Bulk viscous quark matter as a multicomponent fluid

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Based on
Cruz Rojas et al., PRL 133.7 (2024),
Hernandez et al., PRD 113.1 (2026),
& Säppi, PRD 114.1 (2026)

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Finite density and equilibrium conditions

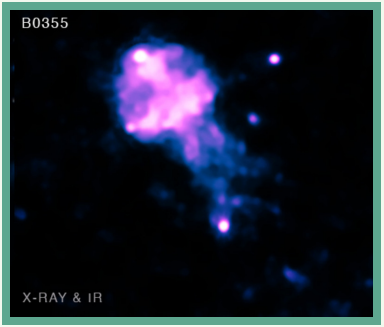
Finite-density ($n > 0$) matter has chemical potentials $\mu > 0$ associated with conserved charges—Baryon number conserved!

'Quiescent' (static, cool, ...) neutron stars in beta and charge equilibrium

for *uds* **quark matter** (and electrons)

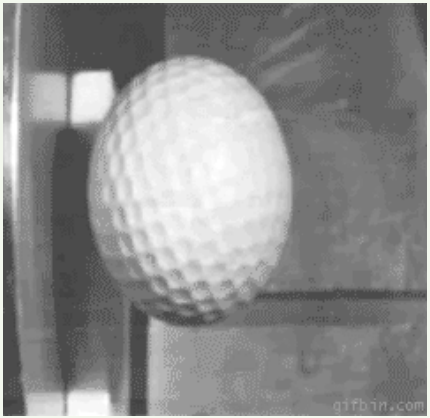
$$\mu_s = \mu_d = \mu_u + \mu_e,$$
$$n_Q = (2n_u - n_d - n_s)/3 - n_e = 0$$

$\longrightarrow$ parametrise with one free chemical potential, usually μ_B



What happens when the equilibrium is disturbed?

Bulk viscous fluids and hydro



Colliding neutron stars go through a compression–decompression cycle

System is (slightly?) out of equilibrium (new chemical potentials!) and chemical processes bring it back to equilibrium

For **non-ideal fluids** this causes energy loss $E'(t) > 0$

In NSs, energy loss primarily due to **bulk viscosity**, restoring processes driven by electroweak interactions[a]

Bulk viscous effects modify GW signature and the dynamics of plasma: Complicated transport eqns[b]

[a] **NOT** the QCD bulk viscosity, which is much smaller

[b] Assume we're only in the pretty green region of hydro validity. Cf. Lorenzo Gavassino's talk

Input for bulk viscosity

Transport coefficients for the hydro evolution depend on

1. Electroweak rates
2. Derivatives of (in-equilibrium) EoS (finite T, μ and also m_s)

Sign problem on the lattice, need some other approach. Using pQCD as a reference now but other methods are available.

Effects of the **strange quark mass** critical for nonzero bulk viscosity in QM!

Quark mass breaks conformality
$\rightarrow$ non-ideal fluid

Small effect for eg. mass–radius curves

Effective expansions for small mass m_s

For bulk viscosity, need to include T, μ and masses m
Even LO thermal pressure not known in closed form with finite particle masses

Numerical determination of mass effects known, but cumbersome and doesn't effect the pressure much

For "soft" masses $m \sim g_s T$ or $g_s \mu$, thermal loop integrals used in pQCD can be consistently expanded for small mass → simpler calcs

$$\sum\!\!\!\!\!\!\int_P \frac{1}{P^2+m^2} = \sum\!\!\!\!\!\!\int_P \frac{1}{P^2} - m^2 \sum\!\!\!\!\!\!\int_P \frac{1}{P^4} \neq 0$$

even in dimreg!

$m_s \sim 100\mathrm{MeV} \lesssim g_s \mu|_{\mathrm{pQCD}}$

m_s is "soft", and the effect is **vital for bulk viscosity**

Add masses to naïve loops, HTL, and DR → pressure at finite μ, T, m_s

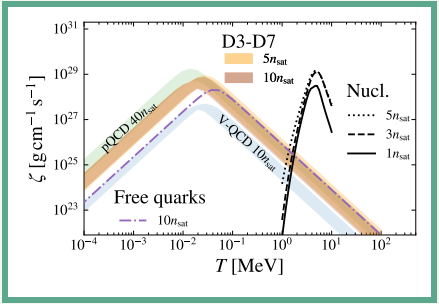
Naïve approach

Simplest example: Process $u + d \leftrightarrow u + s$ with harmonic density oscillations
$\longrightarrow$ frequency-dependent bulk viscosity

Match energy dissipation from the viscosity to the work done by the change in the fluid

$$-\frac{\zeta}{\tau}\int_0^\tau \mathrm{d}t(\boldsymbol{\nabla}\cdot\mathbf{u})^2 = \langle E'(t)\rangle = \frac{n_q}{\tau}\int_0^\tau \mathrm{d}t p(t)V'(t)$$

$$\zeta(\omega) = \frac{\lambda_{EW}A_1^2[\partial_\mu p, \partial_\mu^2 p]}{\omega^2 + (\lambda_{EW}C_1[\partial_\mu p, \partial_\mu^2 p])^2}$$



Qualitative difference between Quark Matter and Nuclear Matter...

Less naïve approach

...but the model is too simplistic, especially if you want to include processes with leptons which have an independent chemical potential.

These are naïvely suppressed in cool matter (say $T \leq m_s \approx 100\mathrm{MeV}$) beceause EW rates scale as $\propto T^4$, not as $\propto T^2$ **but!** careful analysis shows their importance

No need to assume harmonic oscillations: Bulk viscosity equivalently from the evolution equation for $\Pi = p - p_{eq.}$, linearised for $\mu_s - \mu_d$ in a fluid with velocity u, first in the nonleptonic case (single free chemical potential)

$$\tau D_u \Pi + \Pi = -\zeta(\omega = 0)\vartheta,\ \vartheta = \mathrm{div} u \text{ (Israel–Stewart equation)}$$

Same result for ζ as before, but easier to generalise

For the two-DoF case, define

$$\Omega^{-1} = -\hat{M}\lambda, \quad \mathbf{z} = \Omega^{-1}\hat{M}\mathbf{n}$$

as a matrix and vector of

1. **densities** $n_f = \partial p/\partial \mu_f$,
2. **susceptibilities** $\chi_{fg} = \partial^2 p/\partial \mu_f \partial \mu_g$ s.t. $\hat{M} = \hat{M}[\chi]$,
3. electroweak **rates** λ,

and two independent **reaction chemical potentials** (chemical imbalances) μ_1, μ_2 s.t. $\Pi = \Pi_1\mu_1 + \Pi_2\mu_2$. (for definiteness, $\mu_1 = \mu_s - \mu_d, \mu_2 = \mu_u - \mu_d + \mu_e$)
The generalised evolution equation reads [1]

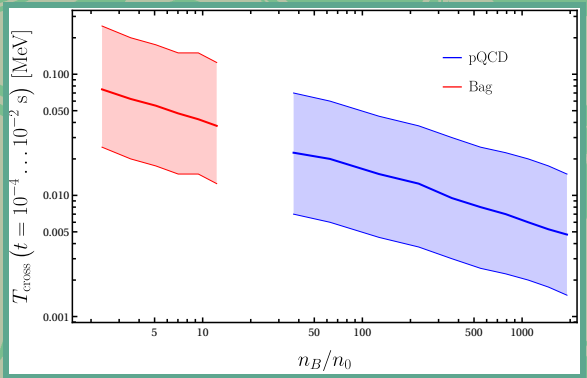
$$\mathrm{Det}\Omega^{-1} D_u^2 \Pi + \mathrm{Tr}\Omega^{-1} D_u \Pi + \Pi = -\vartheta \langle \mathbf{\Pi}, \mathbf{z} \rangle - D_u \vartheta \mathrm{Det}\Omega^{-1} \langle \mathbf{\Pi}, \Omega \mathbf{z} \rangle$$

Frequency-dependent description recovered in the appropriate limit by assuming $\Pi \sim e^{i\omega t}$

[1] To see how this arises in a nuclear-matter analysis see Gavassino, Class.Quant.Grav. 40 (2023) 16

Two-component description

Better description for (unpaired,...) bulk viscous QM as a two-component fluid, with a nonleptonic component dominant at low temperatures and a mixed-channel component dominant at intermediate temperatures (before neutrino trapping)



Crossing temperature between component dominance seems to be a robust quantity

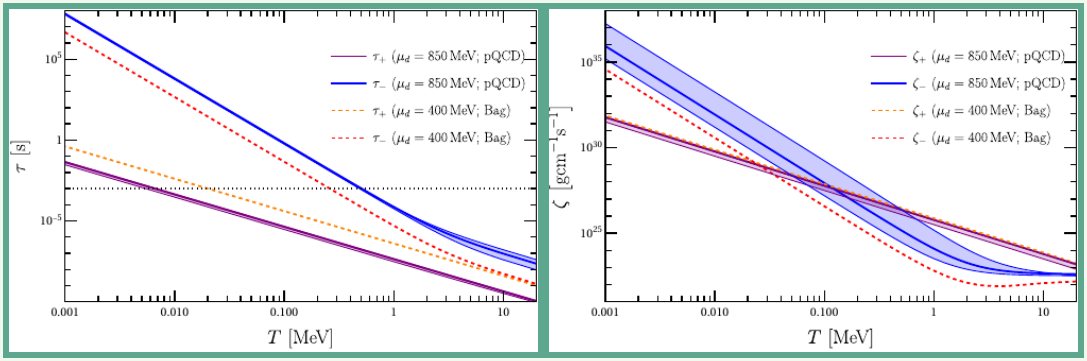
Modified transport coefficients and Green's functions:

$$G(t) = \frac{\zeta_+}{\tau_+} e^{-t/\tau_+} + \frac{\zeta_-}{\tau_-} e^{-t/\tau_-},$$

$\tau_\pm$ eigenvalues of Ω^{-1},
$\zeta_+ = (\zeta\tau_+ - \xi)/(\tau_+ - \tau_-)$,
$\zeta_- = (\zeta\tau_- - \xi)/(\tau_- - \tau_+)$;
$\zeta = \langle \mathbf{\Pi}, \mathbf{z} \rangle, \quad \xi = \tau_+\tau_- \langle \mathbf{\Pi}, \Omega \mathbf{z} \rangle.$

(Extra: Bulk viscosity plots)

Used high-order pQCD at large densities, simple bag model at intermediate densities (NS core)



The transport coefficients $\zeta_\pm, \tau_\pm$ have similar qualitative behaviour: Partial relaxation times well under control and fairly robust, partial bulk viscosities show more variation, but qualitatively similar

Need Green's functions to understand which component dominates (ζ/τ is nontrivial)

n-component generalisation

The discussion generalises "readily" to arbitrary particle content (no (p)QCD necessary), arbitrary chemical reactions with $a \in \{1 \dots n\}$ independent chemical potentials and a linearised (near-equilibrium) bulk viscous fluid with densities evolving according to $D_u n_a = \lambda \mu_a$: Use Cayley–Hamilton to invert the set of differential equations and combine them into

$$p_\Omega(-D_u)\left[\hat{\boldsymbol{\mu}} + (D_u \mathrm{id}_n + \Omega)^{-1}\Omega \mathbf{z}\vartheta\right] = \mathbf{0}$$

Where p_Ω is the characteristic polynomial, $\Pi = \sum_{a=1}^{n} \Pi_a \hat{\mu}_a$, and the equation for Π follows by contracting. Transport coefficients neatly encoded in matrix invariants: n relaxation times from eigenvalues of Ω and n partial viscosities from other matrix-invariant quantities

Greatly simplifies eqns. already for $n = 3$ (used in nuclear matter), sets up equations for future extensions, and shows that the system always couples in the same way

analogs in viscoelastic materials (but in viscoelastic materials these are just model ansatzes! in neutron stars the transport coefficients are related to the microscopic description)

Thank you & Let's all thank Risto and every other organiser for creating a great iteration of SEWM here in Helsinki!

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Outline of a mathsy derivation for general Bulk viscosity

Start with system with k particles $\{p_a\}_{a\in K}$ and reactions $C \in \mathcal{C}$ $\sum_{a\in K} x_a^C p_a \to \sum_{b\in K} \tilde{x}_b^C p_b$ bringing the system to equilibrium.

Assume close-to-equilibrium: Linearised rates $\lambda_C \hat{\mu}_C$ for **small** reaction chemical potentials, assume first n are independent and write $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \dots \hat{\mu}_n)$, $\hat{\mu}_{C_\ell} = \sum_{\alpha=1}^{n} r_{\alpha c_\ell}$ for $l \in \{n+1 \dots |\mathcal{C}|\}$.

Connect with hydro: 4-velocity u, $\vartheta = \partial_\nu u^\nu$, $D_u \equiv u_\nu \partial^\nu$ in some frame $\{\partial^\nu\}$. Treat ϑ as an independent variable (spirit of second order hydro), but take continuity eq. $\partial_\nu n_a u^\nu = D_u n_a + n_a \vartheta = 0 \forall a \in K$.

Incorporate constraints into defn of $\Pi = p_{\mathrm{non-eq}} - p = \langle \boldsymbol{\Pi}, \hat{\boldsymbol{\mu}} \rangle$ — irrelevant until you fix the EoS, here completely general.

Combine into evolution eqn. $D_u \mathbf{n}_{\mathrm{non-eq}} = \lambda_a^\alpha \hat{\mu}_\alpha$ where $\lambda_a^\alpha = -\sum_{C \in \mathcal{C}} (x_a^C - \tilde{x}_a^C) r_a^C \lambda_C$.
Linearise $p_{\mathrm{non-eq}} - p = \sum_a n_a \delta\mu_a$, $n_{a,\mathrm{non-eq}} - n_a = \sum_b \chi_a^b \delta\mu_b$. Then
$D_u \mathbf{n}_{\mathrm{non-eq}} = \chi_a^b D_u \mu_b - \vartheta \mathbf{n}_{\mathrm{non-eq}}$.
The equation can be inverted in the linear regime to give an equation for μ instead. With a suitable basis rotation, we have $D_u \tilde{\mu} = M\lambda \hat{\boldsymbol{\mu}} - \vartheta M \mathbf{n}$, with M is a matrix of inverse susceptibilities (generally nondiagonal!) s.t. $M_{a'}^b = (\chi^{-1})_a^b$ for $a' = a > n$ and $\sum_{a \in K} (x_a^{C_\alpha} - \tilde{x}_a^{C_\alpha})(\chi^{-1})_a^b$ for $a' = \alpha \in \{1 \dots n\}$.
Now restricting to the first n components and defining $\Omega = -\hat{M}\lambda$ and $\mathbf{z} = \Omega^{-1}\hat{M}\mathbf{n}$ as long as Ω is invertible (this is true for nondegenerate systems) we get

$$(\Omega^{-1} D_u + \mathrm{id}_n)\hat{\boldsymbol{\mu}} = -\mathbf{z}\vartheta \iff \Omega\hat{\boldsymbol{\mu}} = -\Omega\mathbf{z}\vartheta - D_u\hat{\boldsymbol{\mu}}.$$

Next we diagonalise the eqns:
Consider the characteristic polynomial $p_\Omega(x) = \sum_{i=0}^{n} c_i[\Omega]x^k = \prod_{\omega\in\Sigma[\Omega]}(x-\omega)$ and an auxiliary variant $\tilde{p}_\Omega = \sum_{i=1}^{n} c_i[\Omega]\Omega^{i-1}$. we use the Cayley–Hamilton theorem to write $\Omega^{-1} = -c_0^{-1}[\Omega]\tilde{p}_\Omega(\Omega)$.
Note the recursion $D_u\Omega^i\hat{\boldsymbol{\mu}} = (-1)^i D_u^{i+1}\hat{\boldsymbol{\mu}} + \sum_{j=0}^{i-1}(-1)^{j+1}D_u^{j+1}\Omega^{i-j}\mathbf{z}\vartheta$.
Insert into evolution equation to get

$$\sum_{i=2}^{n} c_i\left[\Omega\right](-1)^i D_u^i\hat{\boldsymbol{\mu}} - c_1\left[\Omega\right]D_u\hat{\boldsymbol{\mu}} + c_0\left[\Omega\right]\hat{\boldsymbol{\mu}} = -c_0\left[\Omega\right]\boldsymbol{z}\vartheta - \sum_{i=2}^{n} c_i\left[\Omega\right]\sum_{j=0}^{i-2}(-1)^j D_u^{j+1}\Omega^{i-j-1}\boldsymbol{z}\vartheta.$$

$$\begin{aligned}\mathbf{0} &= \sum_{i=0}^{n} c_i\left[\Omega\right](-D_u)^i\hat{\boldsymbol{\mu}} + \frac{1}{D_u\mathrm{id}_n+\Omega}\left(\sum_{i=0}^{n} c_i\left[\Omega\right](-D_u)^i\Omega + D_u\overbrace{\sum_{i=0}^{n} c_i\left[\Omega\right]\Omega^i}^{=0}\right)\boldsymbol{z}\vartheta\\ &= p_\Omega\left(-D_u\right)\left[\hat{\boldsymbol{\mu}} + \left(D_u\mathrm{id}_n+\Omega\right)^{-1}\Omega\boldsymbol{z}\vartheta\right].\end{aligned}$$

Contracting with $\mathbf{\Pi}$, the equation reads $\sum_{\alpha=1}^{n} T_\alpha D_u^\alpha \Pi + \Pi = -\sum_{\alpha=1}^{n} Z_\alpha D_u^{\alpha-1}\vartheta$, with T_α relaxation-time-like coefficients and Z_α bulk-viscosity-like coefficients.

As long as $\partial_t^\alpha \vartheta$ are piecewise continuous, existence and uniqueness is guaranteed for the IVP, the homogeneous equation admits n linearly independent exponential solutions, and in the rest frame $G(t) = \theta(t)\sum_{\alpha=1}^{n} \zeta_\alpha e^{-t/\tau_\alpha}/\tau_\alpha$. Turns out that the eqn. decomposes into n Israel–Stewart ($n = 1$) equations, and $\zeta_\alpha, \tau_\alpha$ are precisely their relaxation times and bulk viscosities!(!)

Easy to show that $\det\Omega^{-1} p_\Omega(\tau_\alpha^{-1}) G_\alpha(t) = 0$ so that for nontrivial, nondegenerate cases $\{\tau_\alpha\}_{\alpha=1}^{n} = \Sigma[\Omega^{-1}]$, $T_\alpha = (-1)^\alpha \det\Omega^{-1} c_\alpha[\Omega]$. Requires a positive-definite Ω.

Slightly more nontrivial to get ζ_α. In the slowly-varying regime $\vartheta(t - t') \approx \vartheta(t)$ we can split $\Pi = \sum_{\alpha=1}^{n} \pi_\alpha$ (not the same as Π_α!) we get $\forall\alpha$ the $n = 1$ case $\tau_\alpha \partial_t \pi_\alpha + \pi_\alpha + \zeta_\alpha \vartheta = 0$. By considering the combination $\sum_{\alpha=1}^{n} \prod_{\beta=1,\beta\neq\alpha}^{n} (\tau_\alpha \partial_t \pi_\alpha + \pi_\alpha + \zeta_\alpha \vartheta) = 0$, the π_α-part reduces to the evolution equation, while the rest reduces to it iff $\zeta_\alpha = \tau_\alpha^{n-1}/(p_{{}_\alpha\mathcal{T}}(\tau_\alpha))\langle \mathbf{\Pi}, q_{{}_\alpha\mathcal{T}}(\Omega)\mathbf{z}\rangle$, where ${}_\alpha\mathcal{T}$ is a diagonal matrix of the eigenvalues of Ω with the αth eigenvalue removed, and q is a reciprocal characteristic polynomial $q_A(x) = \sum_{i=0}^{m} c_{m-i}[A] x^i$.

Any value of n reduces to computing trivial matrix invariants: E.g. $n = 3$

$$\begin{aligned}\mathrm{Det}\Omega^{-1} D_u^3 \Pi + \mathrm{Det}\Omega^{-1}\mathrm{Tr}\Omega D_u^2 \Pi + \frac{\mathrm{Det}\Omega^{-1}}{2}\left(\mathrm{Tr}^2\Omega - \mathrm{Tr}\Omega^2\right) D_u \Pi + \Pi \\ = -\langle \boldsymbol{\Pi}, \boldsymbol{z}\rangle \vartheta - \mathrm{Det}\Omega^{-1}\left(\mathrm{Tr}\Omega\langle \boldsymbol{\Pi}, \Omega \boldsymbol{z}\rangle - \langle \boldsymbol{\Pi}, \Omega^2 \boldsymbol{z}\rangle\right) D_u \vartheta - \mathrm{Det}\Omega^{-1}\langle \boldsymbol{\Pi}, \Omega \boldsymbol{z}\rangle D_u^2 \vartheta.\end{aligned}$$

$$\begin{aligned} Z_1 &= \zeta_1 + \zeta_2 + \zeta_3 = \langle \boldsymbol{\Pi}, \boldsymbol{z}\rangle, \\ Z_2 &= (\tau_2 + \tau_3)\zeta_1 + (\tau_1 + \tau_3)\zeta_2 + (\tau_1 + \tau_2)\zeta_3 = \mathrm{Det}\Omega^{-1}\left(\mathrm{Tr}\Omega\,\langle \boldsymbol{\Pi}, \Omega \boldsymbol{z}\rangle - \langle \boldsymbol{\Pi}, \Omega^2 \boldsymbol{z}\rangle\right), \\ Z_3 &= \tau_2\tau_3\zeta_1 + \tau_1\tau_3\zeta_2 + \tau_1\tau_2\zeta_3 = \mathrm{Det}\Omega^{-1}\,\langle \boldsymbol{\Pi}, \Omega \boldsymbol{z}\rangle. \end{aligned}$$

$$\begin{aligned} \zeta_1 &= \frac{\tau_1^2\left(\langle \boldsymbol{\Pi}, \boldsymbol{z}\rangle - (\tau_2 + \tau_3)\,\langle \boldsymbol{\Pi}, \Omega \boldsymbol{z}\rangle + \tau_2\tau_3\,\langle \boldsymbol{\Pi}, \Omega^2 \boldsymbol{z}\rangle\right)}{(\tau_1 - \tau_2)(\tau_1 - \tau_3)}, \\ \zeta_2 &= \frac{\tau_2^2\left(\langle \boldsymbol{\Pi}, \boldsymbol{z}\rangle - (\tau_1 + \tau_3)\,\langle \boldsymbol{\Pi}, \Omega \boldsymbol{z}\rangle + \tau_1\tau_3\,\langle \boldsymbol{\Pi}, \Omega^2 \boldsymbol{z}\rangle\right)}{(\tau_2 - \tau_1)(\tau_2 - \tau_3)}, \\ \zeta_3 &= \frac{\tau_3^2\left(\langle \boldsymbol{\Pi}, \boldsymbol{z}\rangle - (\tau_1 + \tau_2)\,\langle \boldsymbol{\Pi}, \Omega \boldsymbol{z}\rangle + \tau_1\tau_2\,\langle \boldsymbol{\Pi}, \Omega^2 \boldsymbol{z}\rangle\right)}{(\tau_3 - \tau_1)(\tau_3 - \tau_2)}. \end{aligned}$$

Compare to the expressions of Harris, Fore & Reddy in PRC 111 (2025): Enormous simplification.