The QCD Equation of State at very high temperatures

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Based on

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In collaboration with

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Outline

- The QCD Equation of State and its role
- Determination at high temperatures from lattice QCD
- Results for QCD with 3 flavours up to the electro-weak scale
- Conclusions & Outlook

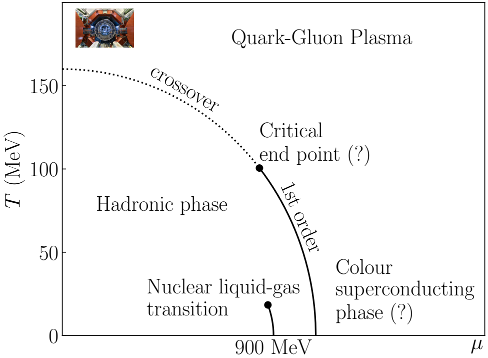
The Equation of State

- The Equation of State (EoS) describes the equilibrium behaviour of the QCD plasma as a function of the temperature T and chemical potential μ

- Composed by e.g.
  - pressure, p
  - entropy density, s
  - energy density, e

  of the QCD plasma

- EoS at $T \lesssim 200$ MeV relevant for
  - Exploration of QCD phase diagram
  - Nuclear physics, neutron stars
  - Heavy-ion collision experiments



- This talk: focus on T-dependence of EoS, at $\mu = 0$

$\mu \neq 0$: [Seppänen, Tue 9:45], [Mandl, Tue 12:00], [Schröder, Tue 14:00], [Cai, Fri 14:45], [Oevermann, poster], [Passarella, poster], [Navarrete, poster]

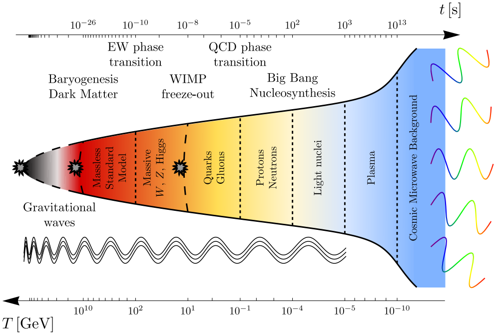
The Equation of State up to the electro-weak scale

- QCD EoS at all temperatures up to the electro-weak scale ($\sim$ 100 GeV) relevant for Cosmology and physics of the Early Universe
  - EoS of Standard Model (SM) determines the expansion rate of the Universe through Friedmann equations

$$\begin{cases} \dot{e} = -3H(e+p) \\ H^2 = \frac{8GH}{3} e \end{cases}$$

    where H Hubble parameter
  - Influence on cosmological observables like primordial GWs, dark matter abundance, ...

[Review: Saikawa, Shirai 18-20]



[Schmitz 22]

The Equation of State up to the electro-weak scale

- For $T \gtrsim 1$ GeV, QCD contributes by $\gtrsim 70\%$ of the SM d.o.f (free-theory estimate)

$$g_{*s}(T) = \left(\frac{2\pi^2 T^3}{45}\right)^{-1} s(T)$$

T (GeV)	g_{*s}^{free}	leptons	QCD	EW
1	76.35	19.36%	78.02%	2.62%
10	86.15	18.27%	79.29%	2.44%
100	102.87	15.31%	74.10%	10.59%
∞	106.75	14.75%	74.00%	11.25%

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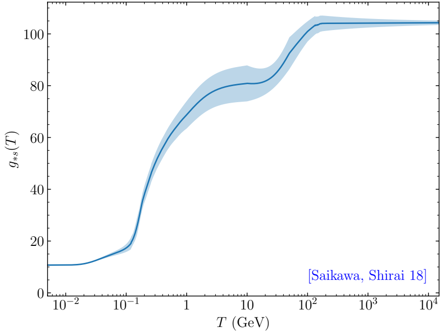

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- Interactive case for g_{*s} estimated by combining perturbative and non-perturbative results
- Main source of uncertainty from QCD EoS, because of
  - lack of lattice results at $T \gtrsim 1$ GeV
  - poor convergence of thermal perturbation theory

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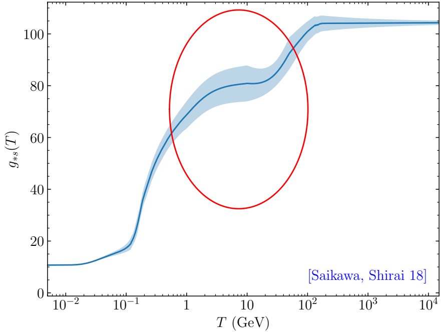


$\Rightarrow$ Call for a non-perturbative study of thermal QCD up to the electro-weak scale

Thermal effective theories of QCD

- Thermal effective theories: high-T expansion of QCD

[Ginsparg 80; Linde 80; Appelquist, Pisarski 81; Braaten, Nieto 95-96; ...]

- Three relevant scales for thermal QCD: ($\hat{g}$ = QCD ren. coupling)

$M = \pi T + \ldots$	Fermions and non-zero Matsubara gluons	hard scale
$m_E \propto \hat{g}T + \ldots$	A_0 zero Matsubara gluons	soft scale
$g_E^2 = \hat{g}^2 T + \ldots$	A_i zero Matsubara gluons	ultrasoft scale

- Asymptotic freedom $\Rightarrow$ hierarchy of scales at asymptotically high T

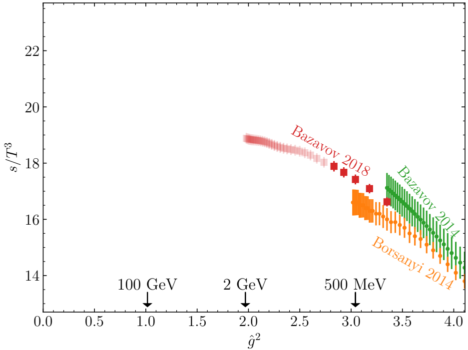
$$\frac{g_E^2}{\pi} \ll m_E \ll \pi T \quad \Leftrightarrow \quad \left(\frac{\hat{g}}{\pi}\right)^2 \ll \frac{\hat{g}}{\pi} \ll 1$$

- Perturbation theory (PT) developed for hard and soft scales (ptQCD, EQCD)
- Non-perturbative (NP) contributions from the ultrasoft scale (MQCD)

NP definition of MQCD: [Catumba, poster]

QCD Equation of State: state of the art

- EoS known from the lattice for $N_f = 2+1$ QCD at $T \lesssim 1$ GeV with $O(1\%)$ precision
- $\hat{g}(T) =$ 5-loop $\overline{\text{MS}}$ strong coupling at renormalization scale $\mu = 2\pi T$



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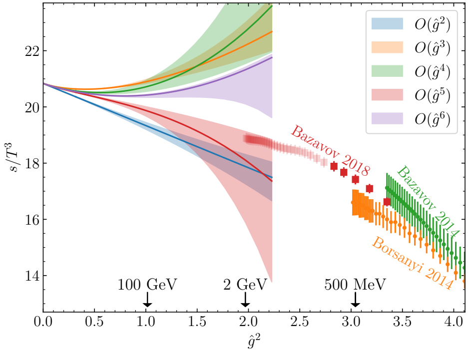
- Thermal PT used at $T \gtrsim 1$ GeV

  [Shuryak 78, ..., Kajantie et al. 03]
- NP ultrasoft contributions are parametrically of $O(\hat{g}^6)$ [Linde 80]
- Slow convergence for $1 \text{ GeV} \lesssim T \lesssim 100 \text{ GeV}$



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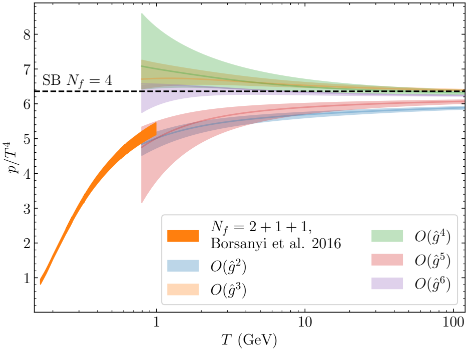

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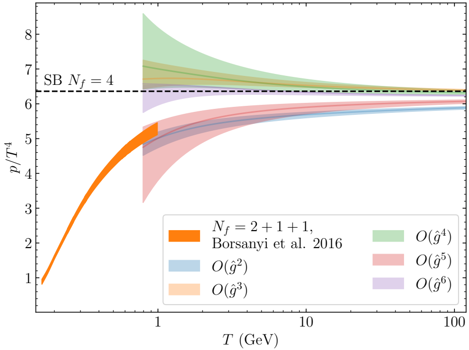

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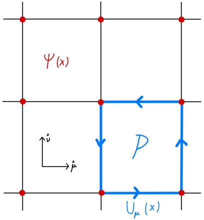
$\Rightarrow$ Our goal: non-perturbative determination of EoS up to the electro-weak scale

Lattice QCD

QCD can be defined on a discretized space-time so that gauge invariance is preserved

- In 4-dim Euclidean spacetime, consider a lattice with $(L_0/a) \times (L/a)^3$ points, $a =$ lattice spacing
- Gauge field: $U_\mu(x) \in \mathrm{SU}(3)$ link variables
- Quark field: $\overline{\psi}(x), \psi(x)$ are N_f triplets in colour space



- Many ways to discretize QCD action; we consider Wilson's lattice action:

$$S = S_G + S_F\,, \quad S_G = \frac{1}{g_0^2}\sum_{\mathcal{P}} \mathrm{tr}\,\{1 - \mathcal{P}\}\,, \quad S_F = a^4 \sum_x \overline{\psi}(x)(D + M_0)\psi(x)$$

- $*$ $g_0^2 =$ bare coupling squared
- $*$ $M_0 =$ bare quark mass matrix
- $*$ $D =$ Wilson-Dirac operator

Temperature of the system $T = 1/L_0$

- $*$ L_0 finite $\Rightarrow$ thermal QCD
- $*$ $L_0 \to \infty \Rightarrow$ vacuum QCD

Lattice QCD

Given a lattice field $\mathcal{O}$, its expectation value is given by the lattice path integral

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int D[U, \overline{\psi}, \psi] \, \mathcal{O}[U, \overline{\psi}, \psi] \, e^{-S} = \int DU \, P[U] \widetilde{\mathcal{O}}[U] \, , \quad P[U] = \frac{1}{\mathcal{Z}} \det(D + M_0) e^{-S_G}$$

1) At given bare lattice parameters $(g_0^2,\, aM_0,\, L_0/a,\, L/a)$, estimate $\langle \mathcal{O} \rangle$ by

- Monte Carlo sampling of U-field configurations according to $P[U]$
- Average over the sampled gauge configurations

$$U^{(1)} \to U^{(2)} \to \cdots U^{(N)} \qquad \langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \widetilde{\mathcal{O}}[U^{(i)}] + O(1/\sqrt{N})$$

$\Rightarrow$ in lattice QCD, errors have a well-defined statistical meaning

2) Repeat the calculation by tuning bare parameters such that a decreases

- QCD is non-perturbatively defined as the continuum limit $(a \to 0)$ and infinite-volume limit $(L \to \infty)$ of lattice QCD
- In practice, the limit is usually taken by extrapolation

EoS from the lattice: state of the art

Hadronic renormalization schemes

Determine the lattice spacing in physical units through

$$a(g_0^2) = [a\mathcal{M}_{\rm had}](g_0^2)/\mathcal{M}_{\rm had}$$

- $\mathcal{M}_{\rm had} \sim \Lambda_{\rm QCD} \sim 200$ MeV (e.g. m_π, m_K, f_π, f_K)
- Calculation carried out in zero-T lattices ($L_0 \to \infty$)
- $a(g_0^2)$ is then used in finite-T lattices (L_0 short and periodic)

Integral methods

Pressure determined as [Boyd et al. 96]

$$\frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_{T_0}^{T} dT' \, \frac{\Theta(T')}{(T')^5}$$

- Trace anomaly $\Theta(T) = e(T) - 3p(T)$ computed on finite-T lattices
- Subtraction at $T_0 \sim \Lambda_{\rm QCD} \ll T$ required to define the EoS

EoS from the lattice: state of the art

Finite-volume effects (FVE) and discretization effects under control when

- $L\Lambda_{\rm QCD} \gg 1$ in zero-T lattices for renormalization and EoS subtraction, **and**
- $aT \ll 1$ in finite-T lattices for calculation of $\Theta(T)$

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Same $a(g_0^2)$ at zero-T and finite-T $\Rightarrow$ Both to be satisfied at the same time. Example:

- $L\Lambda_{\mathrm{QCD}} \gtrsim 10$
- $aT \lesssim 1/5$

$\Rightarrow \quad L/a \gtrsim 50 \times T/\Lambda_{\mathrm{QCD}}$ (for zero-T lattices)

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$\Rightarrow \quad L/a \gtrsim 50 \times T/\Lambda_{\rm QCD}$ (for zero-T lattices)

$*$ $T = 200$ MeV (∼ QCD crossover) $\rightarrow L/a \gtrsim 50$

$*$ $T = 1$ GeV (∼ charm scale) $\rightarrow L/a \gtrsim 250$

$*$ $T = 5$ GeV (∼ bottom scale) $\rightarrow L/a \gtrsim 1250$

Window problem: simulating very different scales $\Lambda_{\rm QCD} \lesssim T_0 \ll T$ at the same lattice spacing is very expensive $\Rightarrow$ for $T \gtrsim 1$ GeV a different strategy is required!

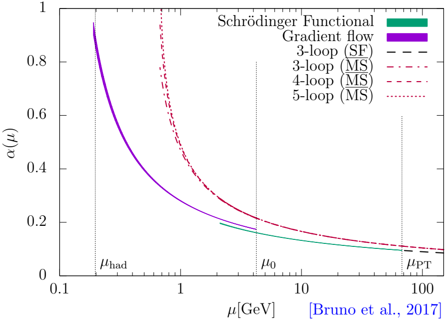
Lattice QCD up to the electro-weak scale

- NP defined coupling $\bar{g}^2$ (e.g. SF scheme)

  [Lüscher et al. 92]

- Relate $\bar{g}^2(\mu_{\mathrm{had}})$ to $\mathcal{M}_{\mathrm{had}}$ using hadronic renormalization schemes
- Compute $\bar{g}^2(\mu)$ up to the electro-weak scale using step-scaling techniques

  [Lüscher et al. 94; Dalla Brida et al. 17]



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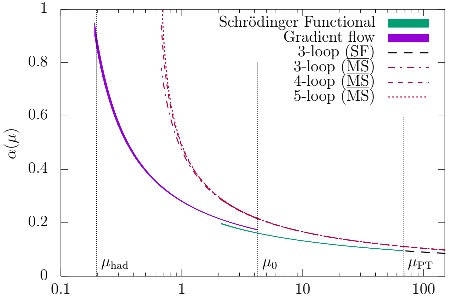


[Bruno et al., 2017]

Lattice spacing $a(g_0^2)$ determined (implicitly) by [Giusti, Pepe 15; Dalla Brida et al. 21]

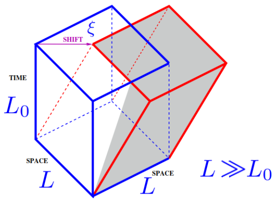
$$\bar{g}^2_{\rm SF}(g_0, a\mu) = \bar{g}^2_{\rm SF}(\mu)\,, \quad \mu = T\sqrt{2}$$

- The relevant scale is $\mu \sim T \Rightarrow$ no window problem [Rescigno, poster]
- $\mu \propto T \Rightarrow a(g_0^2)$ determined at each T separately such that $aT \ll 1$ easily satisfied
- At finite-T, $\mathcal{M}_{\rm gap} \sim \hat{g}^2 T \Rightarrow$ FVE exponentially suppressed as LT for $L \to \infty$

Thermal QCD in a moving frame

Shifted boundary conditions along compact direction, $\boldsymbol{\xi}$ shift parameter [Giusti, Meyer 10-13]

$$A_\mu(x_0+L_0,\boldsymbol{x}) = A_\mu(x_0,\boldsymbol{x}-L_0\boldsymbol{\xi})$$
$$\psi(x_0+L_0,\boldsymbol{x}) = -\psi(x_0,\boldsymbol{x}-L_0\boldsymbol{\xi})$$
$$\overline{\psi}(x_0+L_0,\boldsymbol{x}) = -\overline{\psi}(x_0,\boldsymbol{x}-L_0\boldsymbol{\xi})$$



The SO(4) invariance of QCD implies that

$$p(L_0,\boldsymbol{\xi}) = p(L_0\sqrt{1+\boldsymbol{\xi}^2},\boldsymbol{0})$$

where p is the pressure,

$$p(L_0,\boldsymbol{\xi}) = \lim_{L\to\infty}\frac{1}{L_0L^3}\log\mathcal{Z}(L_0,\boldsymbol{\xi})$$

- The temperature is given by

$$T = \frac{1}{L_0\sqrt{1+\boldsymbol{\xi}^2}}$$

- The entropy s then reads

$$\frac{s}{T^3} = \frac{1}{T^3}\frac{\partial p}{\partial T} = -\frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\partial p}{\partial \xi_k}$$

$\Rightarrow$ by computing the entropy, the zero-temperature subtraction in the EoS is avoided

EoS from the lattice up to the electro-weak scale

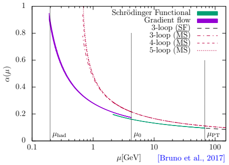
Renormalization via NP running coupling

The lattice spacing $a(g_0^2)$ is determined as a function of the bare coupling through

$$\bar{g}^2_{\mathrm{SF}}(g_0, a\mu) = \bar{g}^2_{\mathrm{SF}}(\mu)\,, \quad \mu = T\sqrt{2}$$

[Dalla Brida et al. 21]

- Renormalization scale μ chosen to be close to the temperature T
- No need to simulate both scales Λ_{QCD} and T at the same $a(g_0^2)$



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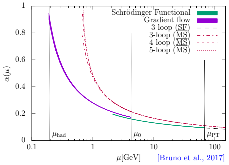


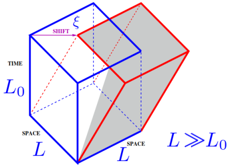
EoS from a moving frame

In thermal QCD with shifted boundary conditions, determine the entropy as

$$\frac{s}{T^3} = -\frac{(1+\boldsymbol{\xi}^2)}{\xi_k}\frac{1}{T^4}\frac{\partial p}{\partial \xi_k}$$

[Giusti, Mayer 10-13]

- Zero-T subtraction in the EoS avoided
- No need to simulate both zero-T and finite-T at the same $a(g_0^2)$



Lattice computation in $N_f = 3$ QCD

- $N_f = 3$ massless $O(a)$-improved Wilson fermions $\rightarrow$ discretization effects of $O(a^2)$

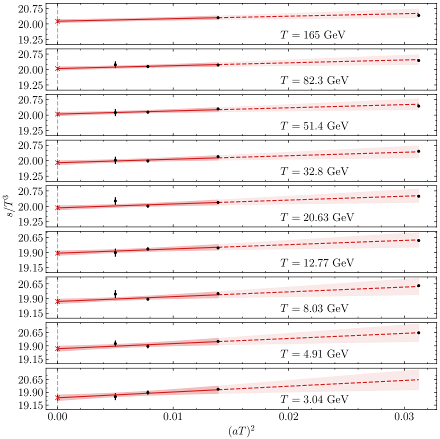
- At each T, compute for several lattice resolutions $0.07 \lesssim aT \lesssim 0.18$

- Spatial sizes L such that $10 \lesssim LT \lesssim 25$ $\rightarrow$ negligible finite-volume effects

- Extrapolation of the data by global fit
$$s(T,a)/T^3 = s(T)/T^3 + d_2(aT)^2$$
where $d_2 = d_2(\bar{g}^2_{\mathrm{SF}})$ (mildly) depends on T

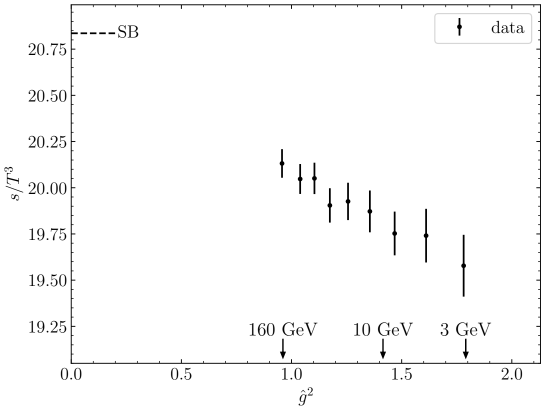
- Fit has $\chi^2/\mathrm{dof} = 0.82$, extrapolated results have $0.5\% - 1.0\%$ accuracy

$$\frac{s(T,a)}{T^3} = -\frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\Delta}{\Delta\xi_k}p(T,a)$$
where $\boldsymbol{\xi} = (1,0,0)$



Equation of State of $N_f = 3$ QCD

- $\hat{g}(T)$ = 5-loop $\overline{\text{MS}}$ strong coupling at renormalization scale $\mu = 2\pi T$. Convenient proxy for T
- At face value, data point straight to Stefan-Boltzmann (SB) limit
- The $T = 165$ GeV point deviates by 3.5% from SB (about 9 sigma)



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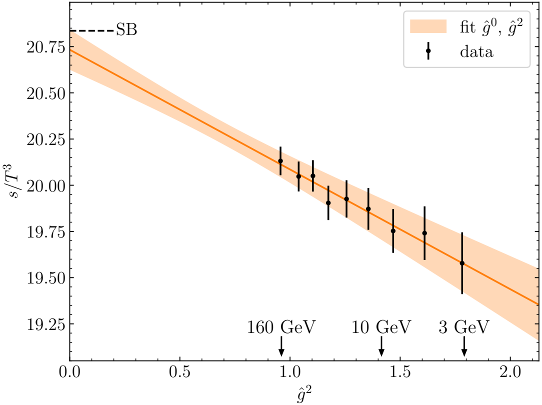
- Linear fit in $\hat{g}^2$:
  $s_0 = 2.954(15)$
  compatible with $s_0^{\mathrm{SB}} = 2.969$



$$\frac{s}{T^3} = \frac{32\pi^2}{45}\left[s_0 + s_2\left(\frac{\hat{g}}{2\pi}\right)^2\right]$$

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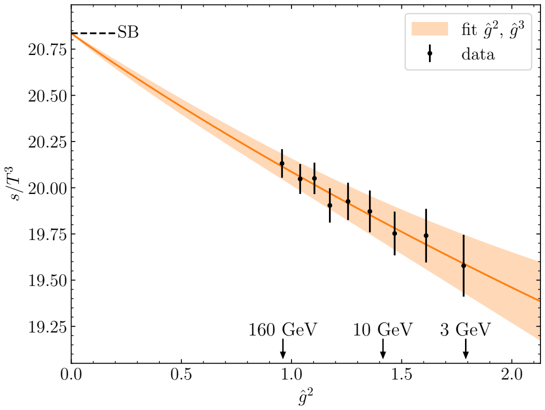
- At face value, data point straight to Stefan-Boltzmann (SB) limit
- The $T = 165$ GeV point deviates by 3.5% from SB (about 9 sigma)
- Enforce s_0^{SB}, add $\sim \hat{g}^3$: $s_2 = -5.1(9)$, $s_3 = 5(5)$ (in PT $s_2 = -8.438$)

$\Rightarrow$ data well described by a simple linear behaviour in $\hat{g}^2$

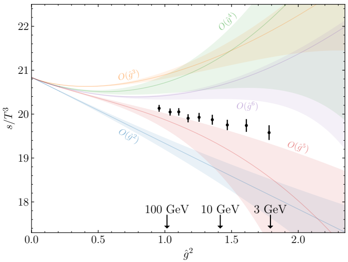


$$\frac{s}{T^3} = \frac{32\pi^2}{45}\left[s_0^{\text{SB}} + s_2\left(\frac{\hat{g}}{2\pi}\right)^2 + s_3\left(\frac{\hat{g}}{2\pi}\right)^3\right]$$

Equation of State of $N_f = 3$ QCD

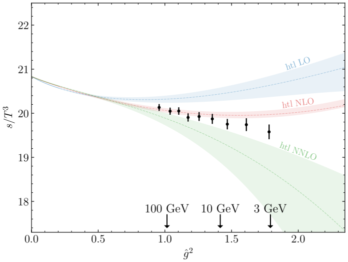
Comparison to thermal PT: [Kajantie et al. 03]

- PT systematic error from renormalization scale variation by the factors 1/2 and 2
- $O(\hat{g}^5)$ and NP point at $T = 165$ GeV agree within ~ 1.5 combined sigma (main contribution from PT error)



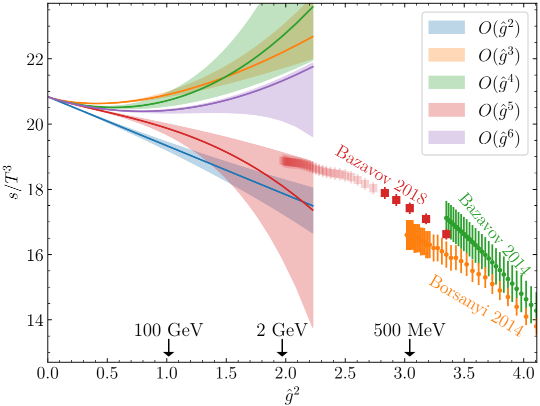
Alternative resummation of PT series with hard-thermal-loop (htl): [Andersen et al. 04-11]

- NLO and NP points agree for $T \gtrsim 50$ GeV
- NNLO and NP point at $T = 165$ GeV agree within ~ 1.5 combined sigma (main contribution from htl error)



Other approaches to thermal PT: [Lowdon, Tue 15:45], [Sandbote, Wed 16:00]

Equation of State of $N_f = 3$ QCD



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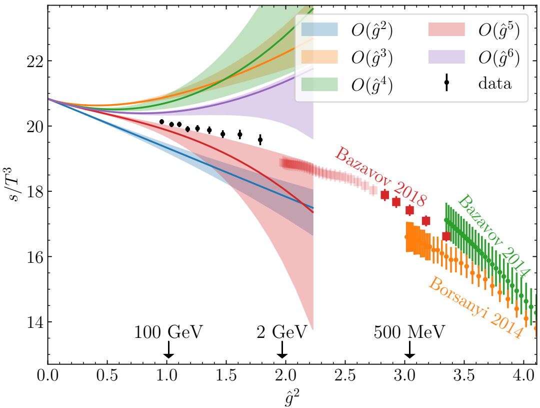


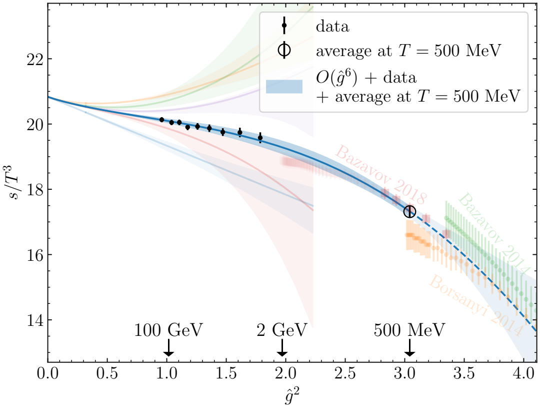
Equation of State of $N_f = 3$ QCD

- Enforce all known PT result, fit q_c and s_7 [Kajantie et al. 03]

  $q_c = -4.0(1.1)\cdot 10^3\,,\quad s_7 = 7(4)\cdot 10^3$

- Fit including $T = 500$ MeV point from $N_f = 2+1$ QCD
  - strange quark mass effect negligible [Laine, Schröder 06]

- Terms beyond known PT give $\sim 40\%$ of interactions at 165 GeV



$$\frac{s}{T^3} = \frac{32\pi^2}{45}\left[\sum_{k=0}^{6} s_k \left(\frac{\hat{g}}{2\pi}\right)^k + q_c \left(\frac{\hat{g}}{2\pi}\right)^6 + s_7 \left(\frac{\hat{g}}{2\pi}\right)^7\right]$$

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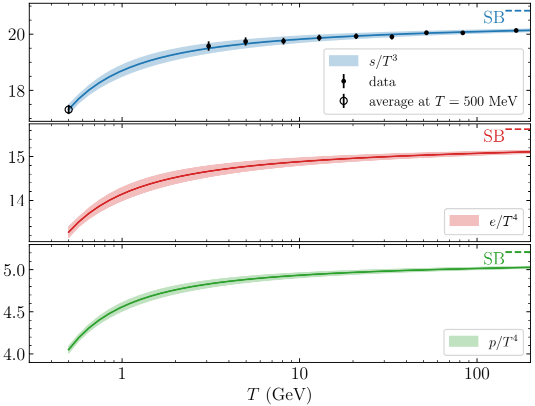
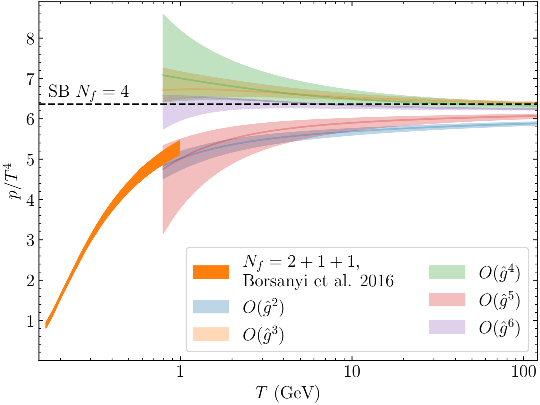


[Bresciani, Dalla Brida, Giusti, Pepe 25]

- Match $p(T)$ to $s(T)$ using $s = \frac{dp}{dT}$
- Energy density $e = Ts - p$

$$\frac{p}{T^4} = \frac{8\pi^2}{45}\left[\sum_{k=0}^{6} p_k \left(\frac{\hat{g}}{2\pi}\right)^k + q_c \left(\frac{\hat{g}}{2\pi}\right)^6 + p_7 \left(\frac{\hat{g}}{2\pi}\right)^7\right]$$

Perturbative insights beyond $N_f = 3$



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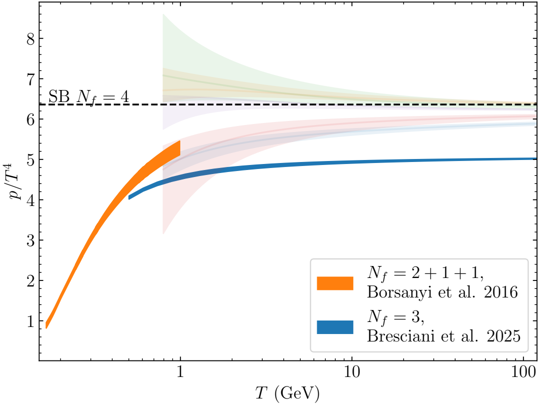
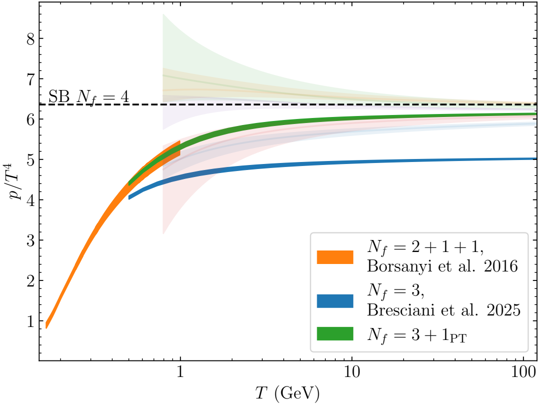


Perturbative insights beyond $N_f = 3$

We can have an idea of the charm quark effect ($m_c = 1.27$ GeV) by including it in PT [Laine, Schröder 06]

$$p^{(3+1_{\rm PT})}(T) = p^{(3)}(T) \times \frac{p^{(3+1)}_{\rm free}(T)}{p^{(3)}_{\rm free}(T)}$$

where $p^{(3)}(T)$ is our NP result for the pressure in $N_f = 3$



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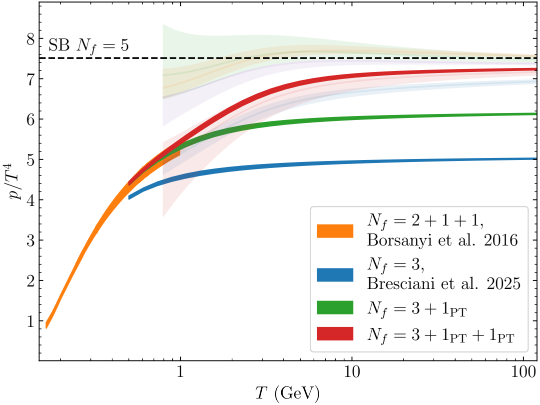
Similarly for the bottom quark ($m_b = 4.19$ GeV)

$$p^{(3+1_{\mathrm{PT}}+1_{\mathrm{PT}})}(T) = p^{(3)}(T) \times \frac{p^{(3+1+1)}_{\mathrm{free}}(T)}{p^{(3)}_{\mathrm{free}}(T)}$$



- A NP study of charm and bottom quark thresholds is required to confirm the perturbative expectations

The QCD energy-momentum tensor $T_{\mu\nu}$

$T_{\mu\nu}$ contains the conserved currents related to Poincaré invariance of the theory

- The EoS is tightly related to $T_{\mu\nu}$, e.g. the entropy is $Ts = -\frac{1+\boldsymbol{\xi}^2}{\xi_k}\langle T_{0k}\rangle_{\boldsymbol{\xi}}$
- n-point functions of $T_{\mu\nu}$ describe transport phenomena in the QCD plasma
- Lattice QCD natural framework to study NP these properties

[Winter, poster]

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The off-diagonal components of the lattice $T_{\mu\nu}$ renormalize as [Caracciolo et al. 89-90]

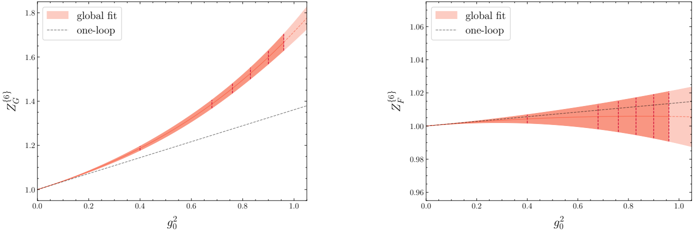
$$T^R_{\mu\nu} = Z_G\, T^G_{\mu\nu} + Z_F\, T^F_{\mu\nu}$$

where $T^G_{\mu\nu}$, $T^F_{\mu\nu}$ are bare lattice fields, and Z_G, Z_F are functions of g_0^2 only

The QCD energy-momentum tensor $T_{\mu\nu}$

Having computed the entropy s, determine the renormalization factors by enforcing on the lattice [Giusti, Pepe 15; Dalla Brida et al. 20]

$$\langle T^R_{0k} \rangle_{\boldsymbol{\xi}} \Big|_{g_0^2, L_0/a} = -\frac{\xi_k}{1+\boldsymbol{\xi}^2}\, T s(g_0^2, L_0/a)$$



[Bresciani et al. 26]

$\Rightarrow$ First NP definition of the traceless components of $T_{\mu\nu}$ in $N_f = 3$ QCD

$\Rightarrow$ This opens the way to the NP study of e.g. transport phenomena in QCD plasma

Conclusions

- With the present theoretical and technical advances in lattice QCD, it is possible to study QCD thermodynamics non-perturbatively up to very high temperatures
- Systematic effects due to the use of perturbation theory can be removed in a controlled way up to the electro-weak scale
- EoS of $N_f = 3$ QCD determined non-perturbatively up to $T \sim 165$ GeV with sub-percent error [Bresciani et al. 25]
- Contributions beyond known PT, including NP effects, still relevant at those very high temperatures
- Results obtained thanks to a new strategy based on
  - Renormalization through running of a NP defined coupling [Dalla Brida et al. 21]
  - Thermal QCD in a moving frame [Giusti, Meyer 10-13]

Outlook

- The strategy works for QCD with four or five flavours as well
- Extend the calculation of EoS to include charm and bottom quark effects up to the electro-weak scale. Preliminary steps:
  - Non-perturbative study of charm and bottom thresholds in the EoS
  - Non-perturbative renormalization of strong coupling and quark masses with 4 or 5 flavours

$\Rightarrow$ The goal is to provide a non-perturbative description of the QCD plasma at equilibrium at the temperatures of the early Universe

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Thank you for your attention!

BACKUP

Electrostatic QCD (EQCD)

- EQCD = 3D effective theory describing physics at energies $\ll \pi T$
- Degrees of freedom: zero Matsubara modes of the 4D gauge fields

$$S_{\mathrm{EQCD}} = \frac{1}{g_E^2} \int d\boldsymbol{x} \left\{ \frac{1}{2} \mathrm{tr}[F_{jk} F_{jk}] + \mathrm{tr}[(D_j A_0)(D_j A_0)] + m_E^2 \mathrm{tr}[A_0^2] \right\} + \cdots$$

- $A_0 \to$ 3D scalar field in adjoint representation of SU(3), mass m_E (Debye mass)
- $A_k \to$ 3D Yang-Mills theory
- Perturbative matching to QCD ($\hat{g}$ = strong coupling of QCD)

$$m_E^2 = \frac{3}{2} (\hat{g} T)^2 + \cdots \,, \quad g_E^2 = \hat{g}^2 T + \cdots$$

Magnetostatic QCD (MQCD)

- MQCD = 3D effective theory describing physics at energies of $O(\hat{g}^2 T)$
- Degrees of freedom: zero Matsubara modes of 4D A_k fields

$$S_{\mathrm{MQCD}} = \frac{1}{g_E^2} \int d\boldsymbol{x} \left\{ \frac{1}{2} \mathrm{tr}[F_{jk} F_{jk}] \right\} + \cdots$$

- NP 3D Yang-Mills theory with coupling increasing with T ($g_E^2 \sim T/\log^2 T$)
- $g_E^2 = \hat{g}^2 T + \cdots$ is the only relevant energy scale
- Dimensionful quantities are proportional to the proper power of g_E^2, with a NP coefficient
- The mass gap of QCD at asymptotically high temperatures is $\mathcal{M}_{\mathrm{gap}} \propto \hat{g}^2 T + \cdots$

Schrödinger functional coupling

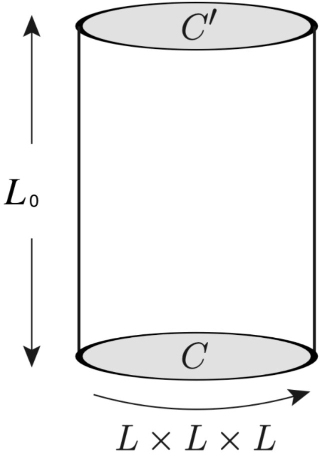
- Schrödinger functional = path integral with Dirichlet boundary conditions C, C' in time
- $\mathcal{Z}[C, C']$ partition function, free-energy

$$\Gamma = -\ln \mathcal{Z}$$

- NP definition of the coupling $(L_0 = L)$

$$\frac{1}{\overline{g}^2_{\mathrm{SF}}(\mu)} = \left\{ \frac{\partial \Gamma}{\partial \eta} \Big/ \frac{\partial \Gamma_0}{\partial \eta} \right\}_{\eta=0}, \quad \mu = \frac{1}{L}$$

where C, C' depend on the dimensionless parameter η



[Lüscher et al. 1991]

Numerical computation of the entropy density

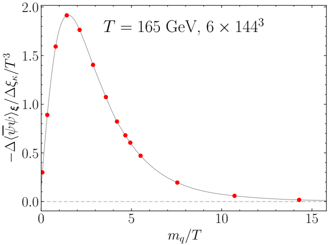
At a given g_0^2 and L_0/a, the entropy density can be decomposed as ($f = -p$)

$$\frac{s}{T^3} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\frac{\Delta f_{\boldsymbol{\xi}}}{\Delta \xi_k} = \frac{1+\boldsymbol{\xi}^2}{\xi_k}\frac{1}{T^4}\left\{\frac{\Delta}{\Delta\xi_k}(f_{\boldsymbol{\xi}} - f_{\boldsymbol{\xi}}^{\infty}) + \frac{\Delta f_{\boldsymbol{\xi}}^{\infty}}{\Delta\xi_k}\right\}$$

where the discrete derivative in the shift is computed with $\xi_k = 1 \pm 2a/L_0$

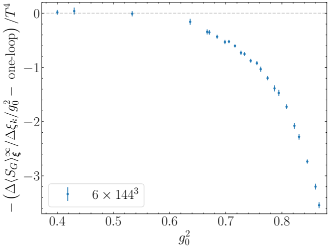
Integral in the mass

$$\frac{\Delta}{\Delta\xi_k}(f_{\boldsymbol{\xi}} - f_{\boldsymbol{\xi}}^{\infty}) = -\int_0^{\infty} dm_q \, \frac{\Delta}{\Delta\xi_k}\langle\overline{\psi}\psi\rangle_{\boldsymbol{\xi}}^{m_q}$$



Integral in the bare coupling

$$\frac{\Delta f_{\boldsymbol{\xi}}^{\infty}}{\Delta\xi_k} = \left.\frac{\Delta f_{\boldsymbol{\xi}}^{\infty}}{\Delta\xi_k}\right|_{\text{1loop}} - \int_0^{g_0^2} du \left[\frac{1}{u}\frac{\Delta}{\Delta\xi_k}\langle S_G\rangle_{\boldsymbol{\xi}}^{\infty} - \text{1loop}\right]_u$$



The integrals are computed with optimized combinations of Gaussian quadratures.

Renormalization of the lattice $T_{\mu\nu}$

On the lattice, SO(4) broken to the hypercubic group. The continuum non-singlet $T_{\mu\nu}$ splits into sextet representation (two fields labelled by G and F),

$$T_{\mu\nu}^{G,\{6\}} = (1-\delta_{\mu\nu})\frac{1}{g_0^2}\left\{F^a_{\mu\alpha}F^a_{\nu\alpha}\right\}\ , \quad T_{\mu\nu}^{F,\{6\}} = (1-\delta_{\mu\nu})\frac{1}{8}\left\{\overline{\psi}\gamma_\mu[\overleftrightarrow{\nabla}^*_\nu + \overleftrightarrow{\nabla}_\nu]\psi + \overline{\psi}\gamma_\nu[\overleftrightarrow{\nabla}^*_\mu + \overleftrightarrow{\nabla}_\mu]\psi\right\}$$

and triplet representation (two fields labelled by G and F),

$$T_{\mu\nu}^{G,\{3\}} = \frac{1}{g_0^2}\left\{F^a_{\mu\alpha}F^a_{\mu\alpha} - F^a_{\nu\alpha}F^a_{\nu\alpha}\right\}, \quad T_{\mu\nu}^{F,\{3\}} = \frac{1}{4}\left\{\overline{\psi}\gamma_\mu\left[\overleftrightarrow{\nabla}^*_\mu + \overleftrightarrow{\nabla}_\mu\right]\psi - \overline{\psi}\gamma_\nu\left[\overleftrightarrow{\nabla}^*_\nu + \overleftrightarrow{\nabla}_\nu\right]\psi\right\}$$

where $\overleftrightarrow{\nabla}_\mu$, $\overleftrightarrow{\nabla}^*_\mu$ are lattice covariant derivatives.
The full symmetry is restored only in the continuum: at finite lattice spacing, $T_{\mu\nu}$ renormalizes as

$$T_{\mu\nu}^{R,\{6\}} = Z_G^{\{6\}}(g_0^2)\,T_{\mu\nu}^{G,\{6\}} + Z_F^{\{6\}}(g_0^2)\,T_{\mu\nu}^{F,\{6\}}\ , \quad T_{\mu\nu}^{R,\{3\}} = Z_G^{\{3\}}(g_0^2)\,T_{\mu\nu}^{G,\{3\}} + Z_F^{\{3\}}(g_0^2)\,T_{\mu\nu}^{F,\{3\}}$$

Renormalization of the lattice $T_{\mu\nu}$

Final parametrizations:

$$Z_G^{\{6\}}(g_0^2) = 1 + Z_G^{\{6\}(1)} g_0^2 + c_1 g_0^4 + c_2 g_0^6$$

$$Z_F^{\{6\}}(g_0^2) = 1 + Z_F^{\{6\}(1)} g_0^2 + c_1 g_0^4$$

$$Z_G^{\{3\}}(g_0^2) = 1 + Z_G^{\{3\}(1)} g_0^2 + c_1 g_0^4 + c_2 g_0^6$$

$$Z_F^{\{3\}}(g_0^2) = 1 + Z_F^{\{3\}(1)} g_0^2 + c_1 g_0^4$$

- Precision at percent-level for $0.4 \leq g_0^2 \leq 0.96$
- $Z_G^{\{6\}}$ and $Z_G^{\{3\}}$ deviate significantly from one-loop result
- $Z_F^{\{6\}}$ and $Z_F^{\{3\}}$ agree with one-loop result