Thermal axion production from the Lattice

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required joke

Q: Is a zebra black with white stripes
or is a zebra white with black stripes?

required joke

Q: Is a zebra black with white stripes
or is a zebra white with black stripes?

A: Yes.

Outline

I will study how many *axions* are produced thermally in hot QCD.

1. Why compute thermal axion production?
2. Relation to spectral function on light-cone
3. Relation to Euclidean quantity at *imaginary momentum*
4. How do you compute that on the lattice?
5. Results and prospects

The theta problem

The Lagrangian of QCD *is*

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu} + \sum_f \bar{\psi}_f(\not{D} + m_f)\psi_f$$

but it *could have been*

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu} + \frac{\Theta}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,G_{\mu\nu}G_{\alpha\beta} + \sum_f \bar{\psi}_f(\not{D} + m_f)\psi_f$$

Why is $|\Theta| < 10^{-10}$?

Axion solution

Several UV-complete models have IR behavior with one new field a:

$$\mathcal{L} = -\frac{1}{2g^2} \,\mathsf{Tr}\, G_{\mu\nu}G^{\mu\nu} + \sum_f \bar{\psi}_f (\not{D} + m_f)\psi_f$$

$$- \frac{1}{2}\partial_\mu a \partial^\mu a + \frac{\left(\Theta + a/f_a\right)}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \,\mathsf{Tr}\, G_{\mu\nu}G_{\alpha\beta}$$

a *automatically dynamically* adjusts to the value where $\Theta + a/f_a = 0$.

- Solves Θ problem
- Candidate for dark matter, mass $m_a^2 = \chi_{\rm top}/f_a^2$
- New radiative degrees of freedom

Today we will think about the radiative degrees of freedom

Thermal axions

Two types of axions are produced in the early Universe:

- Cold nonthermal axions. May be dark matter. See Klaer & GM JCAP 11 (2017) 049, arXiv:1708.07521
- Thermal-bath axions. Topic today

Production rate parametrically Bouzoud Ghiglieri arXiv:2404.06113

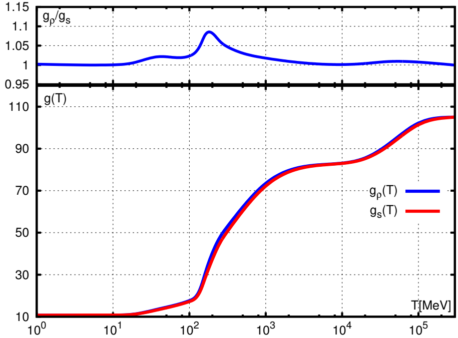
$$\frac{d\Gamma}{d^3k} \sim \alpha_{\mathsf{s}}^3 \frac{T^3}{f_a^2}$$

Equilibration when

$$\frac{d\Gamma}{d^3k} \geq H \sim \frac{T^2}{m_{\mathsf{pl}}} \qquad \Rightarrow \qquad T \gtrsim \frac{f_a^2}{\alpha_{\mathsf{s}}^3 m_{\mathsf{pl}}}$$

Goal: handle strongly-coupled case $T \sim T_c$

What temperatures are important?



Borsanyi et al, Nature, 539(7627):69–71, 2016

Increases radiative DOF from g_* to $g_* + 1$
High freezeout-T: g_* large, small change
Low freezeout-T: g_* small, important change
Key region: 150-300 MeV.

Hi-T rate: perturbation theory Bouzoud Ghiglieri Laine Sakoda JHEP 05:034, 2026
Low-T rate: pions Notari Rompineve Villadoro PRL 131(1):011004, 2023
Most important T-range: right where neither one works!

Rate and Spectral Function

Let's name

$$\frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,G_{\mu\nu}G_{\alpha\beta}(x) \equiv q(x)$$

Interaction $a(x)q(x)/f_a$ can create/annihilate axions.
Axion is almost massless, dispersion $\omega = |\boldsymbol{k}|$.
Standard calculation:

$$\frac{d\Gamma}{d^3\boldsymbol{k}} = -\frac{1}{(2\pi)^3 2k^0 f_a^2} G^<_{qq}(k,k) = -\frac{n_b(k)}{(2\pi)^3 2k^0 f_a^2}\ \sigma_{qq}(k,k).$$

Here $\sigma_{qq}(k,k)$ is the spectral function for the $q(x)$ operator

$$\sigma_{qq}(k,k) = \int d^4x e^{-ix_0|\boldsymbol{k}|+i\boldsymbol{x}\cdot\boldsymbol{k}} \left\langle \Big[q(x)\,,\ q(0) \Big] \right\rangle_\beta$$

Spectral function and Euclidean function

$\sigma_{qq}(k,k) = \mathrm{Im}\, G^{\mathrm{ret}}_{qq}(k,k)$.
Continue G^{ret}_{qq} to complex k, which means *both* ω and $|\boldsymbol{k}|$ are made complex. *Single analytic complex function* in $\omega \in \mathbb{C}$
At *purely imaginary* k, Kramers-Kronig tells us:

$$G^{\mathrm{ret}}_{qq}(i\omega, i\omega\hat{k}) = \int_0^\infty \frac{d\omega'}{\pi}\omega' \frac{\sigma_{qq}(\omega',\omega')}{\omega^2+\omega'^2} \quad \text{(with subtractions?)}$$

For $\omega_n = 2\pi nT$, $G^{\mathrm{ret}}_{qq}(i\omega_n) = G^{\mathrm{E}}_{qq}(\omega_n)$. Therefore

$$G^{\mathrm{E}}_{qq}(\omega_n, i\omega_n\hat{k}) = \int_0^\infty \frac{d\omega'}{\pi}\omega' \frac{\sigma(\omega')}{\omega'^2+\omega_n^2} = -2\pi f_a^2 \int_0^\infty \frac{d\omega'}{n_b(\omega')(\omega'^2+\omega_n^2)} \frac{d\Gamma}{d\omega'}.$$

$G^{\mathrm{E}}_{qq}(\omega_n, i\omega_n\hat{k})$: Euclidean function we can **directly address on the lattice**

Approach pioneered by Harvey Meyer `EPJA 54:192, 2018` for photons

Convergence issues?

After averaging $e^{\omega \hat{k}\cdot \boldsymbol{x}}$ over the direction $\hat{k}$, we want:

$$G^{\mathsf{E}}_{qq}(\omega_n) = \int_0^\beta d\tau \int d^3x \, \langle q(x)q(0)\rangle_{\mathsf{E}} \cos(\omega_n \tau) \frac{\sinh(\omega_n r)}{\omega_n r}$$

oscillating in time but growing in distance which is challenging.
Short-distance: $\langle q(x)q(0)\rangle_{\mathsf{E}} \sim (x^2)^{-4}$ short-dist. divergent.
Contact term at exactly $x = 0$ such that integral is finite.

differences between $G^{\mathsf{E}}_{qq}(\omega_n)$ values are OK! For instance:

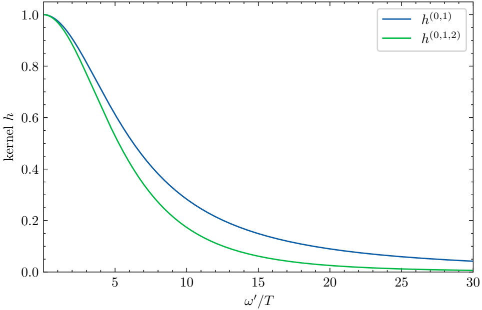
$$\begin{aligned} G^{\mathsf{E}}_{qq}(0) - G^{\mathsf{E}}_{qq}(\omega_1) &= \int_0^\infty \frac{d\omega}{\omega} \frac{(2\pi T)^2}{\omega^2 + (2\pi T)^2} \sigma(\omega) \\ &= \int_0^\beta d\tau \int d^3x \langle q(x)q(0)\rangle \left(1 - \cos(2\pi T\tau)\frac{\sinh(2\pi T r)}{2\pi T r}\right) \end{aligned}$$

Convergent integral over $\sigma(\omega)$ and finite as $|x| \to 0$ with no contact term.
Vanishes if $\langle q(x)q(0)\rangle$ is O(4) invariant: corrections $\propto T^4/x^4$

Correlations we consider

We do *not* have access to $d\Gamma/d^3k$ directly!
We can only learn *integral moments* of $d\Gamma/d^3$ or $\sigma(k,k)$:

$$G^{\mathsf{E}}_{qq}(0) - G^{\mathsf{E}}_{qq}(1)$$
$$= \int_0^\infty \frac{d\omega}{\pi\omega} \frac{(2\pi T)^2 \sigma(\omega)}{(2\pi T)^2 + \omega^2}$$
$$\frac{3G^{\mathsf{E}}_{qq}(0) - 4G^{\mathsf{E}}_{qq}(1) + G^{\mathsf{E}}_{qq}(2)}{3}$$
$$= \int_0^\infty \frac{d\omega}{\pi\omega} \frac{(2\pi T)^2 (4\pi T)^2 \sigma(\omega)}{((2\pi T)^2 + \omega^2)((4\pi T)^2 + \omega^2)}$$



That's the price for not having to analytically continue!
Future: how useful and constraining are these?
Present: Can we actually calculate these?

Let's do it on the lattice!

No show-stoppers on the lattice (?)

1. Integral we want is directly a Euclidean integral
2. Operator q well behaved and free of renormalization
3. Differences short-distance well-behaved so they don't feel severe lattice effects

Two provisos:

- Point 2 above only true after **gradient flow** with $t_{\text{flow}} > 0.4a^2$
  Must take $\lim(t_{\text{flow}} \to 0)$ at the end!
- Data with finite errors, but $\sinh(2\pi Tr)$ grows exponentially...

So I also need to talk about large distances and convergence

Large distance: the good news

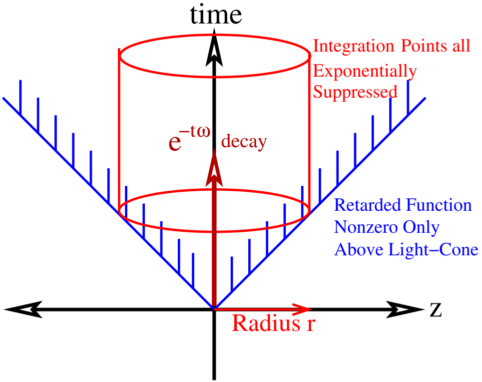
Claim: in Euclidean finite-T, for *any* local operator $\mathcal{O}(x)$ we have

$$\int_0^\beta d\tau \, \langle \mathcal{O}^\dagger(\tau,r)\mathcal{O}(0,0)\rangle \cos(2\pi nT\tau) \underset{r\gg T^{-1}}{\Longrightarrow} \frac{\exp(-mr)}{r}, \quad m \geq 2\pi nT.$$

Why: $G^{\mathsf{E}}(\omega_n,r) = G^{\mathsf{ret}}(i\omega_n,r)$.
In *Minkowski* time, integral over forward light-cone with $e^{-\omega_n t} \leq e^{-\omega_n r}$ factor.

Guaranteed to be $< e^{-\omega_n r}$.

The integrals we need *converge*



Why it's still hard

The correlation function $\int_0^\beta d\tau \cos(2\pi T\tau)\langle q(r,\tau)q(0,0)\rangle \leq e^{-2\pi Tr}$.
But its errors may *not* decay exponentially!
So how do we integrate over all radii?

- start with the pure-glue theory as a test case
- Start with one T: $T = 1.5T_c$ and one lattice spacing $N_\tau = 16$
- Fit the large-r tail and replacing noisy data with a fit
- Find a good doctoral student **or a super-good masters student**

if you need a PhD student, hire Luis Neubauer
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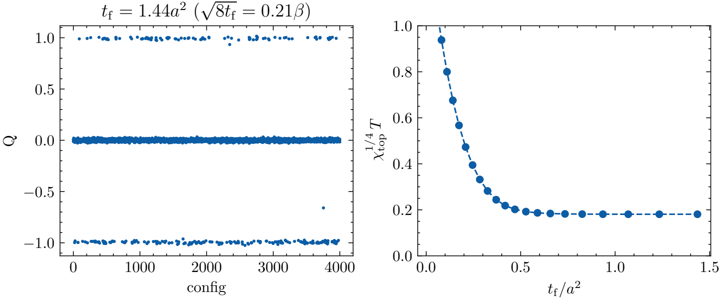
If lattice gauge-fields are smooth, lattice q works well.
If not, severe non-topological fluctuations and large renormalization.
Total topology $Q = \int d^4x q(x)$ and $\langle Q^2 \rangle = \beta V \chi_{\rm top}$:



With gradient flow $t_{\text{flow}} \geq 0.4a^2$, cleanly topological.
More flow *smears* $q(x)$ which will also smear $G(x) \equiv \langle q(x) q(0) \rangle$
But $\int d^4x G(x)$ unchanged.

Computational approach

Plan: Generate independent configurations, and on each:

- Evaluate $q(x)$ (a^2-improved) after various t_{flow} depths
- Use FFT methods to find $G(x) = V^{-1} \sum_y q(y) q(x+y)$
- Fourier-transform in time τ
- Sum points with equal r^2-value to find $G(\omega_n, r^2)$
- Sum over r^2-values including $\sinh(\omega_n r)/\omega_n r$ factor

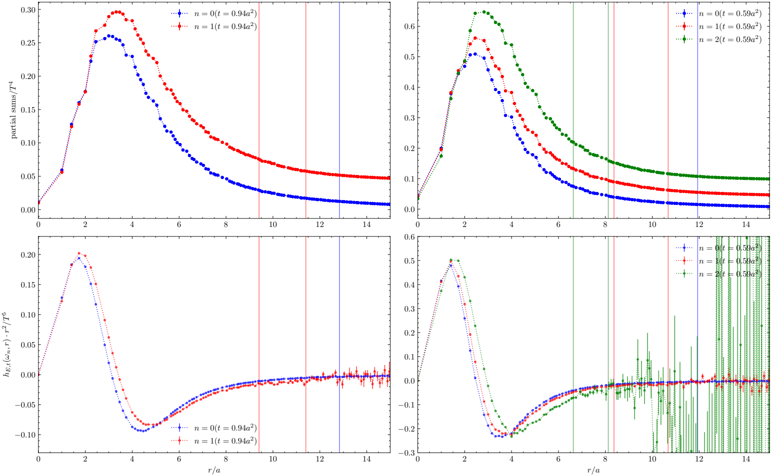
In practice, we have to

- fit the tail of the correlator to an exponential
- replace the largest-r data with the value from this fit

Let's see why.

What the data look like

x-axis: radius r/a. y-axis: Partial Sum: $\int_0^r G(r')r'^2 \sinh(\omega_n r')/(\omega_n r')dr'$



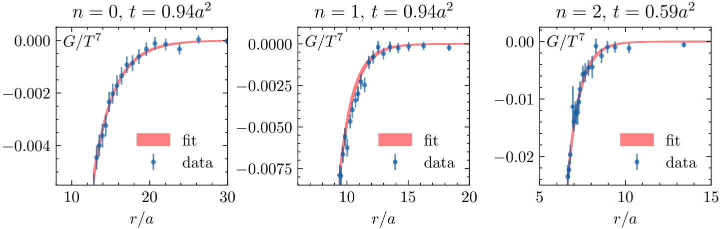
x-axis: radius r/a. y-axis: integrand $r^2 G(r) \sinh(\omega_n r)/(\omega_n r)$

Treating the tail

$G(\omega_n, r) = \int_0^\beta \cos(\omega_n \tau) G(\tau, r)$ has a Lehmann representation

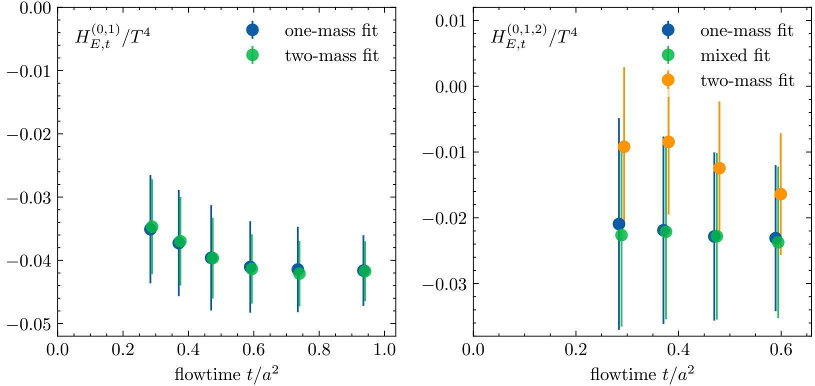
$$G(\omega_n, r) = -\int_{m_0}^{\infty} \rho(m)\, dm\, \frac{e^{-mr}}{r}\,, \qquad\qquad m_0 \geq \omega_n$$

Expect small-m part of $\rho(m)$ to be discrete (delta-functions)
Large-separation: tail fit with e^{-mr}/r (or $c_1 e^{-m_1 r}/r + c_2 e^{-m_2 r}/r$)



Fit to tails works! Harder for larger ω_n

Gradient-flow dependence and limit



Gradient-flow-depth dependence consistent with zero!
(Points below $t_{\text{flow}} = 0.4a^2$ not reliable)
This is actually expected! (backup slide...)

Results (Preliminary)

For *pure-glue* QCD at the *one* temperature we studied:

- Combination $G^{\mathsf{E}}_{qq}(0) - G^{\mathsf{E}}_{qq}(1)$ well determined
- Combination $(3G^{\mathsf{E}}_{qq}(0) - 4G^{\mathsf{E}}_{qq}(1) + G^{\mathsf{E}}_{qq}(2))/3$: 2σ away from zero
- Provides multiple constraints on axion production.
- Calculation actually *easier* at larger coupling
- Most important take-away: this can be done!

Temperature dependence not explored.
Continuum limit not available, but not expected to be severe

Application to full QCD: if you have configurations *with* $N_\tau \geq 16$, *and* you have enough statistics, this should be straightforward.

Conclusions

- Topological-charge correlators on the light-cone $\Rightarrow$ axion production
- **there is no need to analytically continue** but
- one only learns **Integral Moments** of the rate,
  rather than the full differential rate.
- Handling the large-separation tail is tricky but possible
- If you have the configurations, you can do this in full QCD!

and if you need a PhD student, interview Luis Neubauer

Test what you learned

Q: Is a zebra black with white stripes
or is a zebra white with black stripes?

Lehmann representation

$$\text{Consider } \mathcal{O}(z) = \int_0^\beta d\tau \, \cos(\omega_n \tau) \int dx dy \, q(x,y,z,\tau)$$

Treat z-direction as Euclidean continuation of time
(space$=(x,y,\tau) \in \mathbb{R}^2 \times \mathbb{S}^1$) Also note, $\mathcal{O}^\dagger(z) = -\mathcal{O}(z)$

$$\begin{aligned} \text{Correlator } \; -\langle \mathcal{O}(z)\mathcal{O}(0)\rangle &= \langle 0|\mathcal{O}^\dagger e^{-Hz}\mathcal{O}|0\rangle \\ &= \langle \psi | e^{-Hz} | \psi \rangle \end{aligned}$$

$|\psi\rangle = \mathcal{O}|0\rangle$ has an expansion in the energy eigenstates of $\mathbb{R}^2 \times \mathbb{S}^1$ theory:

$$|\psi\rangle = \int_{m_0}^\infty c(m) dm \, |\psi_m\rangle \qquad \text{and therefore}$$

$$\langle \mathcal{O}^\dagger e^{-Hz} \mathcal{O}\rangle = \int_{m_0}^\infty (c^* c(m)) dm \langle \psi_m | e^{-Hz} | \psi_m \rangle = \int_{m_0}^\infty \rho(m) dm e^{-mz}$$

$$e^{-mz} = \int dx dy \frac{\exp(-m\sqrt{z^2+x^2+y^2})}{2\pi\sqrt{z^2+x^2+y^2}} = \int dx dy \frac{e^{-mr}}{2\pi r}$$

therefore correlator associated with separation r is $\int dm \, \rho(m) e^{-mr}/2\pi r$

Flow-time (in)dependence

Gradient-flow changes q *locally* in a way which preserves $Q = \int d^4x\, q(x)$.
Sounds like *smearing operation*

$$q(x) = \int d^4y\, q(x+y) S(y)\,, \qquad 1 = \int d^4y\, S(y) \text{ (Gaussian??)}$$

Also smears two-point correlator $G(x) = \langle q(x) q(0) \rangle$ by S' convolution of two S's

Write $\cos(\omega_n \tau) \sinh(\omega_n r)/\omega_n r = K_n(x)$. Note:

$$G^{\mathsf{E}}_{qq}(\omega_n, t_{\mathsf{flow}}) = \int d^4x\, G(x,\tau) K_n(x) = \int d^4x\, d^4y\, G(x+y) S'(y) K_n(x) \qquad (1)$$

$$= \int d^4x\, G(x) \int d^4y\, S'(y) K_n(x+y) \qquad (2)$$

Same as smearing $K_n(x)$. **BUT IF** $S'(y) = S'(y^2)$ is O(4)-invariant, then

For our $K_n(x)$, turns out: $\int d^4y S'(y) K_n(x+y) = K_n(x)$

Smearing has no effect. Non-O(4) effects are $\propto t_{\mathsf{flow}} T^2$, causing $\propto t^2_{\mathsf{flow}} T^4$ corrections!