SEWM 2026

Towards a consistent approach to perturbation theory at finite temperature

(Based on: P. Ali, PL, O. Philipsen, 2606.14863)

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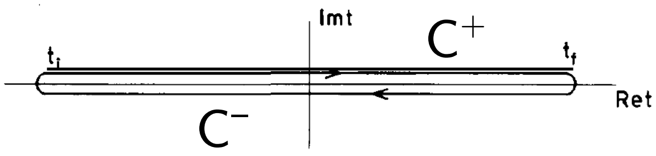
Perturbation theory: setup and subtleties

(i) **Real-time approach**: evolution along Schwinger-Keldish contour $C = C^+ \cup C^-$



"Field doubling": Φ can evolve along each branch $\rightarrow$ Φ^+ and Φ^-

Perturbative calculations involve *four* propagators ($++$, $+-$, $-+$, $--$)

$$\tilde{\tau}^{++}_{(0)}(p) = \frac{i}{p^2 - m^2 + i\epsilon} + \frac{2\pi\delta(p^2 - m^2)}{e^{\beta|p_0|} - 1}$$

(ii) **Imaginary-time approach**: based on "Matsubara" propagators $G(\omega_n, p)$

$$G(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2}$$

$\rightarrow$ Energy integrals replaced with sums $\omega_n = 2\pi n/\beta$

$\rightarrow$ Analytic continuation $i\,\omega_n \rightarrow p_0 + i\,\varepsilon$ taken after

But problems arise: $\rightarrow$ Pinch singularities in the real-time formalism

$\rightarrow$ Infrared divergences in massless theories

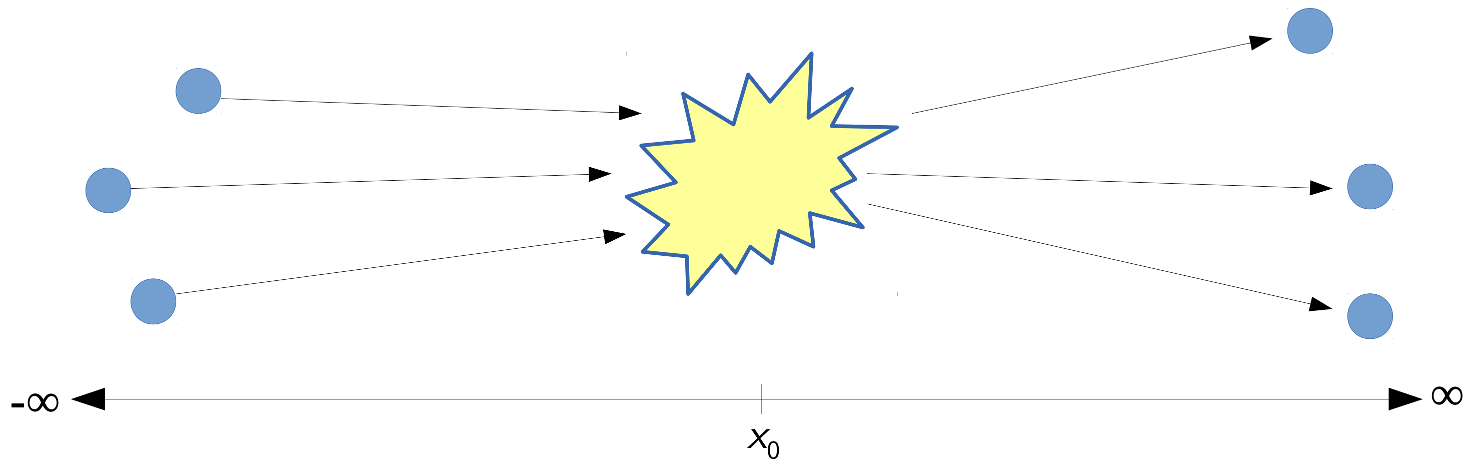
$\rightarrow$ Poor convergence properties (e.g. QCD EoS)

Is this a symptom of a more fundamental inconsistency?

Non-perturbative conflicts

Idea: construct thermal in/out scattering states with fixed **real** dispersion relations $\omega = E(\boldsymbol{p})$

"*Narnhofer-Requardt-Thirring Theorem*" [Commun. *Math. Phys.* 92, 247 (1983)]



Conclusion: *the S-matrix must be trivial!* $\to$ Thermal scattering states can exist, but only when there are no interactions...

Why? Thermal medium is everywhere $\to$ scattering states depend on T

[Landsman, *Ann. Phys.* 186, 141 (1988)]: *propagators with real poles* (e.g. free fields) *cannot form the basis of a finite-temperature perturbative expansion*

Gell-Mann-Low relation

- Gell-Mann-Low (GML) relation* is the basis of perturbation theory

When T=0:

$$\langle\Omega|T\left\{\phi(x_1)\cdots\phi(x_n)\right\}|\Omega\rangle = \frac{\langle\Omega_0|T\left\{\phi_0(x_1)\cdots\phi_0(x_n)e^{i\int d^4z\,\mathcal{L}_I[\phi_0(z)]}\right\}|\Omega_0\rangle}{\langle\Omega_0|T\left\{e^{i\int d^4z\,\mathcal{L}_I[\phi_0(z)]}\right\}|\Omega_0\rangle}$$

- Asymptotic (large-time) fields Φ_0 are non interacting → diagrammatic expansion involves products or convolutions of the free Φ_0 propagator $\dfrac{i}{p^2-m^2+i\epsilon}$

- When $T>0$ require a **thermal** generalisation of the GML relation

Standard T>0 relation:

$$\langle\Omega_\beta|T\left\{\phi(x_1)\cdots\phi(x_n)\right\}|\Omega_\beta\rangle = \frac{\mathrm{Tr}\left(e^{-\beta H}\,T\left\{\phi_0(x_1)\cdots\phi_0(x_n)e^{i\int_C d^4z\,\mathcal{L}_I[\phi_0(z)]}\right\}\right)}{\mathrm{Tr}\left(e^{-\beta H}\,T\left\{e^{i\int_C d^4z\,\mathcal{L}_I[\phi_0(z)]}\right\}\right)}$$

- Assumes ***free*** asymptotic fields → In conflict with the NRT theorem!

- Appears to be the source of the problems... So start with propagators that take thermal effects into account → ***But what form should these take?***

* [Gell-Mann, Low, Phys. Rev. 84 (1951)]

Thermoparticles

When $T>0$ how does the spectral function $\rho(\omega,\vec{p}) = \int d^4x\, e^{i(\omega x_0 - \vec{p}\cdot\vec{x})}\langle[\phi(x),\phi(0)]\rangle$ *change?**

$$\boxed{\rho(\omega,\vec{p}) = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\; \epsilon(\omega)\,\delta\big(\omega^2-(\vec{p}-\vec{u})^2-s\big)\,\widetilde{D}_\beta(\vec{u},s)}$$

$T \to 0$

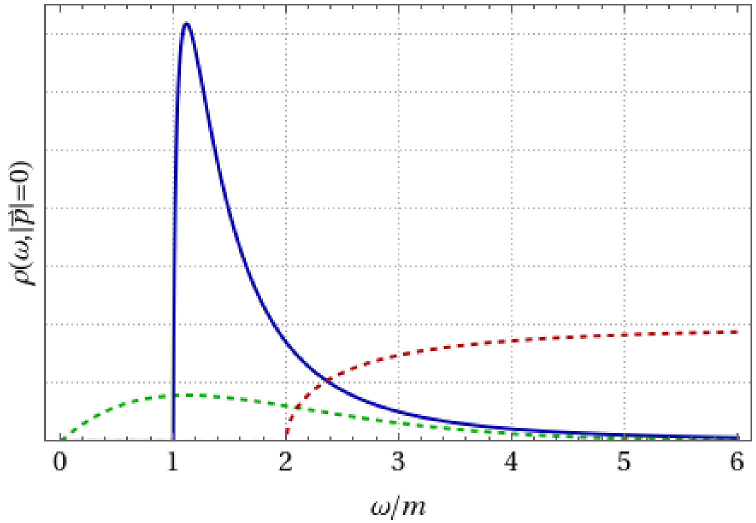
Källén-Lehmann representation

$$\rho(\omega,\vec{p}) = 2\pi\epsilon(\omega)\int_0^\infty ds\,\delta(p^2-s)\,\varrho(s)$$

→ Thermal effects amount to understanding: $\rho(s) \to \widetilde{D}_\beta(\vec{u},s)$

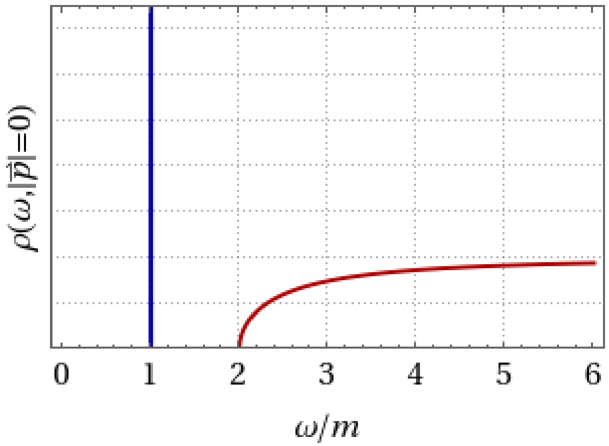
→ Evidence $\widetilde{D}_\beta(\vec{u},s)$ contains components $\widetilde{D}_{m,\beta}(\vec{u})\,\delta(s\text{-}m^2)$ **"Thermoparticles"**

$$\rho(\omega,\vec{p}) = \rho_{\rm coll}(\omega,\vec{p}) + \rho_{\rm TP}(\omega,\vec{p}) + \rho_{\rm thresh}(\omega,\vec{p}) + \cdots$$



$T \to 0$

$\widetilde{D}_{m,\beta}(\vec{u}) \to \delta^3(\vec{u})$



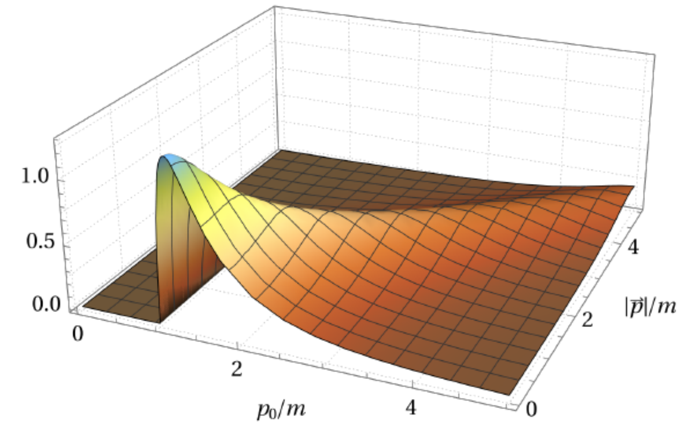
* [J. Bros and D Buchholz, ZPC 55 (1992), Ann. Inst. H.Poincare Phys.Theor. 64 (1996); Nair, Pisarski, PRD 113 (2026)]

- Non-trivial "Damping factor" $\widetilde{D}_{m,\beta}(\vec{u})$ results in thermally-broadened peak in the spectral function $\rightarrow$ *parametrises collisional broadening effects*

Thermoparticles have distinctive characteristics

- Spectral function has a threshold at $\omega = m$

$$\rho_{\mathrm{TP}}(\omega, \vec{p}) = \theta(\omega^2 - m^2)\varrho(\omega, \vec{p})$$



- Two-point function has factorised form

$$\boxed{\mathcal{W}_{\mathrm{TP}}(x_0, \vec{x}) = D_{m,\beta}(\vec{x})\mathcal{W}_{(0)}(x_0, \vec{x})}$$

$\rightarrow$ They dominate the large-time behaviour of $\langle \phi(x_0, \vec{x})\phi(0) \rangle$

$\rightarrow$ Along direction $\vec{x} = \vec{v}\, x_0$ they behave like $D_{m,\beta}(\vec{v}x_0)\,|x_0|^{-3/2}$ hence damping depends on velocity relative to the medium [Bros, Ann. H. Poincare 4, (2003)]

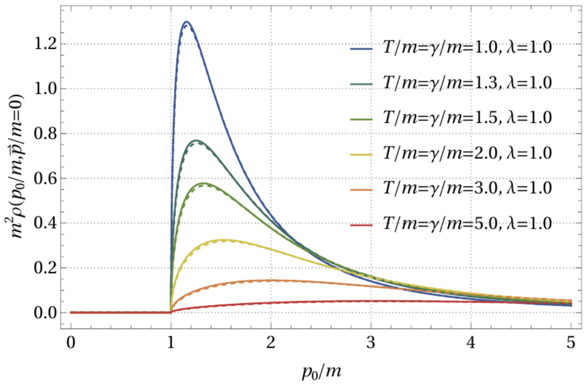
- Propagator singularities can be broader class than simple poles, e.g. when $D_{m,\beta}(\vec{x}) = e^{-\gamma|x|}$

$$\boxed{\widetilde{G}_{\mathrm{TP}}(k_0, \vec{p}) = -\frac{1}{k_0^2 - |\vec{p}|^2 - m^2 - 2\gamma\sqrt{m^2 - k_0^2} - \gamma^2}}$$

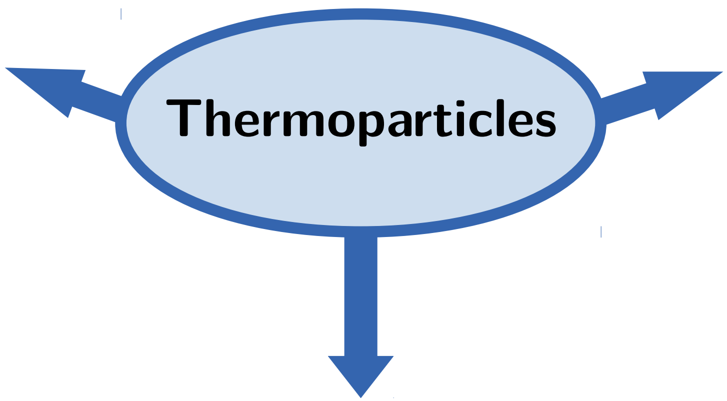
Thermoparticles

There is mounting evidence that thermoparticles play a central role in determining the characteristics of QFTs at finite temperature

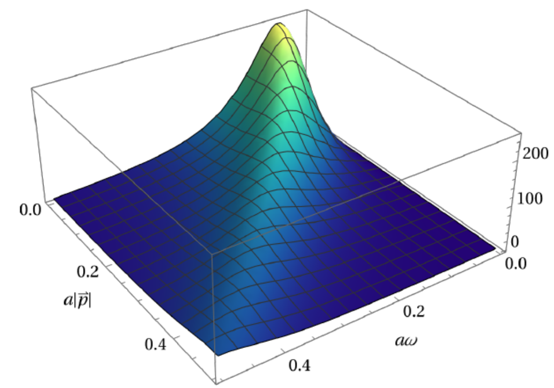
$T > 0$ perturbation theory



[Ali, PL, Philipsen, 2606.14863]
[PL, Philipsen, JHEP 08, 167 (2024)]

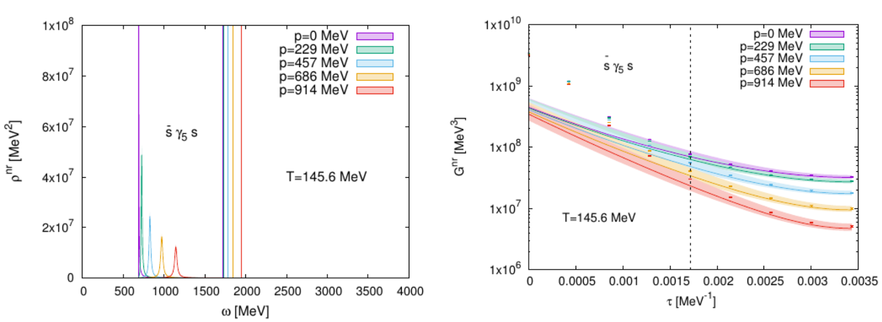


Spontaneous symmetry breaking when $T > 0$



[PL, Philipsen, JHEP 02, 090 (2026)]
[PL, Philipsen, PRD 112, L061701 (2025)]
[Bros, Buchholz, PRD 58 (1998)]

Hadrons at high temperature



[Bala, Kaczmarek, PL, Philipsen, Ueding, JHEP 05, 332 (2024)]
[PL, Philipsen, JHEP 10, 161 (2022)]

Thermoparticle perturbation theory

- Large-time dominance suggests thermoparticles are natural candidates for describing thermal scattering states: determines propagators in perturbative expansion $\rightarrow$ use TP propagators: **"Thermoparticle perturbation theory"**

$$\langle\Omega_\beta|T\left\{\phi(x_1)\cdots\phi(x_n)\right\}|\Omega_\beta\rangle = \frac{\langle\Omega_\beta^0|T\left\{\phi_{\rm TP}(x_1)\cdots\phi_{\rm TP}(x_n)e^{i\int_C \mathcal{L}_I[\phi_{\rm TP}]}\right\}|\Omega_\beta^0\rangle}{\langle\Omega_\beta^0|T\left\{e^{i\int_C \mathcal{L}_I[\phi_{\rm TP}]}\right\}|\Omega_\beta^0\rangle}$$

- The perturbative expansion has the ***same*** structure as the standard approach, but now propagator lines in Feynman diagrams are those of thermoparticles

$$\widetilde{\tau}^{++}_{\rm TP}(p) = \int \frac{d^3\vec{u}}{(2\pi)^3}\left[\frac{i}{p_0^2-(\vec{p}-\vec{u})^2-m^2+i\epsilon}+\frac{\delta\left(p_0^2-(\vec{p}-\vec{u})^2-m^2\right)}{e^{\beta|p_0|}-1}\right]\widetilde{D}_{m,\beta}(\vec{u})$$

Propagators have a *T*-dependent broadening

- ✓ IR divergences in massless theories are regularised
- ✓ "pinch" singularities from $\delta(p^2\text{-}m^2)^2$ type contributions are gone
- ✓ UV divergences are no more severe than at T=0 $\rightarrow$ renormalisable
- ✓ Propagator *uniquely* fixed by dynamics [Bros, Buchholz, NPB 627 (2002)]

Thermoparticle perturbation theory

Lattice test: perform lattice perturbation theory (LPT) calculations in Φ^4 theory using thermoparticle propagators & compare with standard approach

- Data suggests $D_\beta(\boldsymbol{x}) = e^{-\gamma|\boldsymbol{x}|}$, hence modified propagator structure

$$\boxed{G(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2 + \gamma^2 + 2\gamma\sqrt{m^2 + \omega_n^2}}}$$

… which generalises the free field ($\gamma=0$) result: $\boxed{G(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2}}$

In principle: dependence of $\gamma = \gamma(m, T, g)$ is fixed by the theory [Bros, Buchholz, 2002]

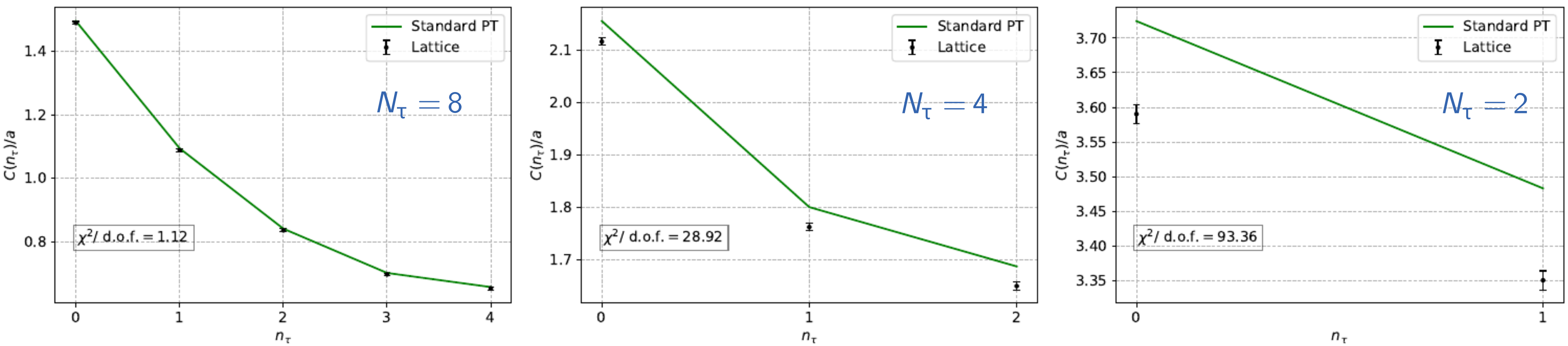
In practice: determine $\gamma = \gamma(am_0, g_0, N_\tau)$ by computing 2-loop LPT predictions of lattice ***spatial*** correlator $C(z) = a^3 \sum_{\tau,x,y} \langle \phi(\tau,\vec{x})\phi(0)\rangle$ until consistent with data

Test: LPT prediction of the ***temporal*** correlator $C(\tau) = a^3 \sum_{x,y,z} \langle \phi(\tau,\vec{x})\phi(0)\rangle$ at different $T=1/aN_\tau$ using extracted γ → ***Non-trivial test!***

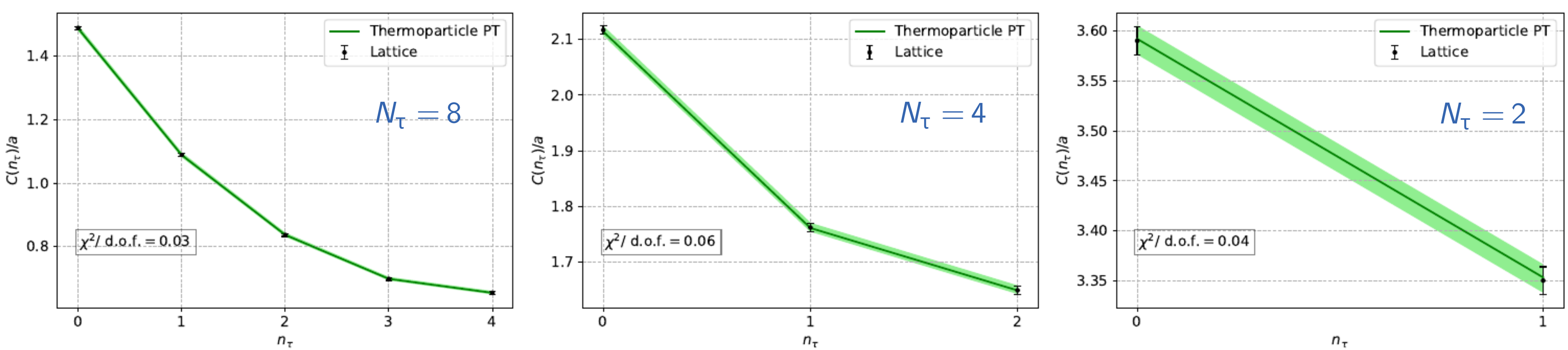
Thermoparticle perturbation theory

Results: $\gamma=0$ predictions consistent with $N_\tau = 16$ (coldest lattice) data

$\rightarrow$ But $\gamma=0$ predictions increasingly deteriorate with temperature



$\rightarrow$ TP predictions consistent with data at all temperatures (lattice errors <1%)



$\rightarrow$ Predictions highly sensitive to propagator parametrisation (e.g. Breit-Wigner not consistent)

Summary & outlook

- Perturbation theory has several known issues $\rightarrow$ realisation of a more fundamental constraint: non-trivial scattering at $T > 0$ is incompatible with real dispersion laws (NRT theorem)

- There is numerical (PL, Philipsen, 2024; Ali, PL, O. Philipsen, 2026) evidence for this, even for massive theories, and at small T

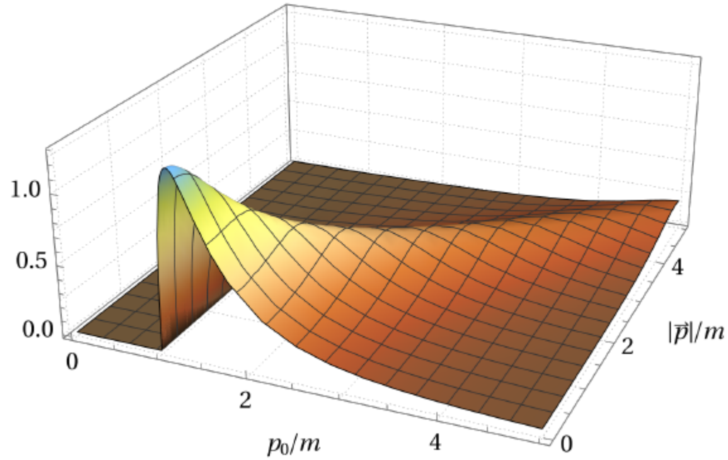
- A resolution of these issues is to use propagators which don't have real dispersion laws $\rightarrow$ ***Thermoparticles*** are natural candidates

- Thermoparticle perturbation theory gives finite-temperature predictions consistent with lattice Φ^4 theory data (Ali, PL, O. Philipsen, 2026)

$\rightarrow$ Framework is generalisable to any QFT

$\rightarrow$ Can compute real-time observables

$\rightarrow$ Study finite-*density* effects... work in progress!



Backup: 1-loop TP perturbation theory

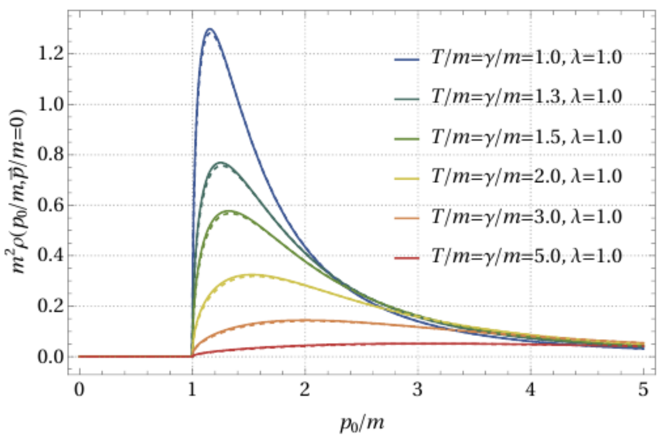
What form do the thermoparticles propagators have? $\rightarrow$ these can be *uniquely* fixed by the dynamical equations of the theory [Bros, Buchholz, 2002]

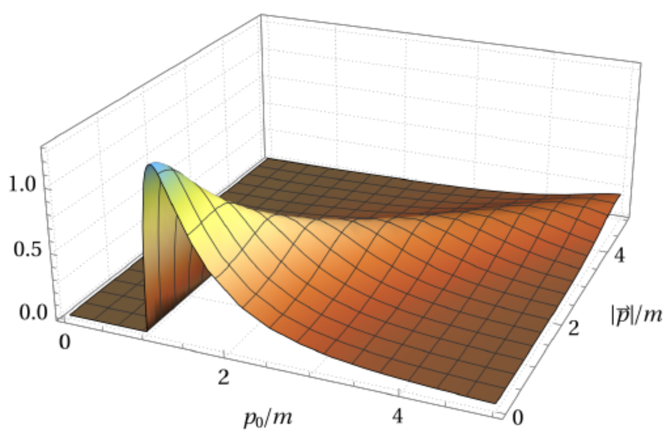
- Given these propagators, one can then perform perturbative calculations

<u>Example</u>: 1-loop calculation of the spectral function $\rho(p_0,\boldsymbol{p})$ in Φ^4 theory with an exponential damping factor $D_\beta(\boldsymbol{x})$ [P. Ali, PL, O. Philipsen, 2606.14863]

$$D_\beta(\boldsymbol{x}) = e^{-\gamma|\boldsymbol{x}|}$$

$$\rho_{\mathrm{TP}}(p_0,\vec{p}) = \epsilon(p_0)\theta(p_0^2-m^2)\,\frac{4\gamma\sqrt{p_0^2-m^2}}{(|\vec{p}|^2+m^2-p_0^2)^2+2(|\vec{p}|^2-m^2+p_0^2)\gamma^2+\gamma^4}$$





Broadening effects already enter at leading order!

... compared with standard approach: $\rho_{(1)}(p_0,\vec{p}) = 2\pi\epsilon(p_0)\left[\delta(p^2-m^2) - \Delta m_+^2\,\delta'(p^2-m^2)\right]$

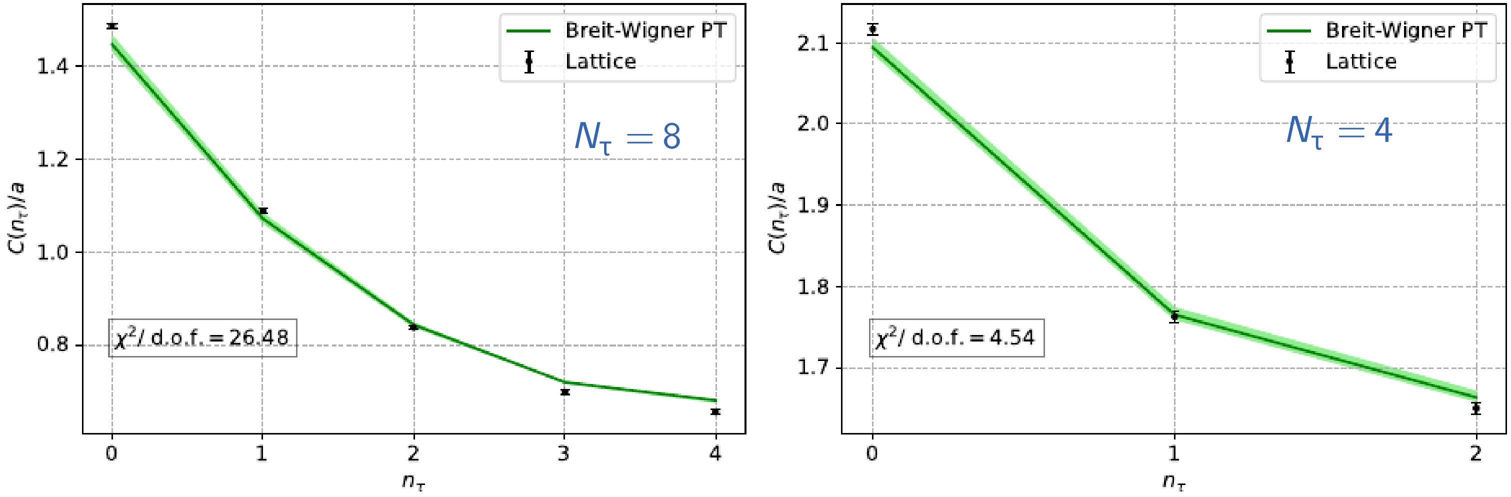
Consistency check: perform the same procedure with different propagator parametrisations, e.g. Breit-Wigner

- In the Breit-Wigner case the spectral function and propagator have the form:

$$\rho_{\mathrm{BW}}(\omega,\vec{p}) = \frac{4\omega\Gamma}{(\omega^2-|\vec{p}|^2-m^2-\Gamma^2)^2+4\omega^2\Gamma^2}$$

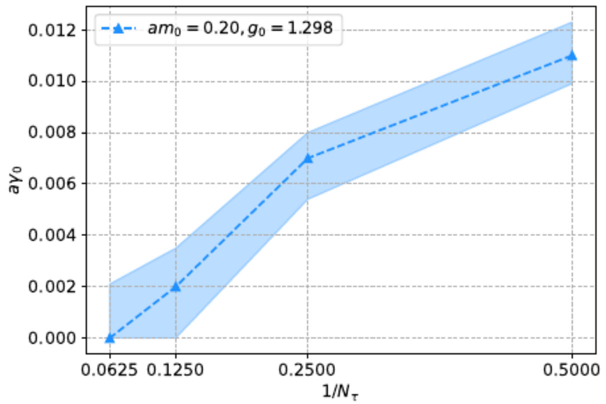
$$G_n(\omega_n,\vec{p}) = \frac{1}{\omega_n^2+|\vec{p}|^2+m^2+\Gamma^2+2\Gamma|\omega_n|}$$

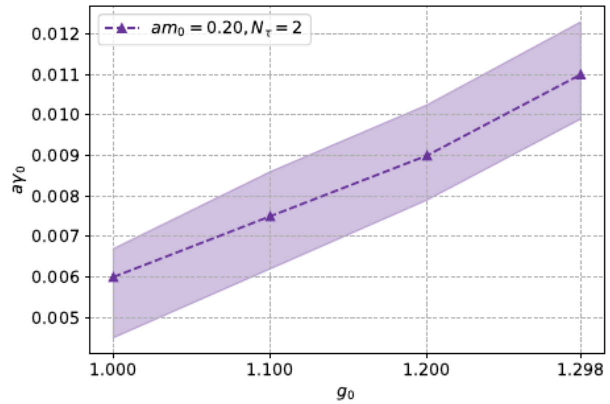
- Extract width Γ for different (am_0, g_0, N_τ) from $C(z)$ and use this to compute 2-loop prediction of $C(\tau)$ $\rightarrow$ no longer describes the data!



Backup: Parametric dependence of damping

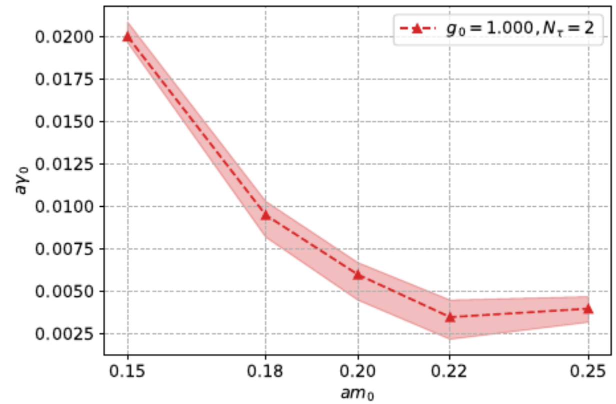
- One can also explore how the damping parameter $a\gamma_0(am_0, g_0, N_\tau)$ depends on the different lattice parameters
- Increasing temperature (i.e. $1/N_\tau$) or the coupling g_0 causes $a\gamma_0$ to **increase**





Thermoparticle scattering states experience greater in-medium effects

- Increasing the lattice mass am_0 results in a **decrease** of $a\gamma_0$



Vacuum mass is larger, hence thermoparticle states less sensitive to properties of the medium

Backup: More perturbative subtleties

- Weldon [*PRD* 65 (2002)] showed that (scalar) perturbation theory is inconsistent at some fixed loop order

  → One can classify the singularities occurring in the n-loop perturbative propagator given the singularities of the basic propagator

  → If the basic propagators have real poles $\omega = E(\boldsymbol{p})$, at some loop order the perturbative propagator will develop a branch point singularity at these poles, e.g. in Φ^4 theory this occurs at 2-loop order

$$G_R(p) = -\frac{1}{(\omega + i\epsilon)^2 - E(\vec{p})^2}$$

- This prohibits corrections to the perturbative thermal propagator pole from being computed, hence the standard approach breaks down

  → This analytic problem is consequence of the NRT theorem and is not restricted to massless or high-temperature regimes

Backup: Φ^4 theory on the lattice

- The implications of these constraints can be looked for numerically by comparing perturbative predictions with lattice calculations

  <u>Idea</u> [PL, O. Philipsen, 2405.02009]: perform lattice simulations of basic field correlators in Φ^4 theory and compare these with the predictions of lattice perturbation theory (LPT) $\rightarrow$ see: Montvay & Münster

- In LPT one uses the discretised propagator $$\boxed{G(p;a,L,L_\tau) = \frac{1}{\sum_\mu \frac{4}{a^2}\sin^2\left(\frac{ap_\mu}{2}\right) + m_0^2}}$$

- Theory is defined on a finite $N_s^3 \times N_\tau$ lattice $\rightarrow$ internal loop integrals are replaced with sums over first Brillouin zone $\mathcal{B}_a = \{p \in \mathbb{R}^4 : -\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}\}$

- In Φ^4 theory the lattice action is: $$\boxed{S = a^4 \sum_{x\in\Lambda}\left[\frac{1}{2}\Delta_\mu^f\phi_0(x)\Delta_\mu^f\phi_0(x) + \frac{m_0^2}{2}\phi_0(x)^2 + \frac{g_0}{4!}\phi_0(x)^4\right]}$$

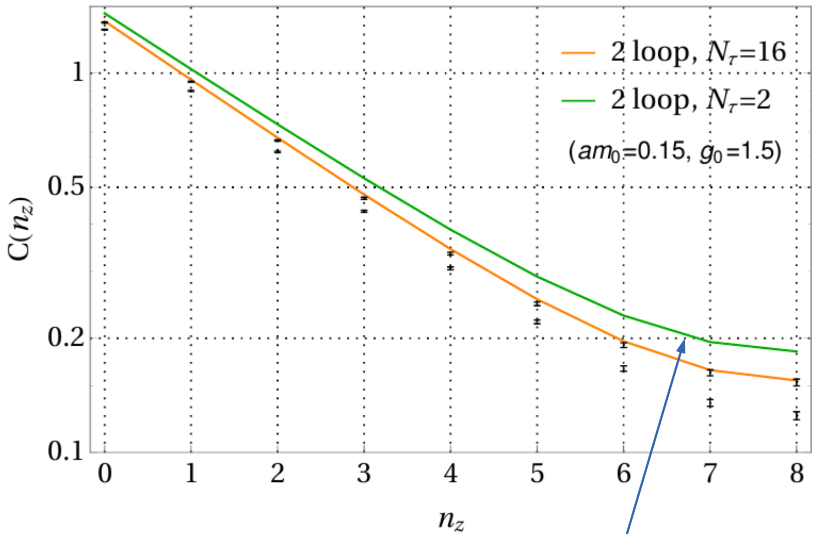
- Here we focus on perturbative calculations of the *spatial* and *temporal* correlators
$$\boxed{\begin{aligned} C(z;N_s,N_\tau) &= a^3 \sum_{\tau,x,y}\langle\phi(\tau,\vec{x})\phi(0)\rangle, \\ C(\tau;N_s,N_\tau) &= a^3 \sum_{x,y,z}\langle\phi(\tau,\vec{x})\phi(0)\rangle. \end{aligned}}$$

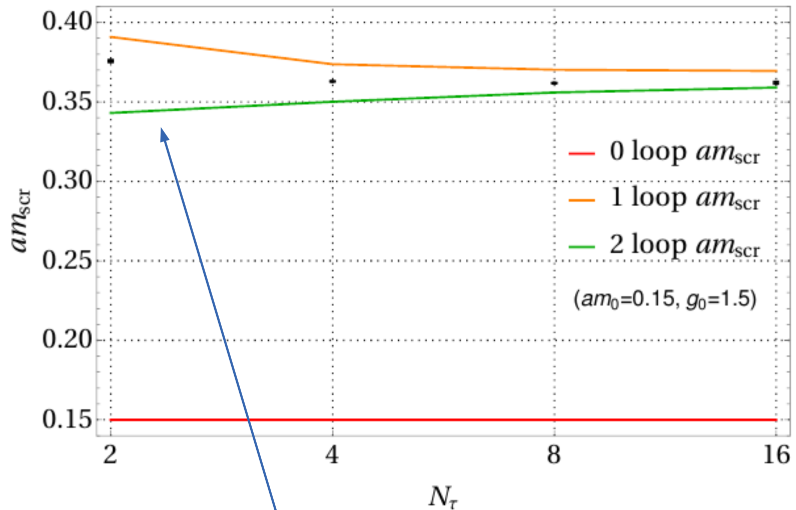
Backup: Φ^4 theory on the lattice

- How do the 2-loop LPT predictions compare with the lattice data as one varies the (lattice) temperature $T = 1/(aN_\tau)$?

  1. For sufficiently small g_0 the perturbative predictions are consistent with the data for all temperatures

  2. For non-negligible g_0 the perturbative predictions for $C(z)$ increasingly deteriorate as T/m increases





For $T/m \sim 1$ the 2-loop predictions deviate drastically from the data

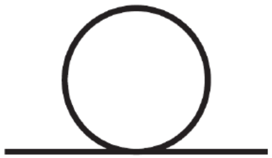
These deviations are also reflected by the screening mass m, defined: $C(z) \sim e^{-m z}$, having the *opposite* T dependence

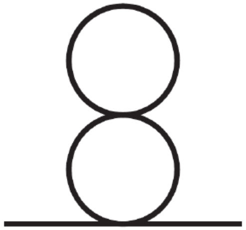
Backup: Φ^4 theory on the lattice

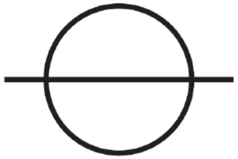
- To establish the origin of these deviations one needs to understand how the 2-loop perturbative predictions are computed

$$\boxed{C(z;a,N_s,N_\tau)=\frac{1}{N_s}\sum_{k_z=0}^{N_s-1}e^{\frac{2\pi i k_z}{aN_s}z}\frac{a}{4\sin^2\left(\frac{\pi k_z}{N_s}\right)+(am_0)^2+a^2\Pi(\omega_E=p_x=p_y=0,p_z=\frac{2\pi k_z}{aN_s};a,N_s,N_\tau)}.}$$

$$\boxed{a^2\Pi_{(2)}(p;a,N_s,N_\tau)=\frac{g_0}{2}J_1(am_0;N_s,N_\tau)-\frac{g_0^2}{4}J_1(am_0;N_s,N_\tau)J_2(am_0;N_s,N_\tau)-\frac{g_0^2}{6}I_3(p,am_0;N_s,N_\tau)}$$







The functions J_n and I_3 are defined:

$$J_n(am_0;N_s,N_\tau)=\frac{1}{N_s^3N_\tau}\sum_{p\in\mathcal{B}_a}\frac{1}{\left[\sum_\mu 4\sin^2\left(\frac{ap_\mu}{2}\right)+(am_0)^2\right]^n},\qquad n=1,2$$

$$I_3(p,am_0;N_s,N_\tau)=\frac{1}{(N_s^3N_\tau)^2}\sum_{q\in\mathcal{B}_a}\sum_{r\in\mathcal{B}_a}\frac{1}{\left[\sum_\mu 4\sin^2\left(p_\mu-\frac{aq_\mu}{2}-\frac{ar_\mu}{2}\right)+(am_0)^2\right]}$$
$$\times\frac{1}{\left[\sum_\nu 4\sin^2\left(\frac{aq_\nu}{2}\right)+(am_0)^2\right]}\frac{1}{\left[\sum_{\Lambda_a} 4\sin^2\left(\frac{ar_\lambda}{2}\right)+(am_0)^2\right]}$$

Backup: Φ^4 theory on the lattice

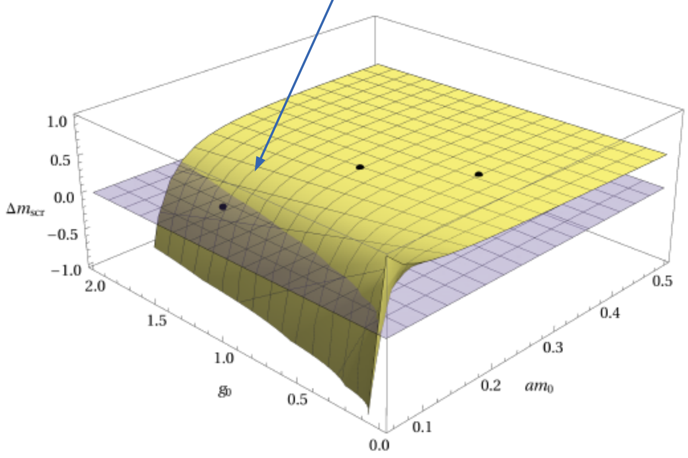
- Numerically, one finds that the sunset diagram contribution is sub-dominant, and that the behaviour of $C(z)$ is largely controlled by the interplay between the tadpole and cactus diagrams

- As one varies (am_0, g_0) there are certain choices for which the self-energy at any fixed N_τ is ***smaller*** than when $N_\tau \to \infty$ (i.e. $T=0$)

  → This implies that the screening mass $m_{\rm scr}$, which is defined via the (continuum) large-z behaviour: $C(z) \sim \exp(-m_{\rm scr}\, z)$, ***decreases*** with T

$$\boxed{\Delta m_{\rm scr} = \frac{\frac{g_0}{2} J_1(am_0; N_s, N_\tau=1) - \frac{g_0^2}{4} J_1(am_0; N_s, N_\tau=1) J_2(am_0; N_s, N_\tau=1)}{\frac{g_0}{2} J_1(am_0; N_s, N_\tau=\infty) - \frac{g_0^2}{4} J_1(am_0; N_s, N_\tau=\infty) J_2(am_0; N_s, N_\tau=\infty)} - 1,}$$

Predictions with (am_0, g_0) closer to the region $\Delta m_{\rm scr} < 0$ deviate further from the data



- When $\Delta m_{\rm scr} < 0$, and the sunset is sufficiently small, then: $m_{\rm scr}\,(T=0) > m_{\rm scr}\,(T>0)$

- This behaviour is ***opposite*** to physical expectations (and the lattice data!)

Backup: Φ^4 theory on the lattice

- The proximity to the region $\Delta m_{\text{scr}} < 0$ depends on the structure of the propagators entering the loop calculations

- For fixed g_0 one *always* move into this region when am_0 is sufficiently small

  $\rightarrow$ This can be understood by the fact that for small am_0 the zero-mode contributions dominate the sums in J_n, and these have the form:

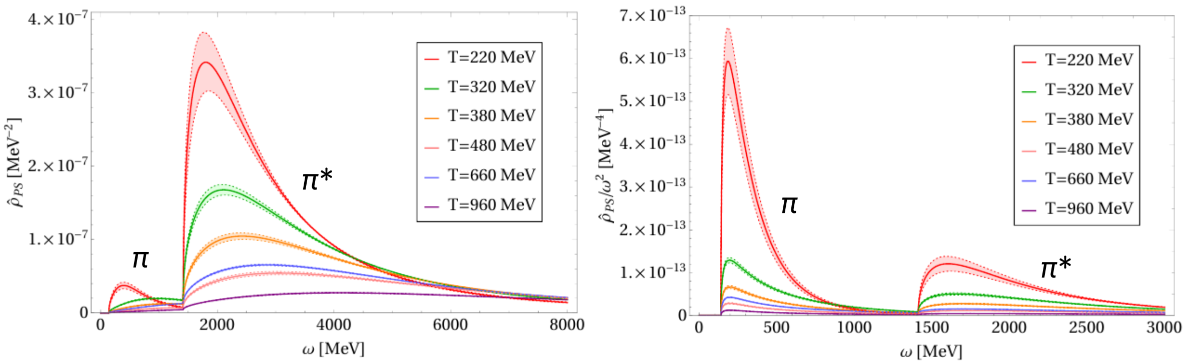
$$\boxed{J_1(am_0; N_s, N_\tau) \sim \frac{1}{N_s^3 N_\tau}\frac{1}{(am_0)^2}, \quad J_2(am_0; N_s, N_\tau) \sim \frac{1}{N_s^3 N_\tau}\frac{1}{(am_0)^4}.}$$

  $\rightarrow$ The cactus diagram outcompetes the tadpole for small enough am_0

Conclusion: the perturbative prediction deviations are driven by the structure of the free field propagators $\rightarrow$ consistent with the constraints of the NRT theorem

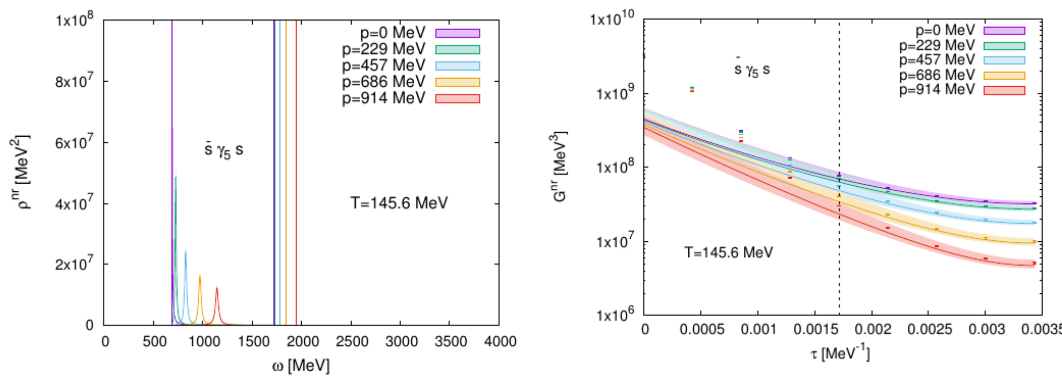
- Need to start with propagators that take thermal effects into account from the outset, i.e. non-real singularities $\rightarrow$ ***But what form should these take?***

- Evidence for low-energy thermoparticle excitations in QCD $\to$ extraction of spectral function for pseudo-scalar meson operator $\boxed{\mathcal{O}^a_{\text{PS}} = \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi}$ in several different quark channels



Light-light pseudo-scalar meson (pion) channel [P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

Light-strange (kaon) and strange-strange (eta) pseudo-scalar meson channels [D. Bala, O. Kaczmarek, P. L., O. Philipsen, and T. Ueding, *JHEP* 05, 332 (2024)]



Data in *all* channels consistent with a thermoparticle-type ground state: suggests light pseudo-scalar mesons (pions, kaons,..) still have a bound-state-like structure, even at high T

Backup: Hadrons at high temperature

Goal: Extract information about the finite T spectral function $\rho_\Gamma(\omega,\boldsymbol{p})$ from data of *Euclidean* correlator $C_\Gamma(\tau,\vec{x}) = \langle O_\Gamma(\tau,\vec{x})\, O_\Gamma(0,\vec{0})\rangle_T$ O_Γ = scalar operator

- Standard approach: extract $\rho_\Gamma(\omega,\boldsymbol{p})$ from temporal correlator $\widetilde{C}_\Gamma(\tau,\boldsymbol{p})$

$$\widetilde{C}_\Gamma(\tau,\vec{p}) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh\left[\left(\frac{\beta}{2} - |\tau|\right)\omega\right]}{\sinh\left(\frac{\beta}{2}\omega\right)} \rho_\Gamma(\omega,\vec{p})$$

→ *Problem is ill-conditioned, need more information!*

- Instead, one can use the ***spatial*** correlator, where one integrates $C_\Gamma(\tau,\boldsymbol{x})$ over $\{\tau,x,y\}$ and fixes a spatial direction z

$$C_\Gamma(z) = \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} e^{ip_z z} \int_0^\infty \frac{d\omega}{\pi\omega}\, \rho_\Gamma(\omega, p_x = p_y = 0, p_z)$$

- It turns out that *if* thermoparticles exist, then they will give a distinct contribution to $C(z)$

$$C(z) \approx \frac{1}{2}\int_{|z|}^{\infty} dR\; e^{-mR} D_{m,\beta}(R)$$

[P.L., *PRD* 106 (2022); P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

→ This component can be extracted *directly* from data

Backup: Hadrons at high temperature

- In particular, one can extract the damping factors $D_\beta(\boldsymbol{x})$ and their temperature dependence from $C(z)$, and then use this to determine the spectral contribution $\rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$

$C(z)$ data $\longrightarrow$ $D_\beta(\boldsymbol{x})$

$+$

$$\boxed{\rho(\omega,\vec{p}) = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2 - (\vec{p}-\vec{u})^2 - s\big)\, \widetilde{D}_\beta(\vec{u},s)}$$

$\rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$

- Can use temporal correlator $\widetilde{C}_\Gamma(\tau,\boldsymbol{p})$ data to check whether extracted $\rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$ components are consistent $\longrightarrow$ a *highly non-trivial test!*

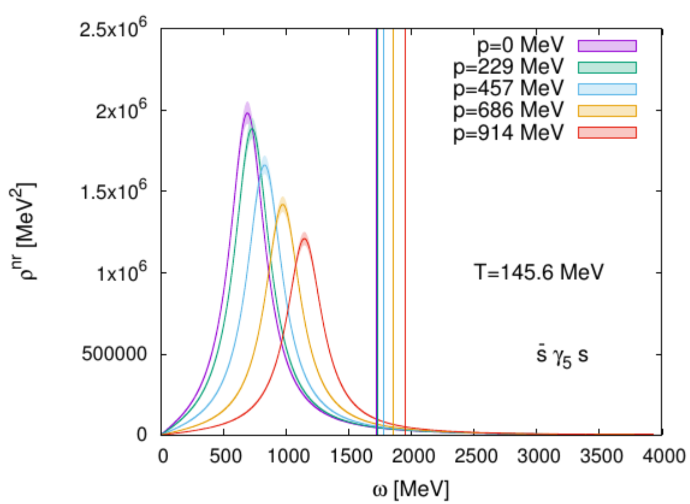
- Test was performed for $\boldsymbol{p}=0$ in [P.L., Philipsen, 2022] (light-light channel), and generalised to $|\boldsymbol{p}|>0$ in [Bala, Kaczmarek, P.L., Philipsen, Ueding, 2024] (light-strange and strange-strange channels)

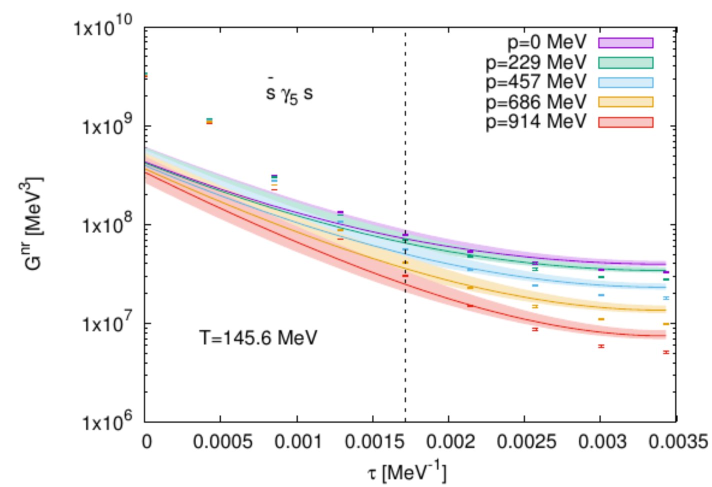
- The robustness of the thermoparticle hypothesis can also be tested by comparing with different causal models, e.g. a Breit Wigner

$$\rho_{\mathrm{BW}}(\omega,\vec{p}) = \frac{4\omega\Gamma}{(\omega^2 - |\vec{p}|^2 - m^2 - \Gamma^2)^2 + 4\omega^2\Gamma^2},$$

$$C_{\mathrm{BW}}(z) = \frac{e^{-\sqrt{m^2+\Gamma^2}|z|}}{2\sqrt{m^2+\Gamma^2}}.$$

- Same procedure as with the thermoparticle case: (i) extract the width parameter Γ and coefficient from the spatial lattice data (ii) use this to predict the corresponding temporal correlator



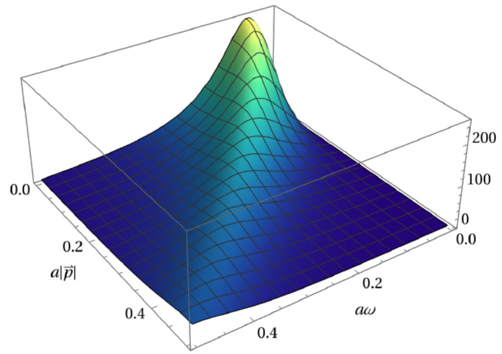


→ Data is *not* consistent with a Breit-Wigner-type ground state!

Backup: SSB when $T > 0$

- Thermoparticles play an essential role in the generalisation of spontaneous symmetry breaking to finite temperature [Bros, Buchholz, *PRD* 58 (1998)]

- Evidence [PL, O. Philipsen, 2501.17120, 2507.14348] that $T > 0$ Goldstone mode in U(1) complex scalar theory is a ***massless*** thermoparticle, and persists for $T > T_{\mathrm{c}}$



$$\boxed{D_{\beta}^{G}(\vec{x}, s) = D_{\beta}^{G}(\vec{x})\delta(s)}$$

$\rightarrow$ *The damping factor determines the phase structure, i.e. value of order parameter q*

"Strong dissipation", $D^{G}(\boldsymbol{x}) \rightarrow 0$ for $|\boldsymbol{x}| \rightarrow \infty$, implies $q = 0$, **Restored phase**

$T = T_{\mathrm{c}}$

"Weak dissipation", $D^{G}(\boldsymbol{x}) > 0$ for $|\boldsymbol{x}| \rightarrow \infty$, implies $q > 0$, **Broken phase**

$T = 0$ $D^{G}(\boldsymbol{x}) =$ const, i.e. no thermal effects, hence $p^2 = 0$ mode

This captures the physics! $\rightarrow$ sufficiently strong thermal effects destroy the long-range order, which leads to symmetry restoration

Backup: SSB when $T > 0$

Goldstone's theorem $\rightarrow$ *What happens when a continuous global symmetry of a QFT is spontaneously broken?*

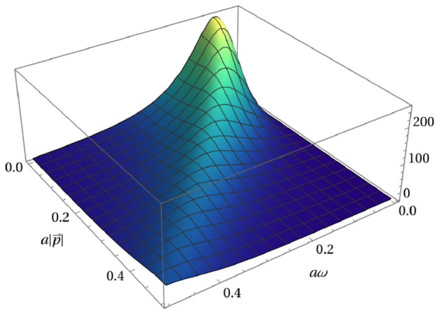
- If j^μ is the conserved current associated with the symmetry, and A is some local field whose transformation δA under the symmetry has a non-trivial expectation value: $<\delta\mathrm{A}> = \lim_{\mathrm{R}\to\infty} <[Q_R, A]> = q \neq 0$, then*

$\rightarrow$ When $T = 0$ the Fourier transform of $<[j^{\,0}(\mathrm{x}), A(0)]>$ contains a $\delta(p^2)$ component, i.e. a massless Goldstone boson

***T* > 0 case**: using spectral representation for $<[j^{\,0}(\mathrm{x}), A(0)]>_\beta$ the SSB condition $<\delta\mathrm{A}>_\beta \neq 0$ implies [Bros, Buchholz, 1998] that the Fourier transform contains a massless **thermoparticle** $D^G_\beta(\vec{x}, s) = D^G_\beta(\vec{x})\delta(s)$

$\rightarrow$ This a **thermal Goldstone boson**: in the $T \to 0$ limit, $D^G(\boldsymbol{x}) \to$ const, and one recovers $\delta(p^2)$

$\rightarrow$ Non-trivial damping factor $D^G(\boldsymbol{x})$ causes broadening of the $p^2 = 0$ Goldstone peak



* See: [F. Strocchi, Symmetry Breaking, Lect. Notes Phys. 732 (2008)]

Backup: SSB when $T > 0$

- In order to define SSB rigorously one needs to define a regularised charged operator Q_R, and this only converges for $R \to \infty$ within commutator $[Q_R, A]$

$$\boxed{Q_{\delta R} = \int d^4x \, \alpha_\delta(t) g_R(\vec{x}) j_0(t, \vec{x})}$$

(i) $g_R(\boldsymbol{x})=1$ for $|\boldsymbol{x}| \leq R$, $g_R(\boldsymbol{x})=0$ for $|\boldsymbol{x}| > R$

(ii) $\alpha_\delta(t)$ has compact support, and $\alpha_\delta(t) \to \delta(t)$

- The condition: $\lim_{R\to\infty} \langle [Q_R, A] \rangle = q$ is then ***always*** well-defined, and $q=0$ is a necessary and sufficient condition for the existence of a charge operator Q, defined by: $\langle u, Qv \rangle := \lim_{R\to\infty} \langle u, Q_R v \rangle$

$\rightarrow$ If this is non-vanishing for *any* A then no such charge exists, i.e. SSB!

- SSB condition: $\boxed{\lim_{\delta\to 0} \lim_{R\to\infty} \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^3} \frac{d^3\vec{p}}{(2\pi)^3} \, \widetilde{D}_\beta^{(+)}(\vec{u}, s) \, \tilde{g}(\vec{p}) \, \tilde{\alpha}\left(\delta R \sqrt{((\vec{p}/R) - \vec{u})^2 + s} \right) = iq\tilde{\alpha}(0)}$

- In the vacuum case this implies: $\widetilde{D}^{(+)}(\boldsymbol{u},s) \to iq \, (2\pi)^3 \, \delta(\boldsymbol{u})\delta(s)$

- Value of q is ***determined*** by Goldstone damping factor $D^{(+)}(\boldsymbol{x})$ for $|\boldsymbol{x}| \to \infty$

Backup: SSB when $T > 0$

- The phase of the theory is determined by the dissipative effects experienced by the thermal Goldstone mode

- Only $D^G(\boldsymbol{x}) \to 0$ for $|\boldsymbol{x}| \to \infty$ is required to ensure the symmetry is restored, but this can happen at any rate

- If the damping factor has the functional form $D^G(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-\varepsilon}$ with some $\varepsilon > 0$ at large $|\boldsymbol{x}|$, the two-point function decays as a pure power-law

> **Interesting possibility**: a symmetry could be restored at high temperatures without there being a finite correlation length on either side of the phase transition

- Damping factors are fixed by the (asymptotic) dynamics of the theory [Bros, Buchholz, (2002)] $\rightarrow$ suggests that finite-temperature phase transitions may not be entirely characterised by universality arguments

Backup: SSB when $T > 0$

- Now we know what the signatures of thermal Goldstone modes are, one can look for them in lattice data

- Consider a simple model with SSB: U(1) complex scalar field theory

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi - \frac{\lambda}{4!}(\phi^\dagger\phi)^2}$$

- In the broken phase at $T = 0$ the model contains a massless Goldstone boson and a resonance-like mode

  → Model expected* to undergo a second-order phase transition: for $T > T_c$ the U(1) symmetry is restored, and $|v|^2 = <\Phi><\Phi^\dagger> = 0$

- Investigate theory on a $L_\tau \times L^3$ lattice ($L_\tau = a\,N_\tau$, $L = a\,N_s$) with action

$$\boxed{S = a^4 \sum_{x\in\Lambda_a}\left[\sum_\mu \left(\frac{1}{2}\Delta_\mu^f\phi^*(x)\Delta_\mu^f\phi(x)\right) + \frac{m_0^2}{2}\phi^*(x)\phi(x) + \frac{g_0}{4!}\left(\phi^*(x)\phi(x)\right)^2\right]}$$

→ Avoid triviality by keeping a>0 fixed, hence $T = (aN_\tau)^{-1}$ is varied in discrete steps

* See e.g. [J. I. Kapusta and C. Gale, *Finite-temperature Field Theory*]

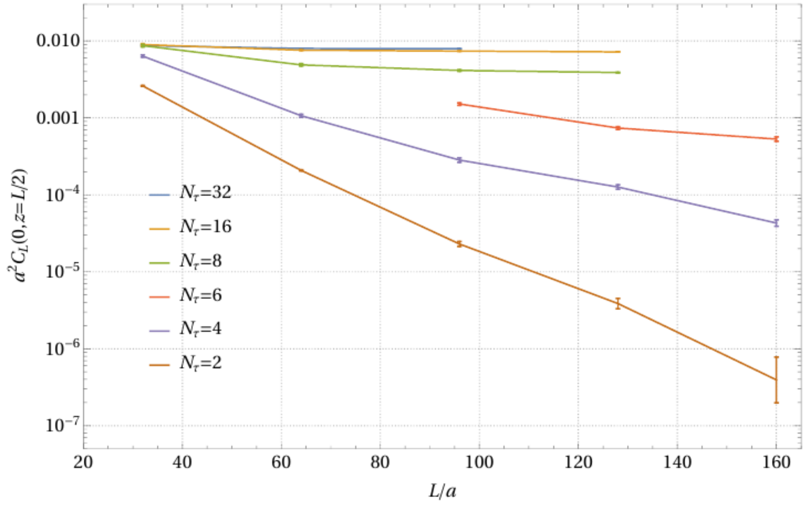
Backup: SSB when $T > 0$

- SSB does not occur in a finite spatial volume $V = L^3$, i.e. there is no notion of a "vev" on the lattice

  $\rightarrow$ Need to perform $L \to \infty$ extrapolation of lattice results!

- For the finite-volume correlator $C_L(\tau, \boldsymbol{x})$: $\boxed{\lim\limits_{L\to\infty} C_L(\tau,\vec{x}) \xrightarrow{|\vec{x}|\to\infty} |v|^2}$

- Based on this property there are different approaches* for extracting $|v|^2$, here we use: $\boxed{|v|^2 = \lim\limits_{L\to\infty} C_L(0, |\vec{x}| = L/2)}$



- $N_\tau = 8, 16, 32$, non-zero for large L $\rightarrow$ **U(1)** ***broken***

- $N_\tau = 2, 4, 6$, vanishing for large L $\rightarrow$ **U(1)** ***restored***

- Infinite-volume extrapolation requires a parametrisation for $C_L(\tau{=}0, |\boldsymbol{x}| = z)$

- Use finite-volume version of $\boxed{C^G(0,\vec{x}) = \frac{\coth\left(\frac{\pi|\vec{x}|}{\beta}\right)}{4\pi\beta|\vec{x}|} D^G_\beta(\vec{x})}$

* See e.g. [H. Neuberger, *PRL* 60,(1988).]

Backup: SSB when $T > 0$

<u>Broken phase</u> ($N_\tau = 8, 16, 32$)

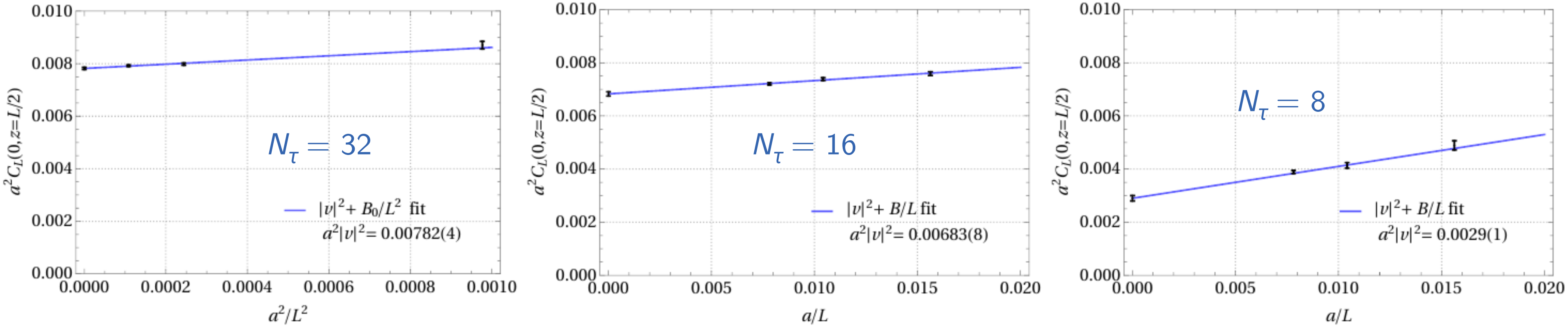
- In this case we assume $D^G(\boldsymbol{x}) \sim$ const and fit the ansatz:

$$\boxed{C_L(0,z) = c_L + b_L \left[\frac{\coth\left(\frac{\pi z}{\beta}\right)}{z} + \{z \to (L-z)\} \right]}$$

Non-zero if in broken phase

Finite L symmetrisation

- Functional form provides very good description of the data for each volume considered ($L/a = 32, 64, 96, 128$). Use this to do $L \to \infty$ extrapolation:



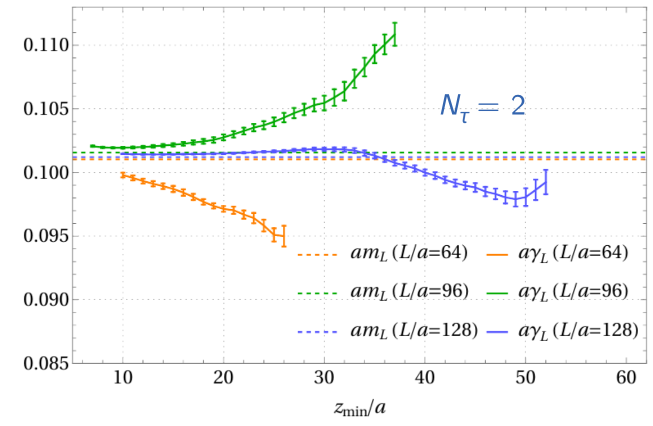
Backup: SSB when $T > 0$

<u>Restored phase</u> ($N_\tau = 2, 4, 6$)

- As outlined previously, the Goldstone mode can still persist in this regime, and would have the properties of a massless thermoparticle
- What is the structure of the damping factor $D^G(\boldsymbol{x})$? Use **spatial** correlator:

$$\boxed{C_L(z) = d_L\left[e^{-m_L z} + \{z \to (L-z)\}\right]} \;\Rightarrow\; \boxed{D_\beta^G(\vec{x}) = \alpha\, e^{-\gamma|\vec{x}|}} \;\Rightarrow\; \boxed{C^G(0,\vec{x}) = \frac{\coth\left(\frac{\pi|\vec{x}|}{\beta}\right)}{4\pi\beta|\vec{x}|}\alpha\, e^{-\gamma|\vec{x}|}}$$

- Spatial correlator fits provide excellent description of data over full range $[0, L/2]$ for each of the (large) volumes considered ($L/a = 64, 96, 128, 160$)
- Test damping by fitting: $\boxed{C_L(0,z) = b_L\left[\frac{\coth\left(\frac{\pi z}{\beta}\right)}{z}e^{-\gamma_L z} + \{z \to (L-z)\}\right]}$
- Consistency of the thermoparticle hypothesis requires that the fit parameters γ_L and m_L approach one another in the infinite-volume $L \to \infty$ limit



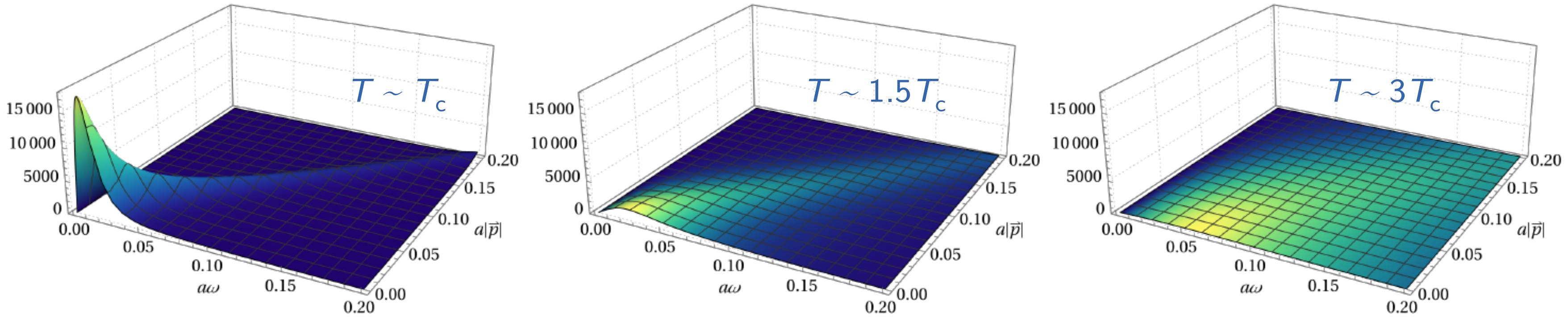
Backup: SSB when $T > 0$

<u>Restored phase</u> ($N_\tau = 2, 4, 6$)

- Data at each N_τ consistent with the correlator being dominated by an exponentially-damped massless thermoparticle $\rightarrow$ **thermal Goldstone**
- One can use the extracted damping factor $D^G(\boldsymbol{x}) = \alpha\, e^{-\gamma|\boldsymbol{x}|}$ to compute the spectral function $\rho_G(\omega,\boldsymbol{p})$ of the Goldstone mode

$$\boxed{\rho_G(\omega,\vec{p}) = \frac{4\alpha\,\omega\gamma}{(\omega^2 - |\vec{p}|^2 - \gamma^2)^2 + 4\omega^2\gamma^2}}$$

- Spectral properties are very different to the $T = 0$ case $\rho_G(\omega,\boldsymbol{p}) \sim \delta(p^2)$



- Broadened peak structure around $p^2 = 0$ increases with temperature

$\rightarrow$ Represents the increasingly strong thermal effects of the medium!

Backup: Asymptotic thermal fields

$\rightarrow$ Given a specific QFT, what form should the thermoparticles take?

Idea: thermal scattering states are defined by imposing an asymptotic field condition [Bros, Buchholz, (2002)]:

Asymptotic fields Φ_0 are assumed to satisfy dynamical equations, but only at large x_0

In Φ^4 theory

$$(\partial^2 + m^2)\phi_0(x) + \frac{\lambda}{3!}\phi_0^3(x) \xrightarrow{|x_0|\to\infty} 0$$

- Since thermoparticles dominate the large-time behaviour of correlators, they are natural candidates for describing such states. It turns out that their damping factors $\widetilde{D}_{m,\beta}(\boldsymbol{u})$ are **uniquely fixed** by the asymptotic condition
- In Φ^4 theory one finds (where κ is a thermal width):

For $\boldsymbol{\lambda} < 0$: $\widetilde{D}^{(-)}_{m,\beta}(\vec{u}) = \frac{2\pi^2}{\kappa^2}\delta(|\vec{u}| - \kappa),$

$$\Downarrow$$

$$\widetilde{G}^{(-)}_{\beta}(k_0,\vec{p}) = \frac{1}{4|\vec{p}|\kappa}\ln\left[\frac{-k_0^2 + m^2 + (|\vec{p}| + \kappa)^2}{-k_0^2 + m^2 + (|\vec{p}| - \kappa)^2}\right]$$

For $\boldsymbol{\lambda} > 0$: $\widetilde{D}^{(+)}_{m,\beta}(\vec{u}) = \frac{4\pi}{\kappa_0\left(|\vec{u}|^2 + \kappa^2\right)},$

$$\Downarrow$$

$$\widetilde{G}^{(+)}_{\beta}(k_0,\vec{p}) = \frac{i}{2|\vec{p}|\kappa_0}\ln\left[\frac{\sqrt{-k_0^2 + m^2} - i|\vec{p}| + \kappa}{\sqrt{-k_0^2 + m^2} + i|\vec{p}| + \kappa}\right]$$