

Thermal nucleation in perturbation theory

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20 Aug 2026

Fig: arXiv:1906.00480

Overview of the talk

0. Introduction
1. Wigner function approach
 - ▶ Real-time quantum foundations
 - ▶ Potential for going further
2. Equilibrium effective field theory approach
 - ▶ Very useful for calculations
 - ▶ See Bogumiła's and Maciej's talks
3. Kinetic theory approach
 - ▶ Backs up equilibrium EFTs
 - ▶ Dynamical prefactor

Fig: arXiv:1906.00480

Nucleation

- System trapped in ϕ_1
- Escaping locally with bubble nucleation to ϕ_2

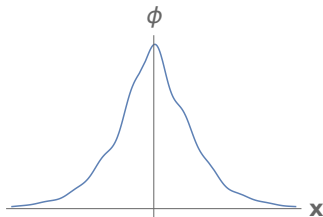


Figure: Cross section of a bubble

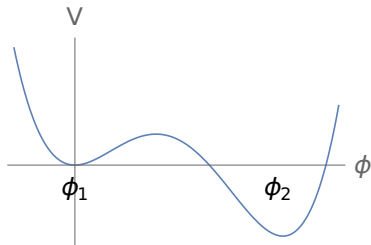


Figure: Potential with a transition

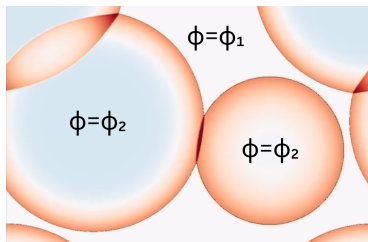


Figure: Growing bubbles [Cutting et al. '20]

Cosmological phase transitions

- Plasma shocks from bubbles
→ Gravitational waves
- Possibly observable with e.g. LISA
 - ▶ BSM signal

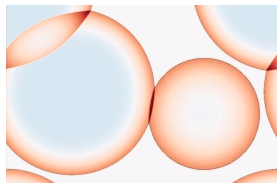


Figure: Growing bubbles ([arXiv:1906.00480](https://arxiv.org/abs/1906.00480))

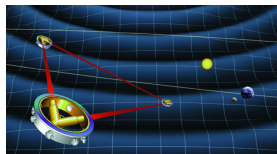


Figure: Gravitational waves and LISA
(ESA)

Nucleation rates and GWs

- Nucleation rate, Γ , changes in time
- Allows to take place
- Duration

$$\Delta t \sim \left(\frac{d}{dt} \ln \Gamma \right)^{-1}$$

- Transition temperature

$$\Delta t^4 \Gamma \sim 1 \Rightarrow T$$

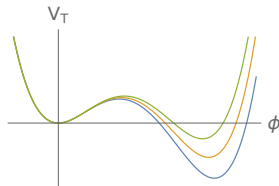


Figure: T -dependent potential

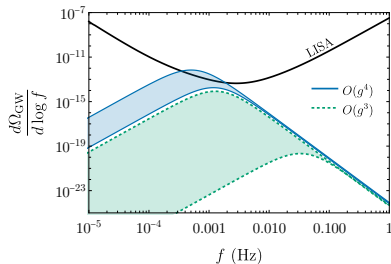


Figure: GW uncertainties [Gould, Tenkanen '21]

Nucleation in laboratories

- Controlled environments
- Testing our theories
- Analogs of the early universe

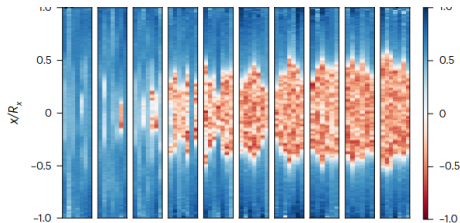


Figure: Nucleation events in ferromagnetic superfluids [Zenesini & al. '23]

Thermal nucleation in QCD

- Hypothetical first order transition in
 - ▶ heavy-ion collisions
 - ▶ neutron star mergers
- Can the nucleation dynamics be understood further?

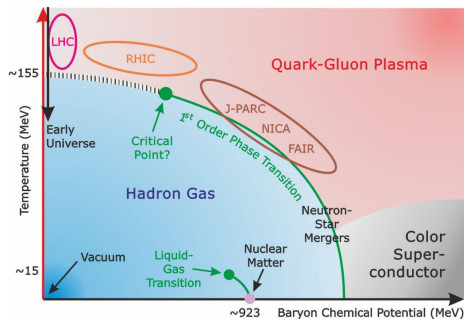


Figure: QCD phase diagram with experiments [A. Steidl, CRC-TR-211]

Nucleation and Wigner functions [Hirvonen '26]

- Defining nucleation rate
 - ▶ Quantum foundation
- Non-perturbative formula
 - ▶ Flux of Wigner function

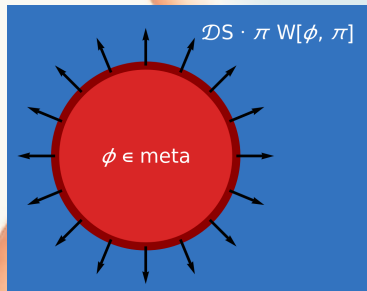


Figure: Flux of Wigner function out of metastable phase

$$\Gamma \propto \int_{\partial \text{meta}} \mathcal{D}S \cdot \pi W[\phi, \pi]$$

Fig: arXiv:1906.00480

Nucleation rate

- System fluctuating around ϕ_m
 - ▶ Metastable phase
- Nucleation rate
 - ▶ Probability P_{meta} leaking

$$\Gamma = -P_{\text{meta}}^{-1} \frac{d}{dt} P_{\text{meta}}$$

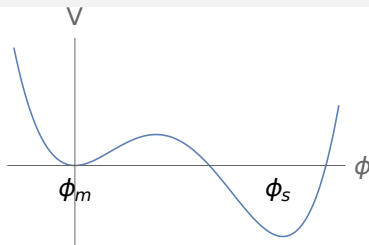


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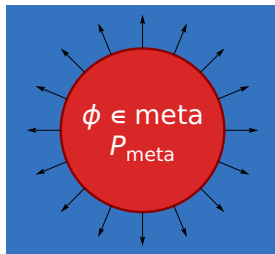


Figure: Metastable phase in configuration space

Statistical description of QFTs

- State: density matrix, $\hat{\rho}$
 - ▶ Field basis, $\rho[\phi, \phi']$
- Probability density:
 - ▶ Diagonal, $\rho[\phi, \phi]$
 - ▶ Yields P_{meta}

$$\rho[\phi, \phi'] = \langle \phi | \hat{\rho} | \phi' \rangle$$

$$P_{\text{meta}} = \int_{\phi \in \text{meta}} \mathcal{D}\phi \rho[\phi, \phi]$$

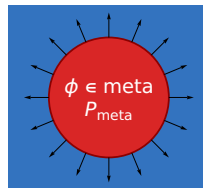


Figure: Metastable phase in configuration space

Wigner function, W

- Transformation

- ▶ ϕ_a ($= \phi - \phi'$)
→ π ($= \partial_t \phi_r$)

- Captures probabilities

$$W[\phi_r, \pi] \\ = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi_a e^{-i\pi \cdot \phi_a} \rho\left[\phi_r + \frac{\phi_a}{2}, \phi_r - \frac{\phi_a}{2}\right]$$

$$\rho[\phi_r, \phi_r] = \int \mathcal{D}\pi W[\phi_r, \pi]$$

$$P_{\text{meta}} = \int_{\phi_r \in \text{meta}} \mathcal{D}\phi_r \mathcal{D}\pi W[\phi_r, \pi]$$

Continuity equation

$$\partial_t \hat{\rho} = i[\hat{\rho}, \hat{H}]$$

- Time evolution: von Neumann
- Integrating over both sides
- Structure
 - ▶ Flow given by π
 - ▶ Local in ϕ_r

$$\frac{1}{\mathcal{N}} \int \mathcal{D}\pi \mathcal{D}\phi_a e^{-i\pi \cdot \phi_a} \bullet$$

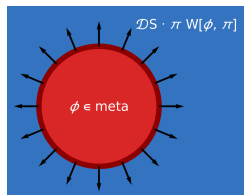
$$\partial_t \rho[\phi_r, \phi_r] = - \int \mathcal{D}\pi \pi \cdot \frac{\delta}{\delta \phi_r} W[\phi_r, \pi]$$

$$\int_{\phi_r \in \text{meta}} \mathcal{D}\phi_r \bullet$$

Nucleation rate with Wigner functions

$$\frac{d}{dt}P_{\text{meta}} = - \int \mathcal{D}\pi \int_{\partial(\phi_r \in \text{meta})} \mathcal{D}S \cdot \pi W[\phi_r, \pi]$$

- Decrease of probability
 - ▶ Flux out of the boundary
- Γ contains all nucleation
 - ▶ Vacuum decay (i.e. quantum tunneling)
 - ▶ Thermal, over-the-barrier escape

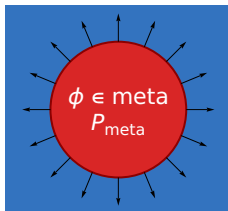


$$\Gamma = \frac{\int \mathcal{D}\pi \int_{\partial(\phi_r \in \text{meta})} \mathcal{D}S \cdot \pi W[\phi_r, \pi]}{\int_{\phi_r \in \text{meta}} \mathcal{D}\phi_r \mathcal{D}\pi W[\phi_r, \pi]}$$

Nucleation from classical to quantum (class. e.g. [Langer '69])

- Rate:

$$\Gamma = -P_{\text{meta}}^{-1} \frac{d}{dt} P_{\text{meta}}$$



	Classical	Quantum
Phase space function	Probability distribution, $\rho[\phi, \pi]$	Wigner function, $W[\phi, \pi]$
P_{meta}	$\int \mathcal{D}\pi \int_{\text{meta}} \mathcal{D}\phi \rho[\phi, \pi]$	$\int \mathcal{D}\pi \int_{\text{meta}} \mathcal{D}\phi W[\phi, \pi]$
$\frac{d}{dt} P_{\text{meta}}$	$\int \mathcal{D}\pi \int_{\partial \text{meta}} \mathcal{D}S \cdot \pi \rho[\phi, \pi]$	$\int \mathcal{D}\pi \int_{\partial \text{meta}} \mathcal{D}S \cdot \pi W[\phi, \pi]$

- Allows to build on classical nucleation theory

Quantum field nucleating over the barrier [Hirvonen '26]

- Over-the-barrier nucleation
 - ▶ (Survives to high T)
 - ▶ from Schwinger–Keldysh
- Nucleating Wigner function
- Rate

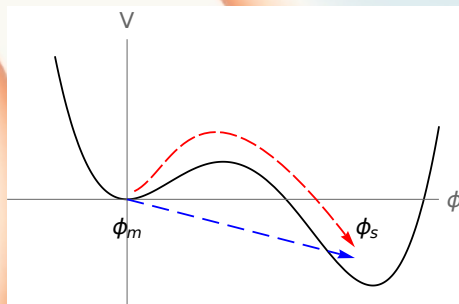


Figure: Two decay channels: quantum tunneling and over-the-barrier escape

Schwinger-Keldysh path integral

- Time dependent expectation values given an initial state $\hat{\rho}(t_i)$
 - Fields “doubled”

$$\langle \hat{\phi}(t_f) \rangle = \int^{\phi_2(t_f)=\phi_1(t_f)} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \rho[\phi_1(t_i), \phi_2(t_i)] e^{iS(\phi_1) - iS(\phi_2)} \phi_1(t_f)$$

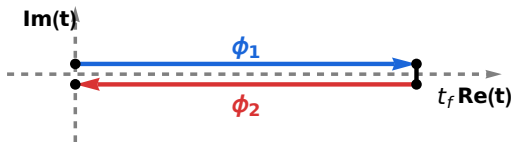


Figure: Schwinger-Keldysh contour

Equation of motion

- Transitioning DoF:

$$\phi_r = \frac{\phi_1 + \phi_2}{2}$$

- ▶ (the continuity equation)

- Expanding in $\phi_a = \phi_1 - \phi_2$

→ Semi-classical EoM

- ▶ Loop expansion around a saddle
- ▶ Low- T limit

$$\begin{aligned} & \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\{i(S[\phi_1] - S[\phi_2])\} \\ &= \int \mathcal{D}\phi_r \mathcal{D}\phi_a \exp\left\{i \int d^D x [-\phi_a (\square \phi_r + V'(\phi_r)) \right. \\ & \quad \left. + \mathcal{O}(\phi_a^3)]\right\} \end{aligned}$$

$$\propto \int \mathcal{D}\phi_r \prod_{0 < t < t_f, \mathbf{x}} \delta(\square \phi_r + V'(\phi_r))$$

$$\left(S[\phi] = \int d^D x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \right)$$

Critical bubble and nucleation dynamics

- Thermal bath: frame
- Static saddle: critical bubble, ϕ_{cb}
 - ▶ Gatekeeper to nucleation
- Linear EoM around ϕ_{cb}
 - ▶ One unstable direction, ϕ_- , with neg. eigenvalue $\lambda_- < 0$

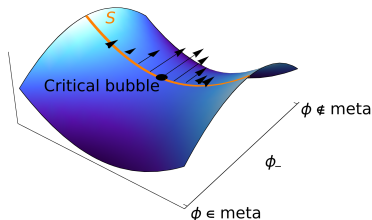


Figure: Critical bubble and negative eigenmode

$$\partial_t^2 \phi_r - \nabla^2 \phi_r + V'(\phi_r) = 0$$

$$-\nabla^2 \phi_{cb} + V'(\phi_{cb}) = 0$$

$$(\partial_t^2 - \nabla^2 + V''(\phi_{cb}))\delta\phi_r = 0$$

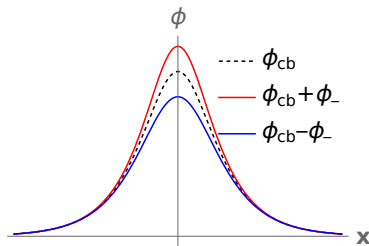


Figure: Critical bubble and negative eigenmode

Steps for the rate

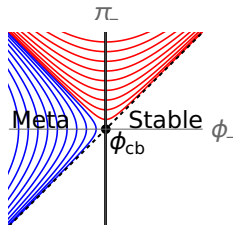
1. Equilibrium density matrix around ϕ_{cb}
 2. Transform to Wigner function
 3. Separate the part originating from meta
 4. Integrate the flux
- Caveat:
 - ▶ Assumes equilibrium flow from meta (Andrey's talk)

$$\rho_{cb}[\phi, \phi'] \propto \int_{\delta\phi_E(0)=\phi}^{\delta\phi_E(\beta)=\phi'} \mathcal{D}\delta\phi_E e^{-S_E[\phi_{cb}+\delta\phi_E]}$$

$$\downarrow$$
$$W_{cb}[\delta\phi_r, \pi]$$

$$\downarrow$$

$$W_{\text{nucl}}[\delta\phi_r, \pi] = \theta(\pi_- - \sqrt{|\lambda_-|}\phi_-) W_{cb}[\delta\phi_r, \pi]$$



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$$\rho_{\text{nucl}}[\phi, \phi'] = \frac{1 - \text{erf} \left[\sqrt{|\lambda_-| \tan\left(\frac{\beta|\lambda_-|}{2}\right)} \left(\phi_- - \frac{i}{2} \cot\left(\frac{\beta|\lambda_-|}{2}\right) \phi_{\text{a-}} \right) \right]}{2} \rho_{\text{cb}}[\phi, \phi']$$

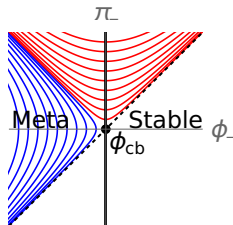
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$$\downarrow$$
$$W_{cb}[\delta\phi_r, \pi]$$

$$\downarrow$$
$$W_{\text{nucl}}[\delta\phi_r, \pi] = \theta(\pi_- - \sqrt{|\lambda_-|}\phi_-) W_{cb}[\delta\phi_r, \pi]$$



Result

- Exp. suppression from ϕ_{cb}
- Equilibrium *quantum* fluctuations

- ▶ Periodic Euclidean time, $\tau \in [0, \beta)$
- ▶ Only the positive modes

- Equiv. to finite-dimensional QM [Affleck '80]

- ▶ Differs from [Linde '80]

$$\Gamma = \frac{V}{2\pi} \left(\frac{S_{\text{E,cb}}}{2\pi} \right)^{\frac{d}{2}} \left(\frac{\det(-\partial_\mu^2 + V''(\phi_{\text{meta}}))}{\det^+(-\partial_\mu^2 + V''(\phi_{\text{cb}}))} \right)^{\frac{1}{2}} e^{-S_{\text{E,cb}}}$$

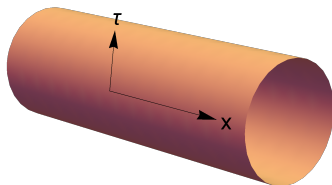


Figure: Periodic Euclidean spacetime of the operators

Time-dependent rate in cosmology

- Rate decreases exponentially:

$$e^{-S_{E,cb}} = e^{-\beta E_{cb}}$$

- ▶ Larger in the past

- Largest rates

- ▶ Require thermal resummations
- ▶ Beyond the linearized EoM

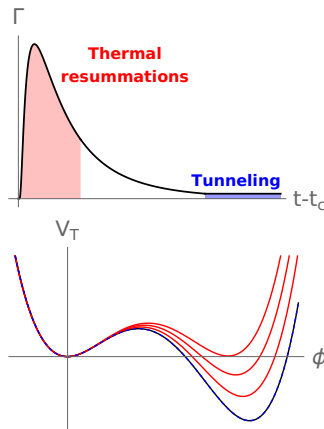


Figure: Time-dependent rate and potential

Effective field theory approach [Gould & Hirvonen '21]

- Effective description, S_{eff} , from high- T QFTs
 - ▶ Leverage scale hierarchy, $m_{\text{eff}} \ll \pi T$
 - ▶ Describe nucleation on the mass scale

Fig: arXiv:1906.00480

Fluctuation scales

$$\Gamma = \frac{V}{2\pi} \left(\frac{S_{E,cb}}{2\pi} \right)^{\frac{d}{2}} \left(\frac{\det(-\partial_\mu^2 + V''(\phi_{\text{meta}}))}{\det^+(-\partial_\mu^2 + V''(\phi_{cb}))} \right)^{\frac{1}{2}} e^{-S_{E,cb}}$$

- Two scales in the determinants
 - ▶ Mass scale, $\sqrt{V''}$
 - ▶ Thermal scale, $T = \beta^{-1}$
- Thermal corrections grow at high- T
 - ▶ $T \gg \sqrt{V''}$
- Nucleation on the mass scale!
 - ▶ Integrate out πT scale

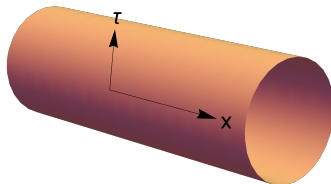


Figure: Periodic Euclidean spacetime

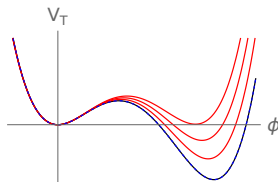


Figure: T -dependent potential

High-temperature dimensional reduction

[Kajantie et al. '95, Braaten & Nieto '95]

- Equilibrium EFT for the mass scale

- ▶ Describes nucleating bubbles

- Integrating out thermal scale

[Hirvonen '22]

- ▶ Usual 1PI action...
- ▶ ...but expand in light masses

[Burgess '20, Hirvonen '22]

$$\frac{1}{P^2 + m_{\text{nucl}}^2} \rightarrow \frac{1}{P^2} - \frac{m_{\text{nucl}}^2}{P^4} + \dots$$

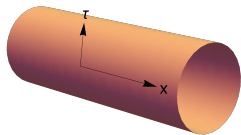
$$Z = \int \mathcal{D}\Phi e^{-S_E}$$

$$= \int \mathcal{D}\Phi_0^{\text{IR}} \mathcal{D}\Phi_0^{\text{UV}} \mathcal{D}\Phi_{n \neq 0} e^{-S_E}$$

$$= \int \mathcal{D}\Phi_0^{\text{IR}} e^{-S_{\text{eff}}[\Phi_0^{\text{IR}}]},$$

$$S_{\text{eff}}[\Phi_0^{\text{IR}}]$$

$$= -\ln \int \mathcal{D}\Phi_0^{\text{UV}} \mathcal{D}\Phi_{n \neq 0} e^{-S_E[\Phi_0^{\text{IR}} + \Phi_0^{\text{UV}} + \Phi_{n \neq 0}]}$$



Nucleation rate

- Effective action, S_{eff}
 - ▶ Critical bubble
 - ▶ Rate

$$\frac{\delta S_{\text{eff}}}{\delta \phi}[\phi_{\text{cb}}] = 0$$

- Determinants with BubbleDet
[Ekstedt, Gould, Hirvonen '23]

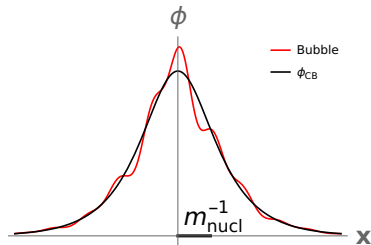
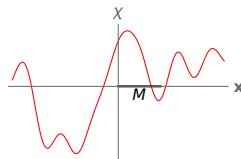
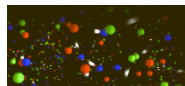
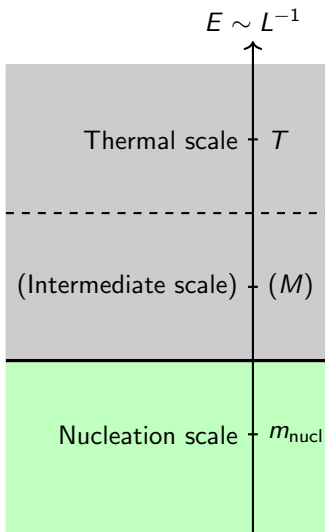
- No
 - ▶ double counting
 - ▶ imaginary parts
 - ▶ diverging derivative expansions

$$\Gamma = \frac{V}{2\pi} \left(\frac{S_{\text{eff,cb}}}{2\pi} \right)^{\frac{3}{2}} \underbrace{\left(\frac{\det(-\nabla^2 + V''_{\text{eff,meta}})}{\det^+(-\nabla^2 + V''_{\text{eff,cb}})} \right)^{\frac{1}{2}}}_{E \sim m_{\text{eff}}} e^{-\overbrace{S_{\text{eff,cb}}}^{E \sim \pi T}}$$

Scales in high-temperature nucleation

- Particles

- Fields
 - ▶ Approx. classical



Kinetic theory approach [Hirvonen '24]

- Real-time, off-equilibrium description
 - ▶ Infrared *classical* field,
 $\phi = \phi_r^{\text{IR}}$
 - ▶ Thermal particles, f_a
 - ▶ Equivalent to contours at leading order
- Equilibrium quite OK!

$$\square\phi + V'_T(\phi) = - \sum_a \frac{dm_a^2}{d\phi} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \delta f_a$$

$$(\partial_t + \mathbf{v} \cdot \nabla + \mathbf{F}_a \cdot \partial_{\mathbf{p}}) f_a = 0$$

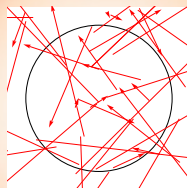


Figure: Bubble in a particle shower

Effective Liouville equation

- EoMs into Liouville equation
- Apply classical nucleation theory

$$\square\phi + V_T'(\phi) = - \sum_a \frac{dm_a^2}{d\phi} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \delta f_a$$

$$(\partial_t + \mathbf{v} \cdot \nabla + \mathbf{F}_a \cdot \partial_{\mathbf{p}}) f_a = 0$$

↓

$$\frac{\partial}{\partial t} \rho[\phi, \pi, \delta f_a] = - \int d^d \mathbf{x} \left(\dot{\phi} \frac{\delta}{\delta \phi} + \dot{\pi} \frac{\delta}{\delta \pi} + \sum_a \int d^d \mathbf{p} \delta \dot{f}_a \frac{\delta}{\delta \delta f_a} \right) \rho[\phi, \pi, \delta f_a]$$

Same potential

- Matching ϕ equilibrium statistics
 - ▶ Kinetic theory, linear order in δf_a
 - ▶ Dimensionally reduced EFT from imaginary time
- Applies to wall velocities too

$$\rho_{\text{eq}}^{\text{kin}}[\phi] = \rho_{\text{eq}}^{\text{EFT}}[\phi]$$

$\Downarrow$

$$V_T(\phi) = V_{\text{eff}}(\phi)$$

$$\Gamma_{\text{kin}} = \frac{\kappa}{\sqrt{|\lambda_-|}} \Gamma_{\text{EFT}}$$

$$\phi = \phi_{\text{CB}} + \delta\phi \exp(\kappa t)$$

$$f_a = f_a^{\text{eq}} + \delta f_a \exp(\kappa t)$$

- Dynamical prefactor
- Exponential growth rate, κ
 - ▶ Modified by off-eq. particles
 - ▶ Approximation from BubbleDet: $\kappa \approx \sqrt{|\lambda_-|}$

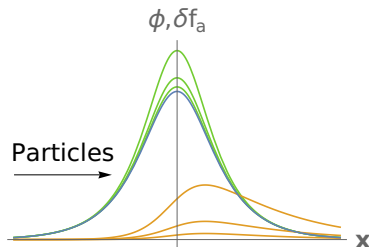
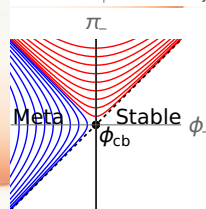
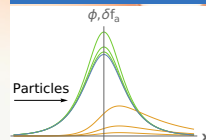
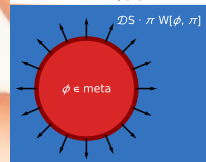
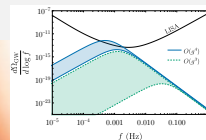


Figure: Exponentially growing bubble, pushes particles out of equilibrium

Summary

- Wigner function approach
 - ▶ Quantum basis
 - ▶ Generalization to high- T
 - ▶ Beyond classical field and collisionless
- EFTs useful in high- T
 - ▶ See Andrii's, Bogumiła's and Maciej's talks
- Kinetic theory
 - ▶ Backs equilibrium EFTs
 - ▶ When is κ important?
- Equilibrium flow assumption [Andrey's talk](#)
 - ▶ When does it hold?
 - ▶ 3+1 dim, high- T , cosmological phase transitions



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Thanks for listening!

- Wigner function approach
 - ▶ Quantum basis
 - ▶ Generalization to high- T
 - ▶ Beyond classical field and collisionless
- EFTs useful in high- T
 - ▶ See Andrii's, Bogumiła's and Maciej's talks
- Kinetic theory
 - ▶ Backs equilibrium EFTs
 - ▶ When is κ important?
- Equilibrium flow assumption [Andrey's talk](#)
 - ▶ When does it hold?
 - ▶ 3+1 dim, high- T , cosmological phase transitions

