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Should relativistic hydrodynamics be causal?

Speaker:

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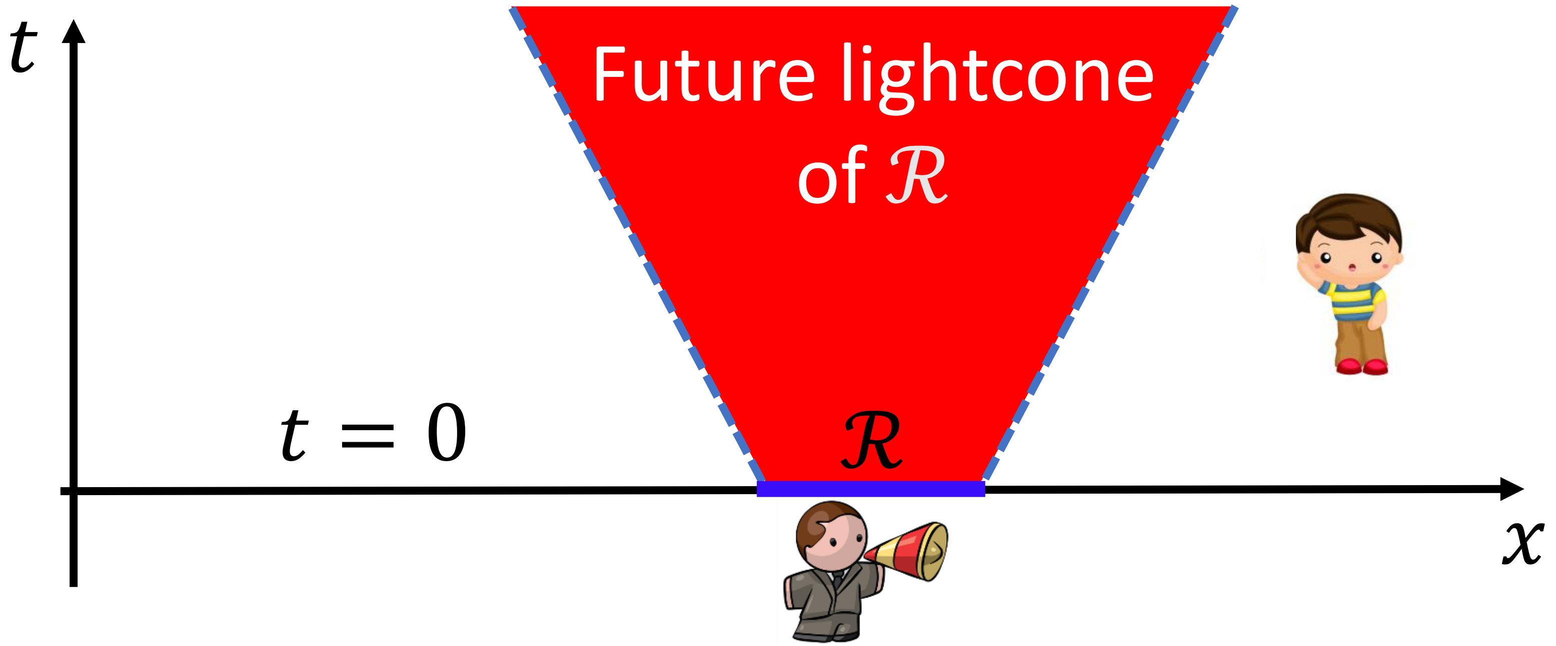
What do we mean by causality?

- As a fundamental principle: We cannot send messages faster than light.
- In hydro PDEs: A change of initial data in a region of space $\mathcal{R}$ does not affect the solution outside the future lightcone of $\mathcal{R}$.



See Wald (*General Relativity*), or Hawking Ellis (*Large scale structure of spacetime*), or Bemfica-Disconzi-Noronha 2018

Outline

Conventions:
$c = 1$
$(-,+,+,+)$

1. **Introduction to causality in hydro** (some math an history)
2. **Should relativistic hydrodynamics be causal <u>in theory</u>?**

   Or, equivalently, are we "morally obliged" to write down causal PDEs?

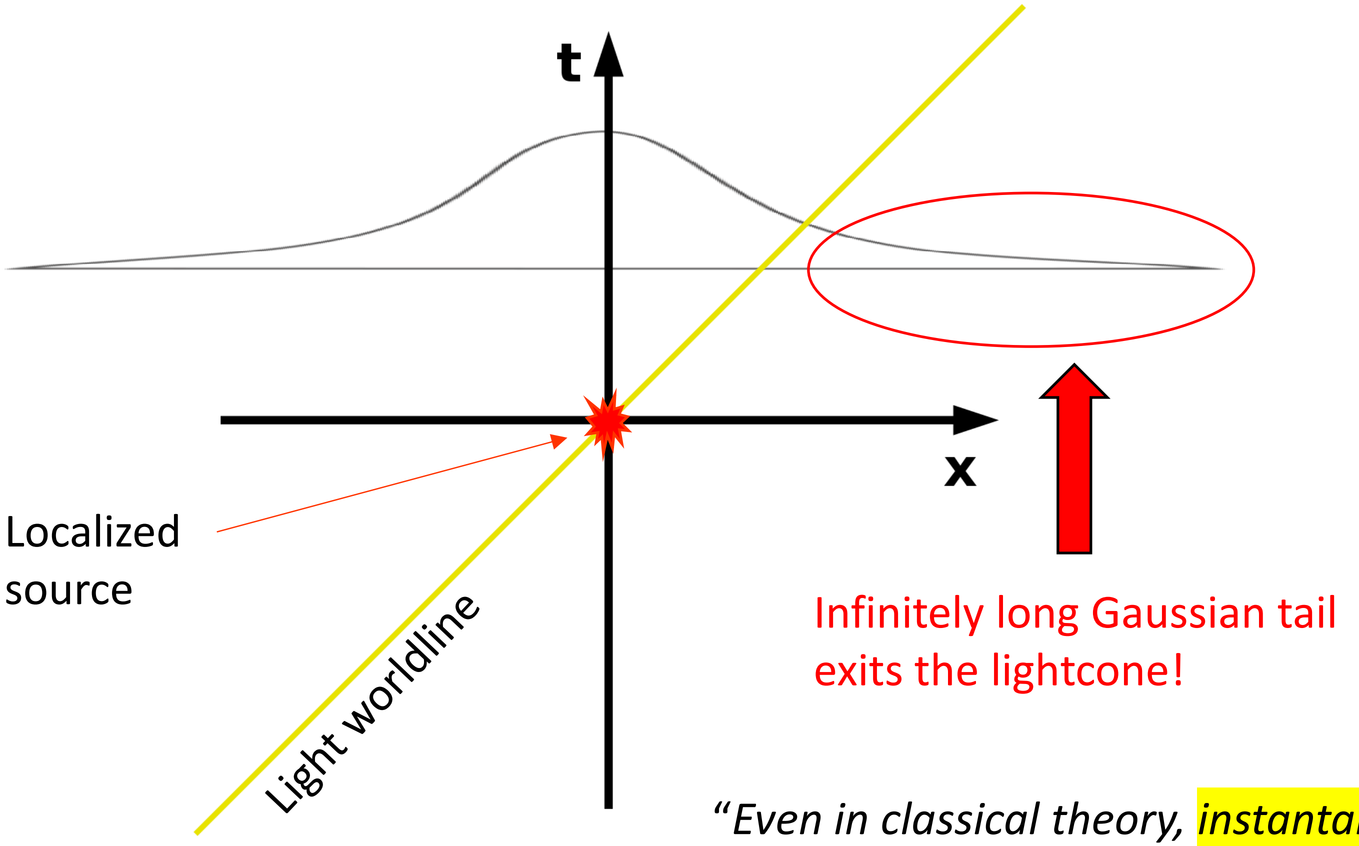
3. **Should relativistic hydrodynamics be causal <u>in practice</u>?**

   Or, equivalently, are we "kindly advised" to write down causal PDEs?

4. **Recent progress towards achieving either side** (the "yes" side and the "no" side)

Introduction to causality in hydro

Standard example of an acausal theory



Diffusion equation

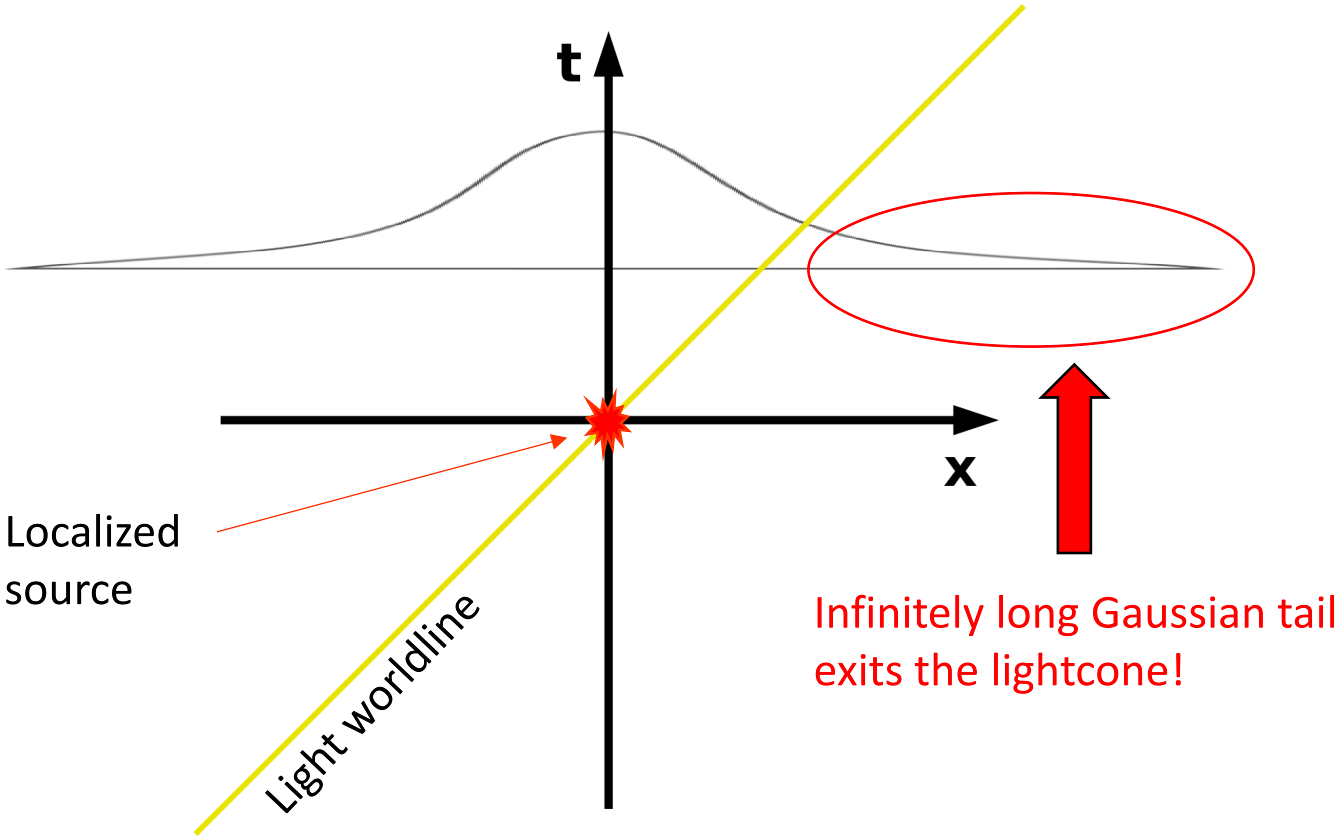
$$\partial_t T = \partial_x^2 T$$

Retarded Green function:

$$G(t,x) = \frac{\Theta(t)}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

"Even in classical theory, instantaneous propagation of heat is an offense to intuition, which expects propagation at about the mean molecular speed; in a consistent relativistic theory it ought to be completely prohibited." (Israel-Stewart 1979)

Standard example of an acausal theory



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Typical causal fix (Cattaneo, 1958)

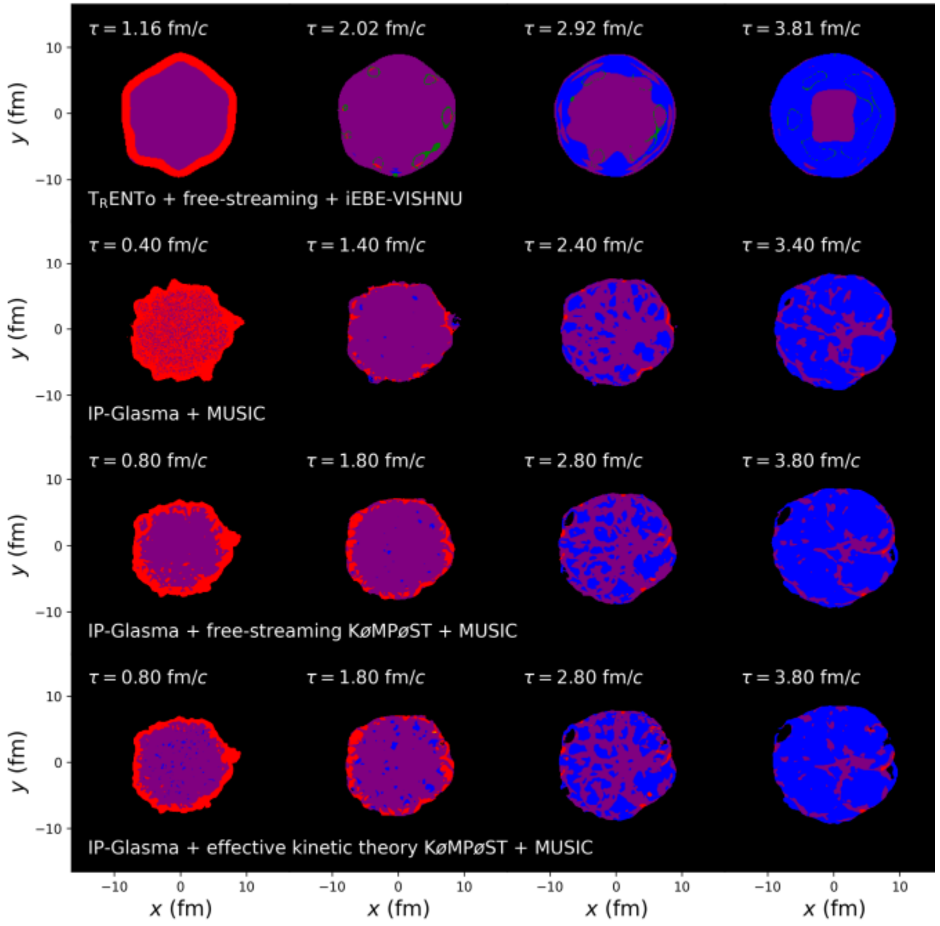
$$\partial_t T = \left(\partial_x^2 - \partial_t^2\right) T$$

Relativistic theories can be acausal

Indeed, when we deal with viscosity, it happens frequently…

Standard textbook	**Incriminating equations**
Weinberg, *Gravitation and cosmology* Section I.2.11 (“Relativistic imperfect fluids”)	$$\Delta T^{\alpha\beta} = -\eta H^{\alpha\gamma}H^{\beta\delta}W_{\gamma\delta} - \chi(H^{\alpha\gamma}U^\beta + H^{\beta\gamma}U^\alpha)Q_\gamma - \zeta H^{\alpha\beta}\frac{\partial U^\gamma}{\partial x^\gamma} \quad (2.11.21)$$
Misner-Thorne-Wheeler, *Gravitation* Exercise 22.7 (“Hydrodynamics with viscosity and heat flow”)	$$T^{\alpha\beta} = \rho u^\alpha u^\beta + (p - \zeta\theta)P^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} + q^\alpha u^\beta + u^\alpha q^\beta. \quad (22.16d)$$ $$q^\alpha = -\kappa P^{\alpha\beta}(T_{,\beta} + Ta_\beta) \quad (22.16i)$$
Landau Lifshitz Vol 6, *Fluid mechanics* Section 136 (“Relativistic equations for flow with viscosity and thermal conduction”)	$$T_{ik} = -pg_{ik} + wu_iu_k + \tau_{ik}, \quad (136.1)$$ $$\tau_{ik} = -c\eta\left(\frac{\partial u_i}{\partial x^k} + \frac{\partial u_k}{\partial x^i} - u_k u^l\frac{\partial u_i}{\partial x^l} - u_i u^l\frac{\partial u_k}{\partial x^l}\right) - c(\zeta - \tfrac{2}{3}\eta)\frac{\partial u^l}{\partial x^l}(g_{ik} - u_iu_k), \quad (136.8)$$

...even in hydro simulations of the QGP



Israel-Stewart framework tries to fix thi (Israel-Stewart 1979, BRSSS 2007, DNMR 2012…):

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \pi^{\mu\nu}$$

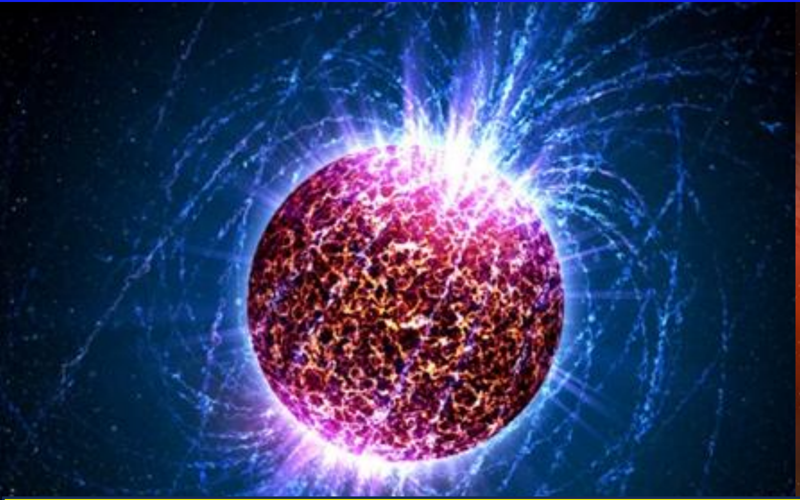
$$\tau_\pi \Delta_{\mu\nu}^{\alpha\beta} u^\lambda \nabla_\lambda \pi_{\alpha\beta} + \pi_{\mu\nu} = -2\eta\sigma_{\mu\nu} \text{ (+ other stuff, possibly)}$$

Causal near equilibrium (Hiscock-Lindblom 1983, Pu-Koide-Rischke 2009,…)
But acausal far from equilibrium (Bemfica et al. 2019, 2021)

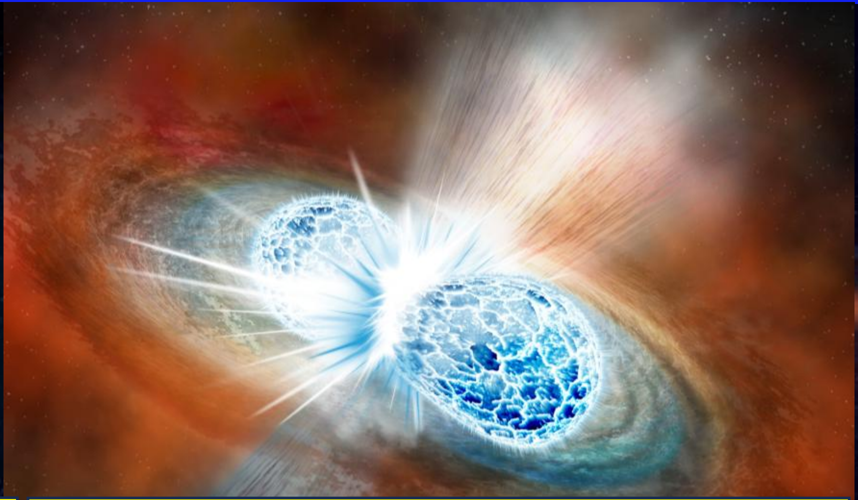
These causality violations show up at early times in simulations! (Plumberg et al. 2022, Krupczak et al. 2023,…)

In this figure: Red=Acausal, Blue=Causal, Purple=We don't know (see Plumberg et al. 2022)

The challenge is with diffusive phenomena



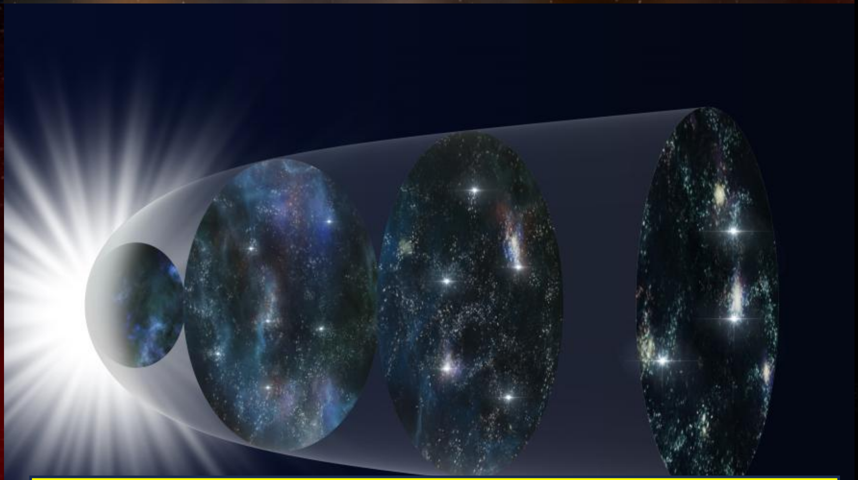
Langlois, Sedrakian & Carter ’98 · Andersson & Comer ’21 · Negreiros, Schramm & Weber ’12



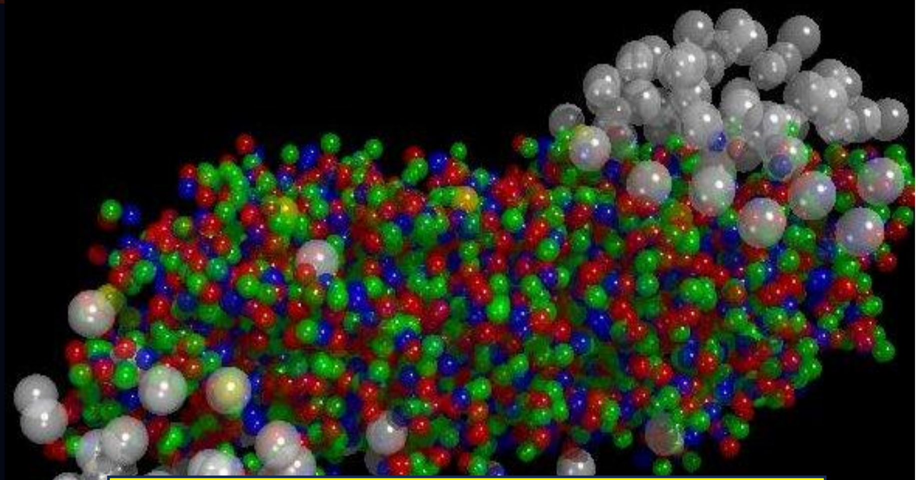
Shibata & Uryū ’00 · Giacomazzo, Rezzolla & Baiotti ’09 · Radice, Rezzolla & Galeazzi ’14

Dimmelmeier, Font & Müller ’02 · Ott et al. ’07 · Müller, Janka & Marek ’12

Gammie, McKinney & Tóth ’03 · De Villiers, Hawley & Krolik ’03 · Mizuno et al. ’18



Steinhardt ’82 · Espinosa, Konstandin, No & Servant ’10 · Leitão & Mégevand ’15



Bjorken ’83 · Kolb, Sollfrank & Heinz ’00 · Romatschke & Romatschke ’07

Should relativistic hydrodynamics be causal <u>in theory</u>?

Or, equivalently, are we "morally obliged" to write down causal PDEs?

Recall: *"Even in classical theory, instantaneous propagation of heat is an offense to intuition, which expects propagation at about the mean molecular speed; in a consistent relativistic theory it ought to be completely prohibited."* (Israel-Stewart 1979)

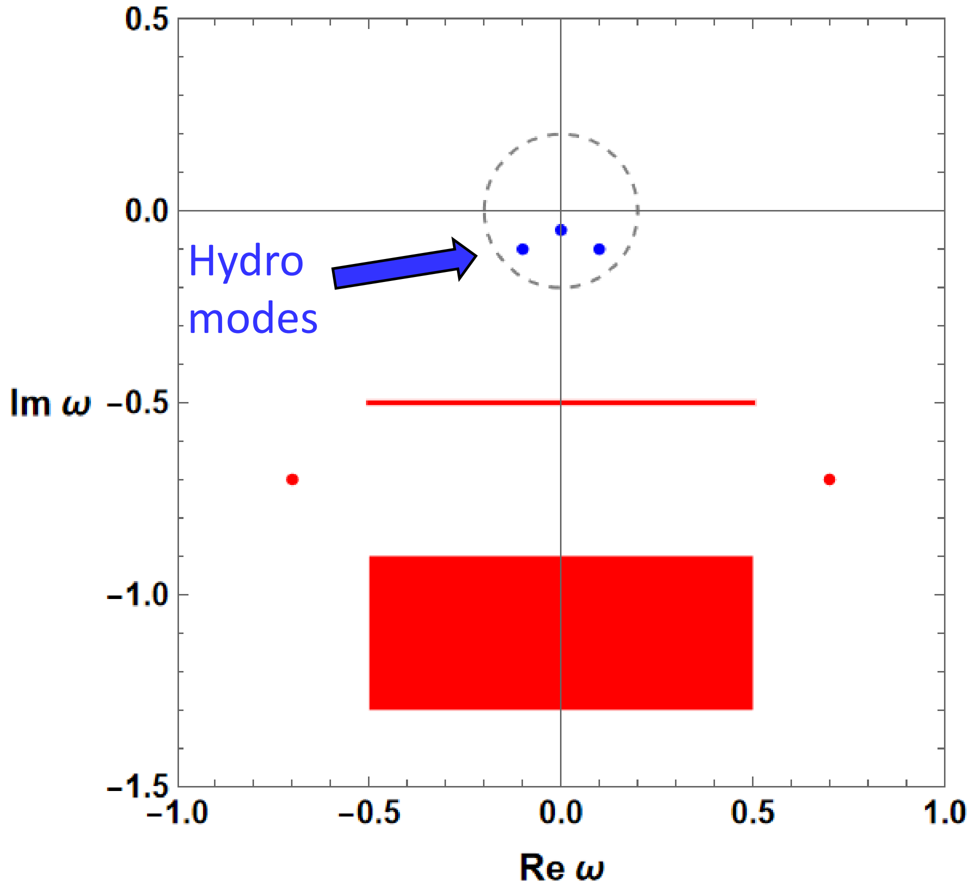
Quasi-normal modes and hydro

Fix a microscopic theory (kinetic theory, holography…);
Look for solutions of the form $e^{ikx-i\omega t}$;
Plot the values of ω for a given k (typically, $k \in \mathbb{R}$);

The frequencies that approach the origin as $k \to 0$ describe hydro (Kovtun-Starinets 2005, Romatschke 2016, Grozdanov-Kovtun-Starinets-Tadic 2019…).

They are:

- Sound waves,
- Shear waves,
- Diffusion modes…



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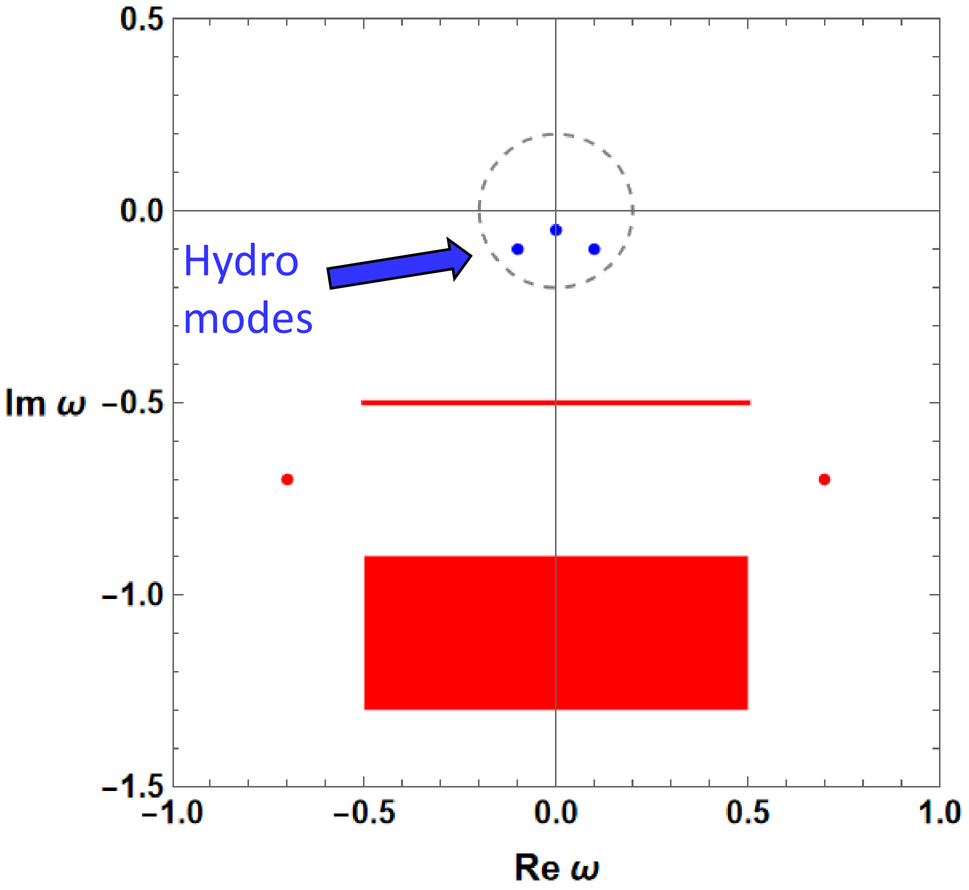


Question: If I split the conserved densities as

$$n(t,x) = n_{Hydro}(t,x) + n_{Non-hydro}(t,x)$$

do n_{Hydro} obey causal PDEs?
(assuming that I can write an exact PDE for them…)

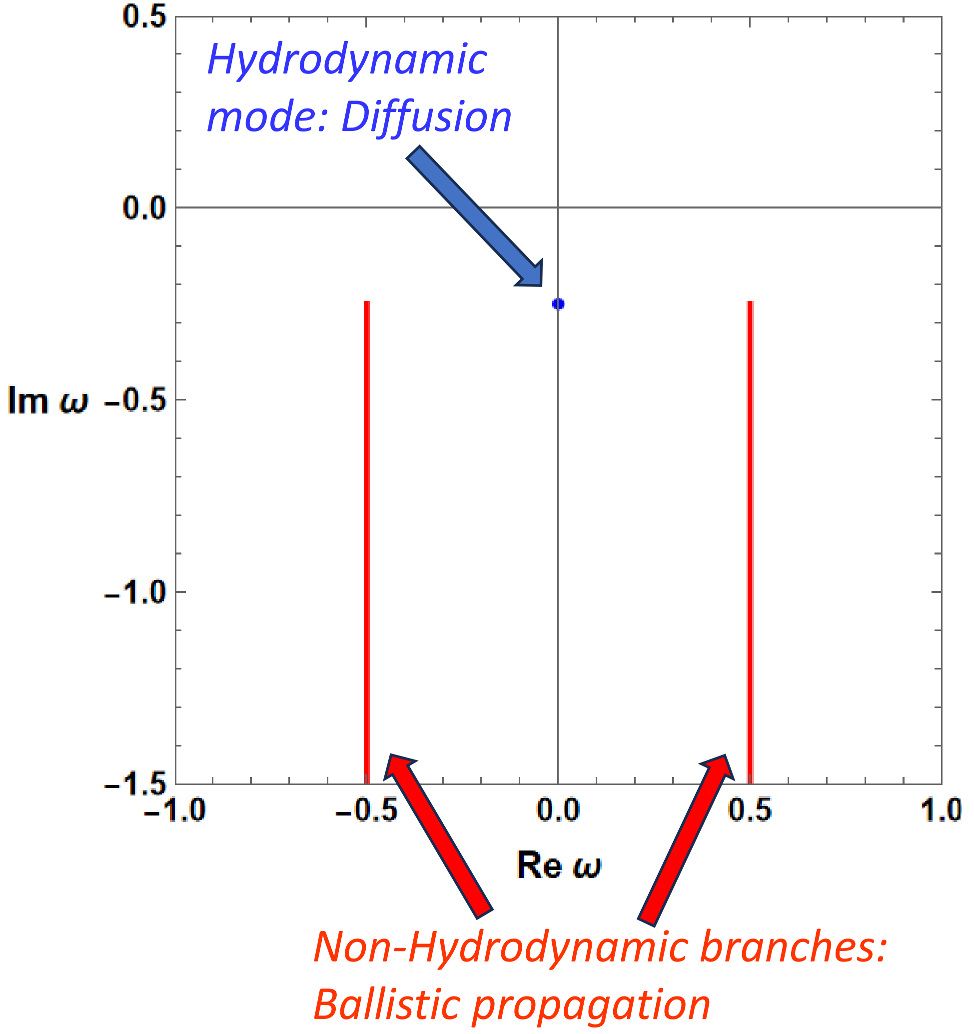
An exactly solvable model

Ultra-Relativistic Fokker-Planck kinetic theory in 1 dimension:

$$(\partial_t + v\partial_x)f = \partial_p(\partial_p f + vf),$$

A perfectly causal system.

$$n(t,x) = n_{Hydro}(t,x) + n_{Non-hydro}(t,x)$$



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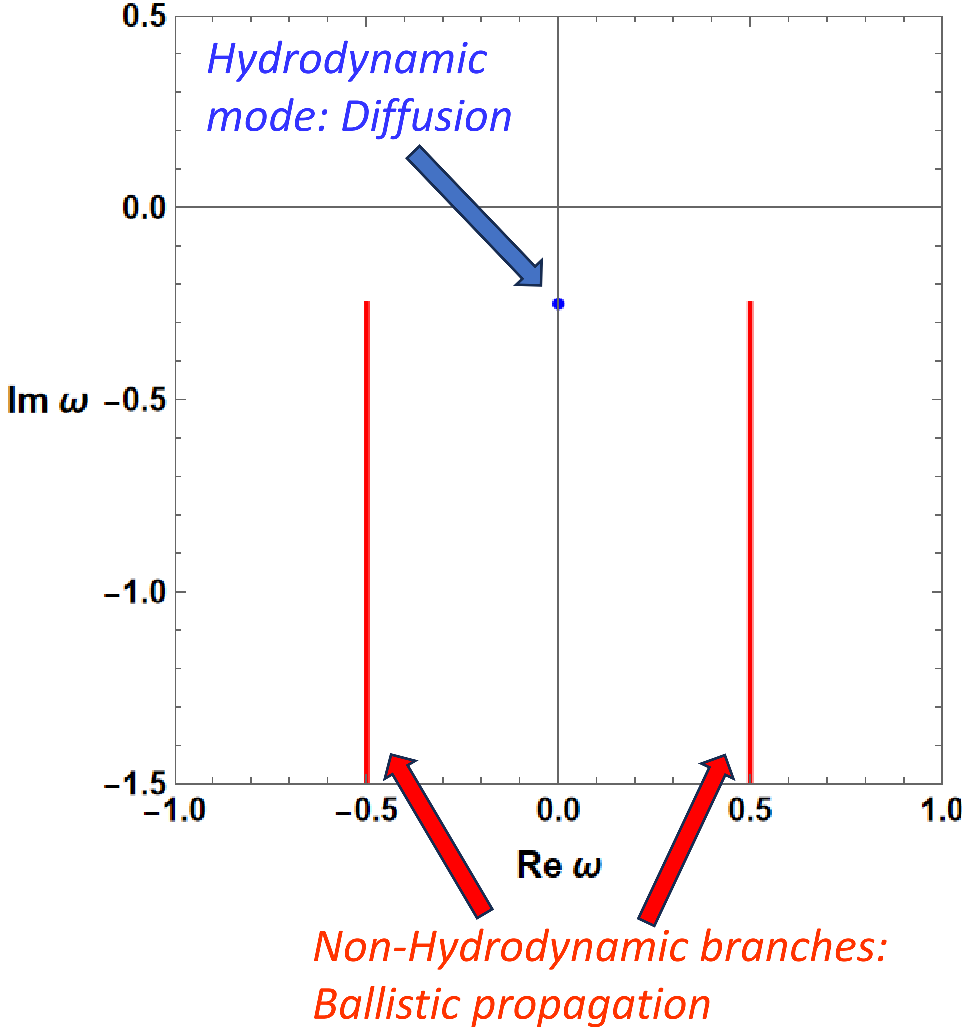
The hydro part obeys the diffusion equation exactly (LG 2026):

$$\partial_t n_{Hydro} = \partial_x^2 n_{Hydro}$$

The hydro sector of a causal microscopic theory is governed (with perfect accuracy) by an acausal PDE!



Hydrodynamic states are non-local

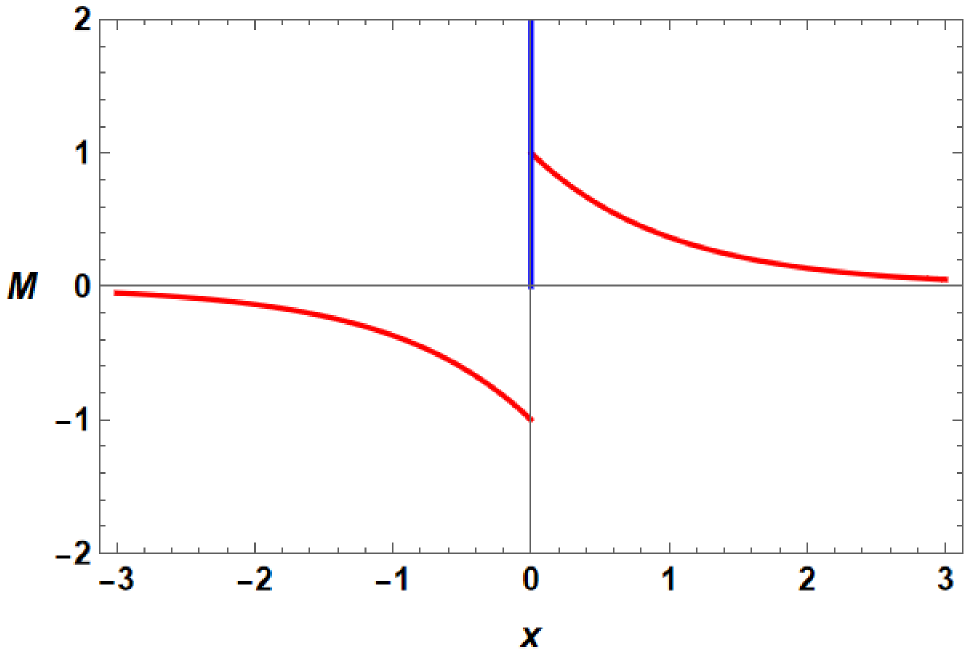
Microscopic initial data that is hydrodynamic always posesses infinitely long tails.

In kinetic theory, these tails appear when we compute higher moments.

In the case of diffusion Green function

$$M_0(x) = \int \frac{dp}{2\pi}\, \delta f(0,x,p) = \delta(x) \quad \text{(the density)}$$

$$M_1(x) = \int \frac{dp}{2\pi} p\, \delta f(0,x,p) = e^{-|x|}\mathrm{sign}(x)$$



Note: This phenomenon has the same mathematical origin as the impossibility to localize particle states in the Dirac equation!

More broadly

1. A subsector of a causal theory need not be causal on its own. (LG 2026)

2. A causal description of viscosity and diffusion must always contain non-hydro modes. (Muller 1967, Israel-Stewart 1979, Geroch 1998, Jou 2010, Grozdanov-Kovtun-Starinets-Tadić 2019, Heller-Serantes-Spalinski-Withers 2023, Brants 2026,…)

3. There is no notion of a causal or an acausal dispersion relation $\boldsymbol{\omega}(\boldsymbol{k})$. The same dispersion relation may be part of a causal or an acausal theory. All historical attempts to bound phase velocity, group velocity, or front velocity have ultimately been falsified, both as necessary and as sufficient conditions of causality. (Aharonov-Komar-Susskind 1969, Fox-Kuper-Lipson 1970, Krotscheck-Kundt 1978, Pu-Koide-Rischke 2009, LG-Disconzi-Noronha 2023, Hoult-Kovtun 2023…)

In summary: The hydro PDEs do not inherit the microscopic causal structure.

Are we morally obliged to wriye down causal PDEs?

<u>**No**</u>

Should relativistic hydrodynamics be causal <u>in practice</u>?

Or, equivalently, are we "kindly advised" to write down causal PDEs?

A useful exercise

$$\partial_t n = \partial_x^2 n$$

$$\boxed{\begin{aligned} \partial_t &= \gamma(\partial_{x'} + v\partial_{x'}) \\ \partial_x &= \gamma(\partial_{x'} + v\partial_{t'}) \end{aligned}}$$

$$(\partial_{t'} + v\partial_{x'})n = \gamma\left(\partial_{x'}^2 + 2v\partial_{x'}\partial_{t'} + v^2\partial_{t'}^2\right)n$$

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Boosted diffusion equation

$$(\partial_{t'} + \cancel{v\partial_{x'}})n = \gamma\left(\cancel{\partial_{x'}^2} + 2v\cancel{\partial_{x'}}\partial_{t'} + v^2\partial_{t'}^2\right)n$$

$$\partial_{t'} n = \gamma v^2 \partial_{t'}^2 n$$

Spatially homogeneous solutions

$$n(t') = \mathbf{A} + \mathbf{B}\,\mathbf{e}^{\frac{t'}{\gamma v^2}}$$

⟵ Very Bad!!!

Boosted diffusion equation

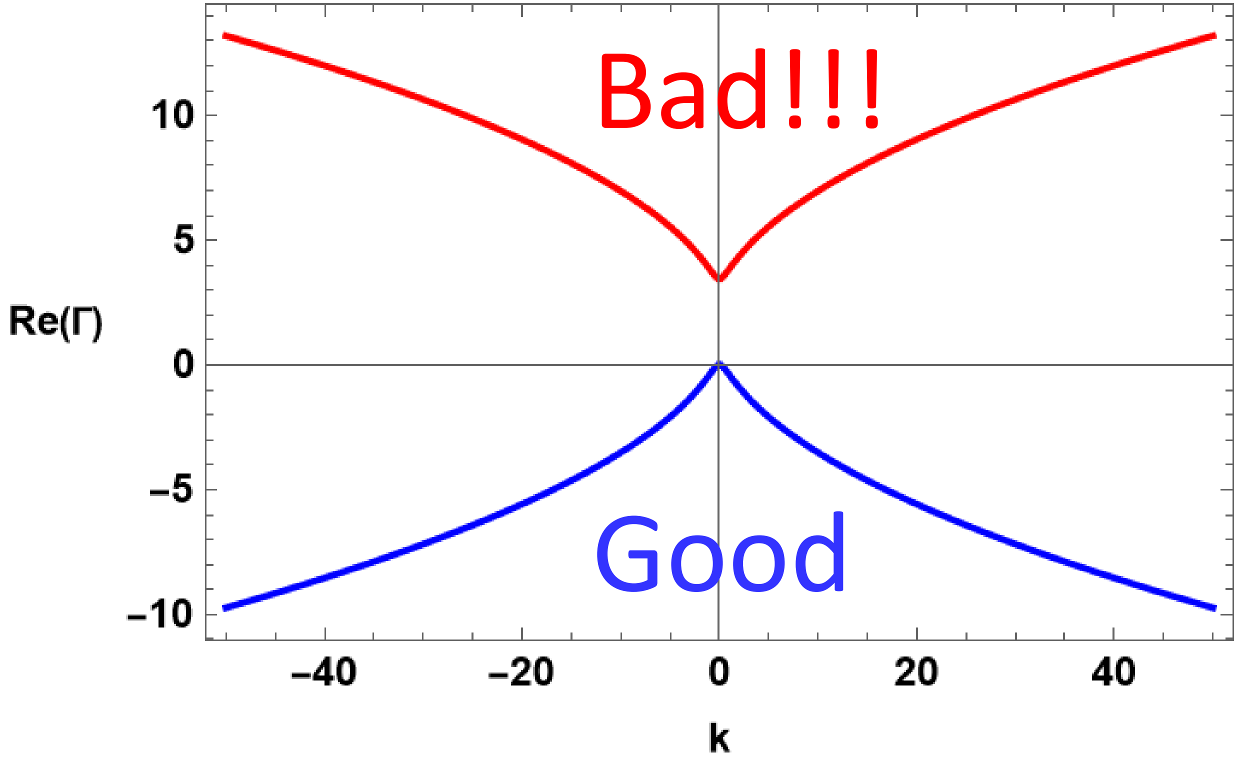
$$(\partial_{t'} + v\partial_{x'})n = \gamma\left(\partial_{x'}^2 + 2v\partial_{x'}\partial_{t'} + v^2\partial_{t'}^2\right)n$$

$$(\partial_{t'} + vik)n = \gamma\left(-k^2 + 2vik\partial_{t'} + v^2\partial_{t'}^2\right)n$$

$$n(t') = \mathrm{A}\mathrm{e}^{\Gamma_1 \mathrm{t}'} + \mathrm{B}\mathrm{e}^{\Gamma_2 \mathrm{t}'}$$

Plane waves
$n(t', x') = n(t')e^{ikx'}$

(Kostaedt-Liu 2000)

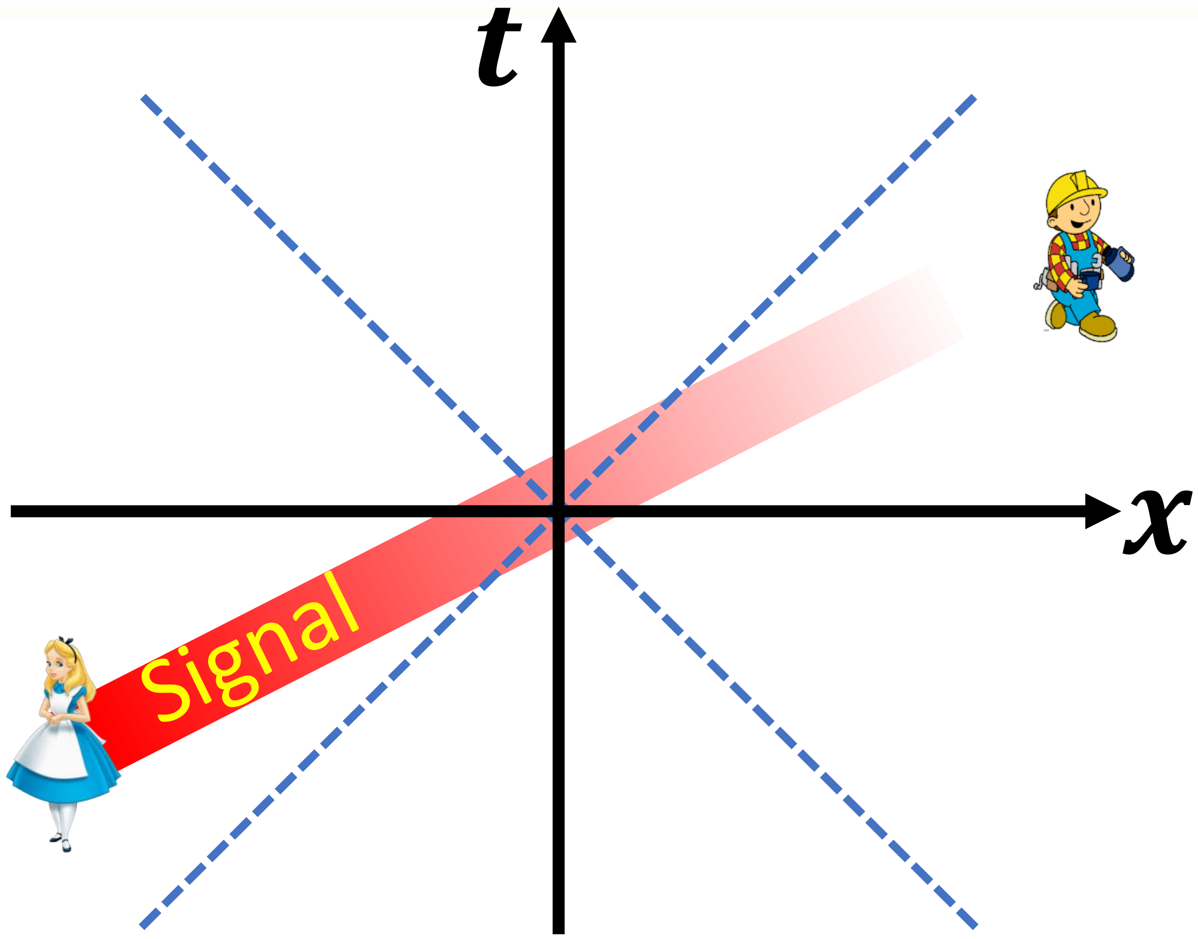


An interesting "coincidence"

Acausal dissipative theories tend to become unstable when you boost them.
Vice versa, causal theories that are stable in one reference frame tend to be stable in all frames.
This was noticed for

1. Relativistic Navier-Stokes (Hiscock-Lindblom 1983)
2. Israel-Stewart theory (Hiscock-Lindblom 1983, Olson 1989, Pu-Koide-Rischke 2009, Roy-Mitra-Singh 2026)
3. Divergence-type theories (Geroch-Lindblom 1990)
4. All theories with an information current (LG-Antonelli-Haskell 2022)
5. BDNK (Bemfica-Disconzi-Noronha 2022)
6. Third-order hydrodynamics (Brito-Denicol 2022)
7. Holographic Gauss-Bonnet (Buchel-Hoult-Kovtun 2026)
8. …

A thought experiment



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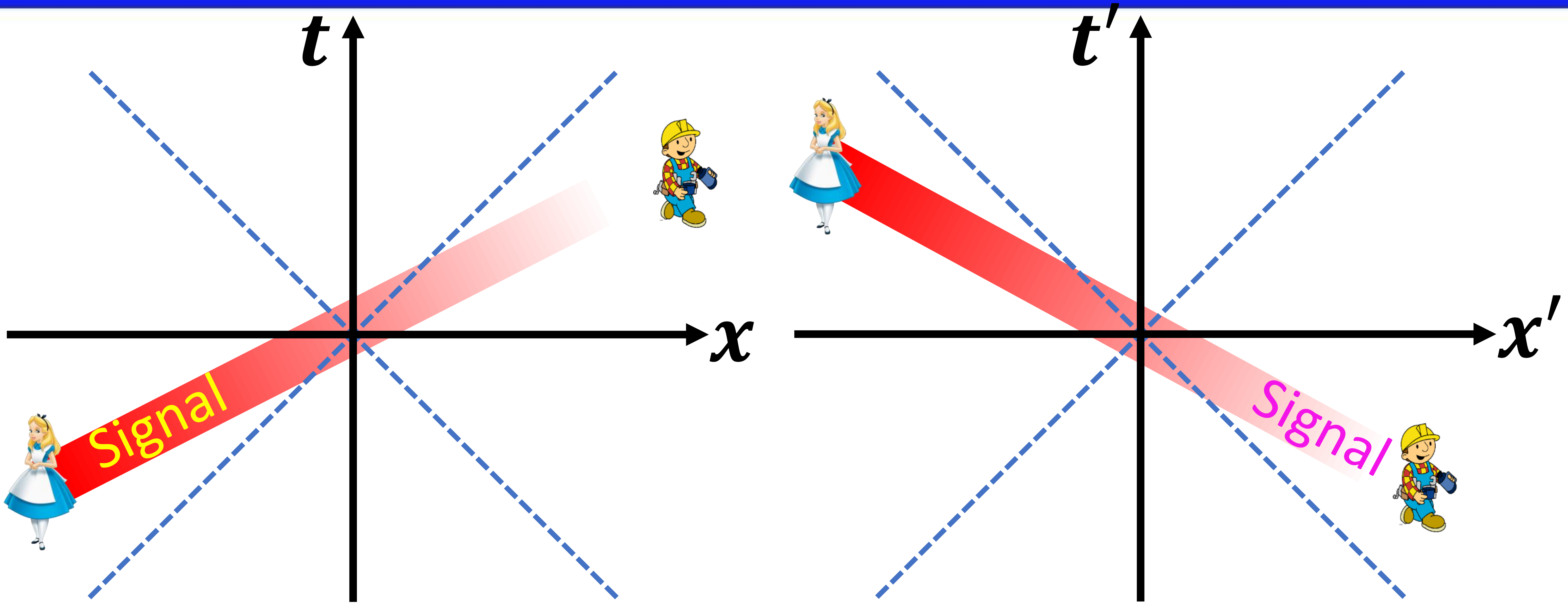
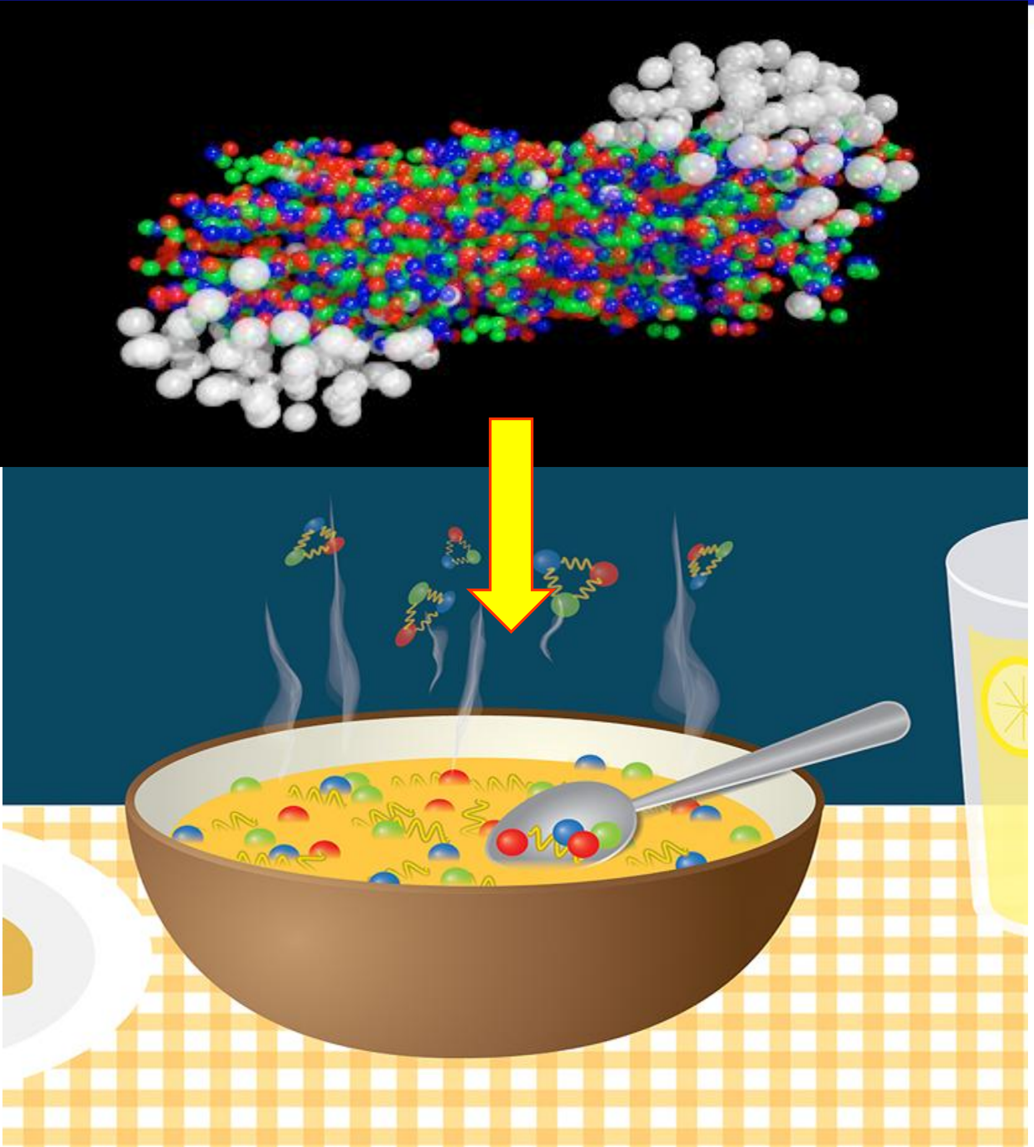


Without causality, the very idea of "stability" becomes observer dependent.

Acausal dissipative systems can be boosted into unstable ones. (LG 2022)

Back to QGP simulations



Recall:

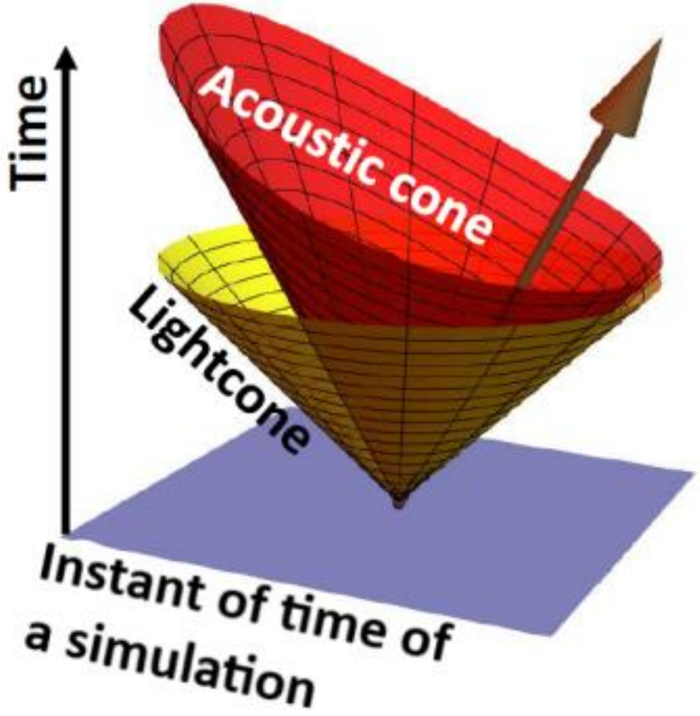
$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \pi^{\mu\nu}$$

$$\tau_\pi \Delta^{\alpha\beta}_{\mu\nu} u^\lambda \nabla_\lambda \pi_{\alpha\beta} + \pi_{\mu\nu} = -2\eta\sigma_{\mu\nu} \text{ (+ “stuff”)}$$

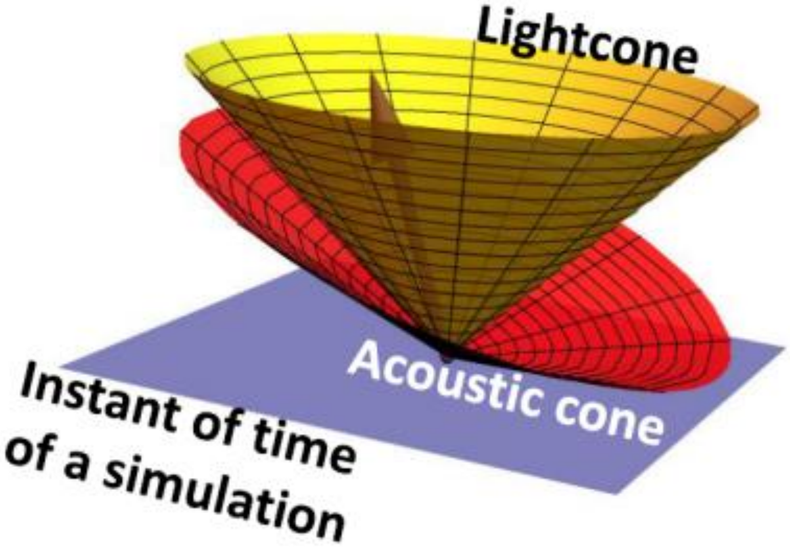
Can be causal or acausal, depending of the value of $\pi_{\mu\nu}$

Three possible scenarios

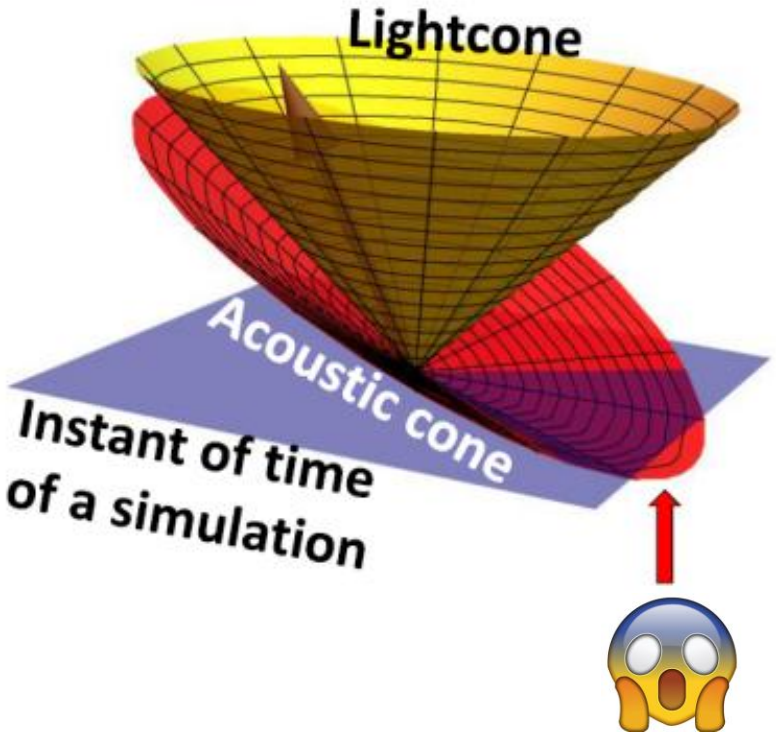
Good: Causal



Bad: Acausal but Stable

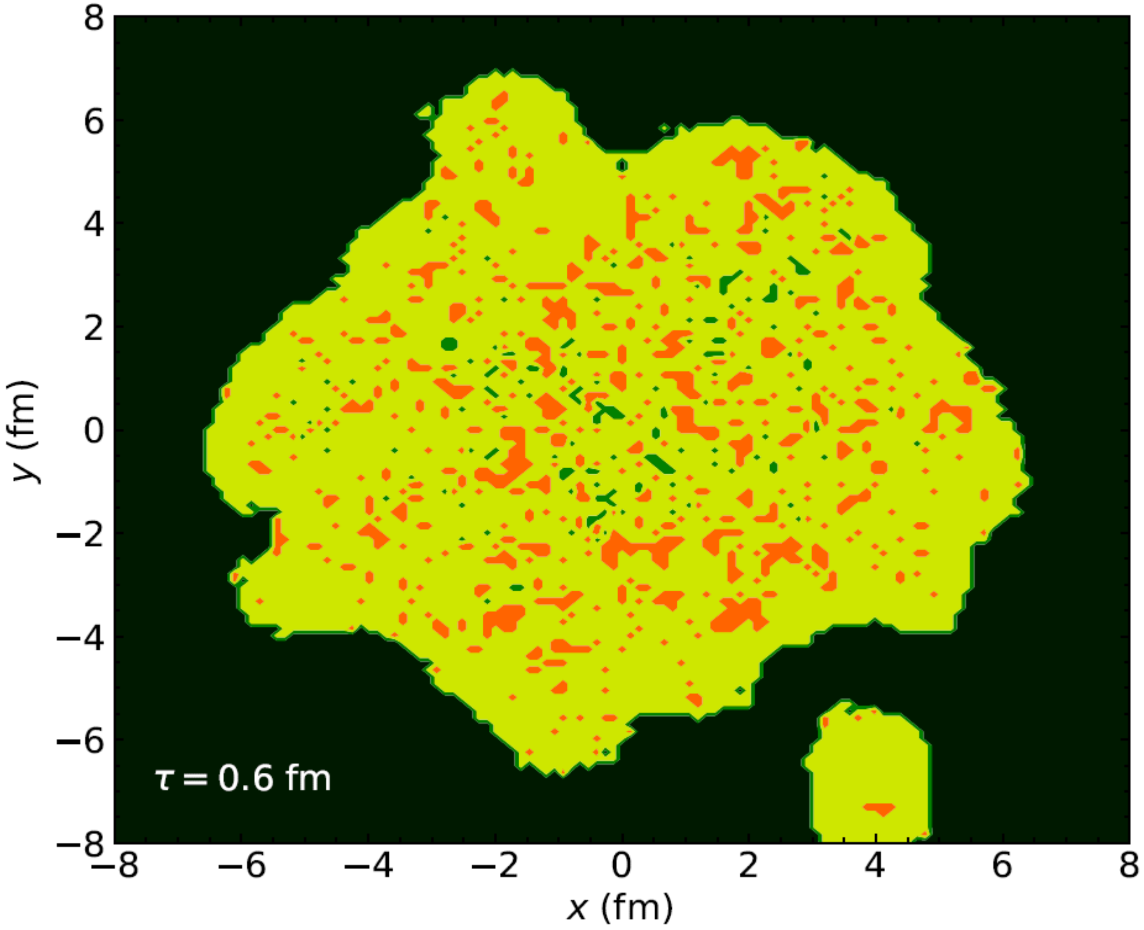


Ugly: Unstable



The good, the bad, and the ugly

Good: Causal **Bad: Acausal but Stable** **Ugly: Unstable**



About this simulation:
IP-Glasma initial state. The input file is publicly available (LG-Hirvonen-Paquet-Singh-SoaresRocha 2025)

Recent progress towards achieving either side

The "Yes" side and the "No" side

BDNK Theory (Yes)

(Bemfica-Disconzi-Noronha 2018,2022, Kovtun 2019,…)

Start from the acausal equation:

$$\partial_t n = D\partial_x^2 n + O(\partial^3)$$

Redefine the effective field using time derivatives, e.g.

$$n = \tilde{n} + \tau\partial_t\tilde{n} + O(\partial^2)$$

Plug into the original equation, and drop order ∂^3:

$$\tau\partial_t^2\tilde{n} + \partial_t\tilde{n} = D\partial_x^2\tilde{n} + O(\partial^3)$$

Taking $\tau > D$, we have causality.

In the linear regime, the result appears similar to Cattaneo and Israel-Stewart. However, in the non-linear regime, it is very different, and one can guarantee full non-linear causality!

Density-frame hydro (No)

(Armas and Jain 2021, Bhambure-Singh-Teaney 2025)

Keep the acausal equation:

$$\partial_t n = D\partial_x^2 n + O(\partial^3)$$

But don't boost it <u>exactly</u>!

Under an exact boost, one has

$$(\partial_{t'} + v\partial_{x'})n = \gamma D(\partial_{x'} + v\partial_{t'})^2 n + O(\partial^3)$$

Use the lower-order equation $(\partial_{t'} + v\partial_{x'})n = O(\partial^2)$ to make all second derivatives purely spacelike:

$$(\partial_{t'} + v\partial_{x'})n = \gamma^{-3}D\,\partial_{x'}^2 n + O(\partial^3)$$

The modified boosted equation is stable despite being acausal!

What BDNK looks like

For simplicity, zero chemical potential:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu}$$
$$-2\eta\sigma^{\mu\nu} - \zeta\, \nabla_\alpha \mathbf{u}^\alpha\, \Delta^{\mu\nu}$$
$$+\kappa_1 \left[(\varepsilon + P) u^\alpha \nabla_\alpha \mathbf{u}_\beta + \nabla_\beta \mathrm{P} \right] \Delta^{\beta(\mu} u^{\nu)}$$
$$+(\kappa_2\, u^\mu u^\nu + \kappa_3 \Delta^{\mu\nu})[u^\alpha \nabla_\alpha \varepsilon + (\varepsilon + P)\nabla_\alpha \mathbf{u}^\alpha]$$

Ideal fluid $[O(\partial^0)]$

Relativistic Navier-Stokes $[O(\partial^1)]$

BDNK UV regulators $[O(\partial^1)$ off-shell, but $O(\partial^2)$ on-shell]

Fix $\boldsymbol{\kappa}_{1,2,3}$ so that the PDEs $\nabla_\mu \mathbf{T}^{\mu\nu} = \mathbf{0}$ are causal and stable.
Note that the extra blue terms are of <u>second</u> order on-shell.
Hence, BDNK describes the same IR physics as relativistic Navier-Stokes.

What density-frame hydro looks like

For simplicity, zero chemical potential:

$$\boldsymbol{T^{0\nu} = (\varepsilon + P)u^0 u^\nu + P g^{0\nu}}$$
$$\boldsymbol{T^{jk} = (\varepsilon + P)u^j u^k + P g^{jk} - T K^{jklh} \partial_l (u_h / T)}$$

$\boldsymbol{K^{jklh}}$ is a complicated tensor that depends only $\boldsymbol{u^\mu}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$.
No gradient corrections to the densities $\boldsymbol{T^{0\nu}}$ (hence why "density frame")
All gradient corrections are space derivatives.

Different observers construct mathematically inequivalent theories, but physical predictions agree to first order.

Comparison between the approaches

	BDNK	Density frame
Non-hydrodynamic modes	Y	N
Unobservable transport coefficients	Y (Many!)	N
Causality	Y	N
Lorentz invariance	Y	Y (but only approximately)
Good with gravity	Y	Y (fixed background), ? (dynamical GR)
Good with stochastic fluctuations	N	Y
Obeys the second law	Y (but only approximately)	Y

Note: Within the formal regime of applicability of relativistic Navier-Stokes, these theories are (and must be!) physically equivalent.

In summary

Should relativistic hydrodynamics be causal <u>in theory</u>?
No, but causality (combined with stability) constrains transport coefficients.

Should relativistic hydrodynamics be causal <u>in practice</u>?
Yes, if Lorentz invariance is enforced exactly.
No, if Lorentz invariance is implemented perturbatively in gradients.

Choose you favorite approach based on your goals:
BDNK: Many unphysical regulators, mathematically solid, causal, Lorentz invariant;
Density frame: Minimal ingredients, good for fluctuations, breaks Lorentz invariance;

…or just you keep using Israel-Stewart, trying to make sure you do not enter the ugly regime.

Appendix

Constraining transport from causality

Hydrohedron program
(Heller-Serantes-Spalinski-Withers 2023)

Assume causality + stability. Then, you find:

$$Im(\omega) \le |Im(k)|$$

Assume further that

$\omega_{Hydro}(k) = \sum_n c_n k^n$ for $|k| \le R$ (validity range)

Then, some bounds on c_n, and thus on transport coefficients, in terms of R follow. For example:

$$0 \le D \le \frac{16}{3\pi R}$$

Bounds from kinetic theory
(LG 2026)

Assume that

$$i\omega \in \mathbf{Spectrum}(R + ikV)$$

with R and V self-adjoint operators on a Hilbert space (as in kinetic theory).

Causality and stability imply $R \ge 0$ and $|V| \le 1$.

One then obtains bounds on transport coefficients, such as

$$0 \le D \le \tau \text{ (non-hydrodynamic gap)}$$