complex Langevin

Motivation*

- Lattice simulations of quantum chromodynamics (QCD) at nonzero baryon chemical potential suffer from a sign (or complex-action) problem: the path integral measure e^{-S_E}, with S_E denoting the Euclidean QCD action after integrating out the fermions, is not real and positive definite, which prevents the straightforward application of conventional lattice methods based on importance sampling.
- Our method of choice to tackle the sign problem is the complex Langevin approach.

Complex Langevin dynamics

Toy example

- One-variable toy model $S(x) = \frac{\lambda}{4}x^4$, $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$
- Complexify: $x \to z = x + \mathrm{i}y$
- Auxiliary (Langevin) time τ
- Evolve in $\tau \to$ generate ensemble
- Compute observables in the limit $\tau \to \infty$

Complex Langevin equation

$$\frac{dz}{d\tau} = -\underbrace{\frac{\partial S(z)}{\partial z}}_{\text{drift term}} + \eta(\tau)$$

[Parisi '83]
[Klauder '83]

drift term

(real) Gaussian noise:

$$\langle \eta(\tau) \rangle = 0$$
$$\langle \eta(\tau)\eta(\tau') \rangle = 2\delta(\tau - \tau')$$

Complex Langevin dynamics*

Toy example

- The complex Langevin equation is a stochastic differential equation describing the evolution of complexified degrees of freedom in an auxiliary time dimension.
- For a theory with a single real degree of freedom x, the complex Langevin approach introduces the complexified variable $z = x + iy$ and studies its evolution in the Langevin time τ.
- This process is used to generate configurations, which, after equilibration, are averaged over to compute estimators of expectation values of observables.

Complex Langevin dynamics

Lattice QCD

- Dynamical degrees of freedom: gauge links $U_{x\mu} \in \mathrm{SU}(3)$
- Complexification: $U_{x\mu} \in \mathrm{SL}(3,\mathbb{C})$
- Discrete τ-evolution (step-size $\Delta\tau$):

Complex Langevin update for QCD

$$U_{x\mu}^{(\tau+\Delta\tau)} = \exp\left[i\lambda_a\left(-\Delta\tau D_{ax\mu} S[U] + \sqrt{\Delta\tau}\eta_{ax\mu}^{(\tau)}\right)\right] U_{x\mu}^{(\tau)}$$

Gell-Mann matrices

left-derivative

$$D_{ax\mu} f[U] = \frac{\partial}{\partial\alpha} f\left[e^{i\alpha\lambda_a} U_{x\mu}\right]_{\alpha=0}$$

$$S[U] = S_g[U] - \ln\det D[U]$$

Complex Langevin dynamics*

Lattice QCD

- In lattice QCD, the dynamical degrees of freedom are the SU(3)-valued gauge links $U_{x\mu}$, which in the complex Langevin approach are complexified to SL(3,$\mathbb{C}$).
- The formula on the previous slide shows how to evolve the complexified gauge links in τ for a finite step $\Delta\tau$. Important constituents are the left-derivative replacing the usual derivative and $\eta_{ax\mu}^{(\tau)}$, which describes independent Gaussian variables.

Simulation details

[M.M., Sexty, Unterhuber '26]

- 2+1+1 flavors of staggered quarks
- Tree-level improved Symanzik gauge action, four stout-smearing steps
- $N_t \in \{10, 12, 16\}$, aspect ratio $\frac{N_s}{N_t} = 2$, finite-size effects under control
- Continuum-extrapolation along line of constant physics by [Bellwied et al., '15]
- Strangeness- and charm-neutrality, isospin symmetry
- First ever complex Langevin simulations with physical pion mass

Simulation details*

- All previous complex-Langevin studies considered larger-than-physical pion masses. Our simulations, however, for the first time work at the physical point and we also perform a continuum extrapolation. Despite the relatively small aspect ratio, a handful of cross-checks with larger volumes indicate that finite-size effects are under control in our simulations.
- On the next slides, our results for the baryon number density n_B and pressure p are shown as functions of the baryon chemical potential μ_B for different temperatures T. Moreover, comparisons with hybrid-Monte-Carlo (HMC) simulations, as well as with a free theory of quarks and gluons and hard-thermal-loop perturbation theory (HTL) are shown for certain temperatures. The shaded regions indicate the presence of saturation effects, which complicate the continuum extrapolation.

QCD equation of state

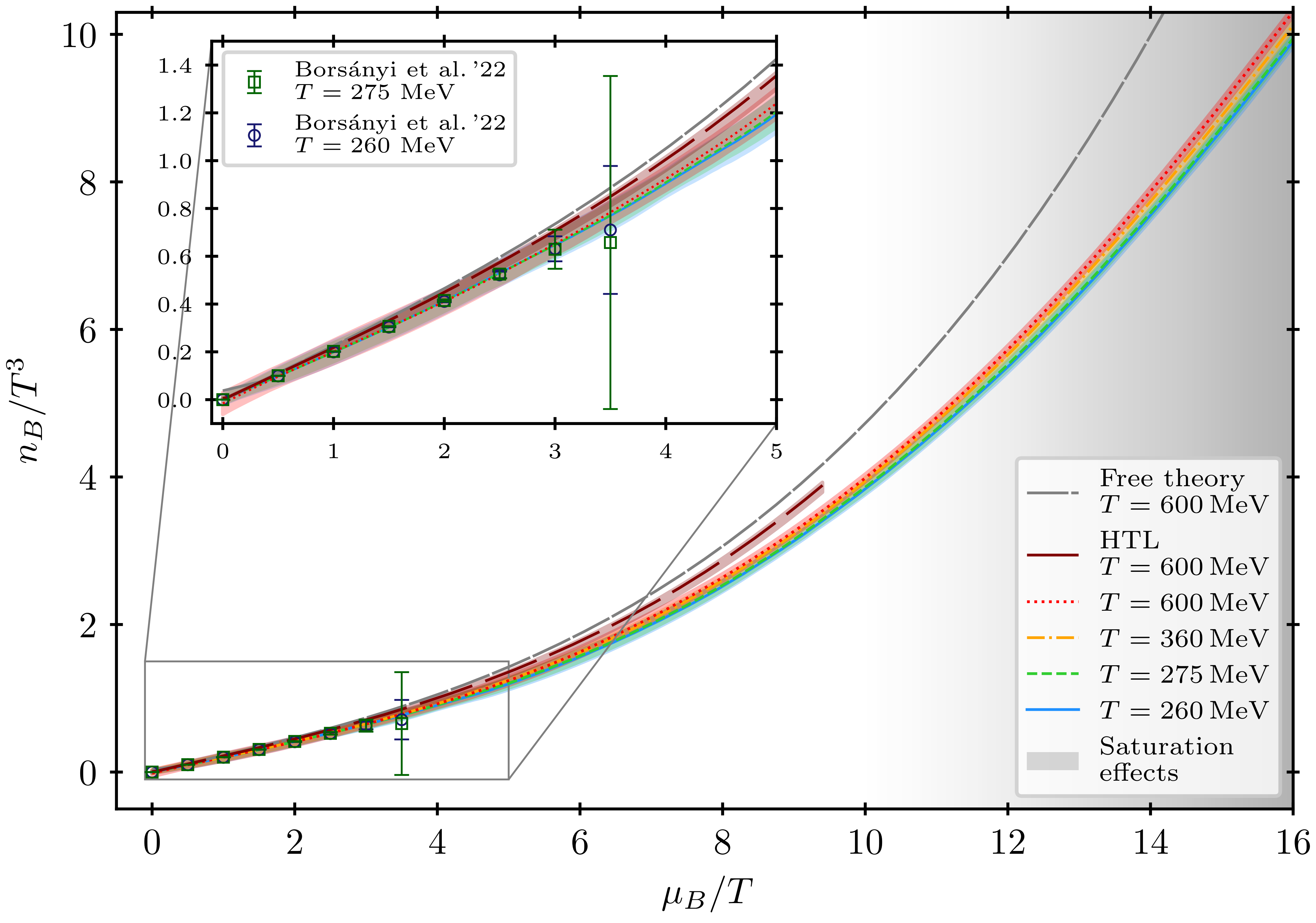
[M.M., Sexty, Unterhuber '26]

Baryon number density

$$n_B(T, \mu_B) = \frac{T}{V} \frac{\partial}{\partial \mu_B} \ln Z(T, \mu_B)$$

QCD equation of state

[M.M., Sexty, Unterhuber '26]



HTL: [Haque et al. '14]

Pressure difference

$$\Delta p(T,\mu_B) = \int_0^{\mu_B} d\mu_B'\, n_B(T,\mu_B')$$

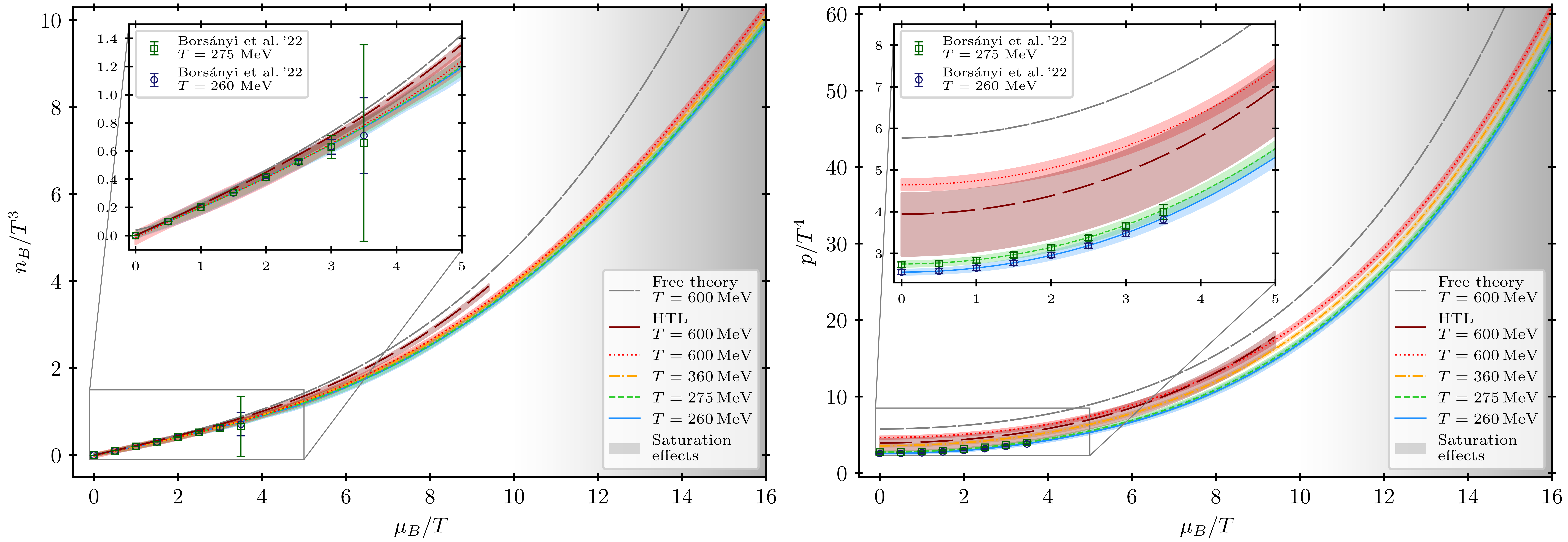
Pressure

$$p(T,\mu_B) = \Delta p(T,\mu_B) + p(T,0)$$

taken from [Borsanyi et al. '16]

QCD equation of state

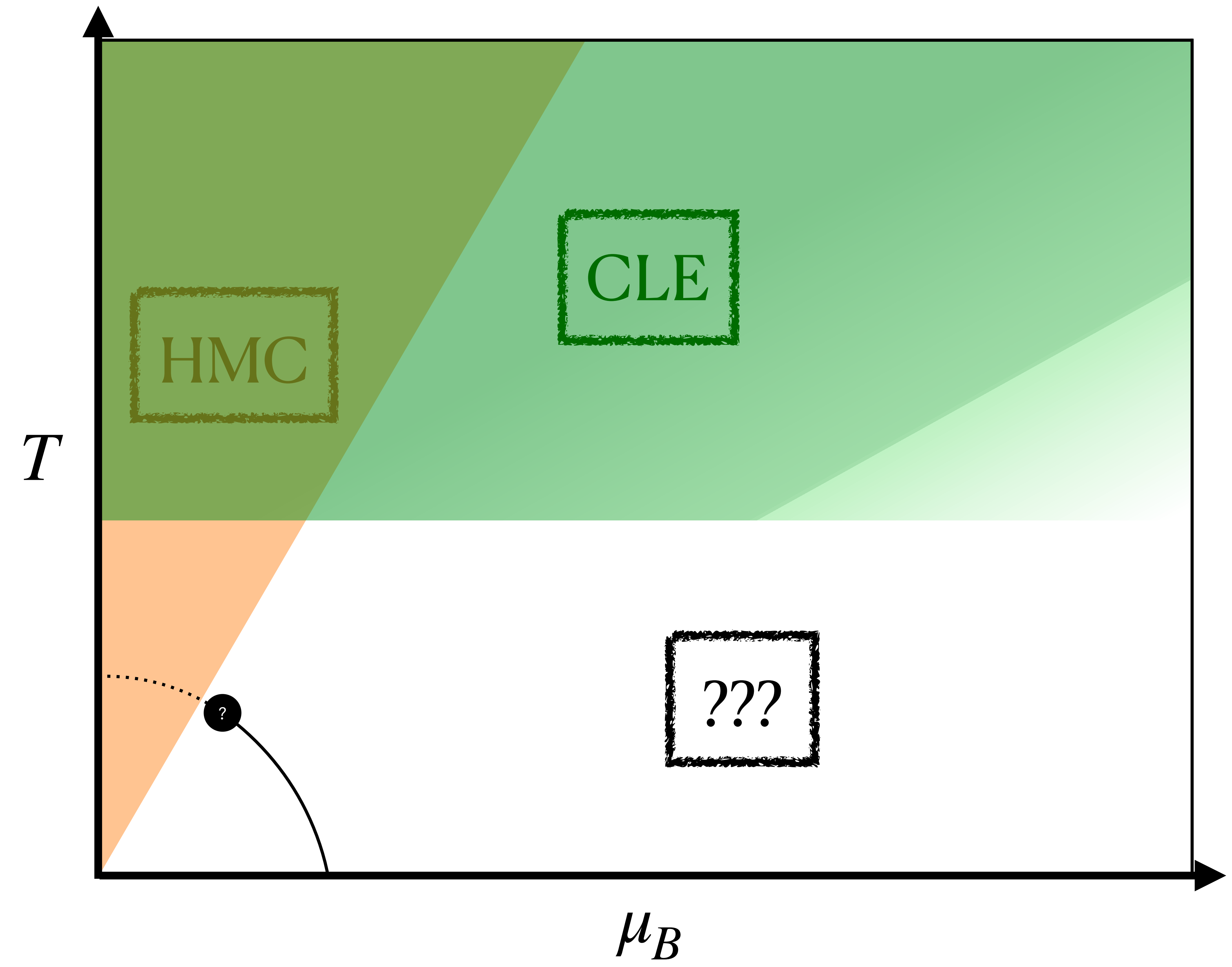
[M.M., Sexty, Unterhuber ’26]



HTL: [Haque et al. ’14]

QCD phase diagram

What is possible with complex Langevin



QCD phase diagram*

What is possible with complex Langevin

- While HMC-based methods are typically limited by baryon-chemical-potential—temperature ratios of around $\mu_B/T \approx 3$ to 4, in our simulations we can comfortably reach $\mu_B/T = 10$ and most likely even higher. Our simulations, however, are limited to rather high temperatures, $T \geq 260\,\mathrm{MeV}$, well above the QCD crossover.
- To go to lower temperatures, as shown on the next slide, we are experimenting with so-called kernels, which can be thought of as nontrivial diffusions coefficients, which we train using machine learning methods with the so-called unitarity norm N_U as the loss function. Loosely speaking, if N_U becomes too large, the complex Langevin evolution samples irrelevant regions of the extended field space and one usually tries to avoid such a situation in practice if possible. Indeed, our training approach leads to a stabilization of the τ-dependence of N_U. So far, however, our numerical tests are restricted to the heavy-dense limit of QCD at high temperatures.

In other news ...

Machine-learning kernels

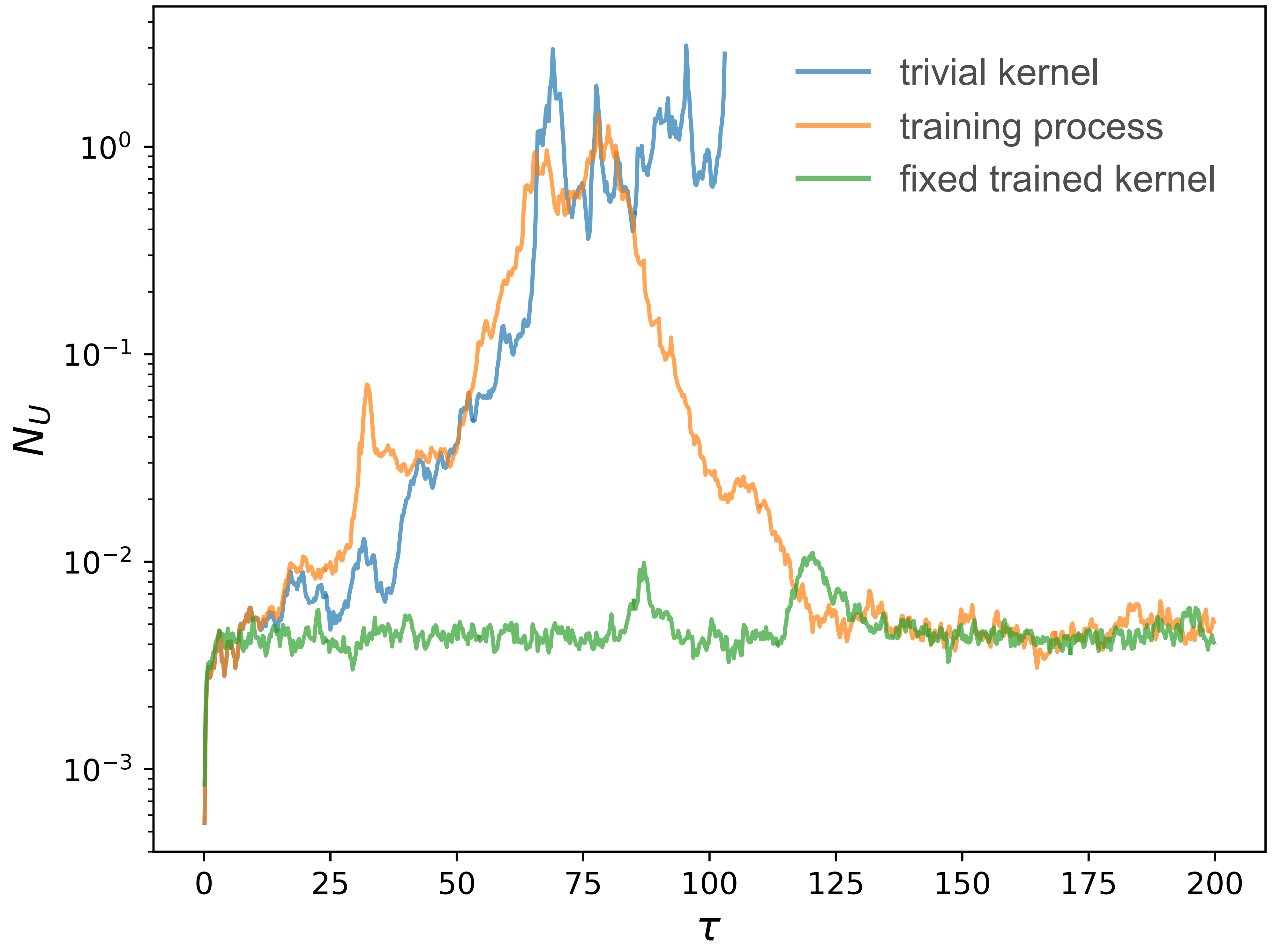
see also [Alvestad, Larsen, Rothkopf '23]
[Lampl, Sexty '23]
[Alvestad, Rothkopf, Sexty '24]

Complex Langevin update for QCD

$$U_{x\mu}^{(\tau+\Delta\tau)} = \exp\left[\mathrm{i}\lambda_a M_{ax\mu}^{(\tau)}[U]\right] U_{x\mu}^{(\tau)}$$

Kernel-dependent $M_{ax\mu}^{(\tau)}[U]$

$$M_{ax\mu}^{(\tau)}[U] = -\Delta\tau\, H_{ax\mu,by\nu} H_{cz\rho,by\nu} D_{cz\rho} S[U] + \sqrt{\Delta\tau}\, H_{ax\mu,by\nu}\eta_{by\nu}^{(\tau)}$$



[M.M., Seiler, Sexty '26]

Summary & Outlook

First complex Langevin study at the physical point.

Equation of state for unprecedentedly high baryon densities.

Restricted to temperatures well above the QCD crossover.

Progress in the theoretical understanding of the method.

Outlook: Lower temperatures (e.g., via kernels, maybe use machine-learning).

Backup

In other news ...

Progress in theoretical understanding

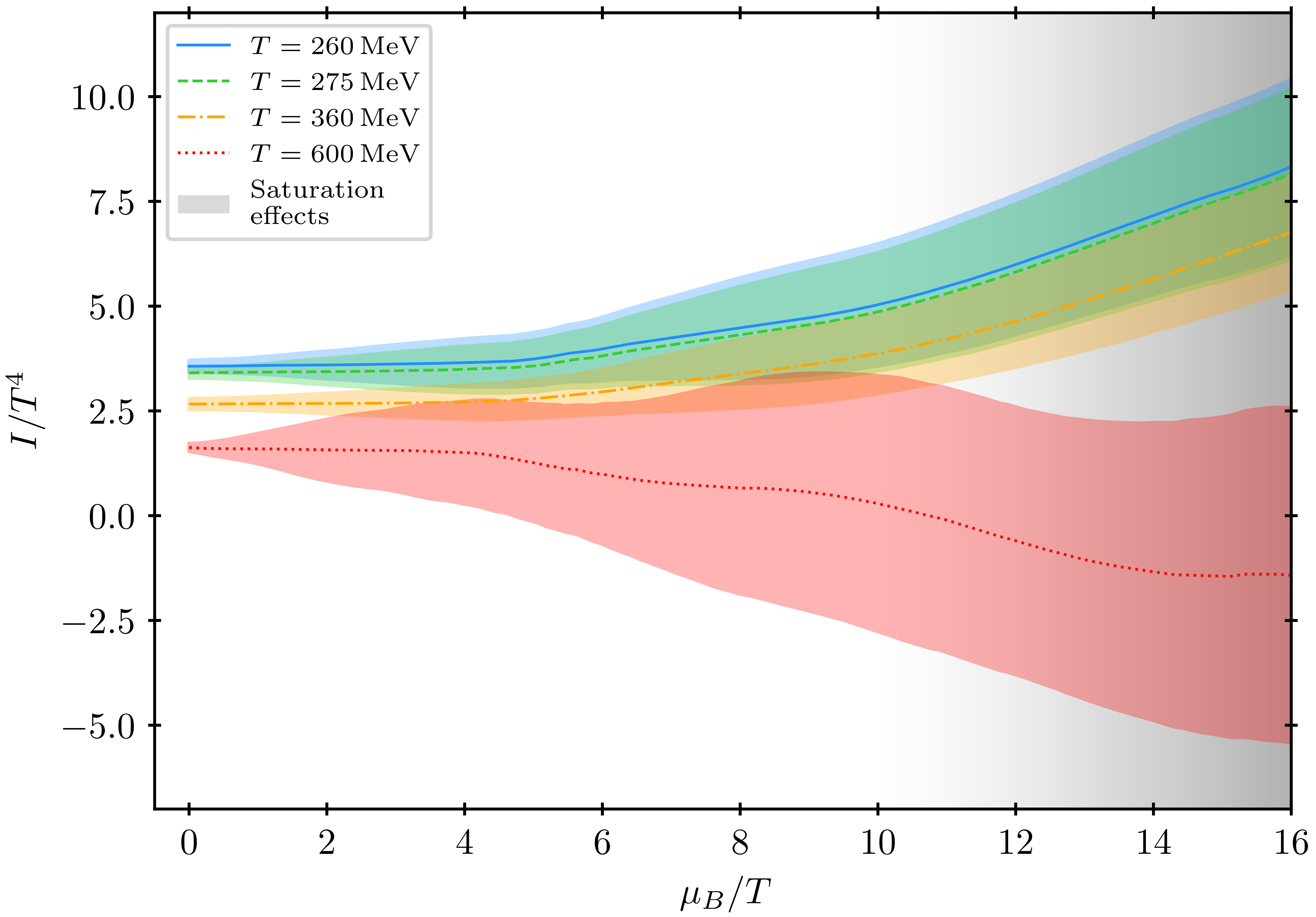
- Relation between kernels and integration cycles [Hansen, M.M., Seiler, Sexty '25]
- Necessary and sufficient criterion for correctness [M.M., Seiler, Sexty '25]
- Systematic comparison of diagnostic tools [M.M. '26]

In other news ...*

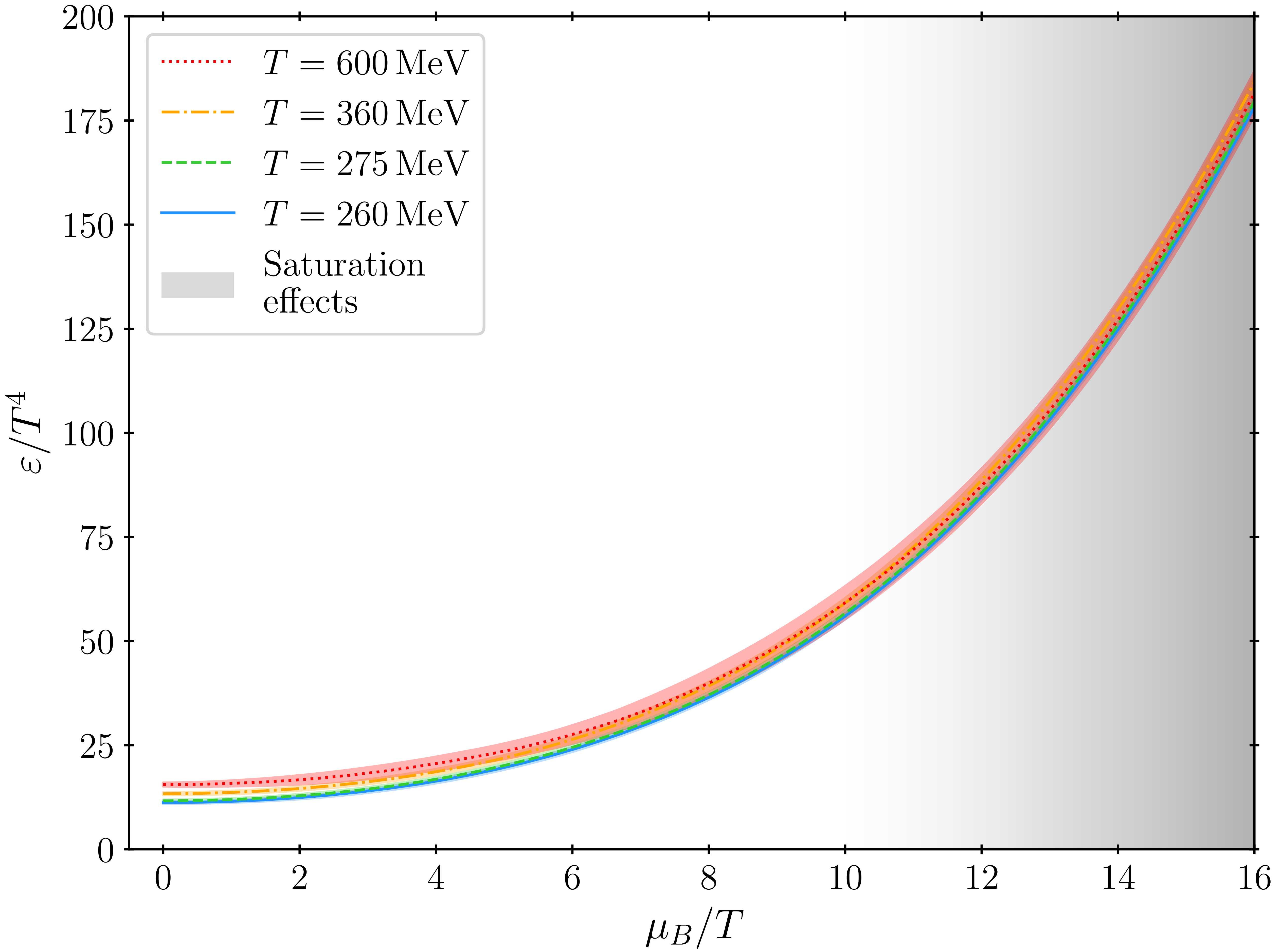
Progress in theoretical understanding

- Complex Langevin dynamics may receive contributions from unwanted so-called integration cycles, which correspond to unphysical solutions to the theory's Dyson—Schwinger equations which arise due to the complexification. In a recent paper, we have quantified the relation between kernels and the integration cycles sampled by complex Langevin simulations in the context of toy models.
- Moreover, we have recently put forward the (to date) only correctness criterion for complex Langevin that is both necessary and sufficient. Its downside, however, is its limited practical applicability.
- Finally, the many different available correctness criteria for complex Langevin were recently compared and contrasted in a systematic way.
- The next few slides show additional thermodynamic observables as a function of μ_B/T for different temperatures.

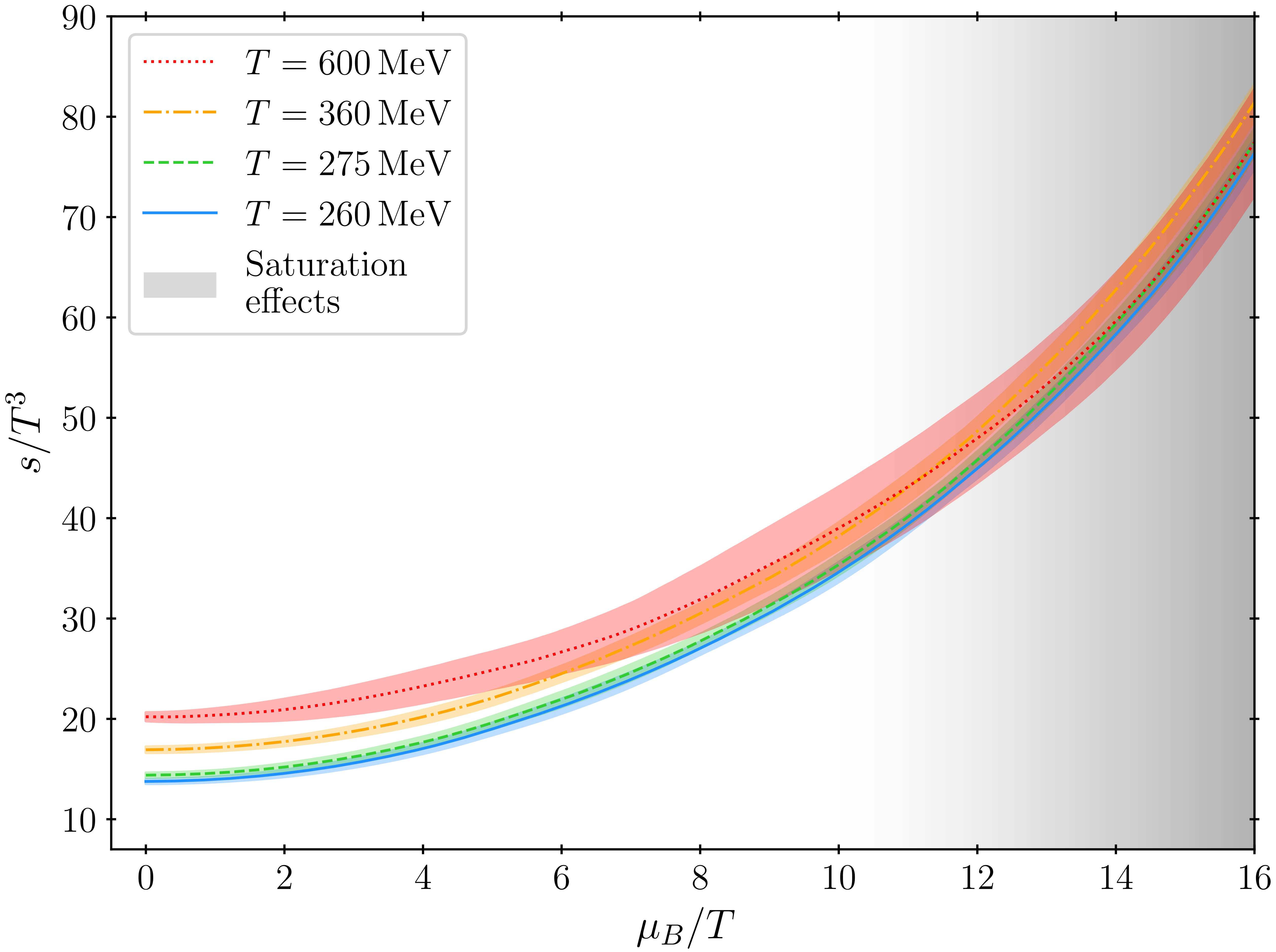
Trace anomaly



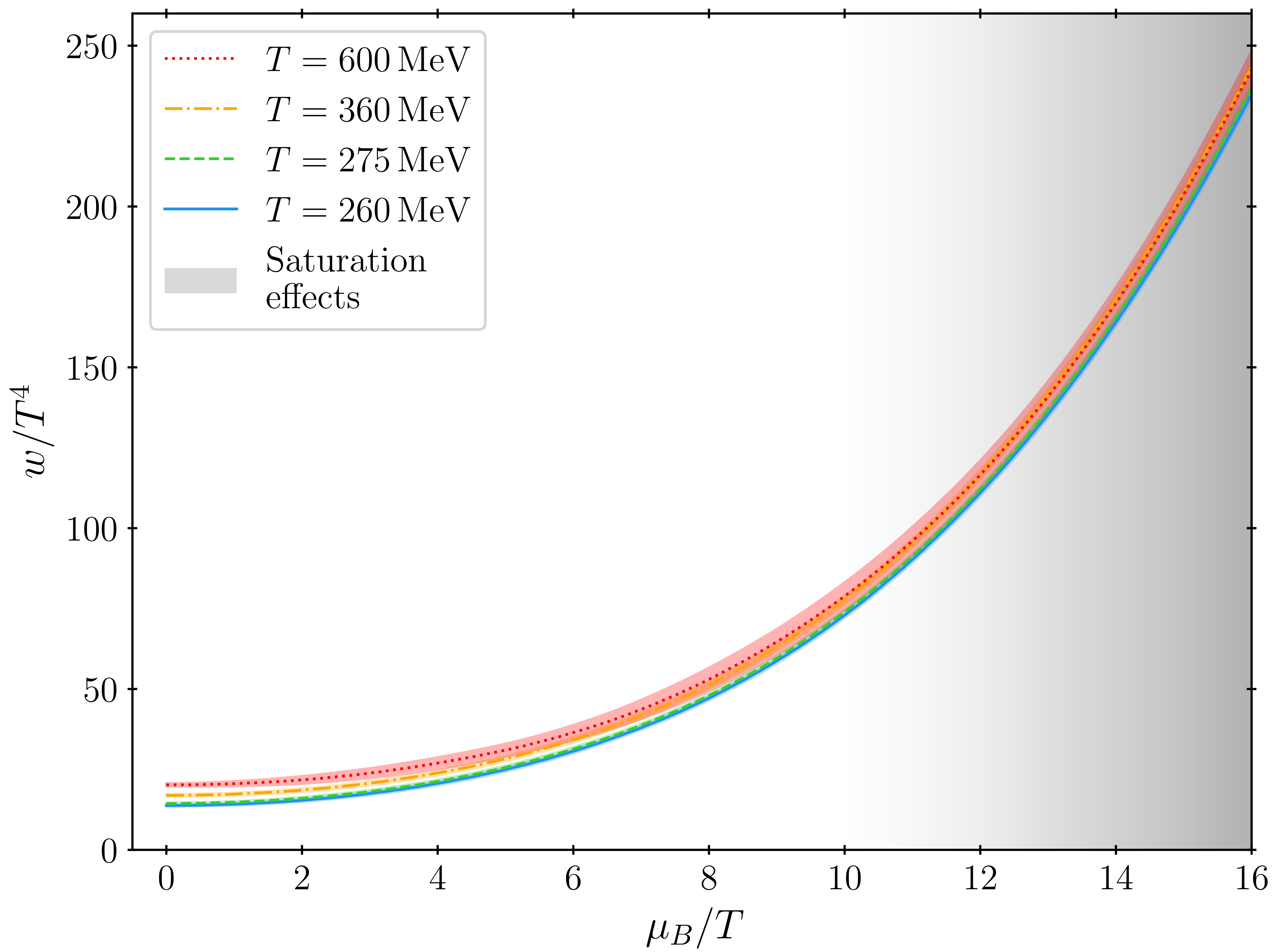
Energy density



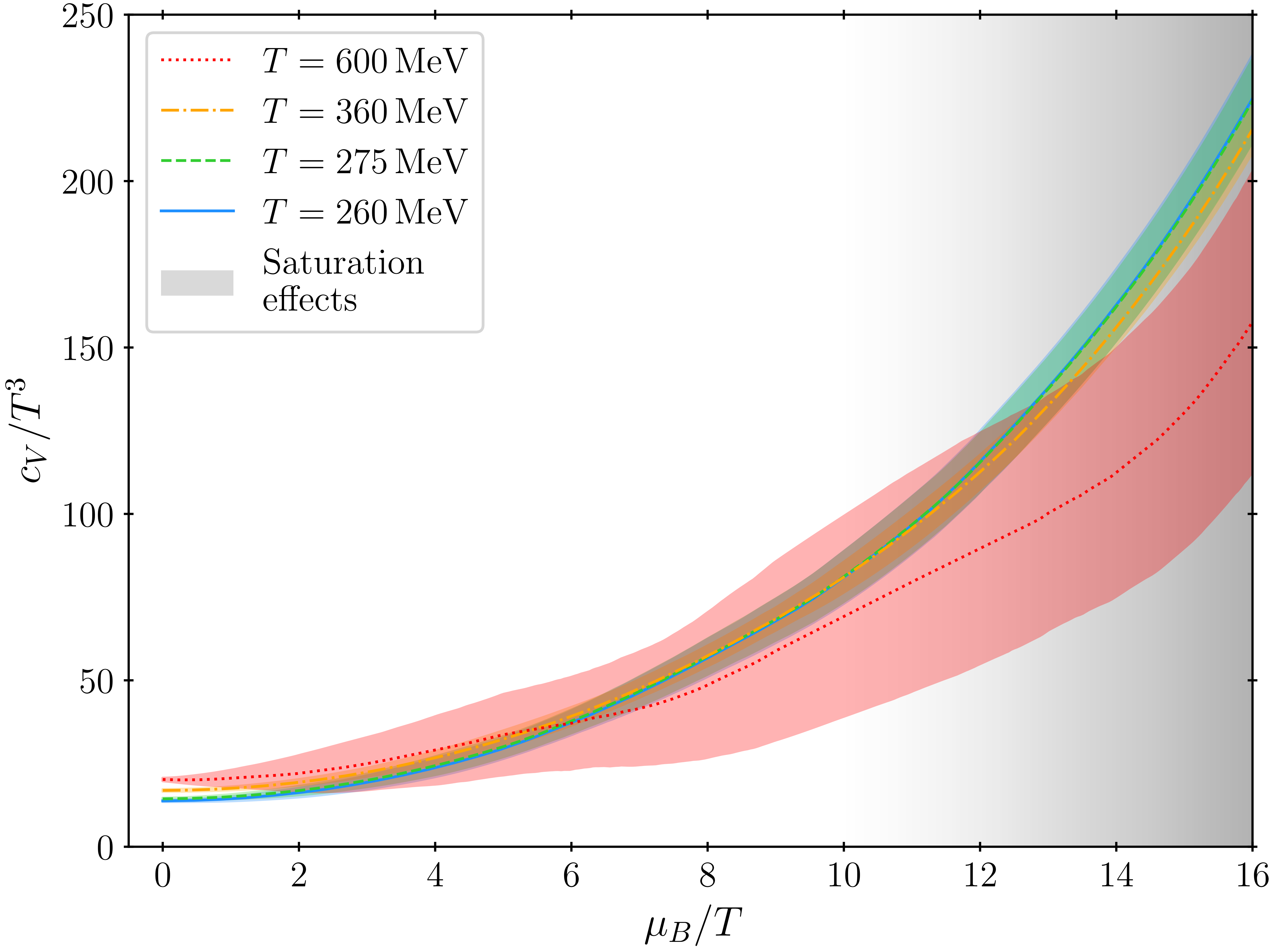
Entropy density



Enthalpy density



Specific heat capacity



Contact*

- For any questions/discussion, please do not hesitate to contact the author via michael.mandl@uni-graz.at .