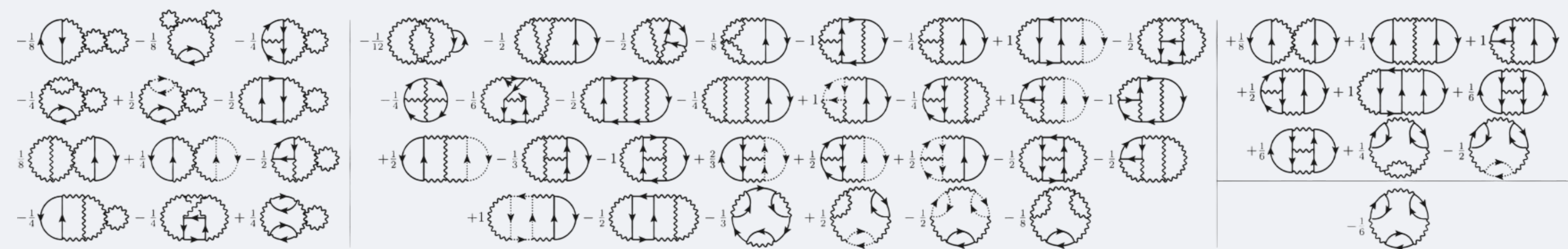
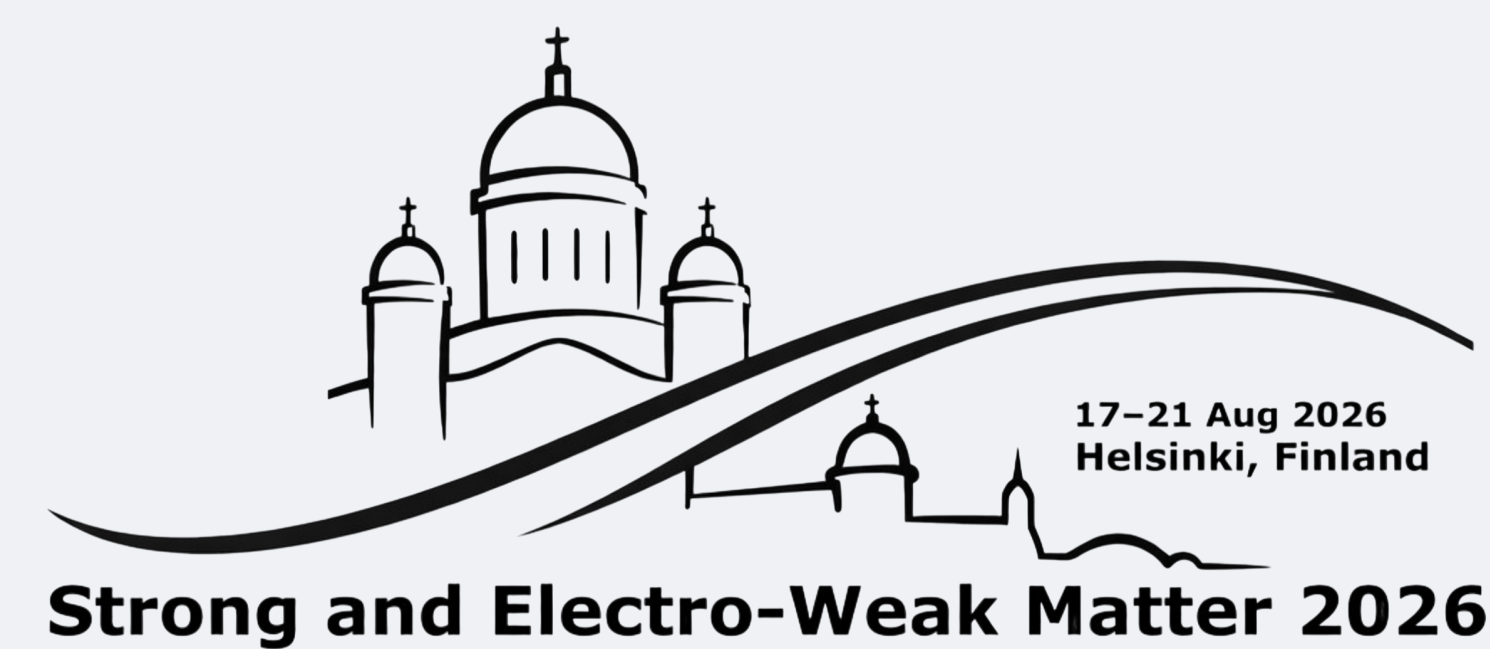


Four-loop Equation of State of Cold Quark Matter



Kaapo Seppänen
18.08.2026

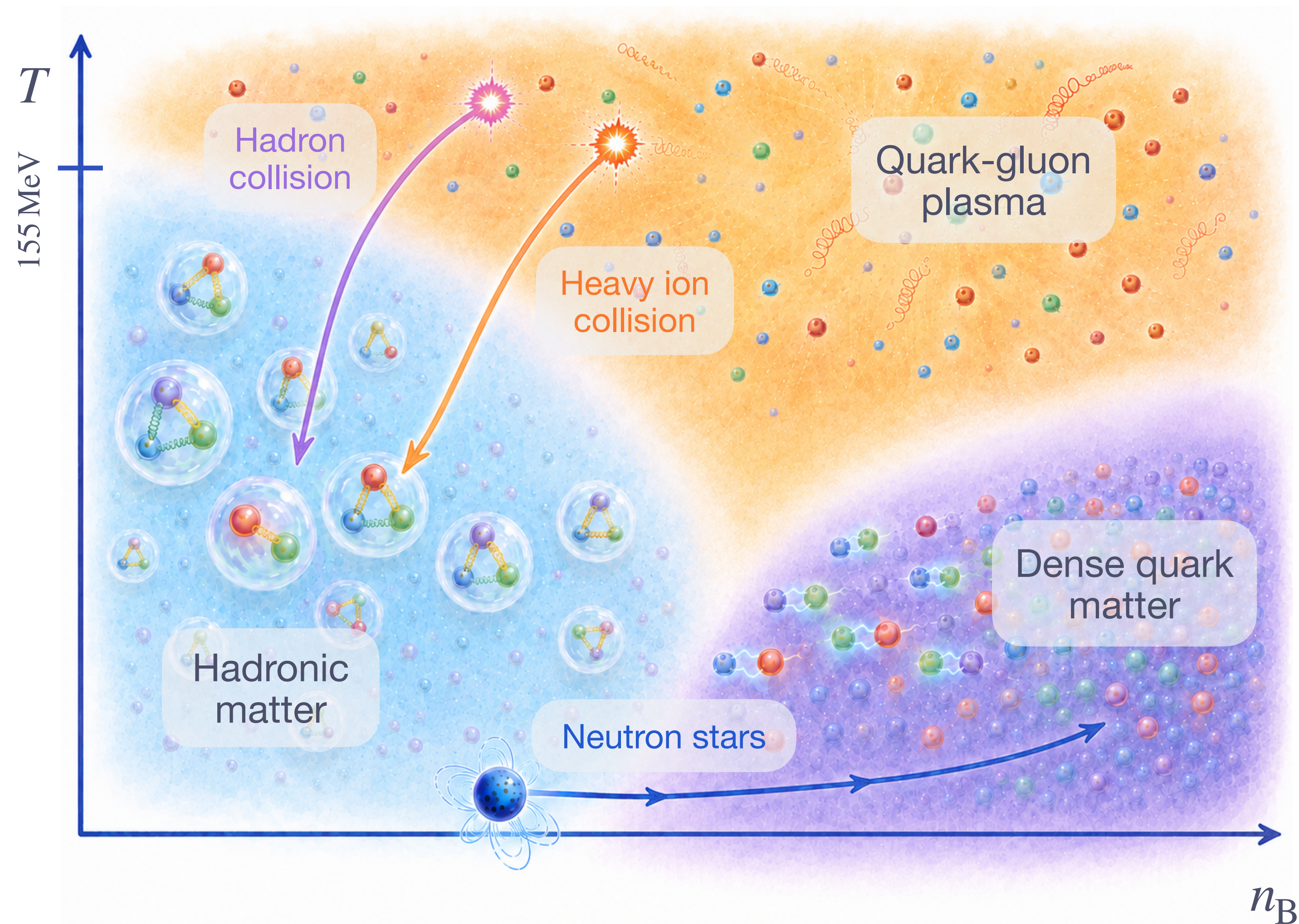
u^b **UNIVERSITÄT
BERN**

Outline

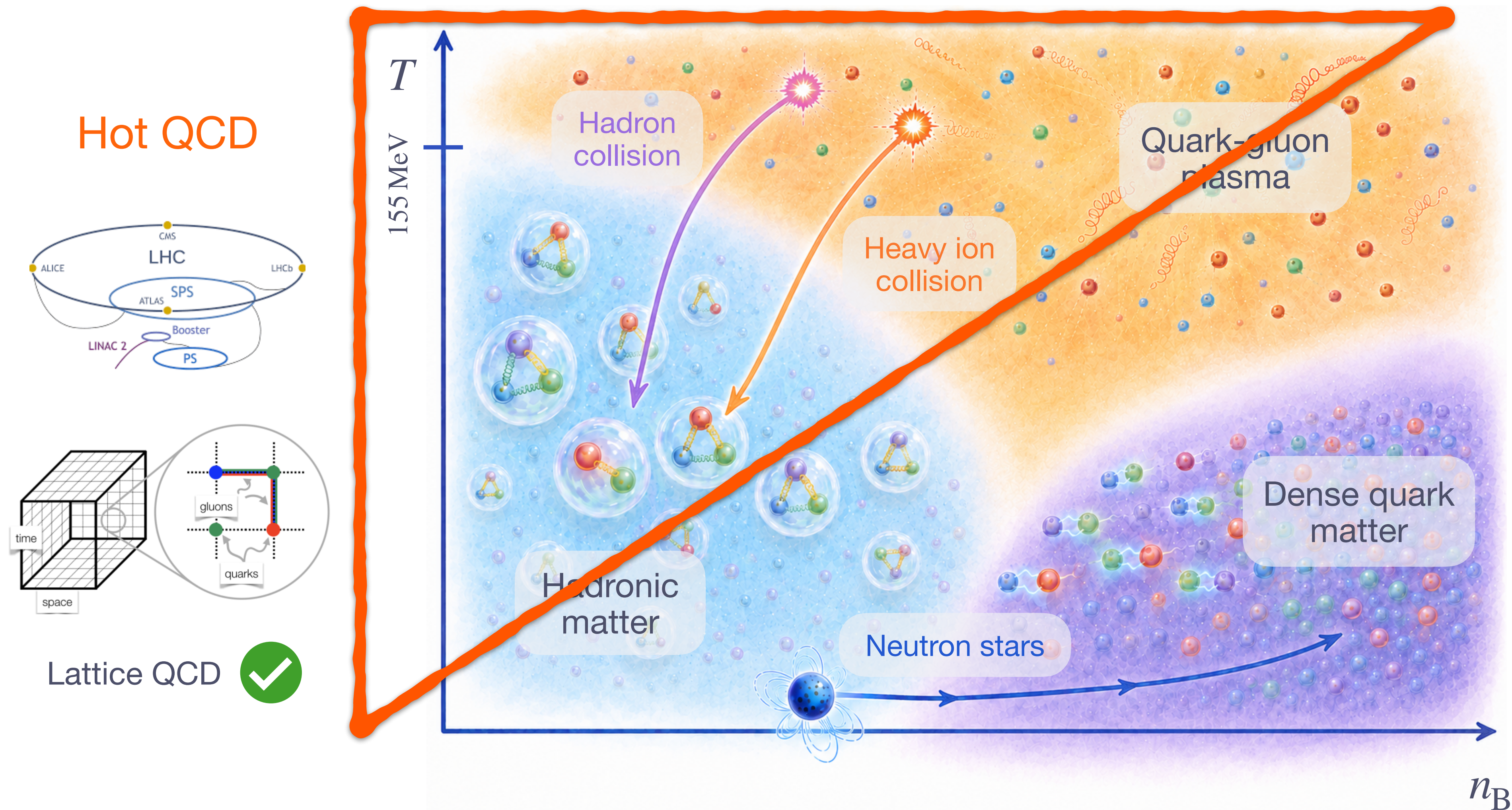
- I. QCD phase diagram and neutron stars
- II. Quark matter equation of state from pQCD
- III. Computation of c_0 
- IV. Thermal Loop-Tree Duality
- V. Summary

QCD phase diagram and neutron stars

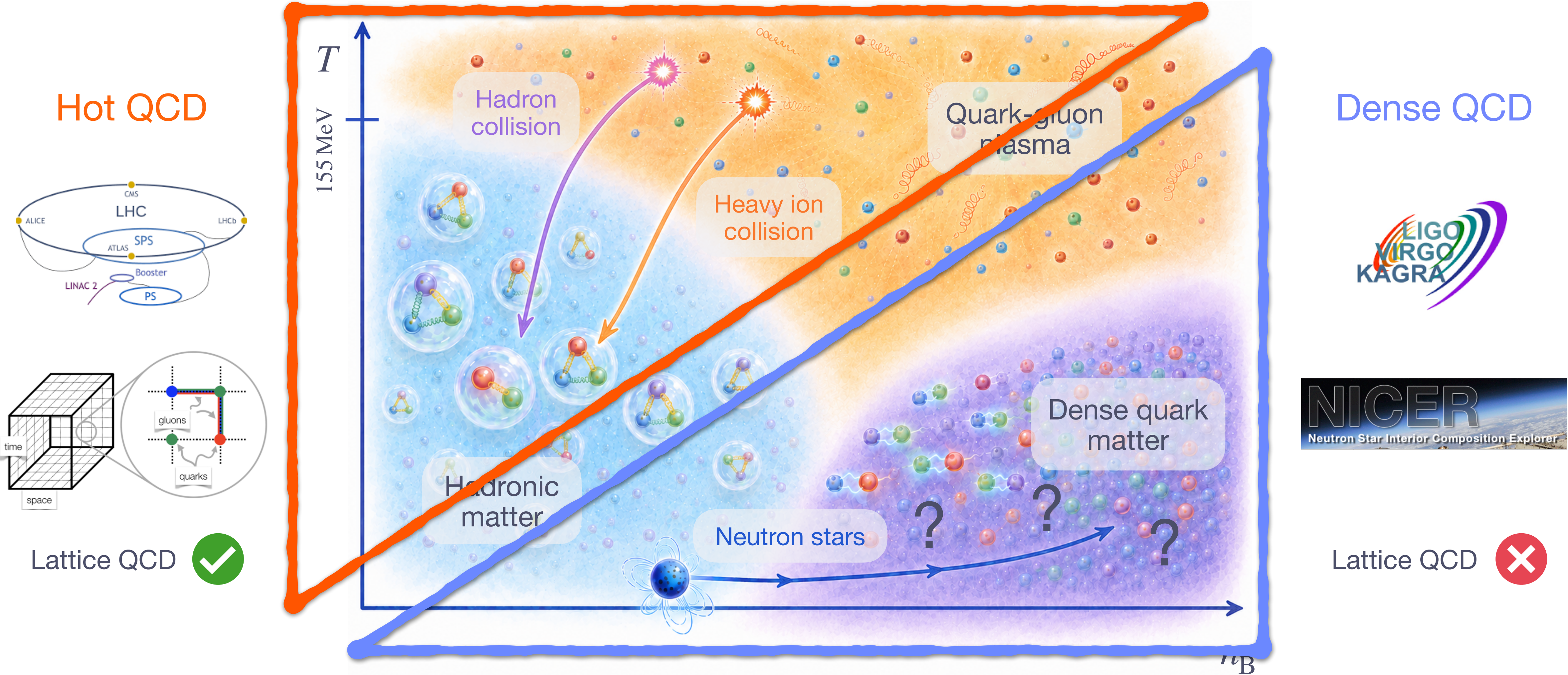
QCD matter under extreme conditions



QCD matter under extreme conditions



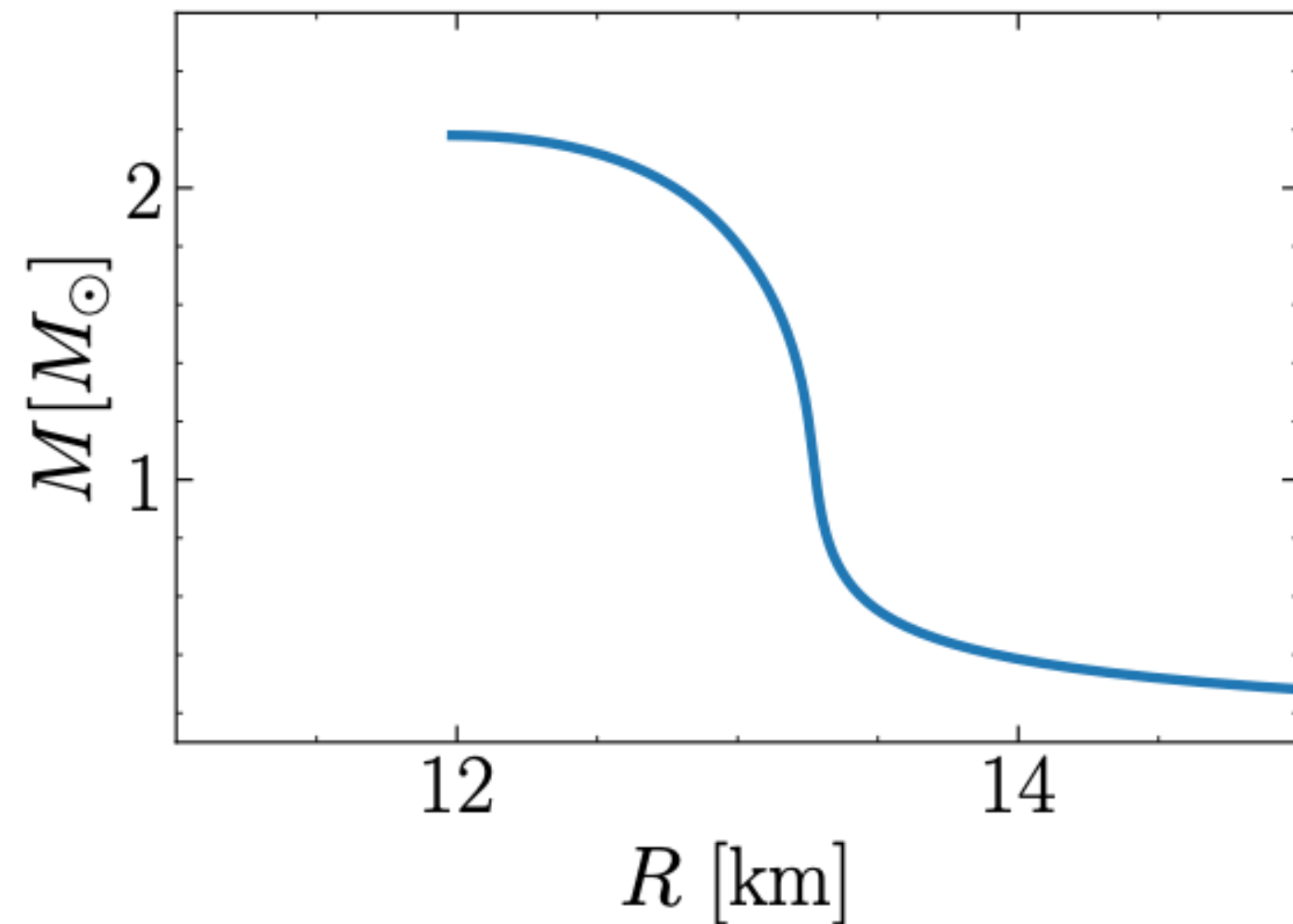
QCD matter under extreme conditions



Neutron stars

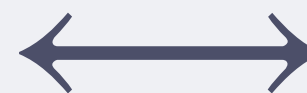
Laboratories of dense matter

Macroscopic properties of the star

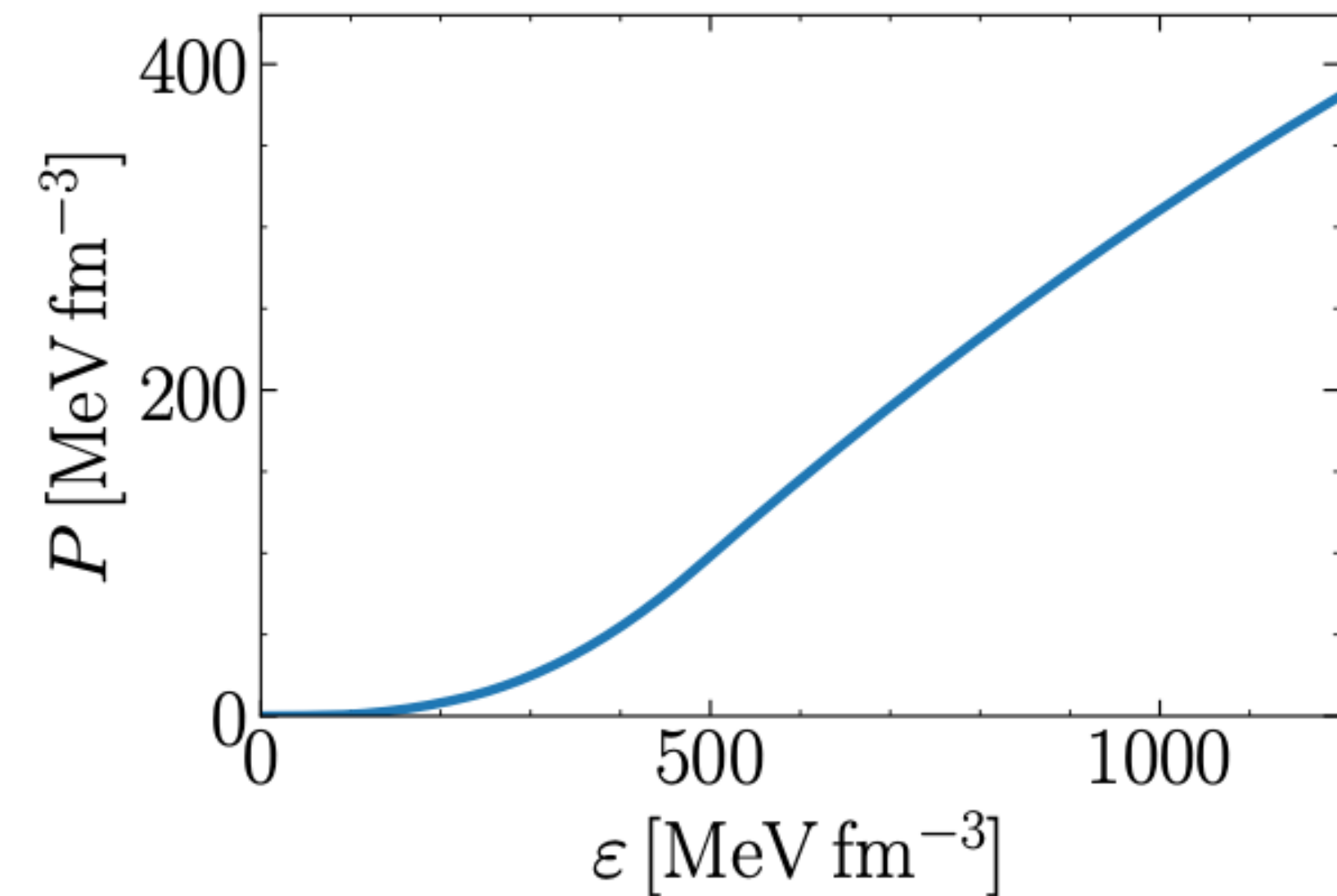


Mass-radius relation

TOV



Microscopic properties of matter



Equation of State (EoS)

Neutron star EoS

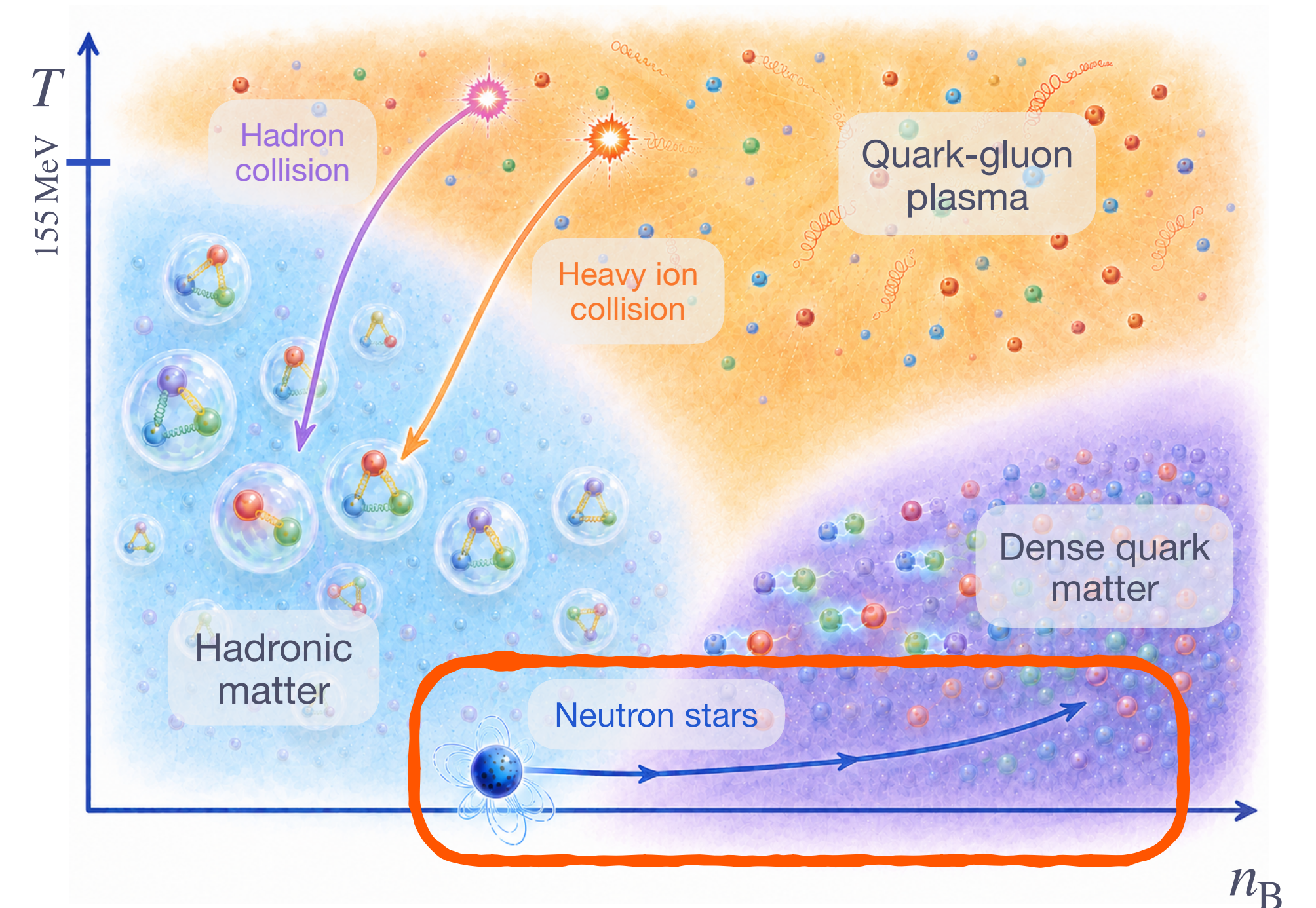
What we know

Neutron stars are strongly interacting cold and dense systems

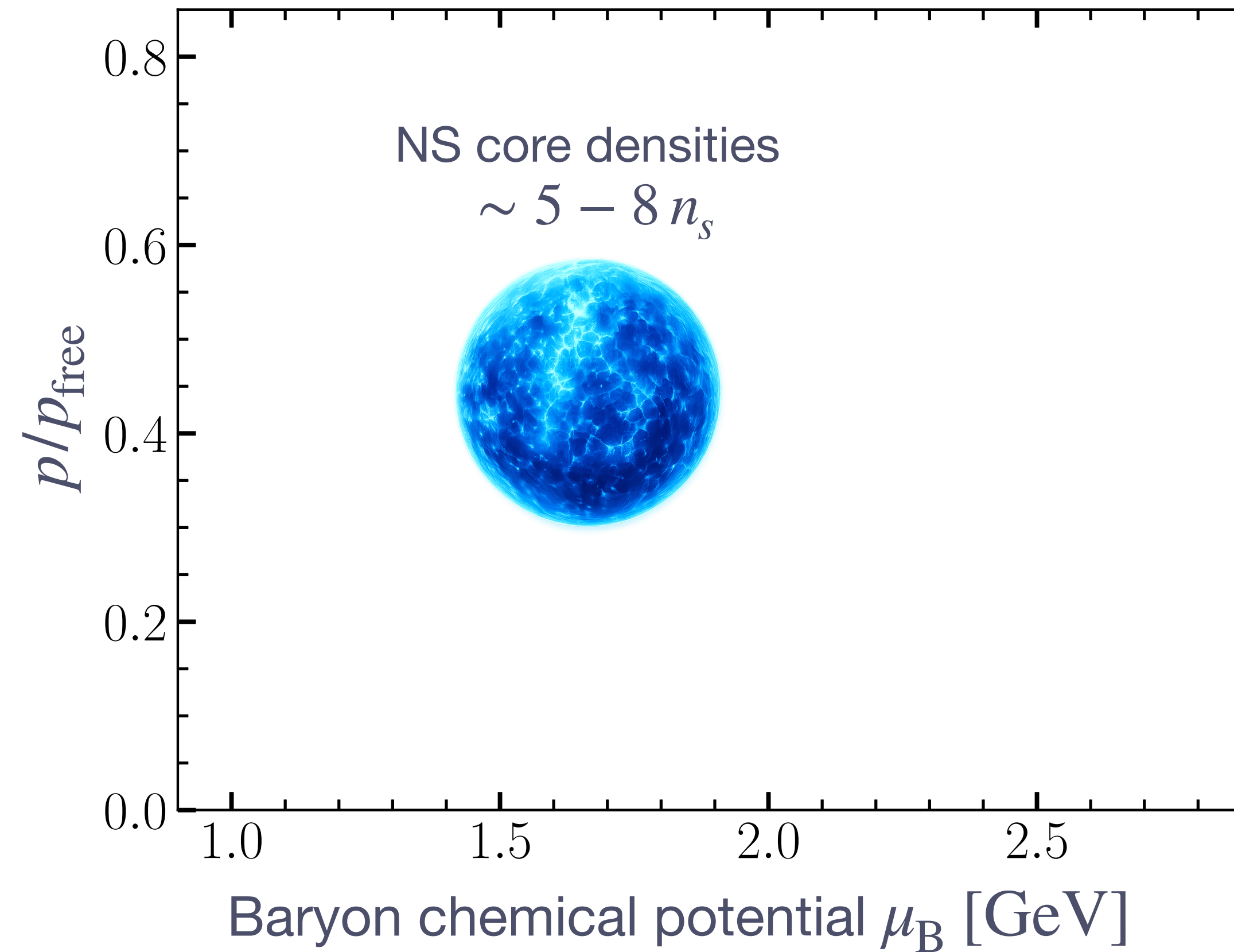
Neutron star EoS $\approx$ dense QCD matter EoS

with

- $T \simeq 0, \mu_B > 0$
- β -equilibrium and charge neutrality
- d.o.f. from hadrons to 3-flavor quark matter

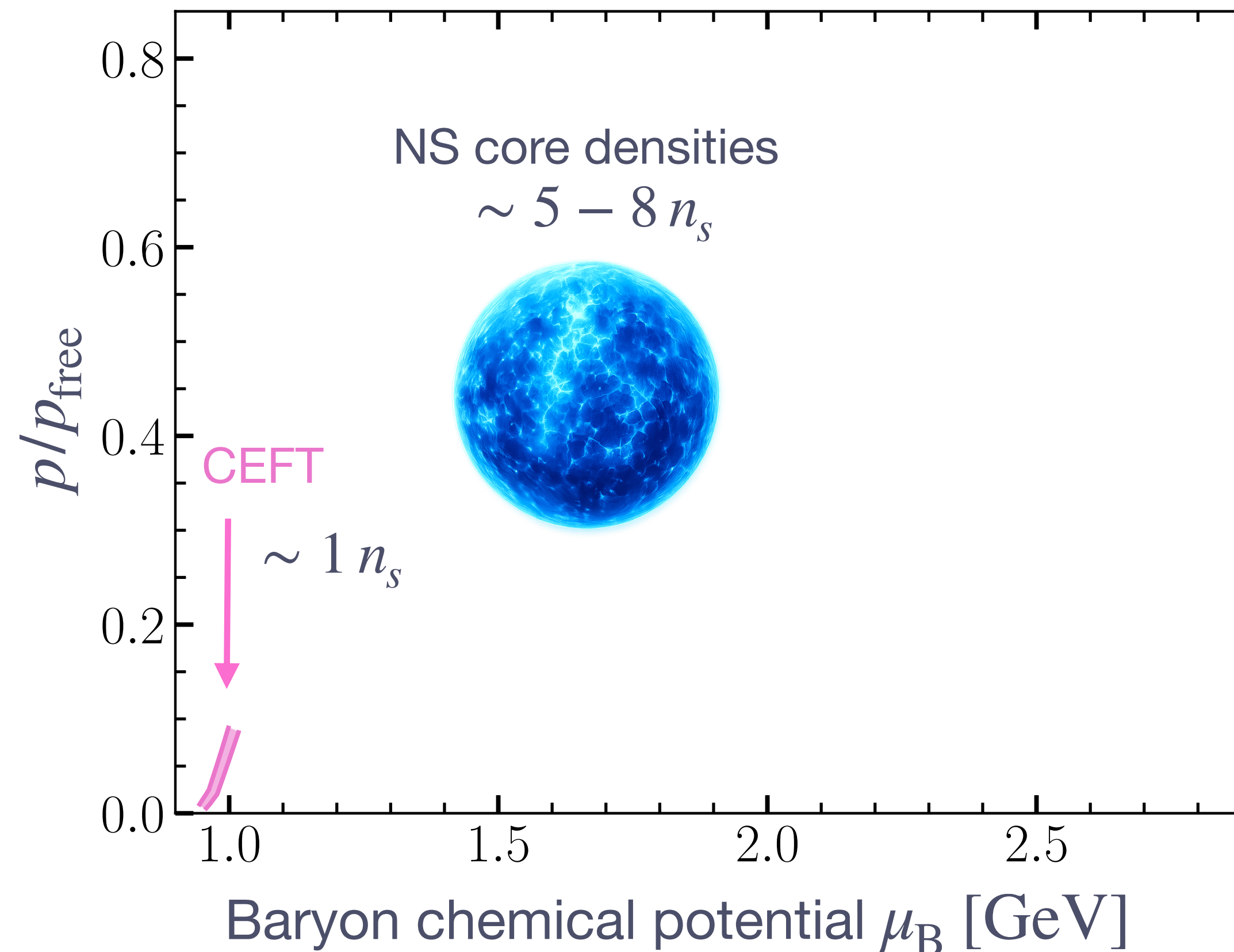


Dense QCD matter EoS



Dense QCD matter EoS

First-principles QCD information only at low and high densities

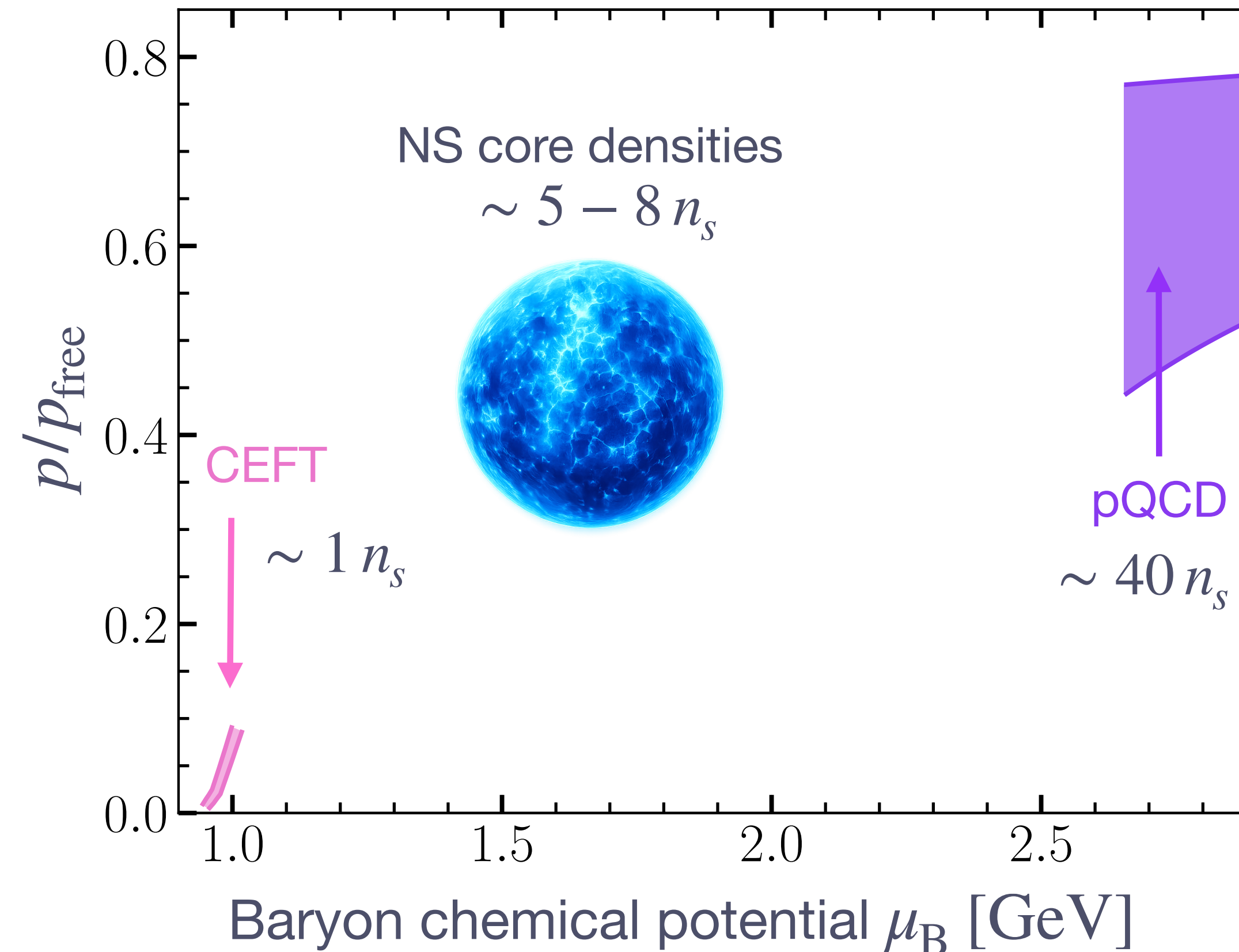


Chiral effective field
theory (CEFT)
nucleons and pions

Dense QCD matter EoS

First-principles QCD information only at low and high densities

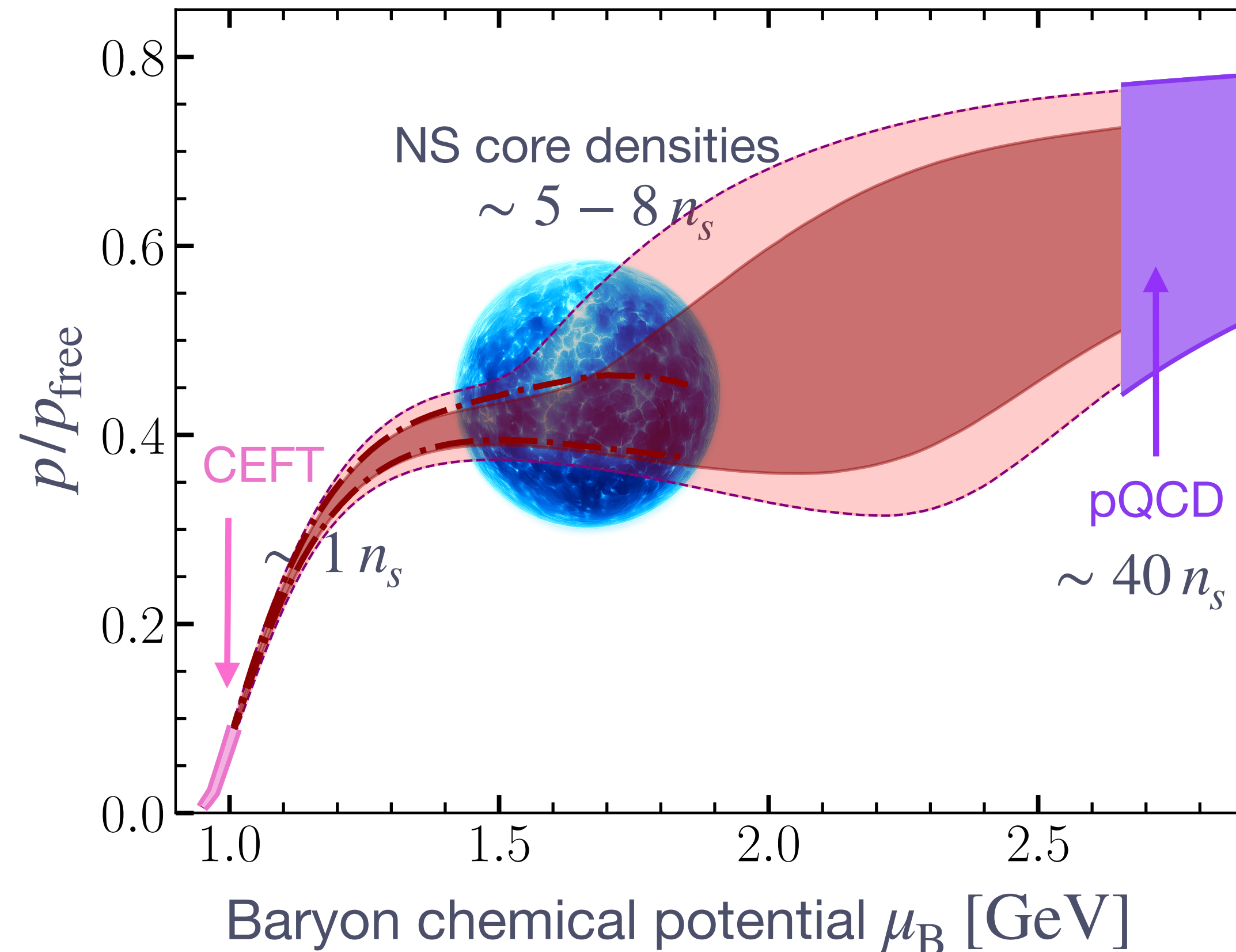
Chiral effective field
theory (CEFT)
nucleons and pions



Perturbative QCD (pQCD)
quarks and gluons

Dense QCD matter EoS

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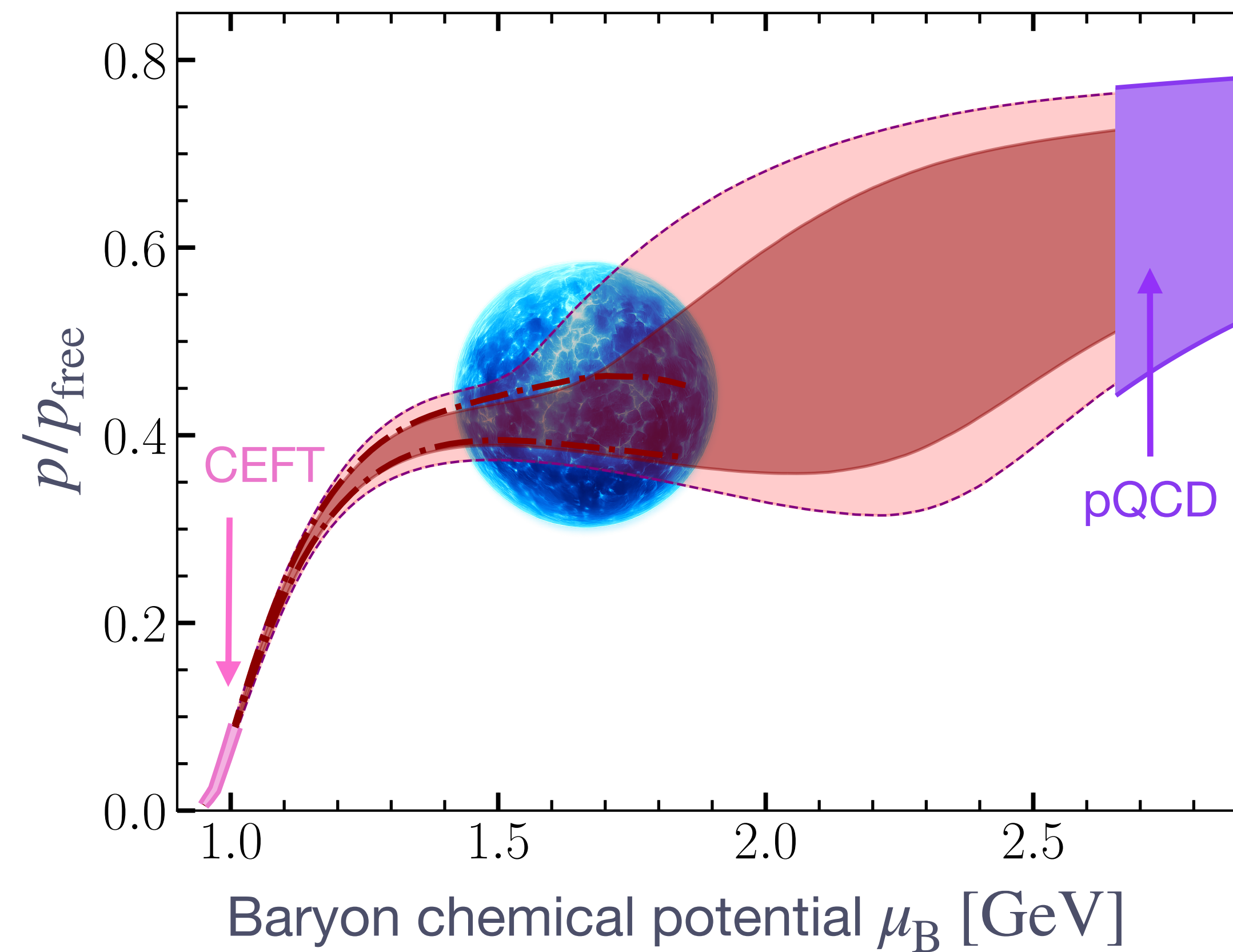


Chiral effective field theory (CEFT)
nucleons and pions

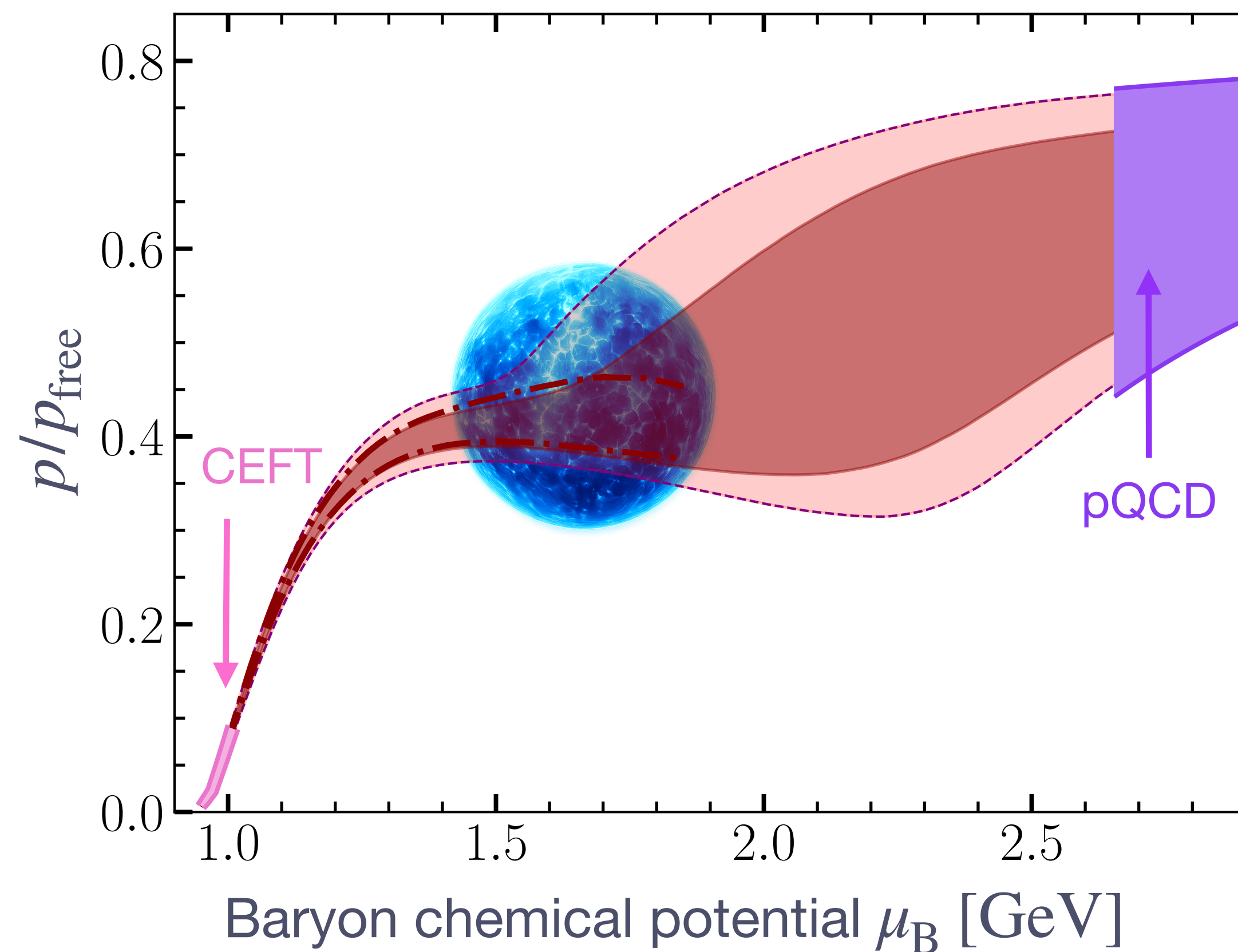
Perturbative QCD (pQCD)
quarks and gluons

Talk by
Oleg Komoltsev

Dense QCD matter EoS



Dense QCD matter EoS



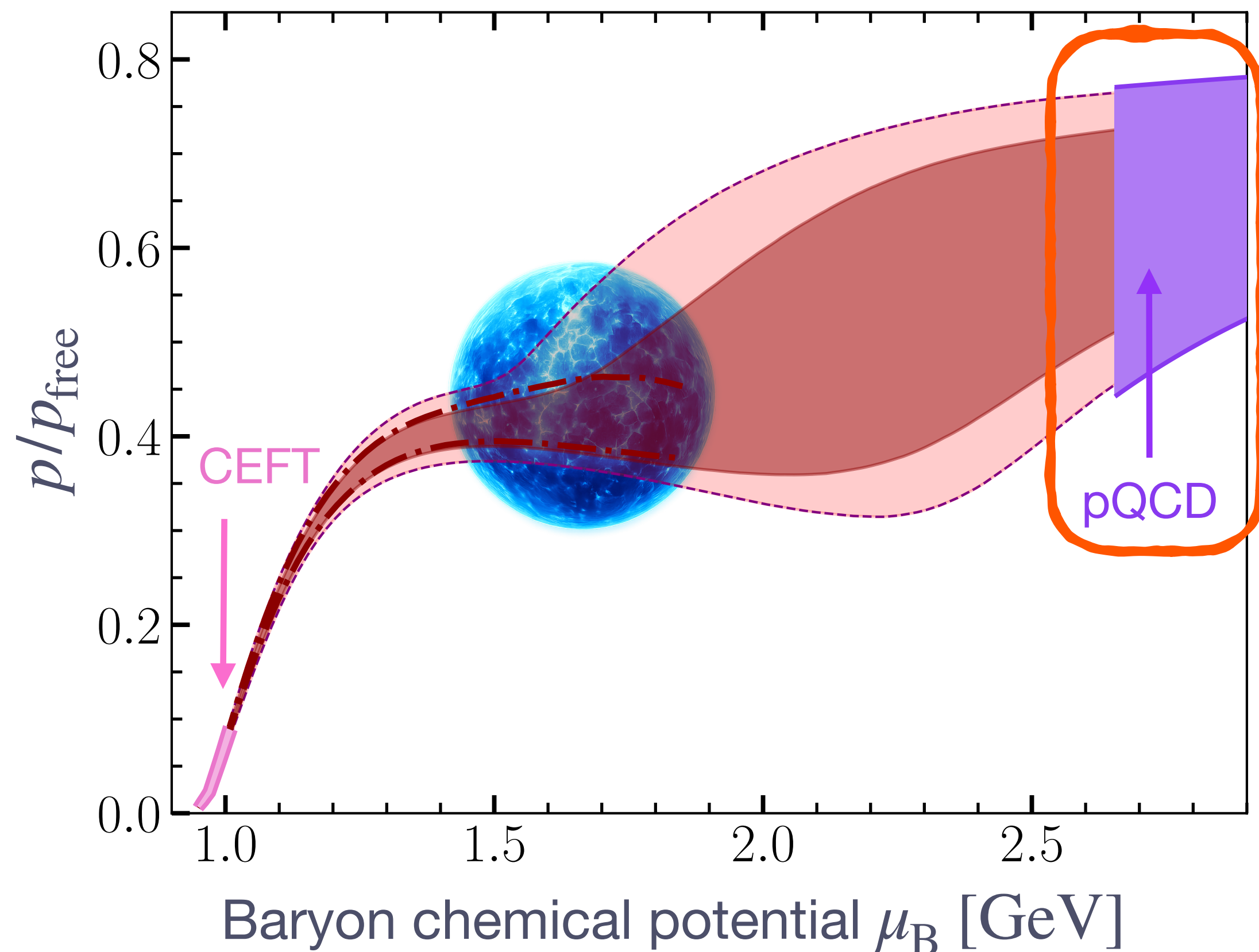
- Model-independent Bayesian inference:

[Annala, Gorda, Hirvonen, Komoltsev, Kurkela, Nättilä, Vuorinen Nat. Comm. 14 \(2023\)](#)

(and many more refs...)

Dense QCD matter EoS

Rest of this talk



- Model-independent Bayesian inference:

[Annala, Gorda, Hirvonen, Komoltsev, Kurkela, Nättilä, Vuorinen Nat. Comm. 14 \(2023\)](#)

(and many more refs...)

- **Current bottleneck:**
pQCD constraint has a large renormalization scale uncertainty
 $\Rightarrow$ Push pQCD result to higher order to decrease the uncertainty

Quark matter equation of state from pQCD

Quark matter EoS from pQCD

Quark matter EoS from pQCD

- **Quantity:** massless unpaired 3-flavor QCD pressure at $T = 0$ and $\mu_B > 0$

Quark matter EoS from pQCD

- **Quantity:** massless unpaired 3-flavor QCD pressure at $T = 0$ and $\mu_B > 0$
- **Framework:** small- α_s expansion of QCD partition function

$$p(\mu_B) \sim \ln Z = \ln \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A e^{-S_{\text{QCD}}[\bar{\psi}, \psi, A]}$$

$$= p_{\text{free}} + p_1 \alpha_s + p_2 \alpha_s^2 + p_3 \alpha_s^3 + \dots$$

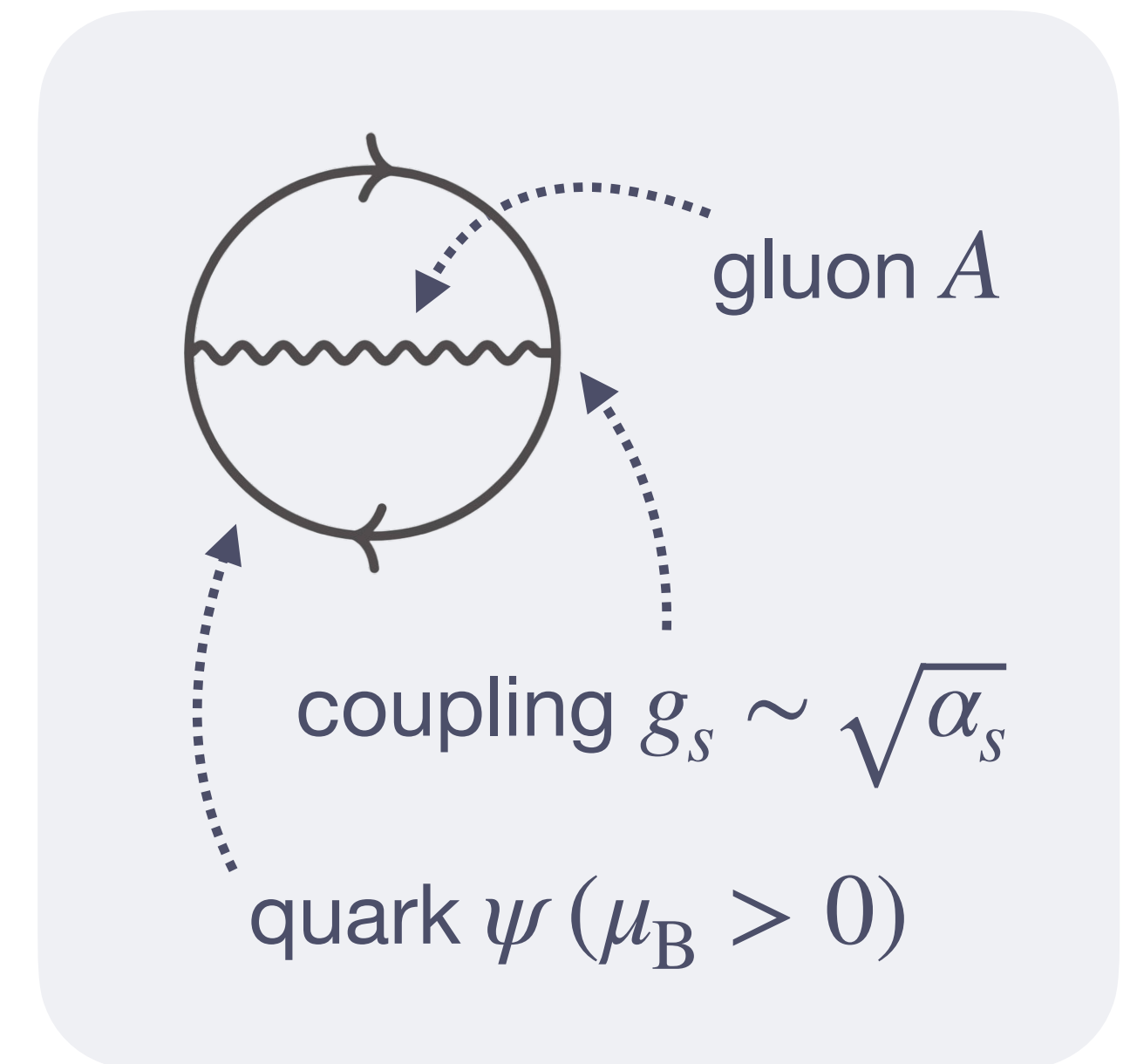
$$= \text{LO} + \text{NLO} + \left(\begin{array}{c} 7 \times \\ \text{3-loop} \end{array} \right) + \left(\begin{array}{c} 52 \times \\ \text{4-loop} \end{array} \right) + \dots$$

“LO”

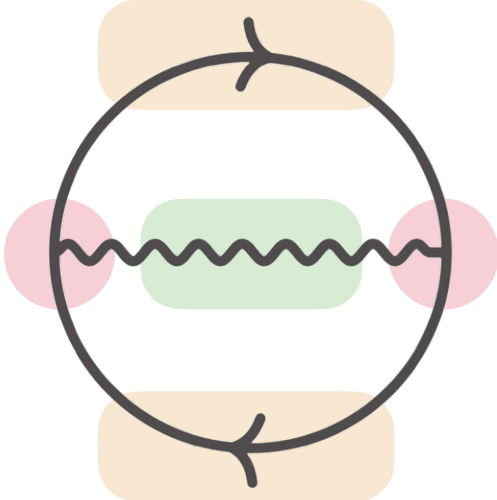
“NLO”

“N2LO”

“N3LO”



Loop integrals

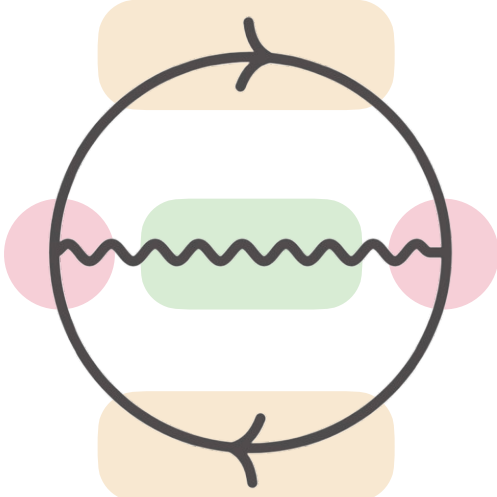


The diagram on the left shows a bubble loop. A horizontal wavy line (gluon) connects two vertices on the left. From each vertex, a fermion line (solid line with arrows) goes up and around the top of the loop, and another goes down and around the bottom. The top and bottom segments of the loop are highlighted with light orange rounded rectangles. The wavy line is highlighted with a light green rounded rectangle.

$$= - \int_{K_1 K_2} \text{Tr} \left[\left(-ig_s \gamma^\mu T^a \right) \frac{-i K_1}{K_1^2} \left(-ig_s \gamma^\nu T^b \right) \frac{-i K_2}{K_2^2} \right] \frac{\delta^{ab} g_{\mu\nu}}{(K_1 - K_2)^2}$$

The trace part of the equation is enclosed in a large dark blue bracket. Inside the bracket, the terms are highlighted with colored rounded rectangles: the vertex factors $(-ig_s \gamma^\mu T^a)$ and $(-ig_s \gamma^\nu T^b)$ are in pink, the propagator denominators $\frac{-i K_1}{K_1^2}$ and $\frac{-i K_2}{K_2^2}$ are in orange, and the final tensor structure $\frac{\delta^{ab} g_{\mu\nu}}{(K_1 - K_2)^2}$ is in green.

Loop integrals

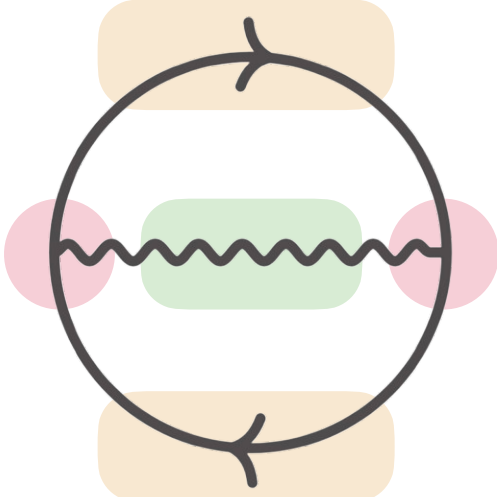


$$= - \int_{K_1 K_2} \text{Tr} \left[\begin{array}{c} (-ig_s \gamma^\mu T^a) \\ \frac{-i K_1}{K_1^2} \end{array} \begin{array}{c} (-ig_s \gamma^\nu T^b) \\ \frac{-i K_2}{K_2^2} \end{array} \right] \frac{\delta^{ab} g_{\mu\nu}}{(K_1 - K_2)^2}$$

$$K_i = \left(k_i^0 + i \mu_B / 3, \vec{k}_i \right)$$

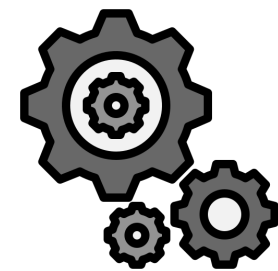
Chemical potential μ_B
shifts loop energies k_i^0

Loop integrals



$$= - \int_{K_1 K_2} \text{Tr} \left[\left(-ig_s \gamma^\mu T^a \right) \frac{-i K_1}{K_1^2} \left(-ig_s \gamma^\nu T^b \right) \frac{-i K_2}{K_2^2} \right] \frac{\delta^{ab} g_{\mu\nu}}{(K_1 - K_2)^2}$$

= [intermediate steps]



$$= - \frac{\mu_B^4}{54\pi^3} \alpha_s$$

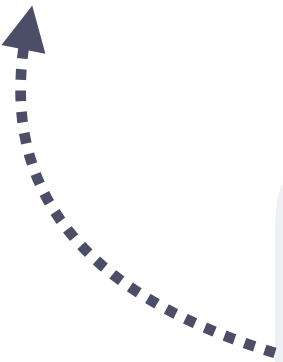
$$K_i = \left(k_i^0 + i \mu_B / 3, \vec{k}_i \right)$$

Chemical potential μ_B
shifts loop energies k_i^0

Renormalization scale uncertainty

$$p(\mu_B) = p_{\text{free}} + p_1 \alpha_s(\bar{\Lambda}) + p_2(\bar{\Lambda}) \alpha_s^2(\bar{\Lambda}) + p_3(\bar{\Lambda}) \alpha_s^3(\bar{\Lambda}) + \dots$$

Uncertainty comes from missing higher-order terms
 $\Rightarrow$ estimated by varying the unphysical renormalization scale $\bar{\Lambda}$


$$\frac{dp}{d\bar{\Lambda}} = 0$$

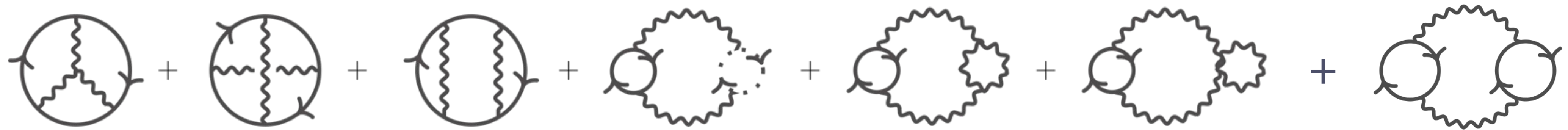
Soft resummations

Caveat starting at 3-loops α_s^2

Soft resummations

Caveat starting at 3-loops α_s^2

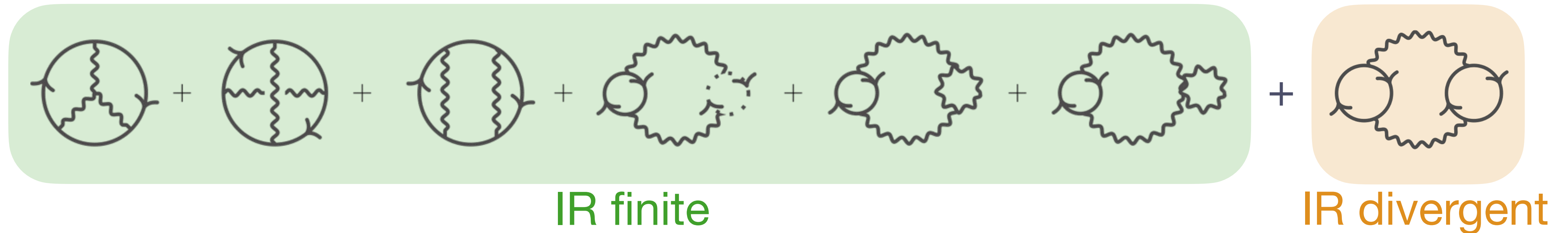
1. **Hard** scale $\sim \mu_B$: Full theory diagrams within naive α_s expansion



Soft resummations

Caveat starting at 3-loops α_s^2

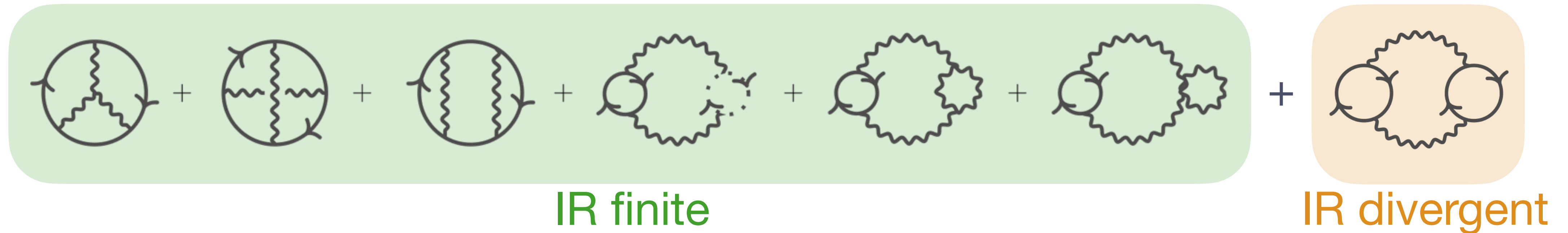
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Soft resummations

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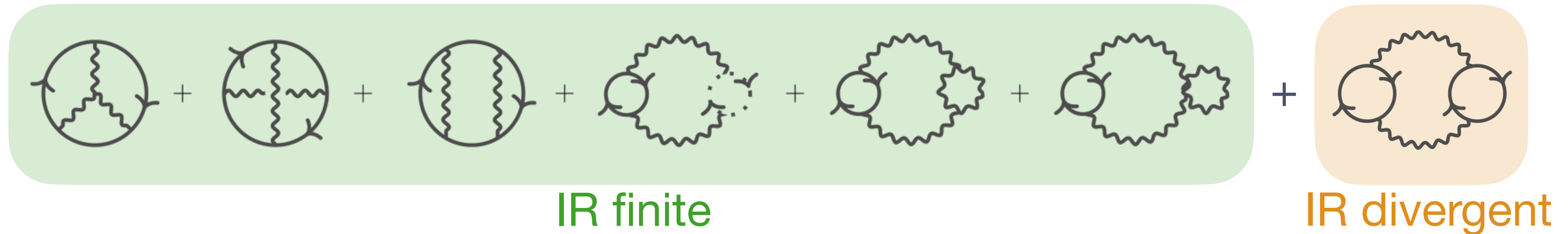


2. **Soft scale** $\sim \sqrt{\alpha_s} \mu_B$: Hard Thermal Loop (HTL) effective theory [Braaten, Pisarski NPB 337 \(1990\)](#)

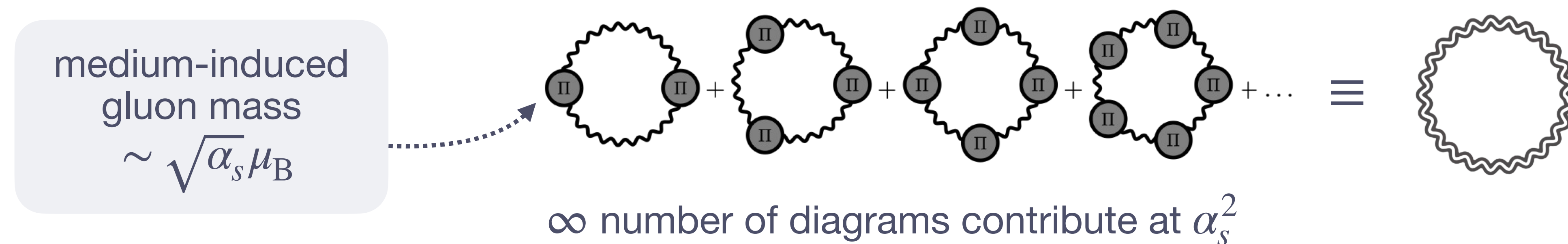
Soft resummations

Caveat starting at 3-loops α_s^2

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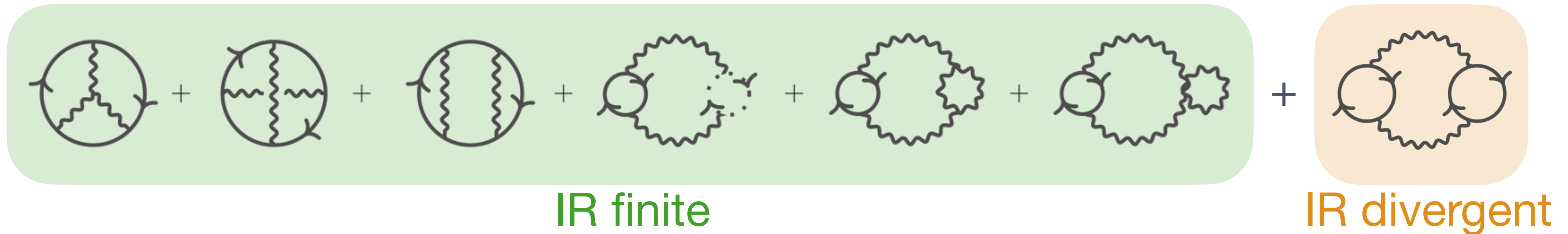
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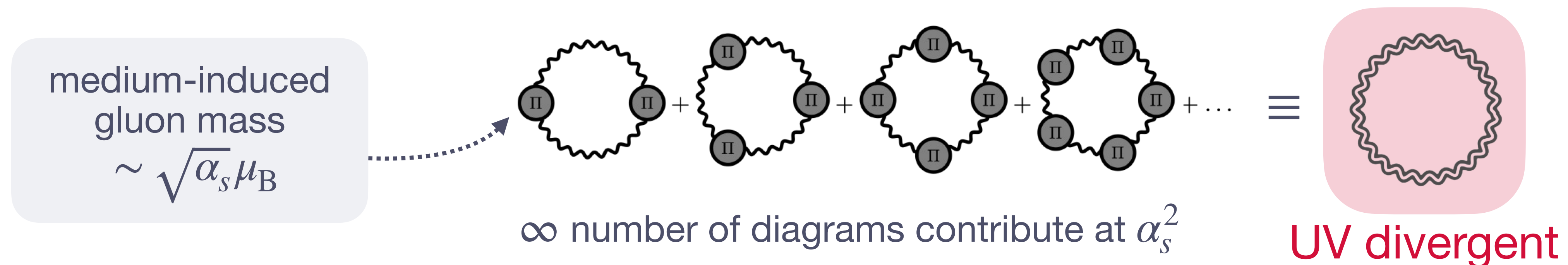
Soft resummations

Caveat starting at 3-loops α_s^2

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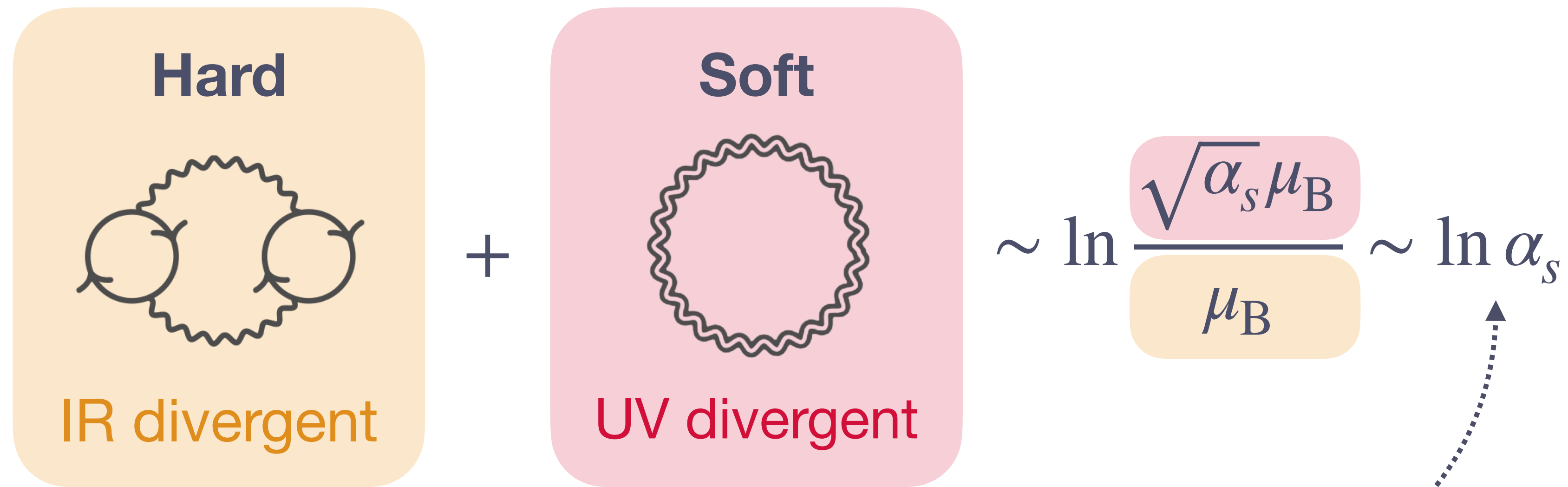


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Soft resummations

Caveat starting at 3-loops α_s^2



Divergences cancel but non-analytic terms remain!

Quark matter EoS from pQCD

History and progress

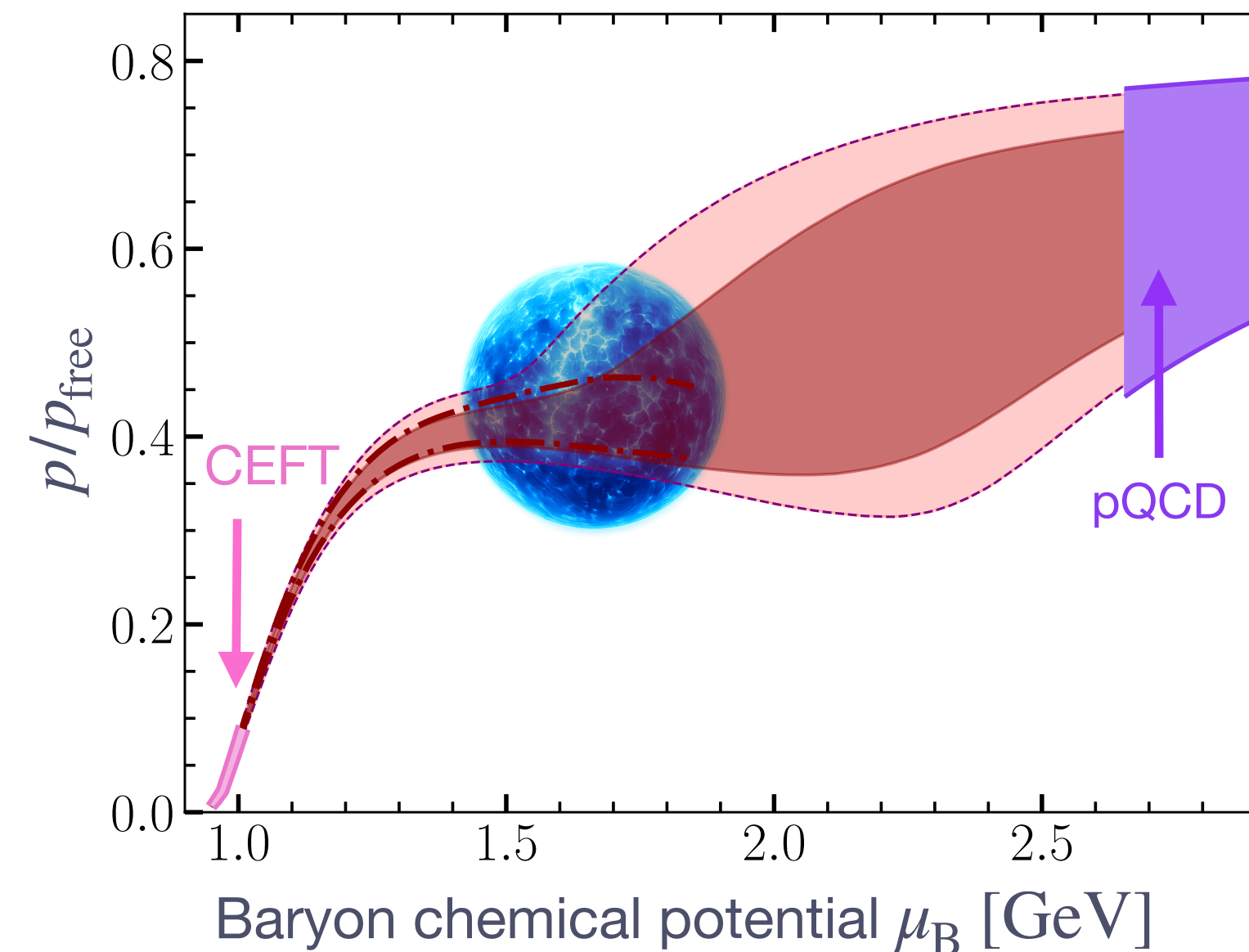
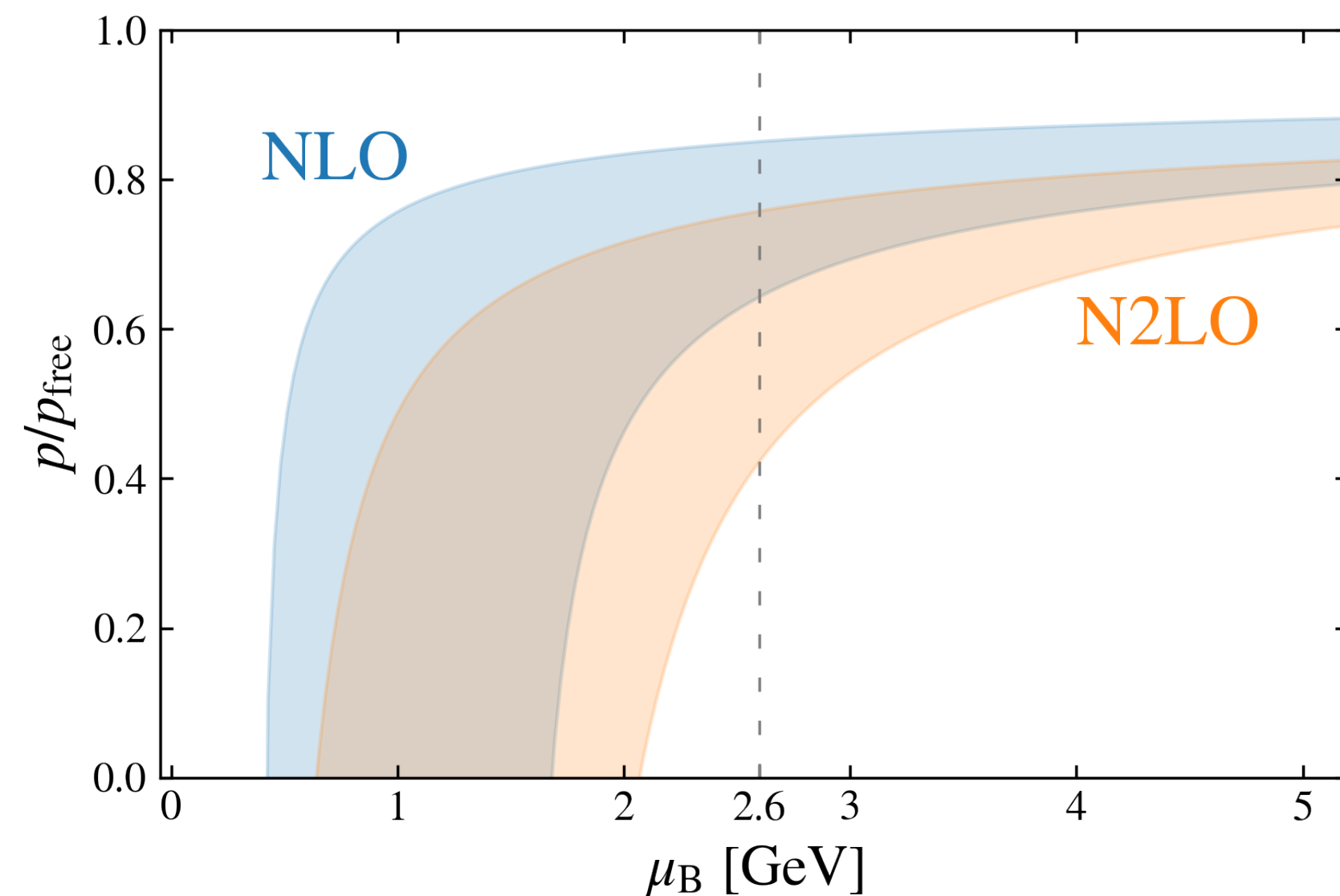
$$\frac{p(\mu_B)}{p_{\text{free}}} = 1 + p_{1,1}\alpha_s + (p_{2,1}\ln\alpha_s + p_{2,0})\alpha_s^2 + (p_{3,2}\ln^2\alpha_s + p_{3,1}\ln\alpha_s + p_{3,0})\alpha_s^3 + \dots$$

Quark matter EoS from pQCD

History and progress

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N2LO (3-loop): [Freedman, McLerran PRD 16 \(1977\)](#)



Quark matter EoS from pQCD

History and progress

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Improvements:

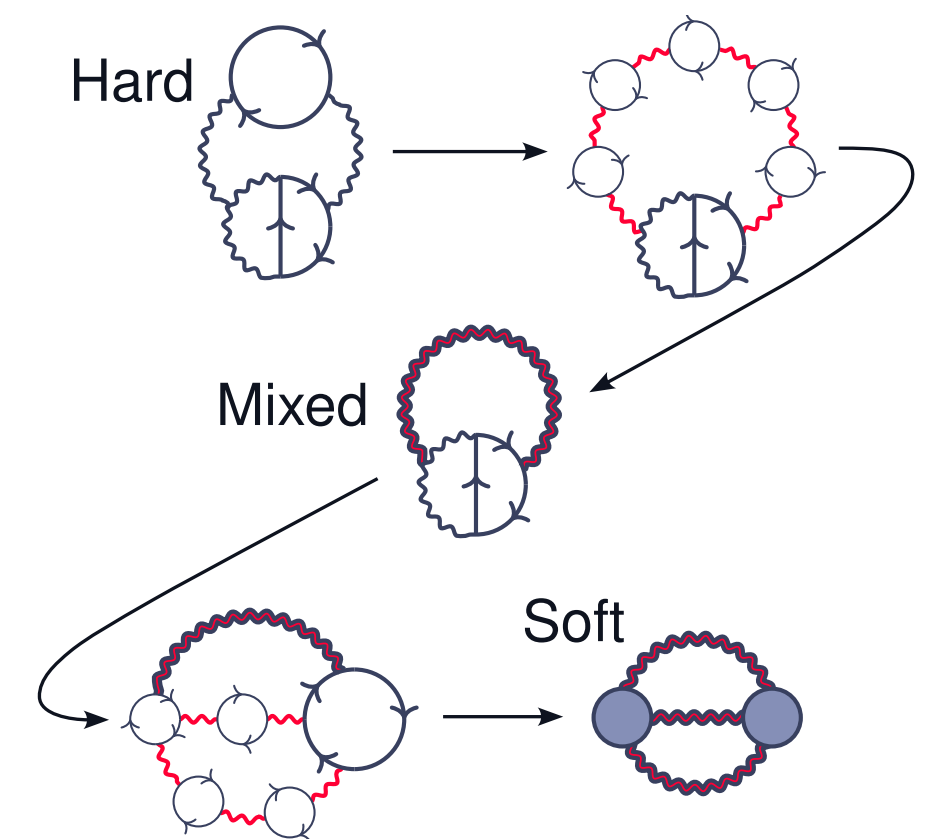
- $\overline{\text{MS}}$ scheme and analytic result: [Vuorinen PRD 68 \(2003\)](#)
- Quark mass: [Kurkela, Romatschke, Vuorinen PRD 81 \(2010\)](#)
[Gorda, Säppi PRD 105 \(2022\)](#)
[Fernandez, Kneur PRD 111 \(2025\)](#)
- Temperature: [Kurkela, Vuorinen PRL 117 \(2016\)](#)

Quark matter EoS from pQCD

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N3LO (4-loop):



Quark matter EoS from pQCD

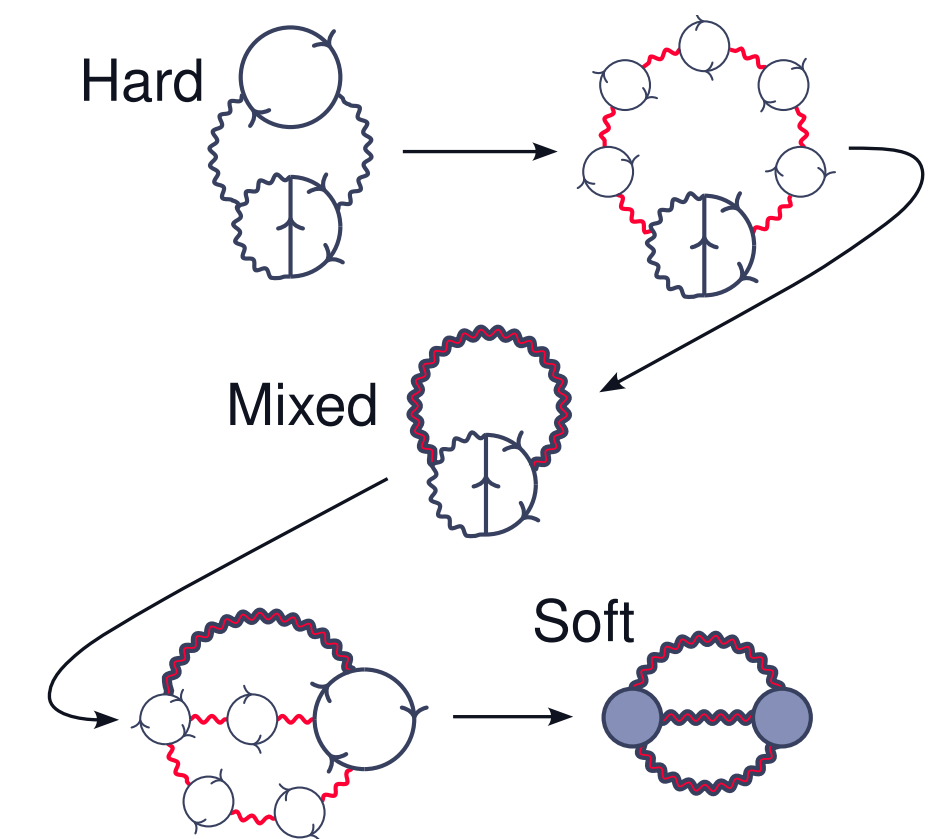
History and progress

$$\frac{p(\mu_B)}{p_{\text{free}}} = 1 + p_{1,1}\alpha_s + (p_{2,1}\ln\alpha_s + p_{2,0})\alpha_s^2 + (p_{3,2}\ln^2\alpha_s + p_{3,1}\ln\alpha_s + p_{3,0})\alpha_s^3 + \dots$$

N³LO (4-loop):

- HTL-resummed **soft** and **mixed** contributions:

- $p_{3,2}$: [Gorda, Kurkela, Säppi, Romatschke, Vuorinen PRL 121 \(2018\)](#)
[Gorda, Kurkela, Paatelainen, Säppi, Vuorinen PRL 127 & PRD 104 \(2021\)](#)
[Fernandez, Kneur PRL 129 \(2022\)](#)



Quark matter EoS from pQCD

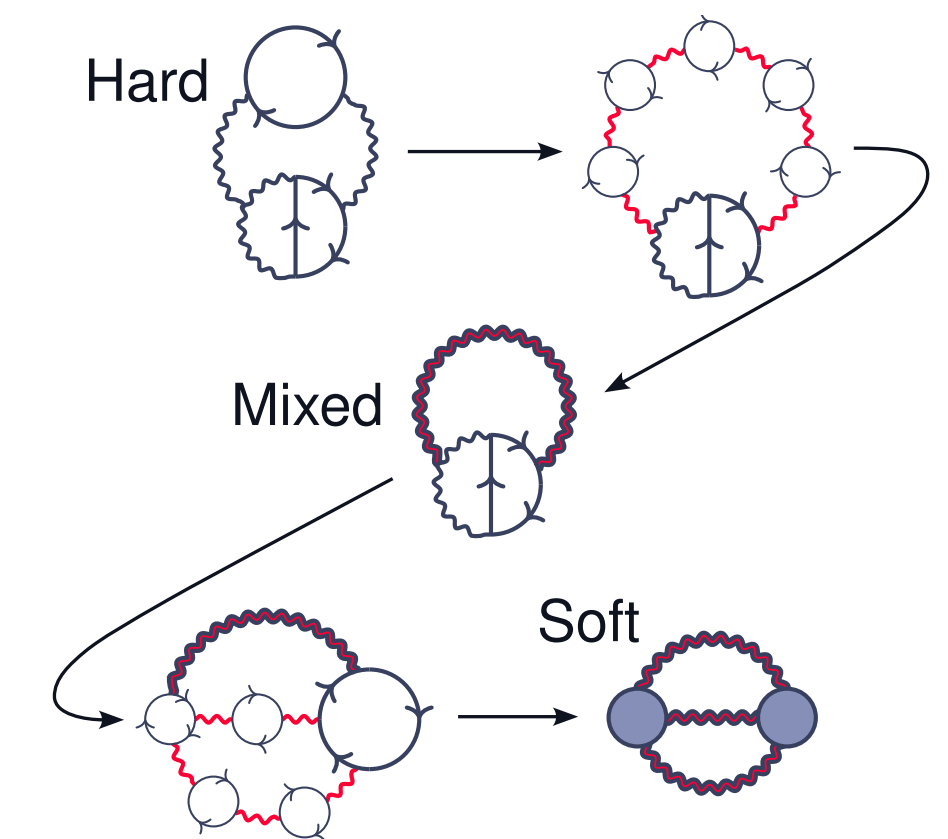
History and progress

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[Fernandez, Kneur PRL 129 \(2022\)](#)
- $p_{3,1}$: [Gorda, Paatelainen, Säppi, KS PRL 131 \(2023\)](#)



Brings result on par with its
high- T counterpart:
[Kajantie et al. PRD 67 \(2003\)](#)

Quark matter EoS from pQCD

History and progress

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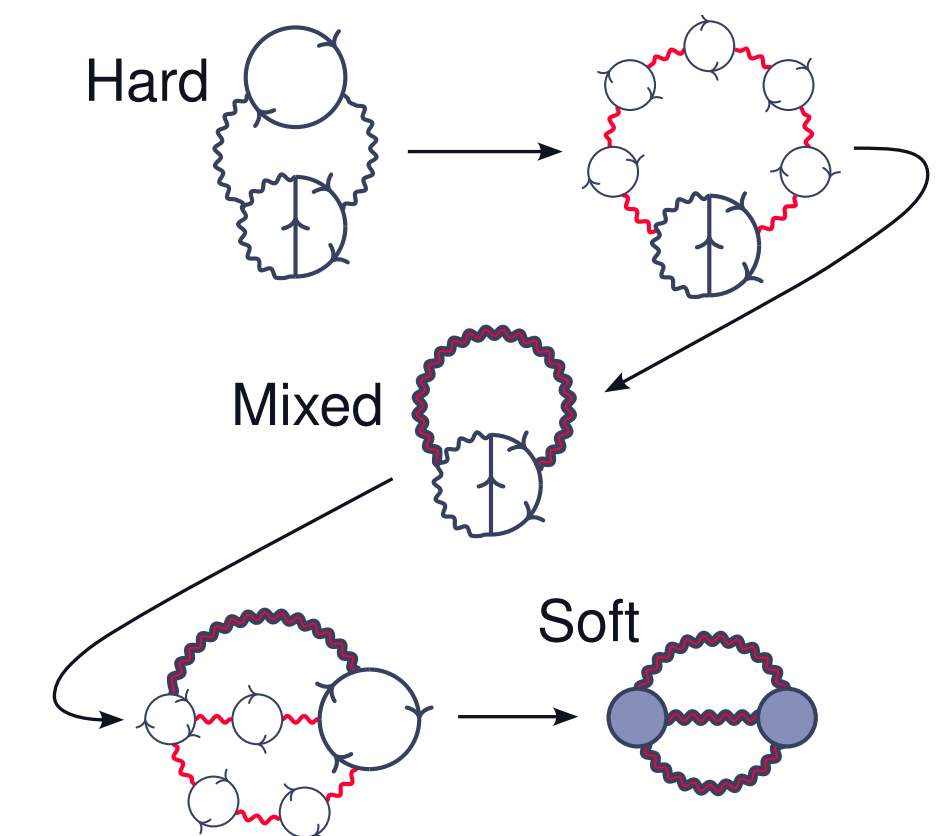
N³LO (4-loop):

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[Fernandez, Kneur PRL 129 \(2022\)](#)
- $p_{3,1}$: [Gorda, Paatelainen, Säppi, KS PRL 131 \(2023\)](#)

- Hard** contributions:

- $p_{3,0}$: known up to a single number c_0

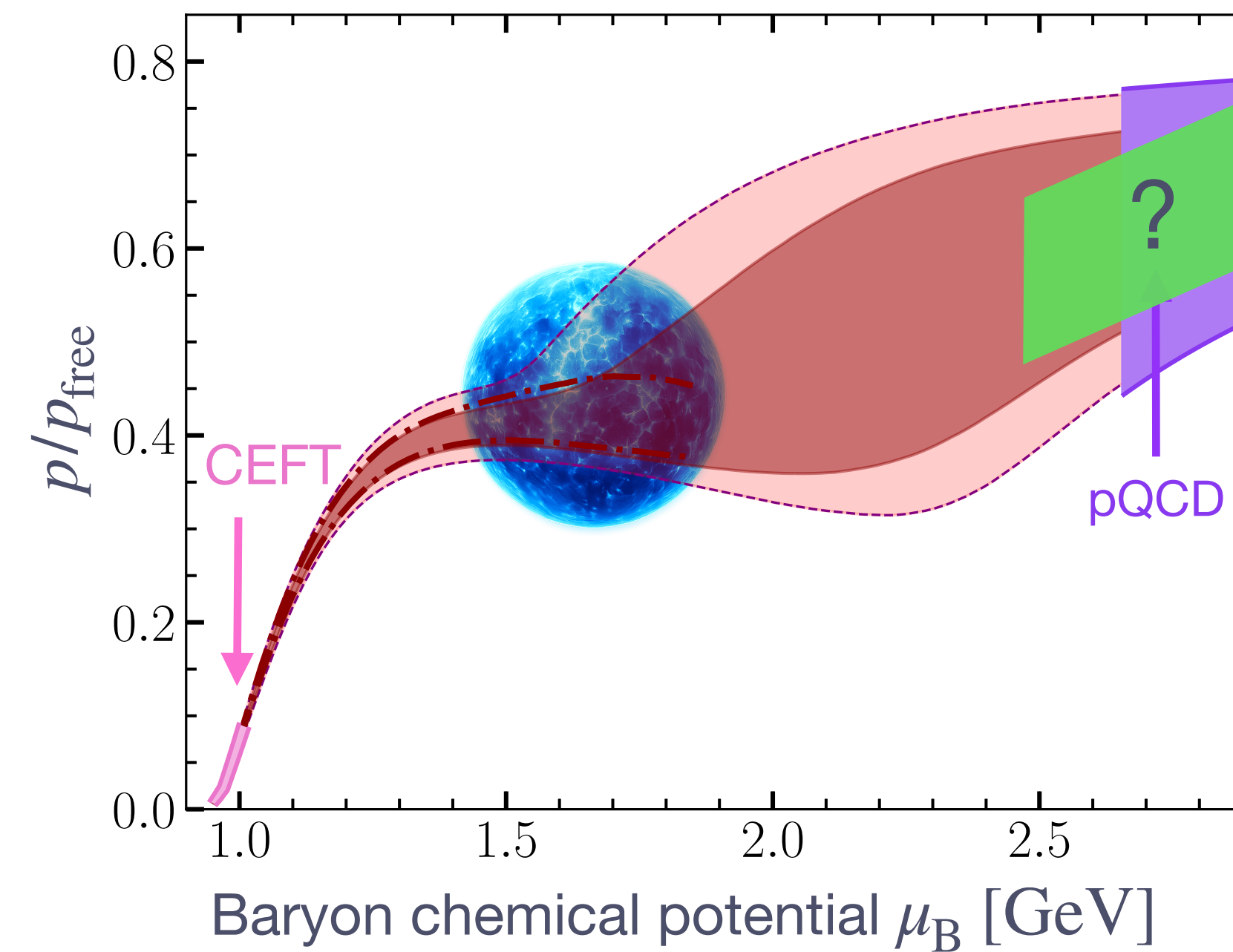
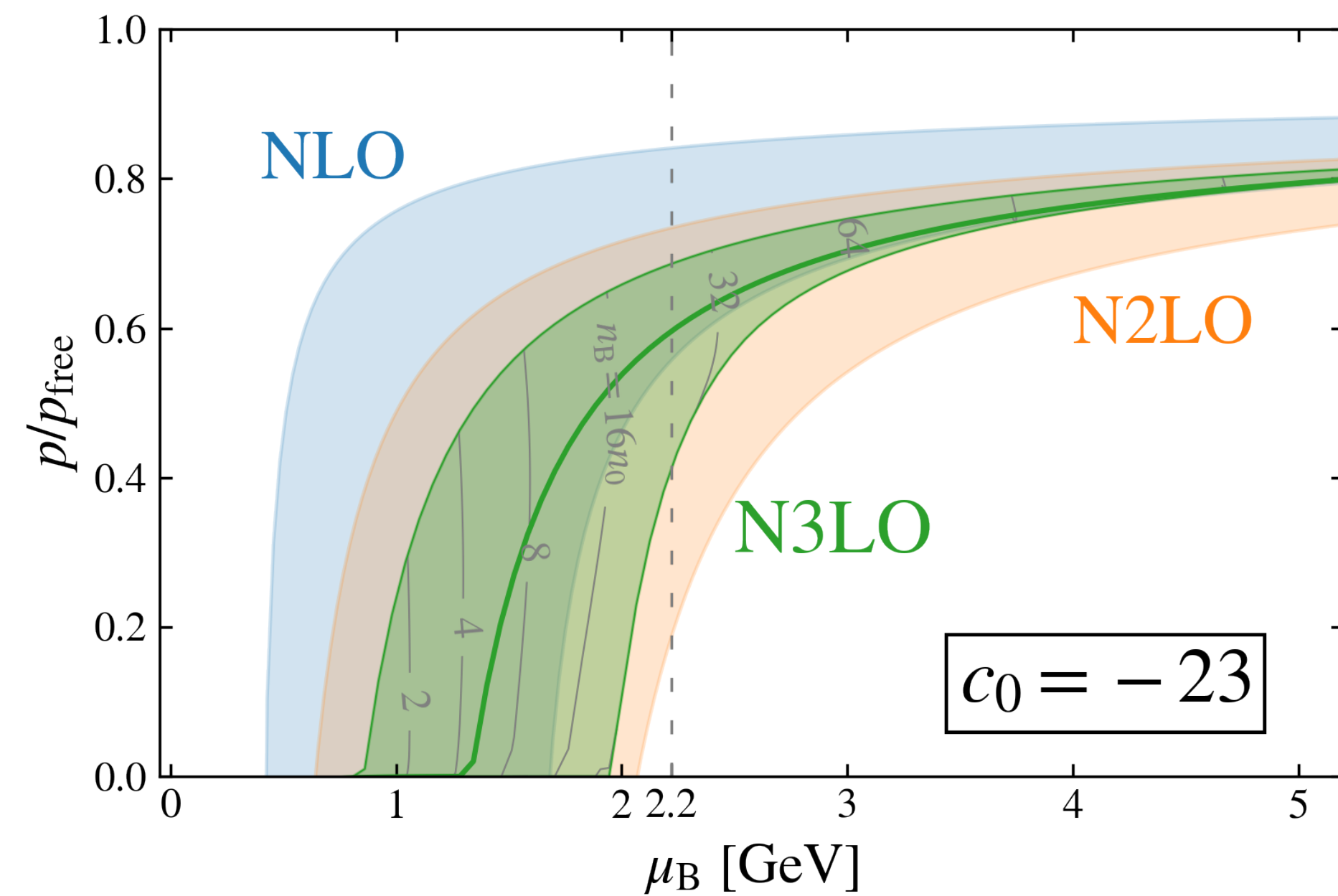


Brings result on par with its high- T counterpart:
[Kajantie et al. PRD 67 \(2003\)](#)

Possibility for dramatic improvement!

Bayesian estimate of c_0 based on lower order results:

Gorda, Paatelainen, Säppi, KS PRL 131 (2023)



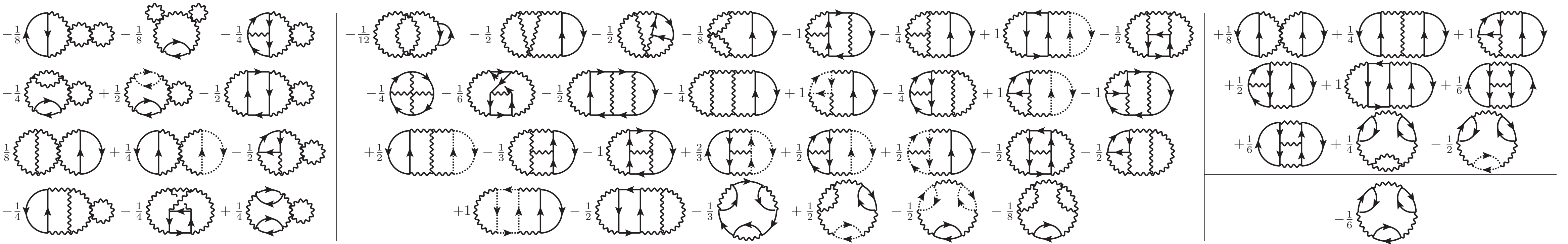
⇒ Completing the N3LO calculation could significantly improve the neutron star EoS!

Computation of c_0



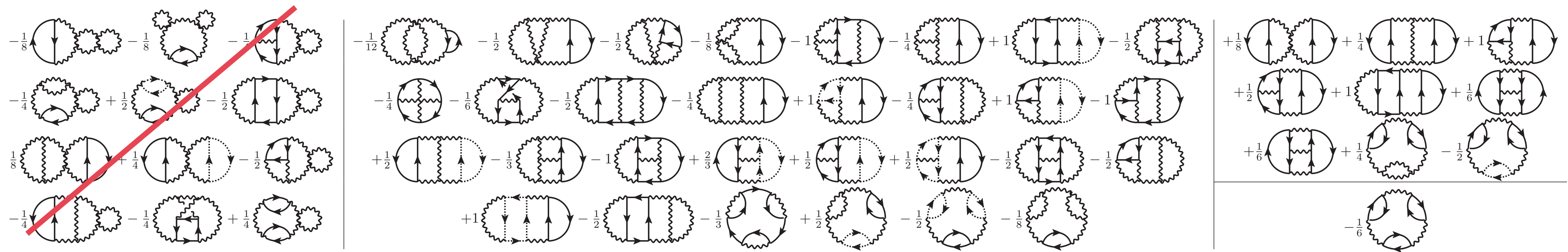
Huge challenge

Evaluation of 52 four-loop QCD Feynman diagrams with nonzero μ_B



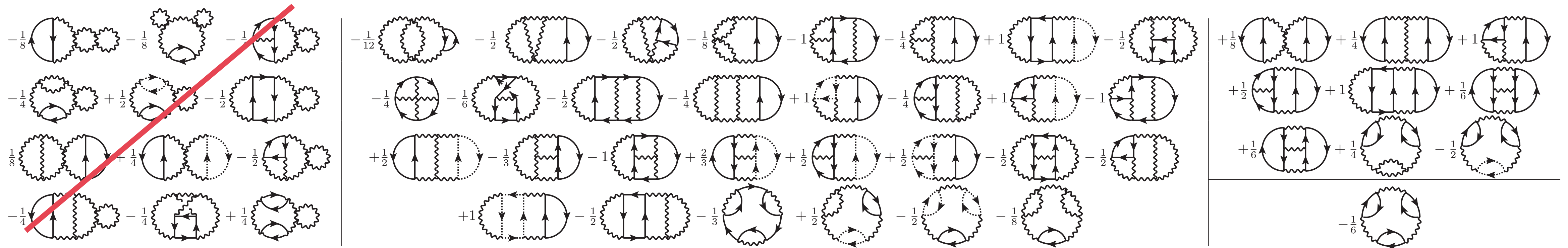
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Huge challenge

Evaluation of 52 four-loop QCD Feynman diagrams with nonzero μ_B



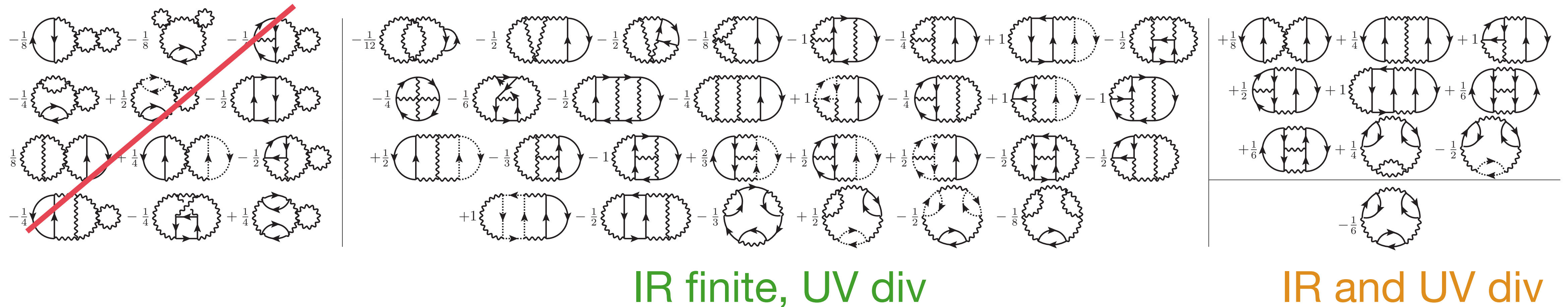
IR finite, UV div

IR and UV div

$$\mathcal{C}_0 \sim (30 \text{ diagrams}) N_f + (9 \text{ diagrams}) N_f^2 + (1 \text{ diagram}) N_f^3$$

Huge challenge

Evaluation of 52 four-loop QCD Feynman diagrams with nonzero μ_B



$$\mathcal{C}_0 \sim (30 \text{ diagrams}) N_f + (9 \text{ diagrams}) N_f^2 + (1 \text{ diagram}) N_f^3$$

UV div's: renormalization of α_s

IR div's: should cancel against soft HTL parts

Organization

Kärkkäinen, Navarrete, Nurmela,
Paatelainen, KS, Vuorinen PRL 135 (2025)



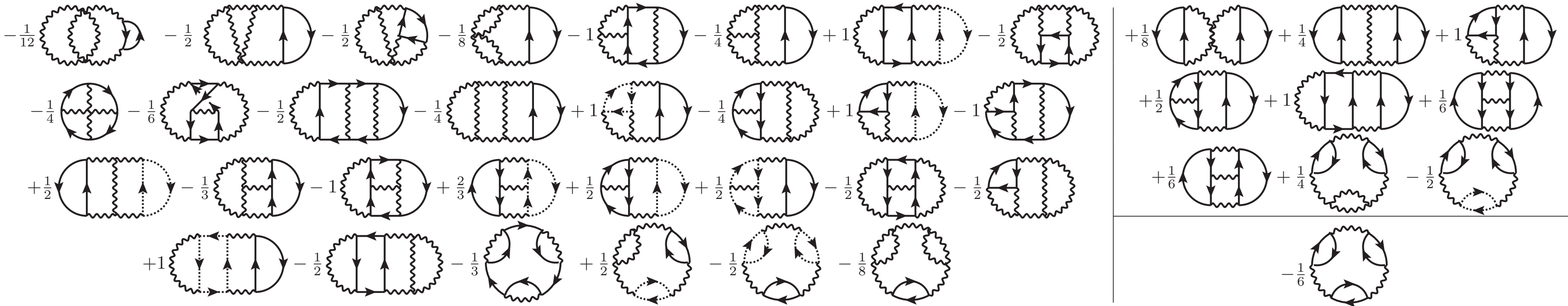
EDITORS' SUGGESTION

Quark Matter at Four Loops: Hardships and How to Overcome Them

Aapeli Kärkkäinen, Pablo Navarrete, Mika Nurmela, Risto Paatelainen, Kaapo Seppänen, and Aleksi Vuorinen

Phys. Rev. Lett. **135**, 021901 (2025) - Published 7 July, 2025

A new theoretical framework simplifies the four-loop Feynman diagrams for dense QCD to few infrared-finite integrals, which brings the higher-order calculation of the pressure inside cold and dense quark matter within reach.



Organization

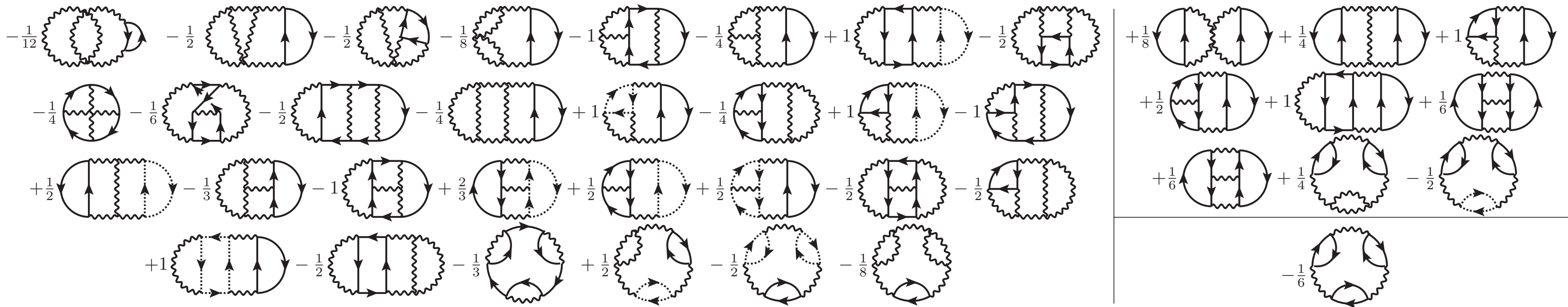
Kärkkäinen, Navarrete, Nurmela,
Paatelainen, KS, Vuorinen PRL 135 (2025)



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We have demonstrated:

Organization

Kärkkäinen, Navarrete, Nurmela,
Paatelainen, KS, Vuorinen PRL 135 (2025)



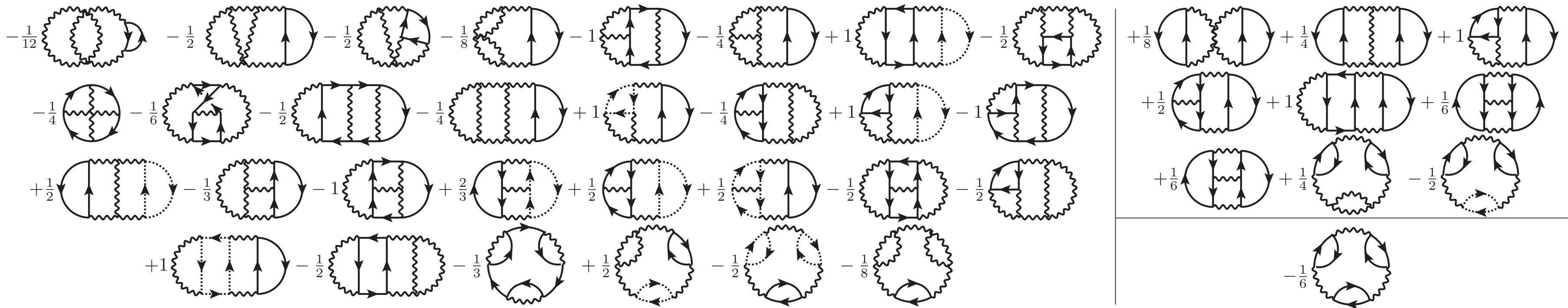
EDITORS' SUGGESTION

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Aapeli Kärkkäinen, Pablo Navarrete, Mika Nurmela, Risto Paatelainen, Kaapo Seppänen, and Aleksi Vuorinen

Phys. Rev. Lett. **135**, 021901 (2025) - Published 7 July, 2025

A new theoretical framework simplifies the four-loop Feynman diagrams for dense QCD to few infrared-finite integrals, which brings the higher-order calculation of the pressure inside cold and dense quark matter within reach.



We have demonstrated:

- Reduction of $O(10^5)$ scalarized integrals to $O(100)$ 4-loop masters

Organization



Kärkkäinen, Navarrete, Nurmela,
Paatelainen, KS, Vuorinen PRL 135 (2025)

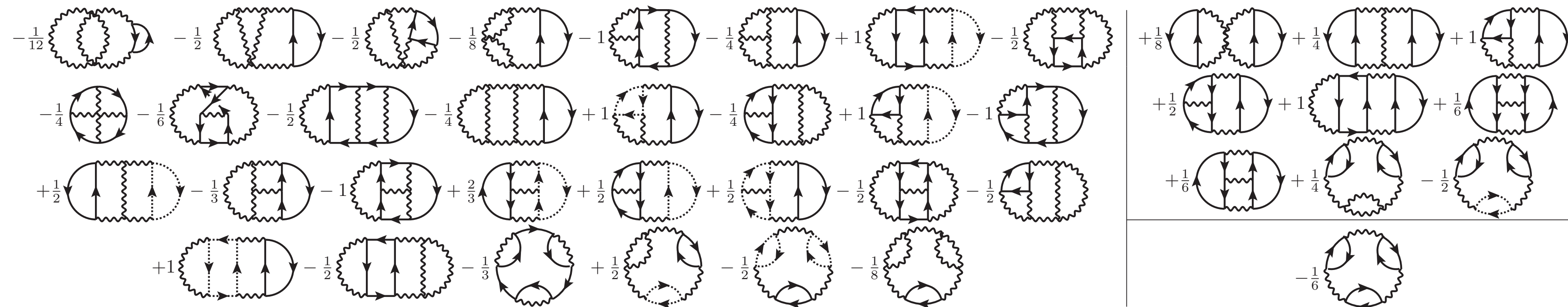
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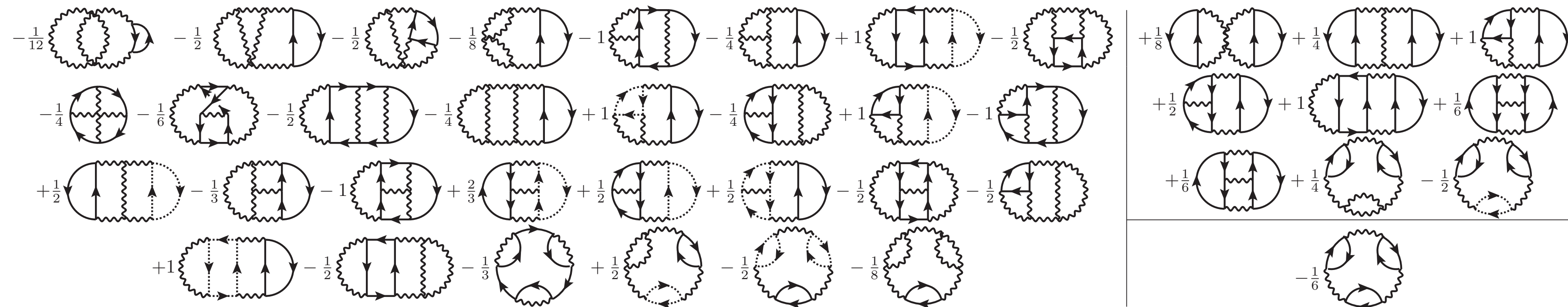
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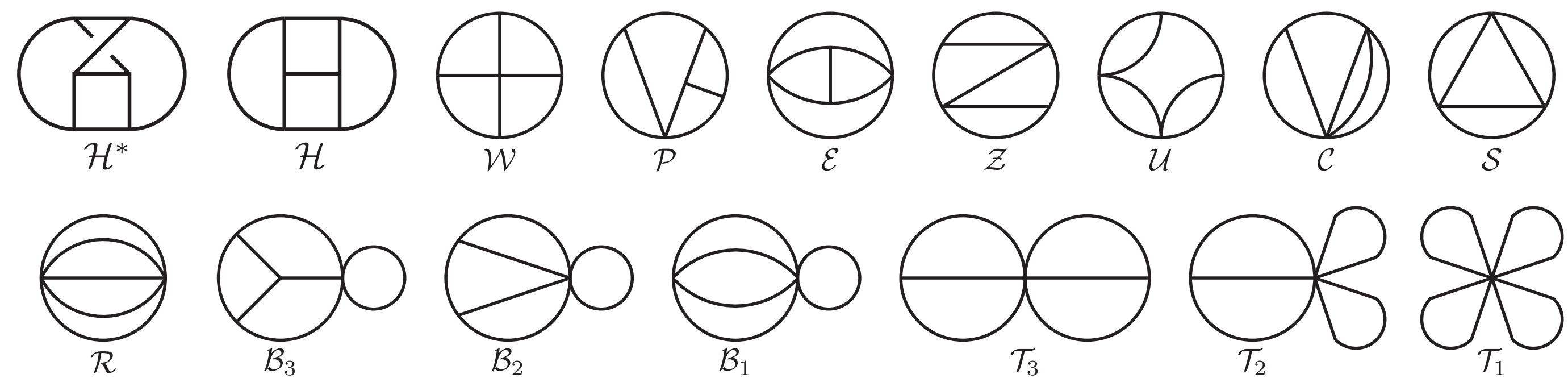


We have demonstrated:

- Reduction of $O(10^5)$ scalarized integrals to $O(100)$ 4-loop masters
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- Cancellation of IR divergences against already-computed HTL parts

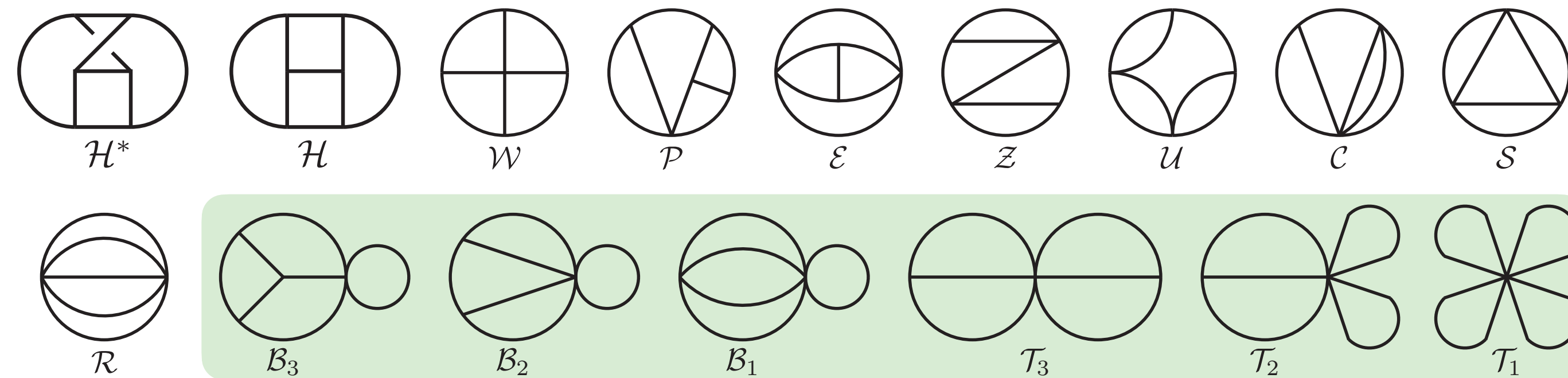
Remaining computational barrier

Computation of $\mathcal{O}(100)$ IR finite but UV divergent 4-loop master integrals



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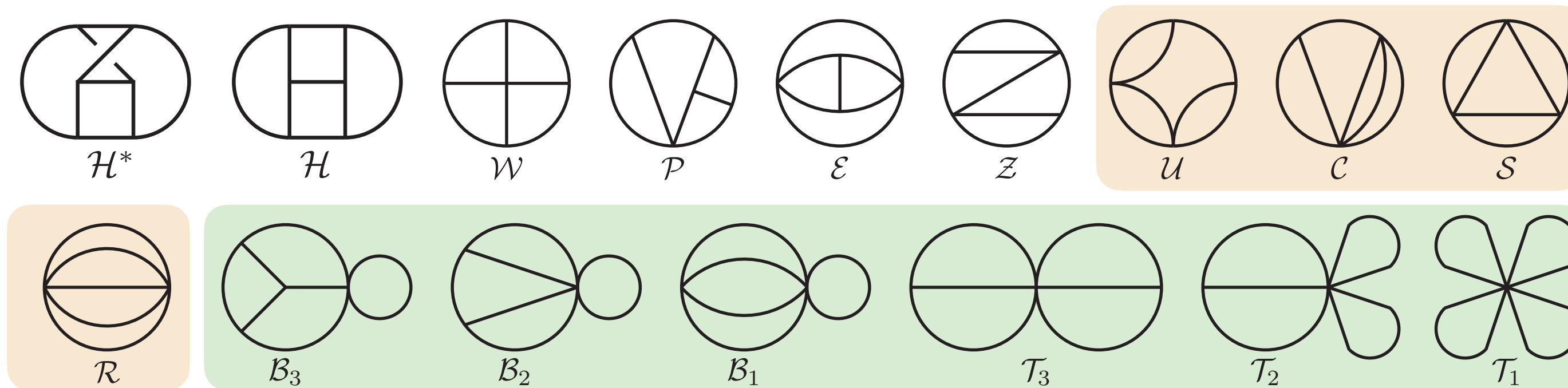


“Difficulty”:

- **“Easy”**: factorized topologies

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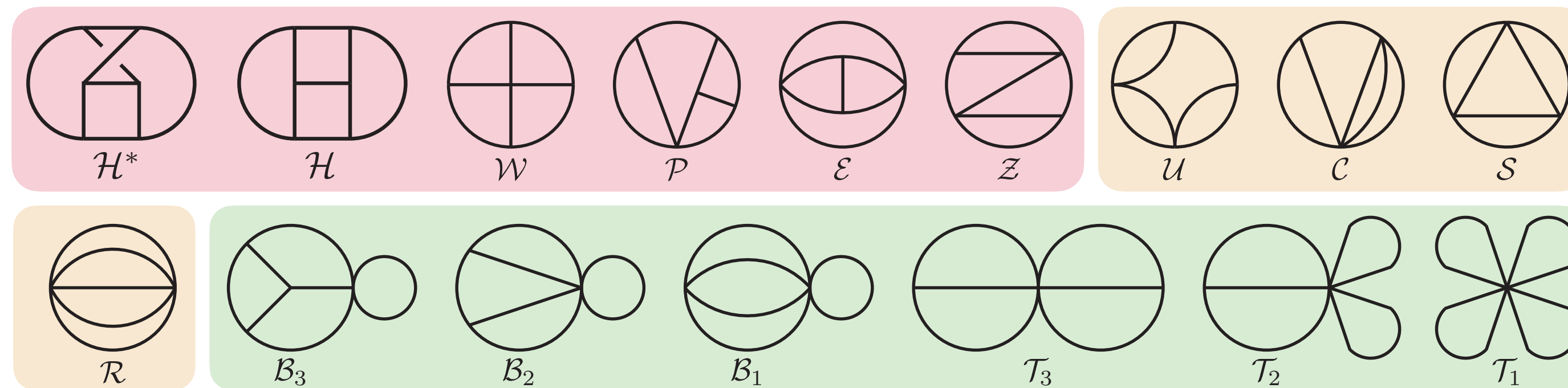


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“Difficulty”:

- **“Easy”:** factorized topologies
- **Medium:** doable analytically but very tedious
- **Hard:** traditional analytic methods not enough in practice ? ? ?

Lack of powerful analytic tools at $\mu > 0$

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- State-of-the-art **vacuum** QFT multi-loop methods (e.g. IBP and differential equations) hard to generalize to finite μ due to “broken” Lorentz symmetry
 $k^0 \rightarrow k^0 + i\mu$

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- **Cutting rules** traditionally used at $\mu > 0$:

$$\begin{aligned}
 \text{Bubble} &= \text{Bubble}_{\mu=0} - 2 \int_{\vec{p}} \frac{\Theta(\mu - E_{\vec{p}})}{2E_{\vec{p}}} \text{CutBubble}_{\mu=0} + \int_{\vec{p}} \int_{\vec{q}} \frac{\Theta(\mu - E_{\vec{p}})\Theta(\mu - E_{\vec{q}})}{4E_{\vec{p}}E_{\vec{q}}} \text{CutCutBubble}_{\mu=0}
 \end{aligned}$$

$\mu=0$
 $p^0 \rightarrow iE_{\vec{p}}$
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3d phase-space integrals over on-shell $\mu = 0$ amplitudes

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 \mu=0
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 =
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3d phase-space integrals over on-shell $\mu = 0$ amplitudes

↪ but involve spurious divergences between cuts

↪ analytic computation typically hopeless at 4-loops

Stuck with analytic methods?

Go fully numerical!



Inspiration from collider physics

Numerical Loop-Tree Duality

Direct numerical integration in momentum space via Loop-Tree Duality (LTD)

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$$\int_K f(K)$$

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1. **Subtract divergences** to make integrand locally finite in $D = 4$

$$\int_K f(K) = \int_K \underbrace{[f(K) - \text{CT}(K)]}_{\equiv f_{\text{fin}}(K)} + \int_K \text{CT}(K)$$

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Inspiration from collider physics

Numerical Loop-Tree Duality

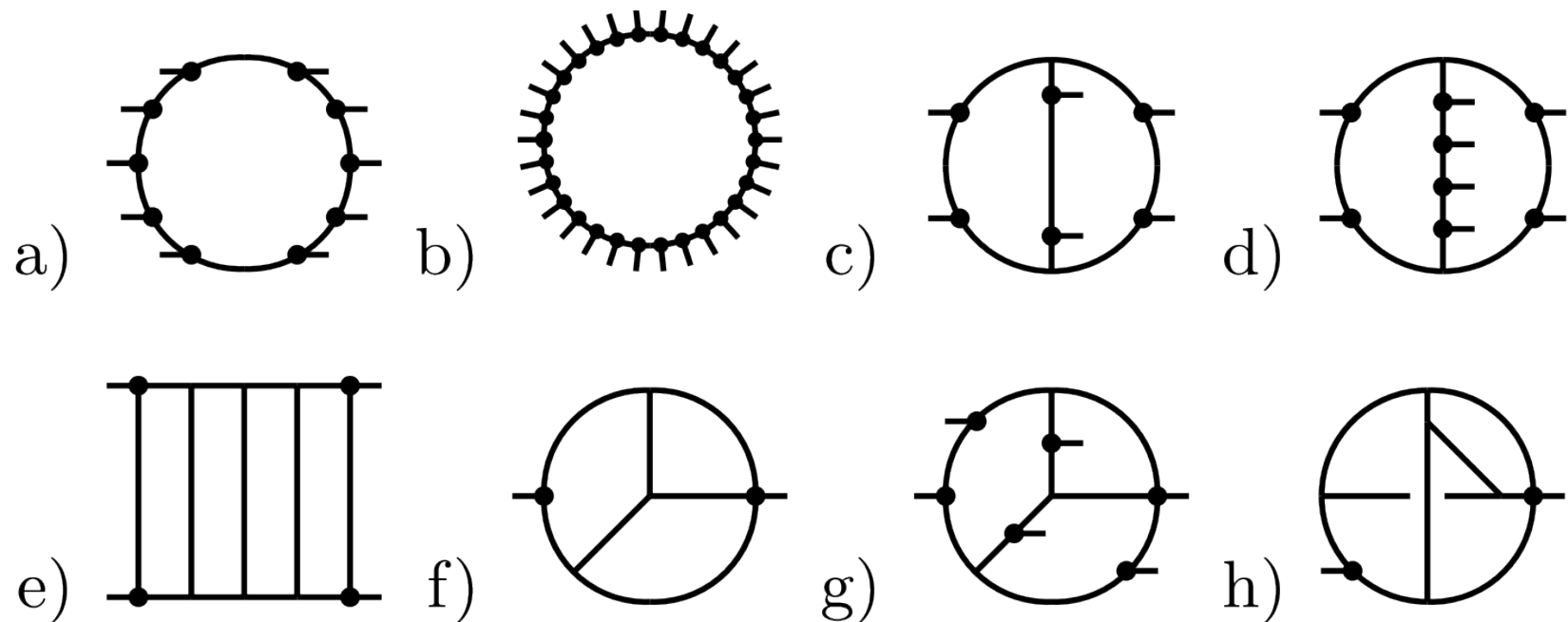


In collider context:

- Highly developed tool for numerical computation of multi-loop multi-scale scattering amplitudes
- One step further: direct numerical integration of differential cross sections using Local Unitarity (γ Loop)

Capatti, Hirschi, Kermanschah, Ruijl PRL 123 (2019)

Capatti, Hirschi, Pelloni, Ruijl JHEP 04 (2021)



G	Reference	Numerical LTD		N [10^6] [μ s]	
a)*	[33]	$i 4.31638 \cdot 10^{-7}$	$i 4.31637(19) \cdot 10^{-7}$	110	1.1
b)	[33]	$i 0.358640$	$i 0.358646(29)$	210	5.9
c)	[7]	$1.1339(5) \cdot 10^{-4}$	$1.133719(58) \cdot 10^{-4}$	5500	2.5
c)*	[7]	$4.398(1) \cdot 10^{-8}$	$4.39825(17) \cdot 10^{-8}$	5500	2.5
d)*	[7]	$2.409(1) \cdot 10^{-8}$	$2.40869(27) \cdot 10^{-8}$	5500	3.5
e)	[34]	$-1.433521 \cdot 10^{-6}$	$-1.4338(18) \cdot 10^{-6}$	1500	27.4
f)	[35]	$i 5.26647 \cdot 10^{-6}$	$i 5.236(38) \cdot 10^{-6}$	7000	3.3
g)*	[7]	$i 1.7790(6) \cdot 10^{-10}$	$i 1.77648(48) \cdot 10^{-10}$	22000	11
h)	[35]	$-8.36515 \cdot 10^{-8}$	$-8.309(31) \cdot 10^{-8}$	7000	15.8

Dense Loop-Tree Duality

Natural generalization

Extends to finite μ_B naturally

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{dk^0}{2\pi} f(k^0, \vec{k}) \longrightarrow \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{dk^0}{2\pi} f(k^0 + i\mu_B/3, \vec{k})$$

LTD

dLTD

$\Rightarrow$ dense Loop-Tree Duality (dLTD)

Navarrete, Paatelainen, KS PRD 110 (2024)

Dense Loop-Tree Duality

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Navarrete, Paatelainen, KS PRD 110 (2024)

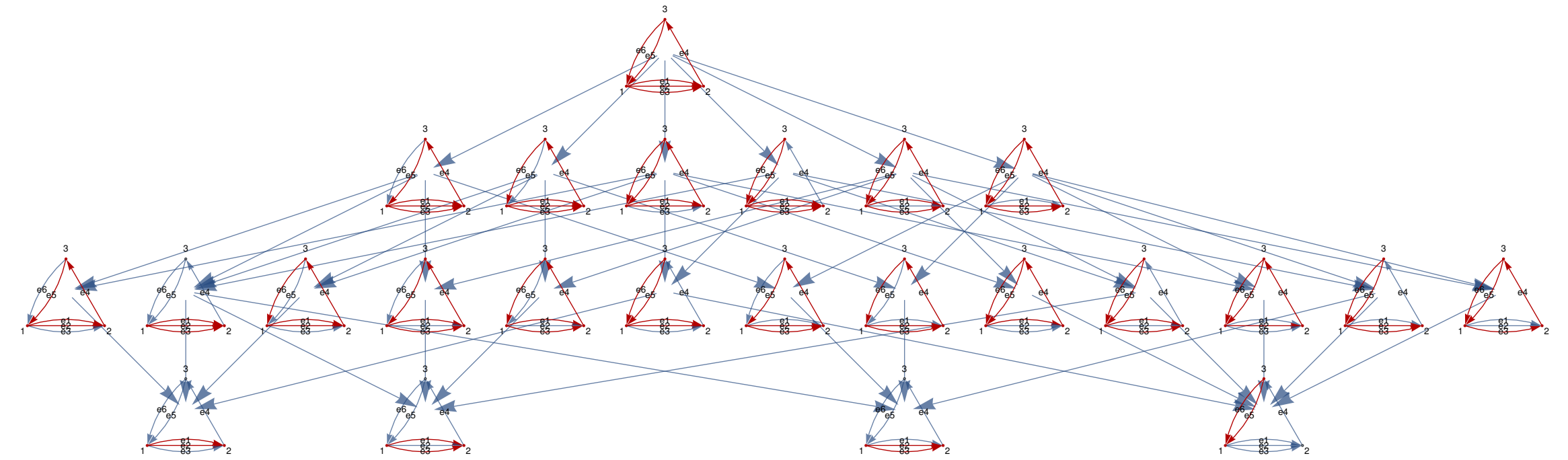
Dense Loop-Tree Duality

Fully automated pipeline

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Fully automated pipeline

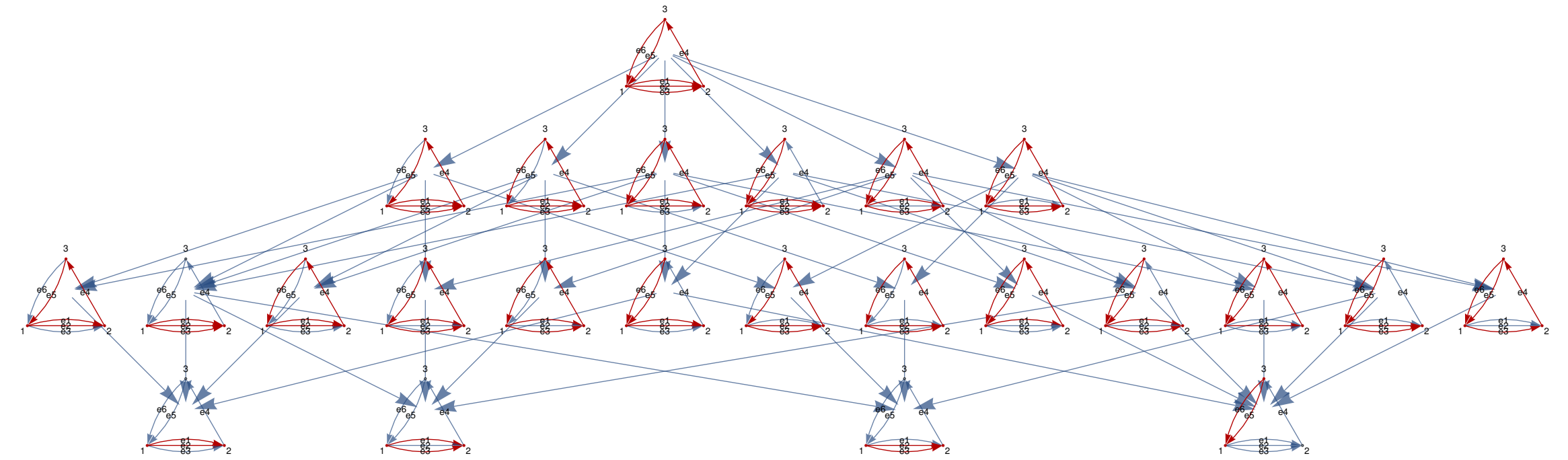
1. All-order UV local subtraction via BPHZ R-operation



Dense Loop-Tree Duality

Fully automated pipeline

1. All-order UV local subtraction via BPHZ R-operation
2. All-order automated analytic energy integration (=LTD) involving Laplace transforms of graph polytopes



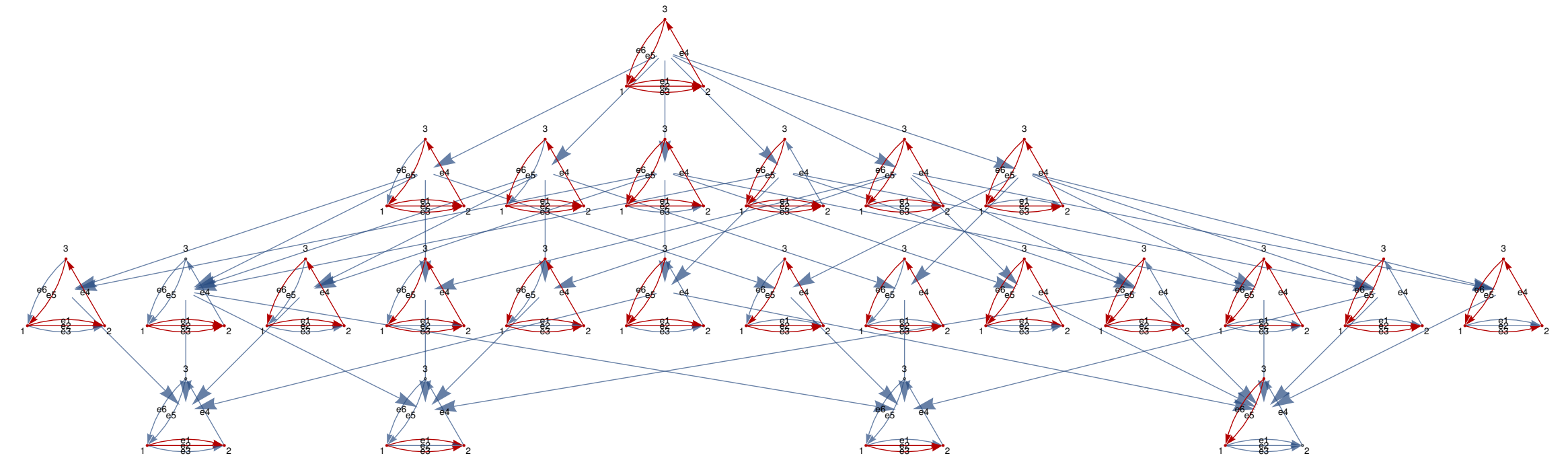
$$\Omega^\infty \left(\text{Diagram 1} \right) = \frac{1}{E_1 + E_2 + E_3} \Omega^\infty \left(\text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right)$$

The diagram shows a sequence of four Feynman diagrams. The first diagram on the left has vertices labeled v_1, v_2, v_3, v_4, v_5 and a red curved line. A red dotted arrow points from this diagram to the right-hand side of the equation. The right-hand side consists of three diagrams added together, each with vertices v_1, v_2, v_3, v_4, v_5 and different internal line configurations. The first diagram on the right has a blue curved line, the second has a black curved line, and the third has a blue curved line.

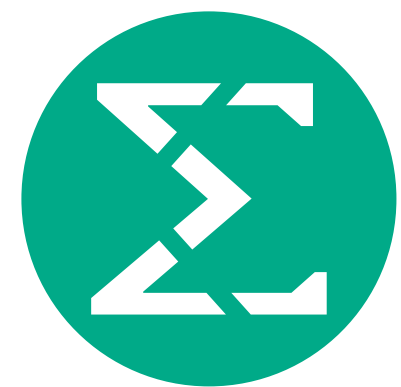
Dense Loop-Tree Duality

Fully automated pipeline

1. All-order UV local subtraction via BPHZ R-operation
2. All-order automated analytic energy integration (=LTD) involving Laplace transforms of graph polytopes
3. Monte Carlo sampling using efficient compiled Symbolica evaluators



$$\Omega^\infty \left(\text{Diagram with vertices } v_1, v_2, v_3, v_4, v_5 \right) = \frac{1}{E_1 + E_2 + E_3} \Omega^\infty \left(\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \right)$$



Symbolica

Input expression

```
x^5*y + cos(x) +  
a*b^3 - sqrt(z)  
+ sin(a)^2 + ...
```

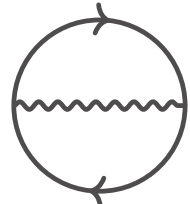
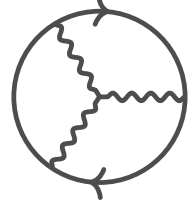
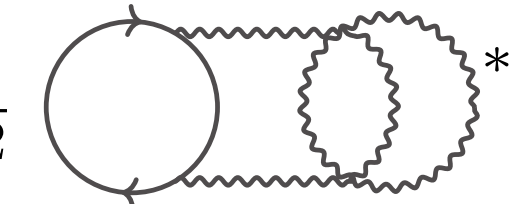
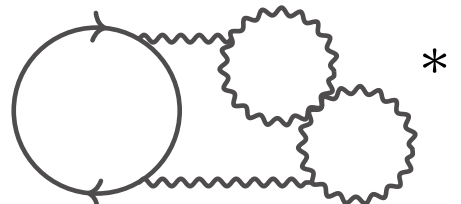


Optimized kernel

```
ldr d0, [x0]  
fmul d1, d0, d0  
fadd d0, d1, d2
```

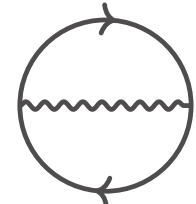
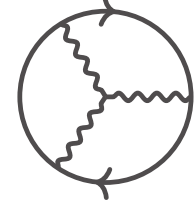
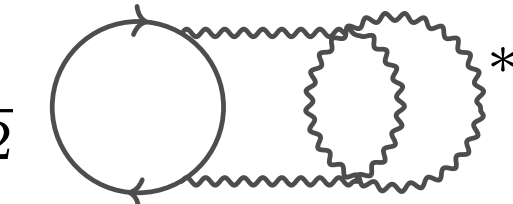
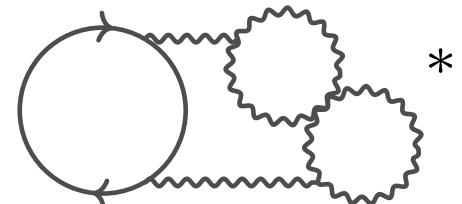
Dense Loop-Tree Duality

Benchmarks

Diagram	$\varepsilon^{-2} [10^{-3}]$	$\varepsilon^{-1} [10^{-3}]$	$\varepsilon_{\text{traditional}}^0 [10^{-3}]$	$\varepsilon_{\text{dLTD}}^0 [10^{-3}]$	$N [10^6]$	$[\mu\text{s}]$
$-\frac{1}{2}$ 	0	0	$-3.849\,743$	$-3.8495(12)$	30	5.8
$-\frac{1}{3}$ 	0	$-0.219\,409$	$-1.682\,136$	$-1.682\,10(23)$	200	12.9
$-\frac{1}{12}$ 	0	0.001 563	0.023 186	0.023 18(6)	500	21.0
$-\frac{1}{8}$ 	0.014 068	0.173 504	1.174 161	1.174 23(10)	500	13.0

Dense Loop-Tree Duality

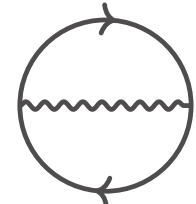
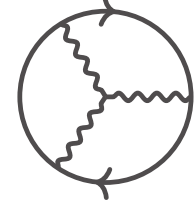
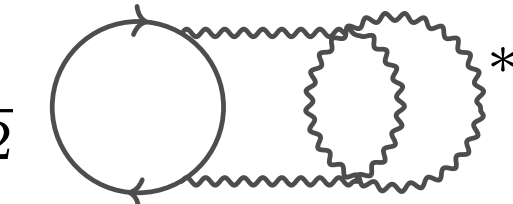
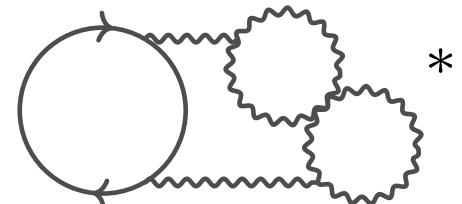
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UV poles analytically from BPHZ R-operation: $\int_K \text{CT}(K)$

Dense Loop-Tree Duality

Benchmarks

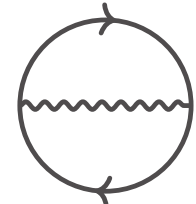
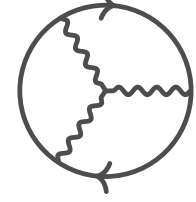
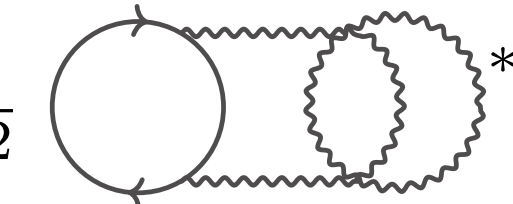
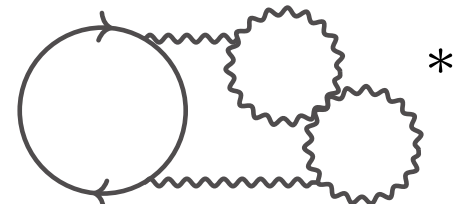
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Numerical finite part matches with literature value

Dense Loop-Tree Duality

Benchmarks



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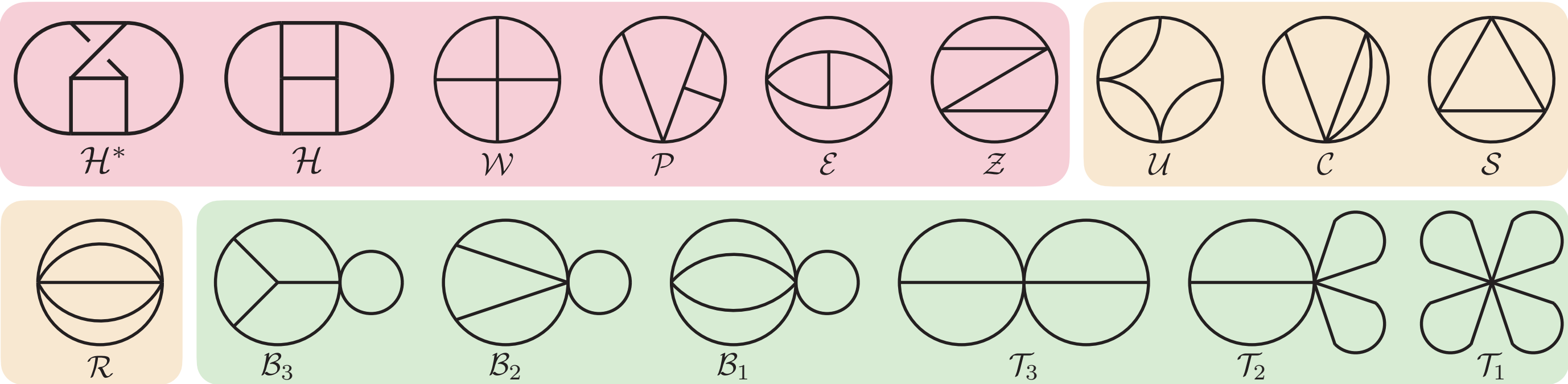
Performance scales well with number of loops

Ready to evaluate c_0



Evaluation of 4-loop masters

56 ints

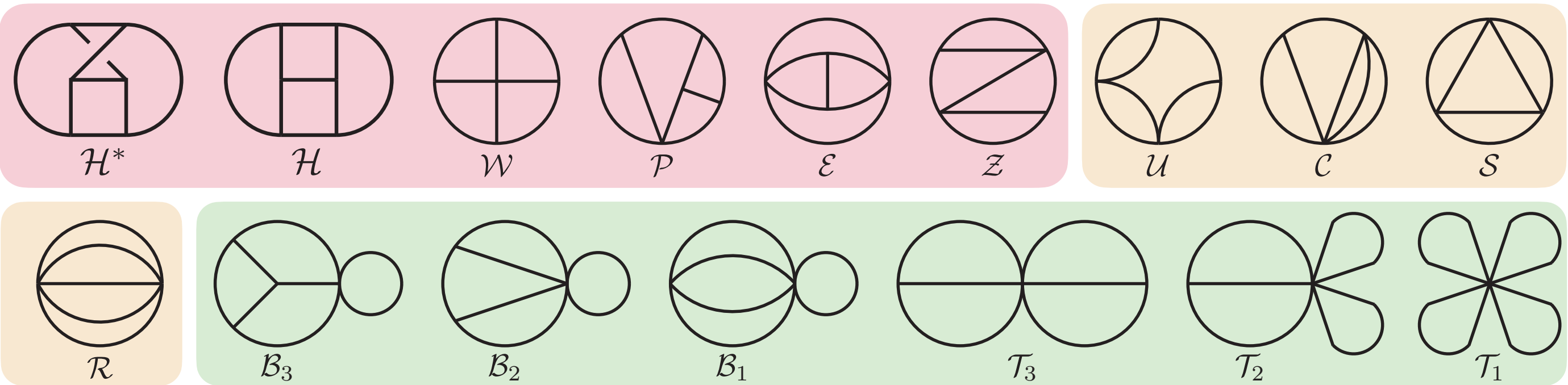


35 ints

26 ints

Evaluation of 4-loop masters

56 ints



35 ints

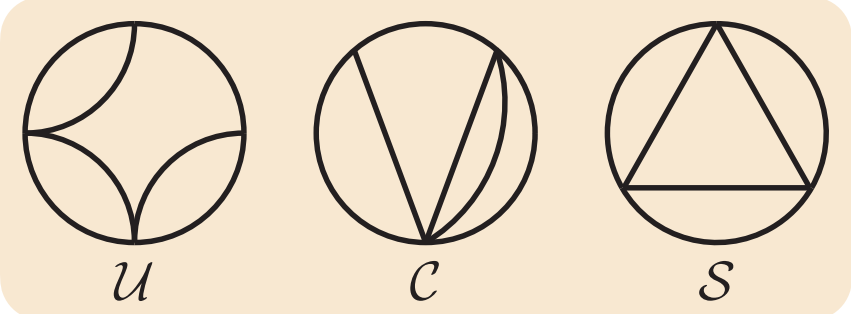
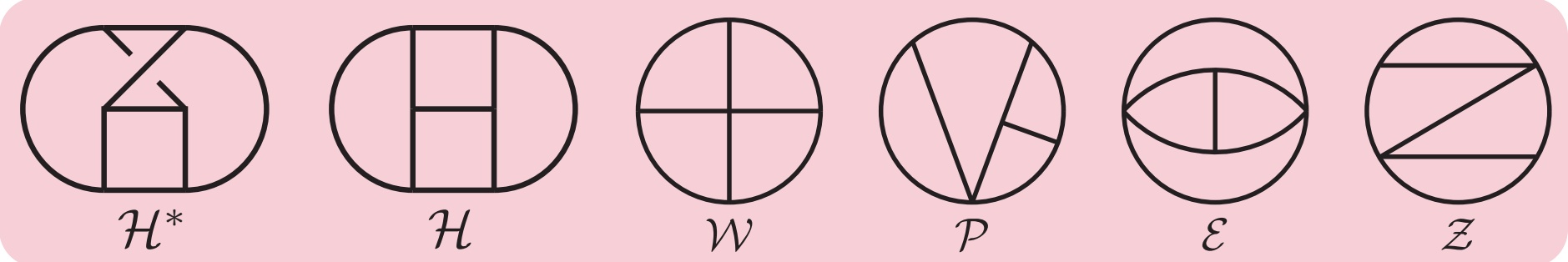
26 ints

Cutting rules:

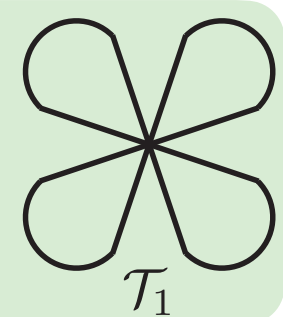
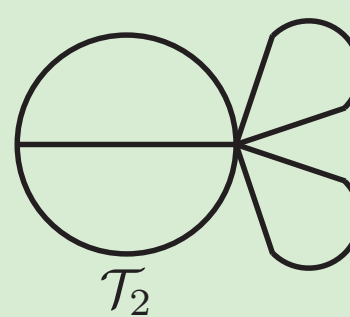
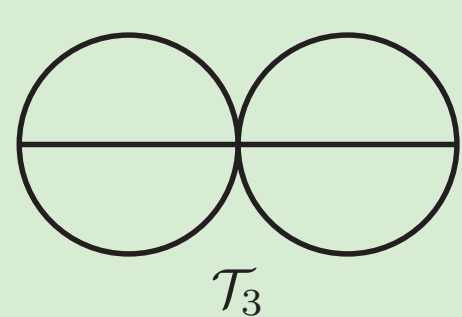
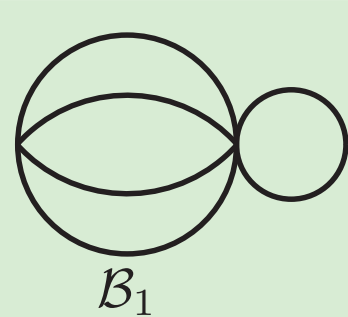
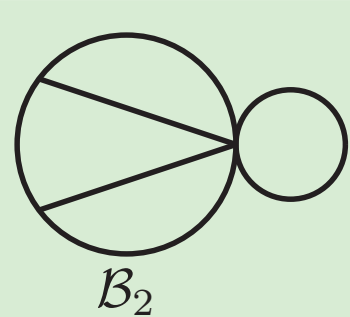
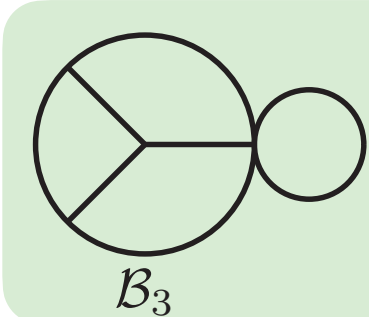
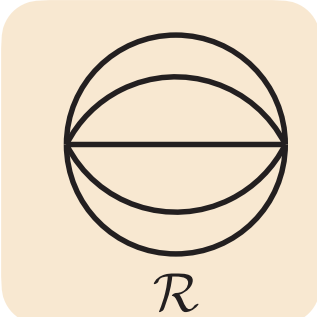
- All ●
- IR div. of ●

Evaluation of 4-loop masters

56 ints



35 ints



26 ints

Cutting rules:

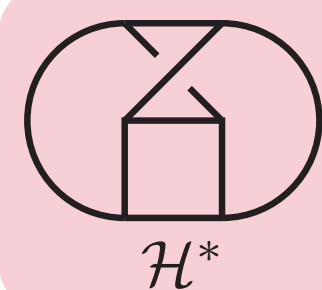
- All 
- IR div. of 

dLTD:

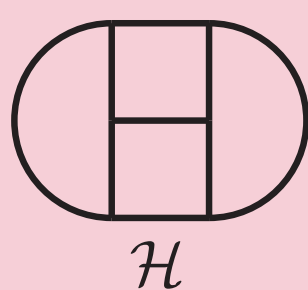
- All 
- IR fin. of 

Evaluation of 4-loop masters

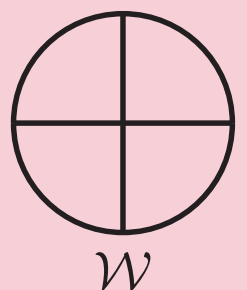
56 ints



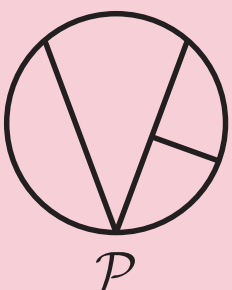
$\mathcal{H}^*$



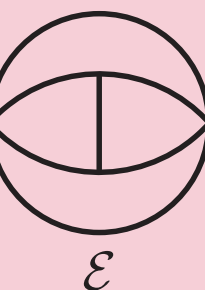
$\mathcal{H}$



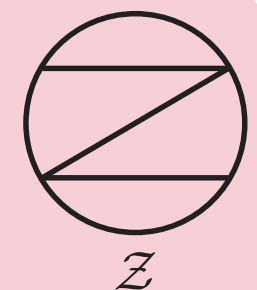
$\mathcal{W}$



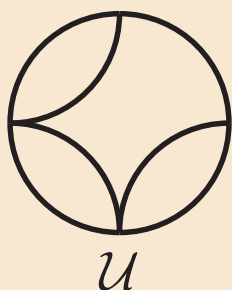
$\mathcal{P}$



$\mathcal{E}$



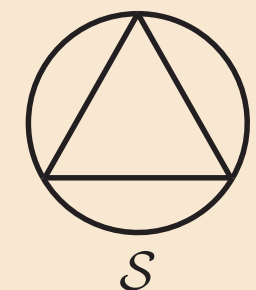
$\mathcal{Z}$



$\mathcal{U}$

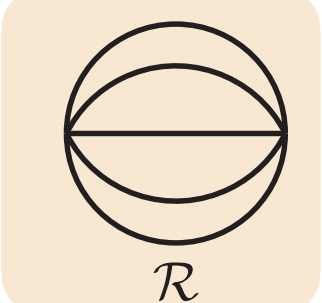


$\mathcal{C}$

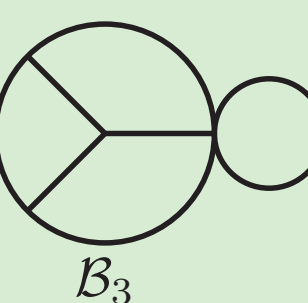


$\mathcal{S}$

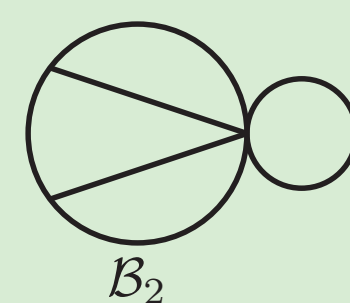
35 ints



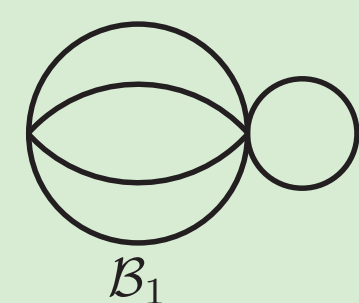
$\mathcal{R}$



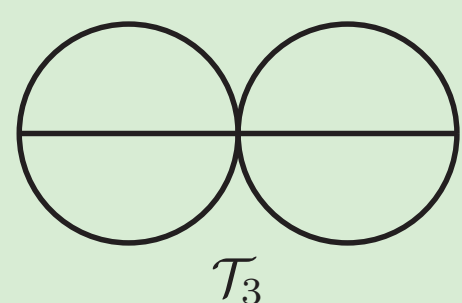
$\mathcal{B}_3$



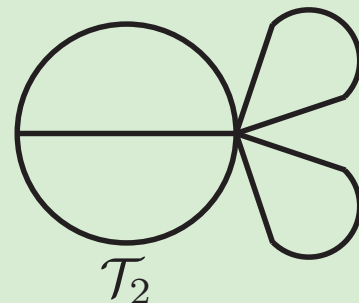
$\mathcal{B}_2$



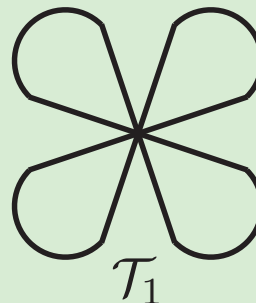
$\mathcal{B}_1$



$\mathcal{T}_3$



$\mathcal{T}_2$



$\mathcal{T}_1$

26 ints

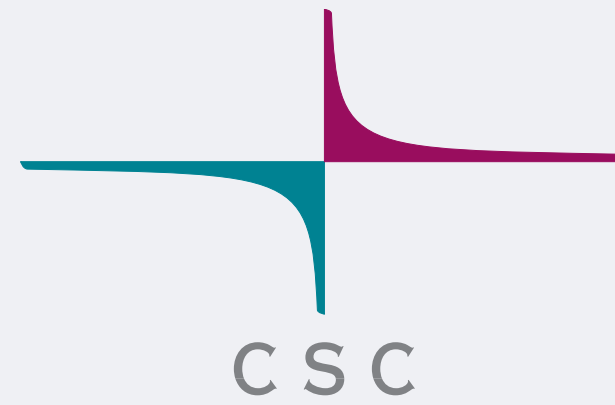
Cutting rules:

- All 
- IR div. of 

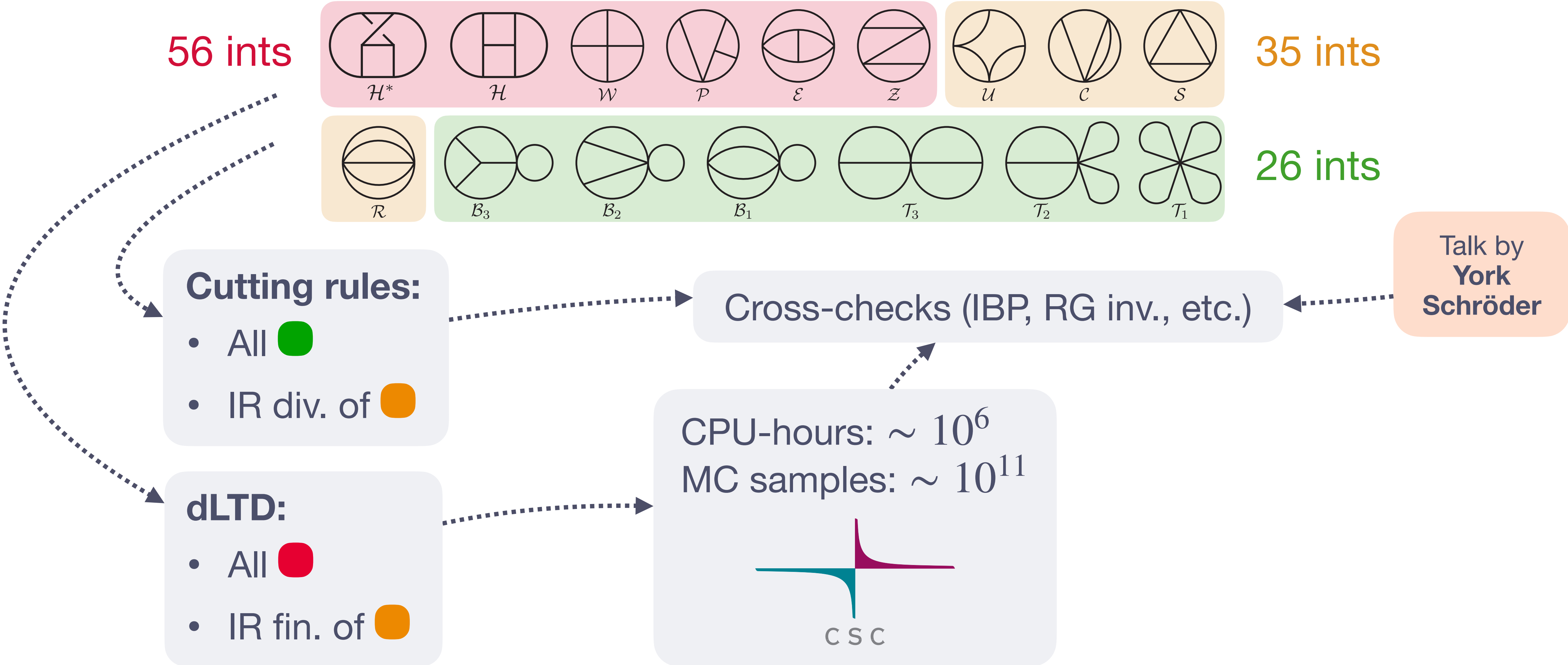
dLTD:

- All 
- IR fin. of 

CPU-hours: $\sim 10^6$
MC samples: $\sim 10^{11}$

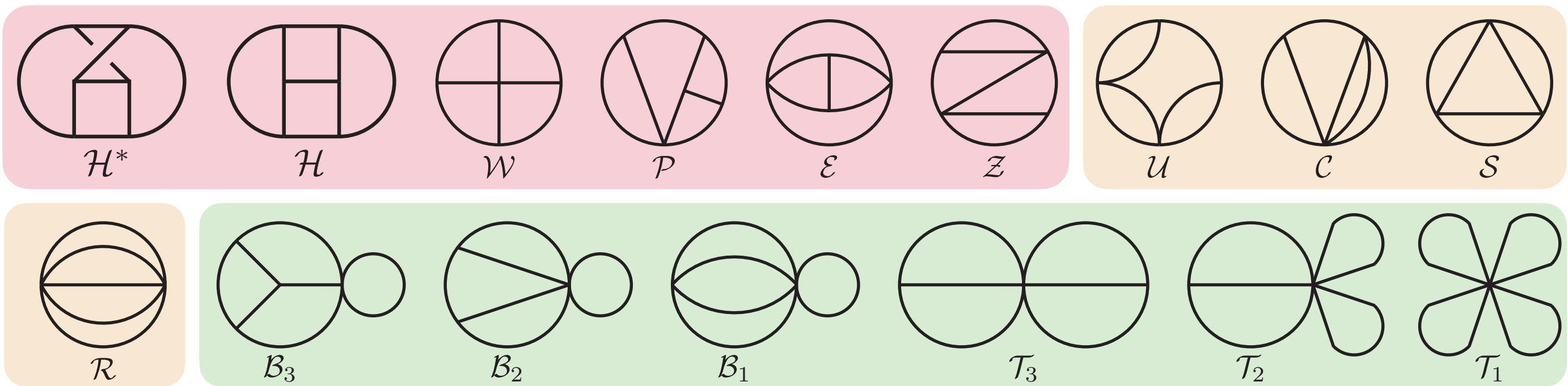


Evaluation of 4-loop masters



Evaluation of 4-loop masters

56 ints



35 ints

26 ints

Cutting rules:

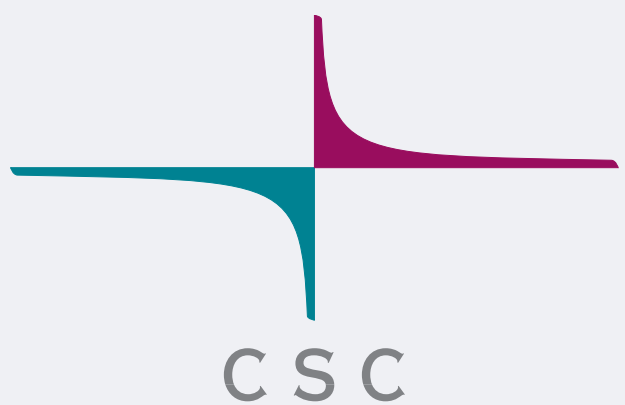
- All ●
- IR div. of ●

dLTD:

- All ●
- IR fin. of ●

Cross-checks (IBP, RG inv., etc.)

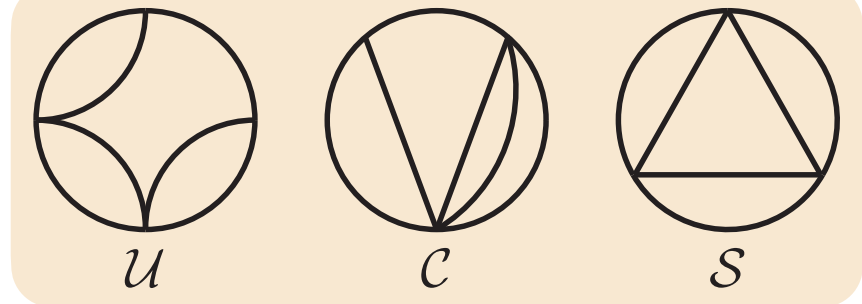
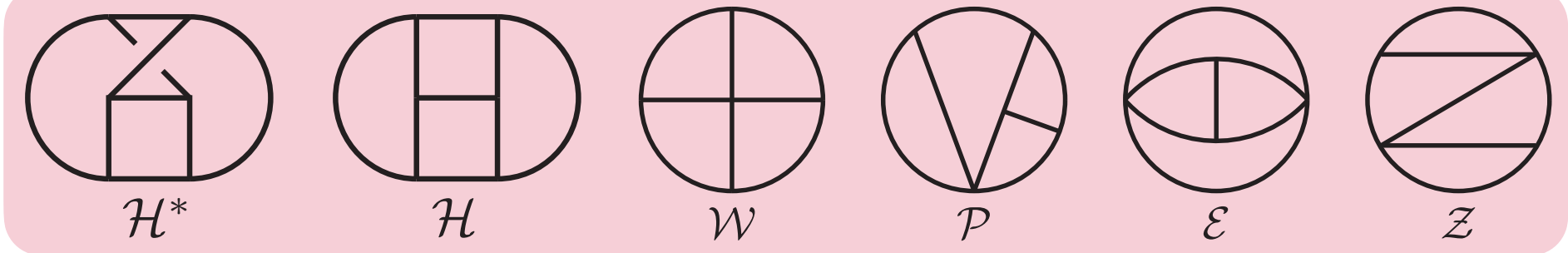
CPU-hours: $\sim 10^6$
MC samples: $\sim 10^{11}$



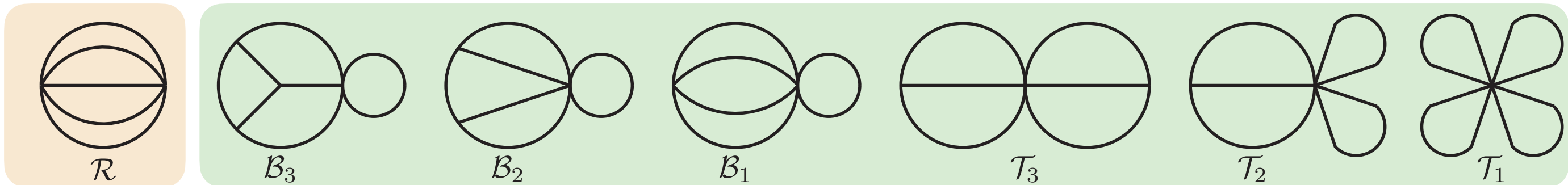
c_0 computed
up to $\sim 1\% \sigma$

Evaluation of 4-loop masters

56 ints



35 ints



26 ints

Cutting rules:

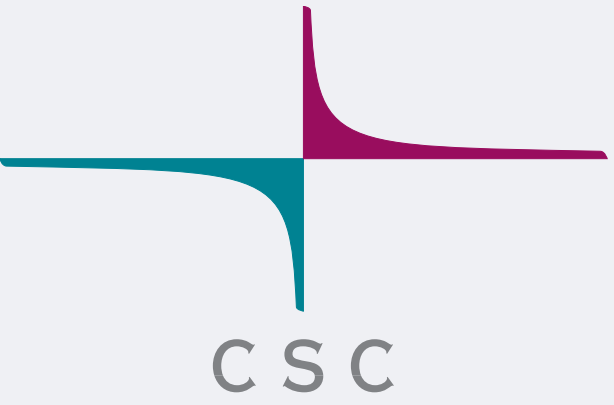
- All ●
- IR div. of ●

dLTD:

- All ●
- IR fin. of ●

Cross-checks (IBP, RG inv., etc.)

CPU-hours: $\sim 10^6$
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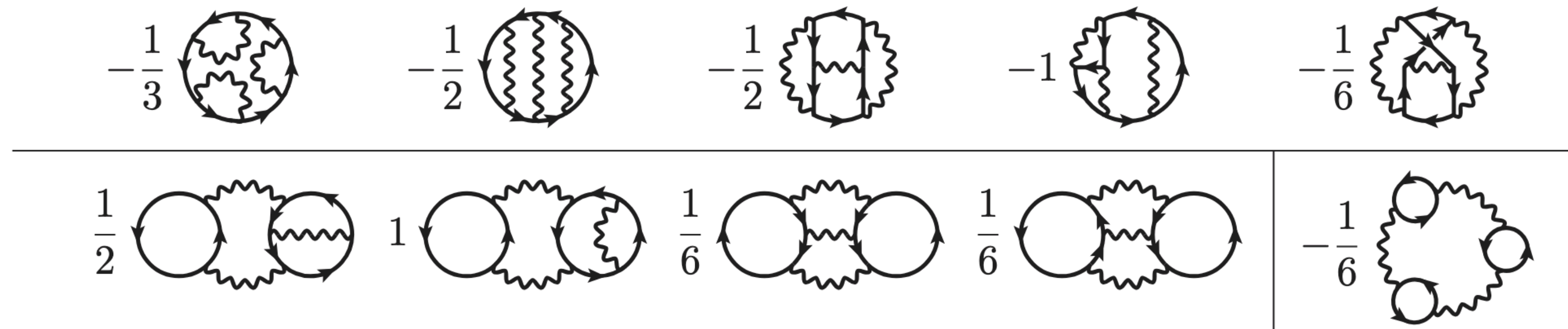


QED limit

$$\frac{p_{\text{QED}}(\mu_e)}{p_{\text{free}}} = 1 + p_{1,1} \left(\frac{\alpha_e}{\pi} \right) + \left[p_{2,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{2,0} \right] \left(\frac{\alpha_e}{\pi} \right)^2 + \left[p_{3,2} \ln^2 \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,0} \right] \left(\frac{\alpha_e}{\pi} \right)^3 + \dots$$

QED limit

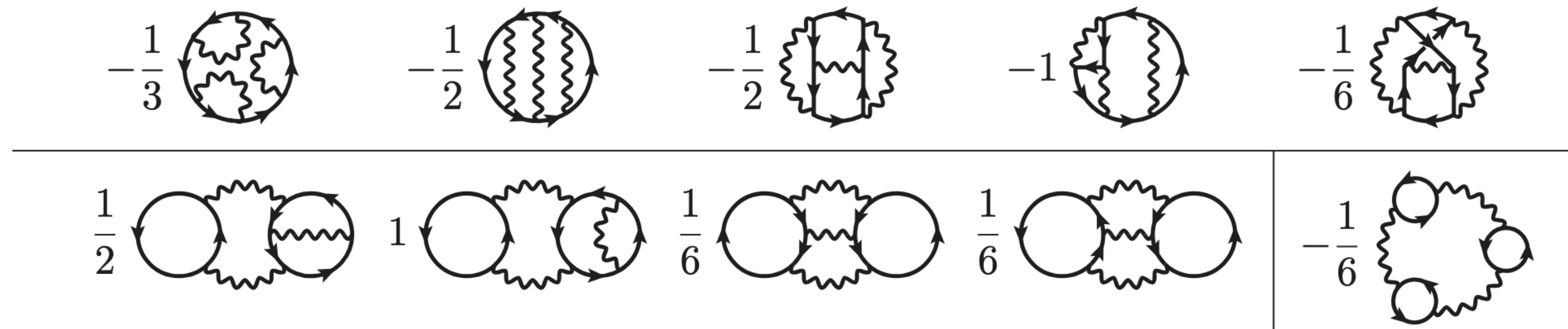
$$\frac{p_{\text{QED}}(\mu_e)}{p_{\text{free}}} = 1 + p_{1,1} \left(\frac{\alpha_e}{\pi} \right) + \left[p_{2,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{2,0} \right] \left(\frac{\alpha_e}{\pi} \right)^2 + \left[p_{3,2} \ln^2 \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,0} \right] \left(\frac{\alpha_e}{\pi} \right)^3 + \dots$$



- $p_{3,0}$: hard contribution reduces to 75 4-loop masters

QED limit

$$\frac{p_{\text{QED}}(\mu_e)}{p_{\text{free}}} = 1 + p_{1,1} \left(\frac{\alpha_e}{\pi} \right) + \left[p_{2,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{2,0} \right] \left(\frac{\alpha_e}{\pi} \right)^2 + \left[p_{3,2} \ln^2 \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,0} \right] \left(\frac{\alpha_e}{\pi} \right)^3 + \dots$$

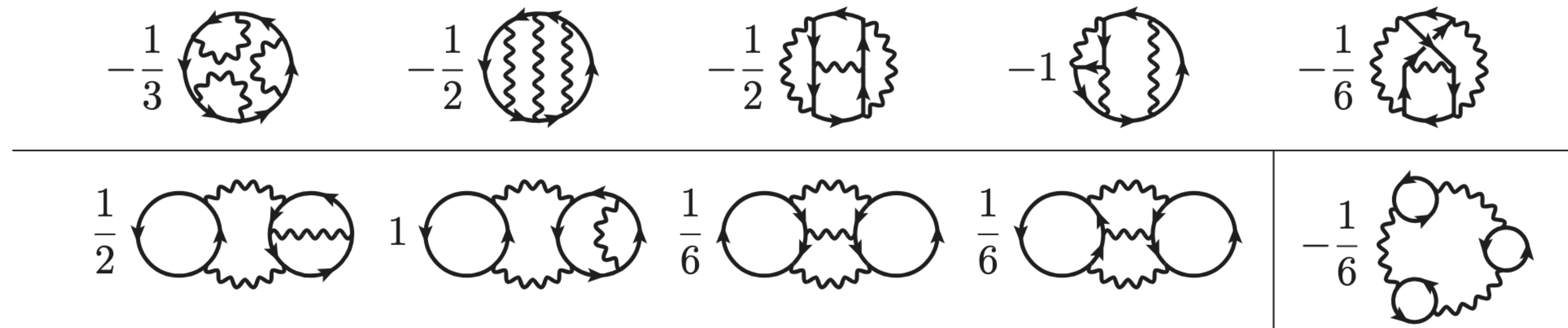


- $p_{3,0}$: hard contribution reduces to 75 4-loop masters
- $p_{3,1}$: HTL-resummed contribution already computed before

Gorda, Kurkela, Österman, Paatelainen, Säppi, Schicho, KS, Vuorinen PRD 107 (2023)

QED limit

$$\frac{p_{\text{QED}}(\mu_e)}{p_{\text{free}}} = 1 + p_{1,1} \left(\frac{\alpha_e}{\pi} \right) + \left[p_{2,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{2,0} \right] \left(\frac{\alpha_e}{\pi} \right)^2 + \left[\cancel{p_{3,2} \ln^2 \left(N_f \frac{\alpha_e}{\pi} \right)} + p_{3,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,0} \right] \left(\frac{\alpha_e}{\pi} \right)^3 + \dots$$



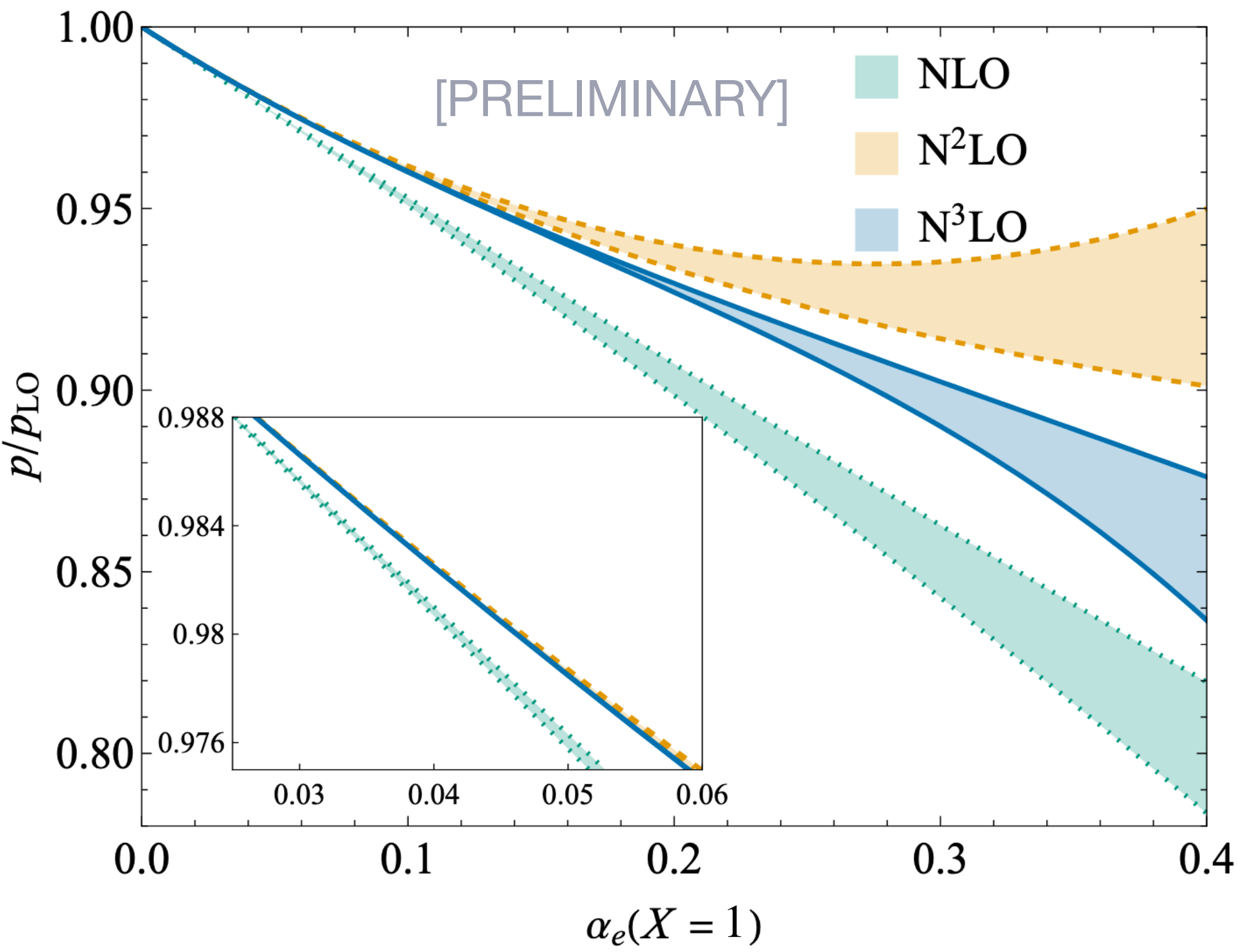
- $p_{3,0}$: hard contribution reduces to 75 4-loop masters
- $p_{3,1}$: HTL-resummed contribution already computed before
Gorda, Kurkela, Österman, Paatelainen, Säppi, Schicho, KS, Vuorinen PRD 107 (2023)
- $p_{3,2}$: vanishes in QED

QED limit

$$\frac{p_{\text{QED}}(\mu_e)}{p_{\text{free}}} = 1 + p_{1,1} \left(\frac{\alpha_e}{\pi} \right) + \left[p_{2,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{2,0} \right] \left(\frac{\alpha_e}{\pi} \right)^2 + \left[p_{3,2} \ln^2 \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,1} \ln \left(N_f \frac{\alpha_e}{\pi} \right) + p_{3,0} \right] \left(\frac{\alpha_e}{\pi} \right)^3 + \dots$$

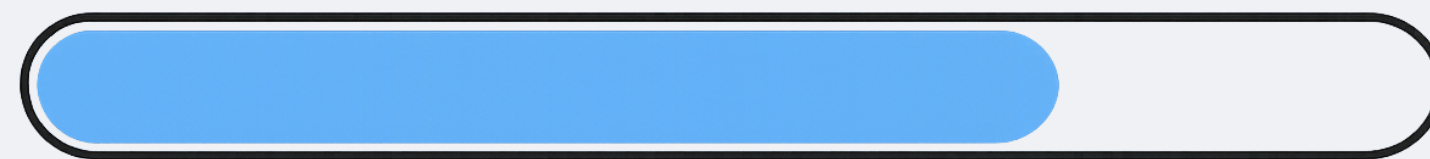
$p_{1,1}$	$-3/2$	
$p_{2,1}$	$-3/2 N_f$	
$p_{2,0}$	$(0.45717 + \ln X) N_f + 51/16$	$X = \frac{\bar{\Lambda}}{2\mu_e}$
$p_{3,2}$	0	
$p_{3,1}$	$[0.18931 + 2 \ln X] N_f^2 + 4.16299 N_f$	
$p_{3,0}$	$[1.02270(2) + 0.39042 \ln X - 2/3 \ln^2 X] N_f^2$ $- [9.57(8) + 7/2 \ln X] N_f - 8.41(11)$	

“ c_0 in QED”



Thermal Loop-Tree Duality

New precision era in thermal perturbation theory



work in progress...

Thermal LTD

New precision era in thermal perturbation theory

- dLTD extends naturally to finite temperature: $\int \rightarrow \oint$ [KS PhD Thesis \(2025\)](#)
- Also particle masses and external momenta can be included
- Algorithmic procedure independent of loop order

Goal: Automated evaluation of perturbative corrections in thermal field theory



Thermal LTD

Proofs of principle

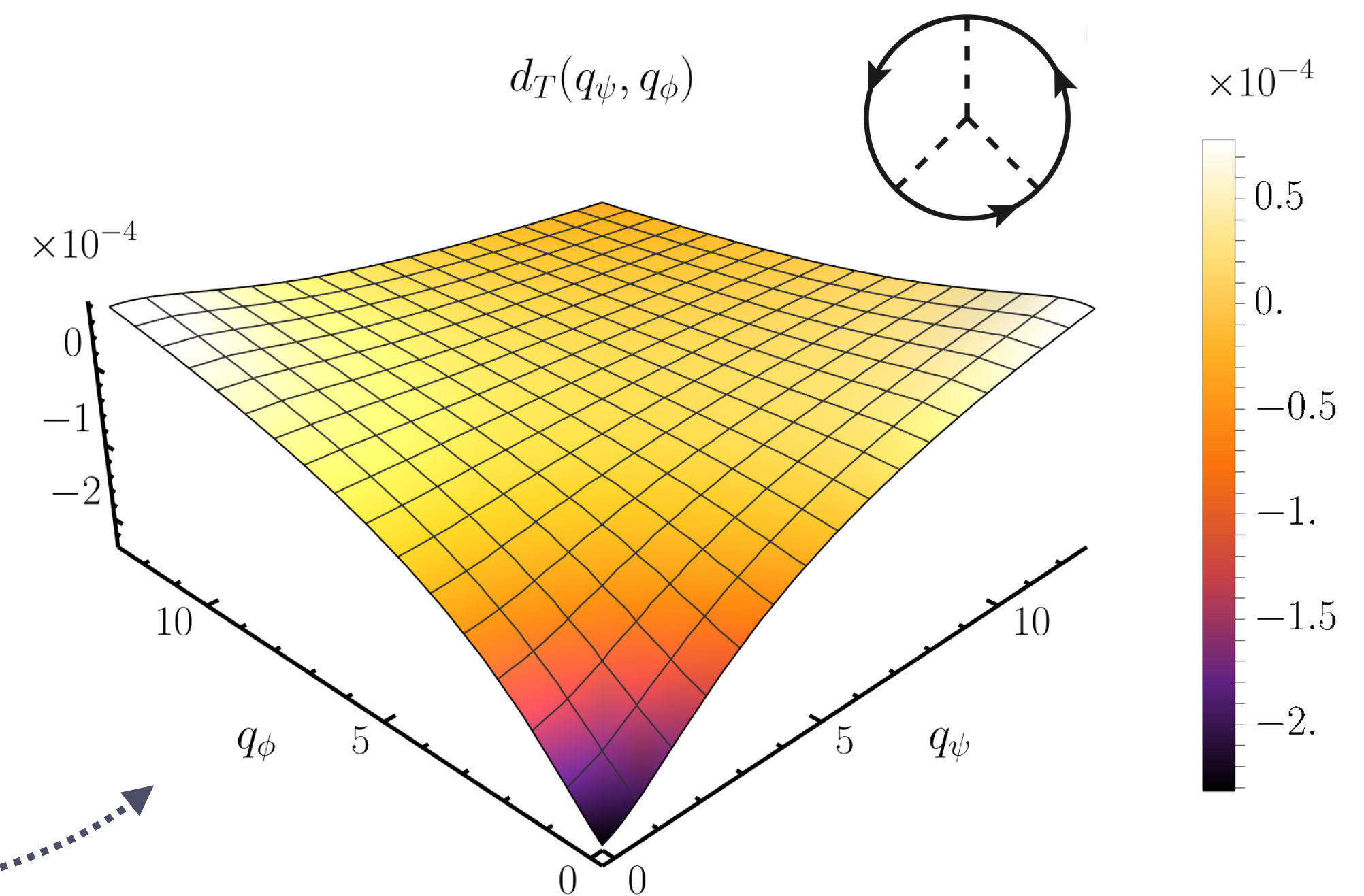
Already successfully used in many applications including $T > 0$:

- Thermal effective potential without high- T expansion:

Navarrete, Paatelainen, KS, Tenkanen
JHEP 01 (2026)

$$V_{\text{eff}}(\varphi, T) = V_{\text{tree}}(\varphi) - \frac{T}{\mathcal{V}} \ln \int \mathcal{D}\Phi e^{-\int_0^\beta d\tau \int_{\mathcal{V}} \mathcal{L}_E(\Phi)}$$

$\int \int \int$ with two masses



Thermal LTD

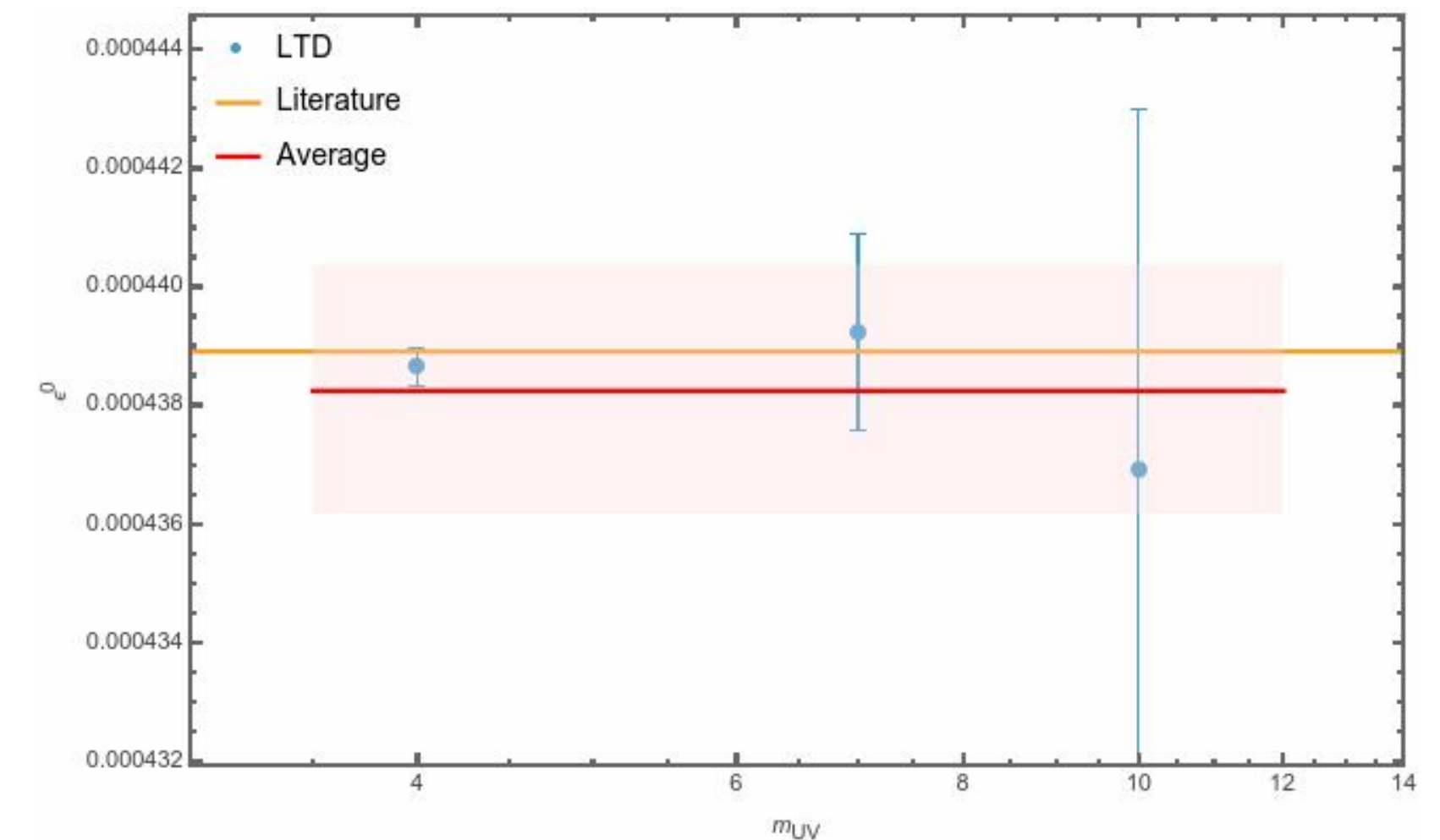
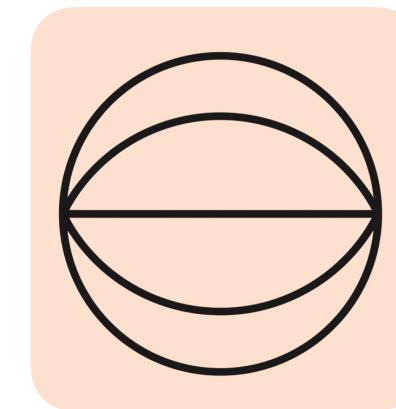
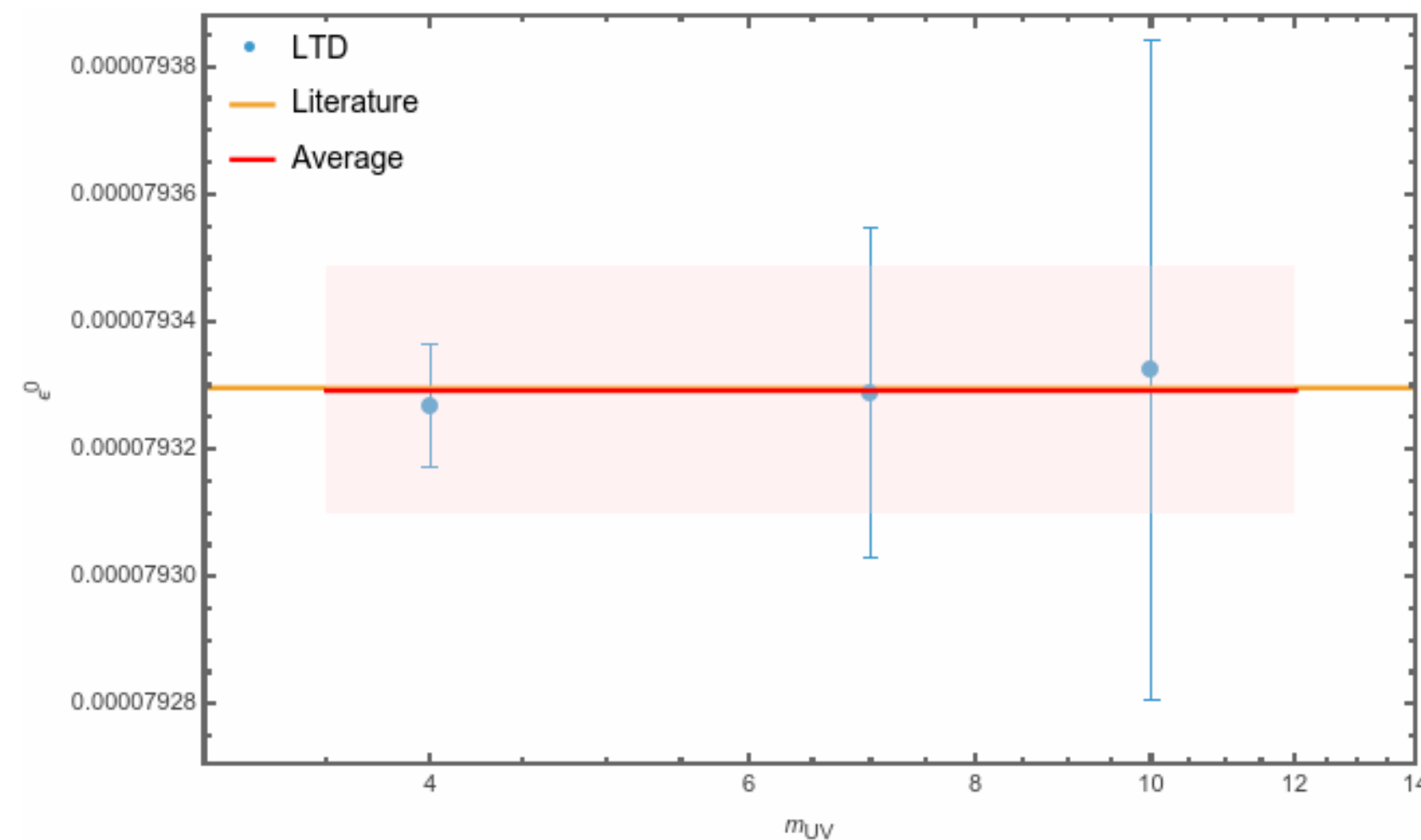
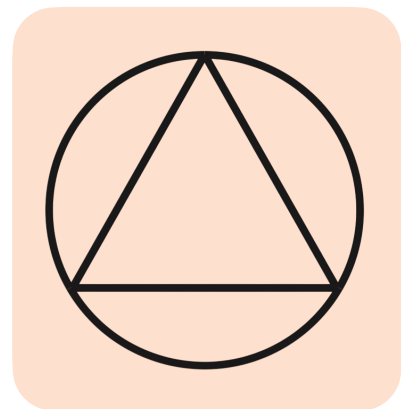
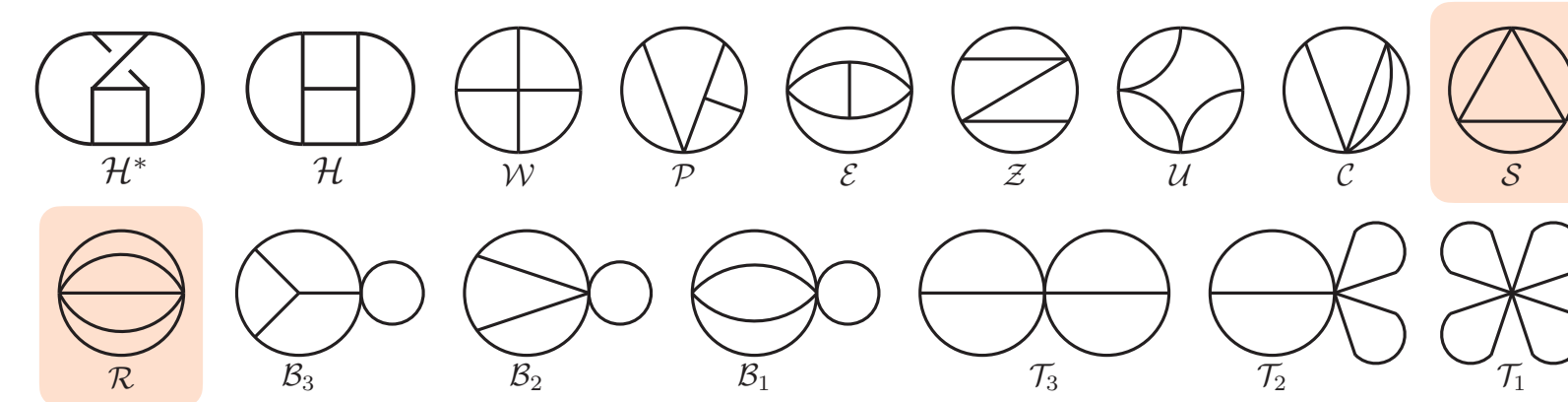
Proofs of principle

Poster by **Leon Sandbote**

Already successfully used in many applications including $T > 0$:

- Four-loop hot Yang-Mills pressure:

In preparation (2026/2027)



Thermal LTD

Future directions

Thermal LTD

Thermal LTD

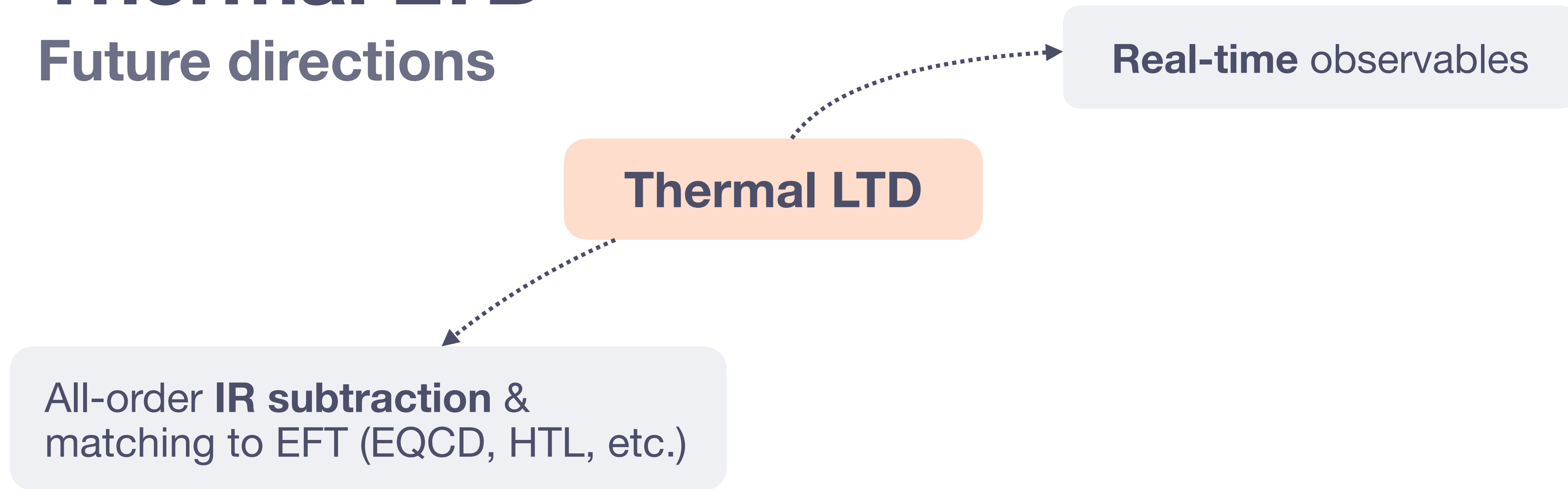
Future directions

Thermal LTD

All-order **IR subtraction** &
matching to EFT (EQCD, HTL, etc.)

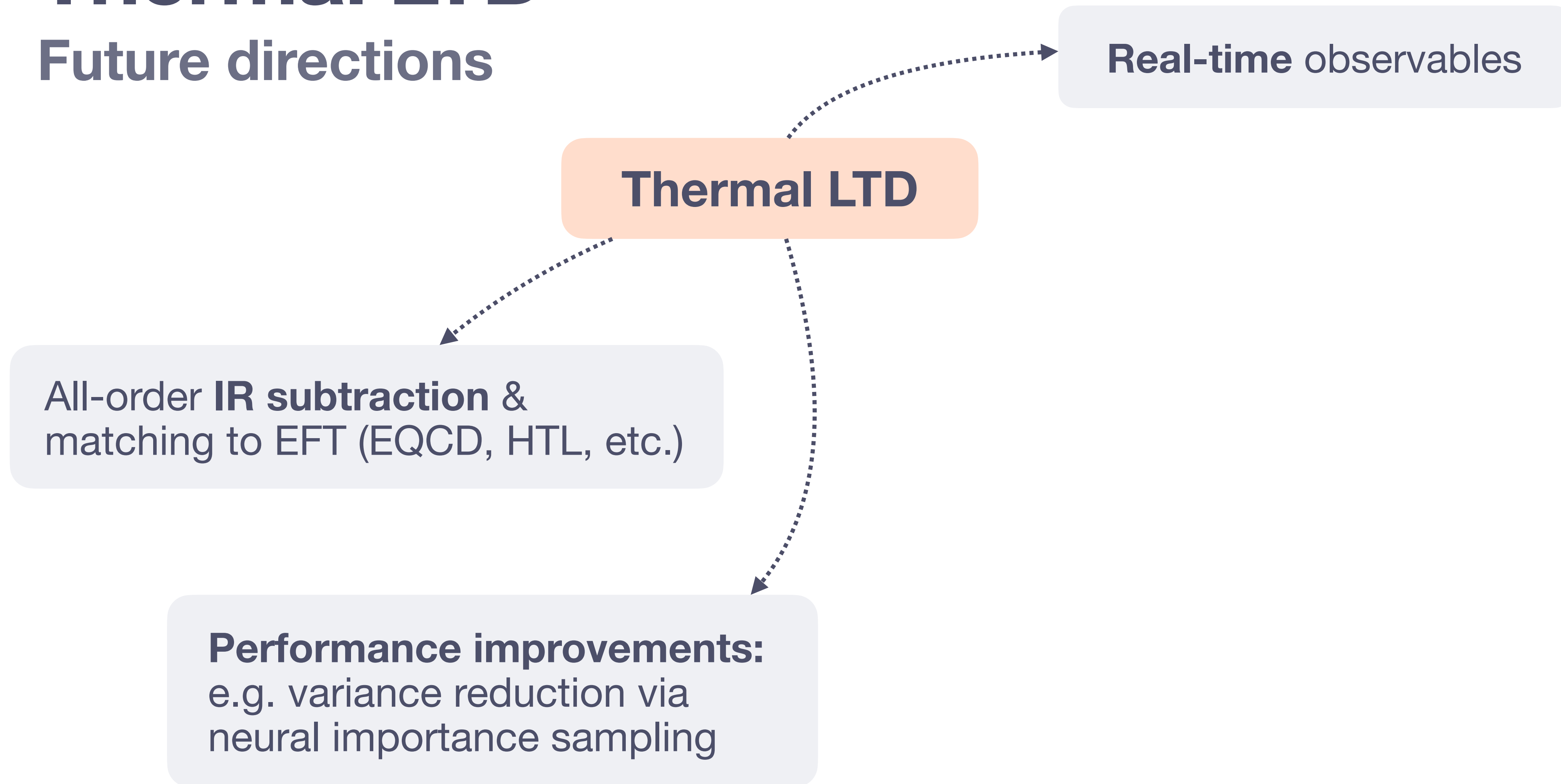
Thermal LTD

Future directions



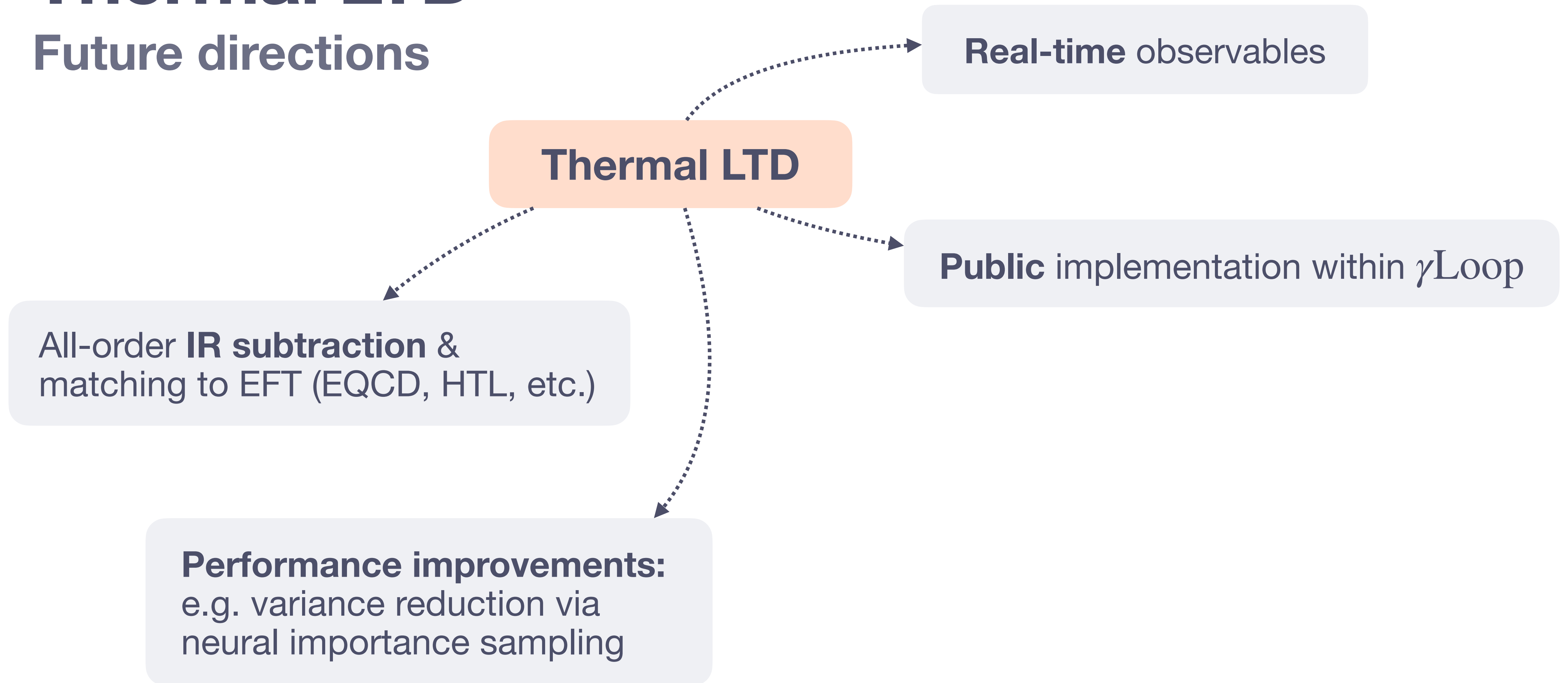
Thermal LTD

Future directions



Thermal LTD

Future directions



Thermal LTD

Future directions

Thermal LTD

```
graph TD; A[Thermal LTD] -.-> B[Real-time observables]; A -.-> C[Public implementation within \gamma Loop]; A -.-> D[Performance improvements: e.g. variance reduction via neural importance sampling]; A -.-> E[All-order IR subtraction & matching to EFT (EQCD, HTL, etc.)];
```

Real-time observables

Public implementation within γ Loop

All-order **IR subtraction** & matching to EFT (EQCD, HTL, etc.)

Performance improvements:
e.g. variance reduction via neural importance sampling

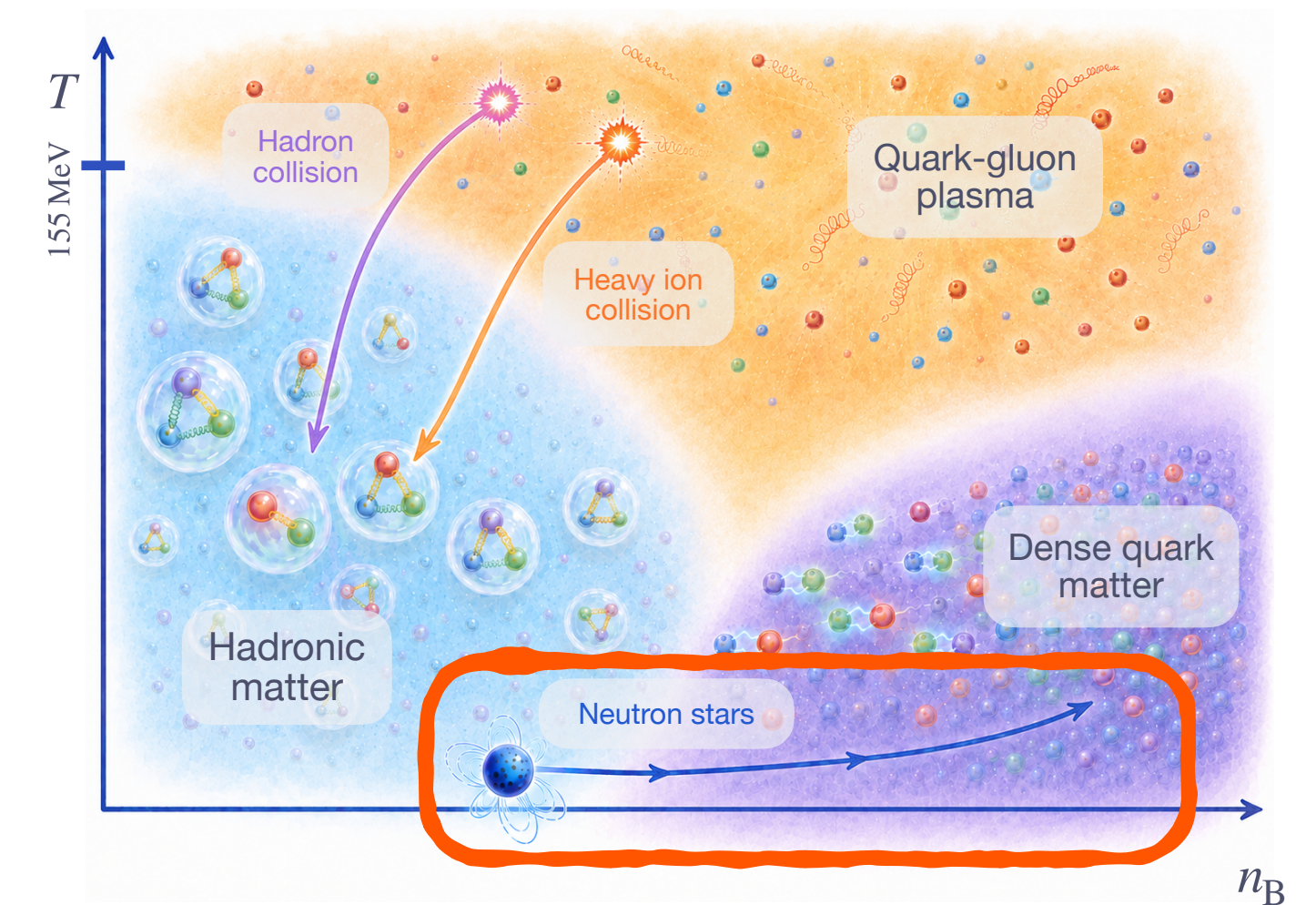
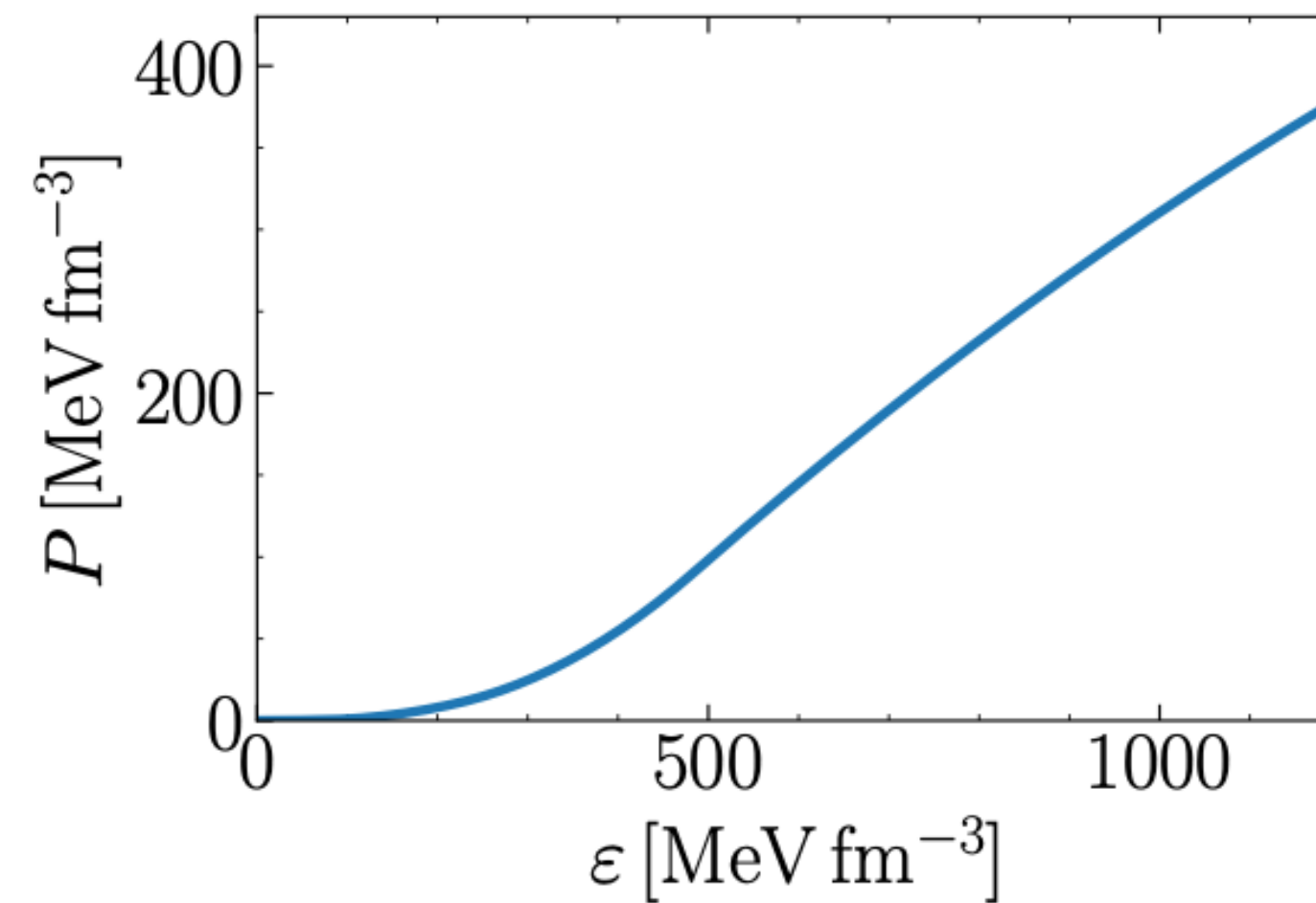
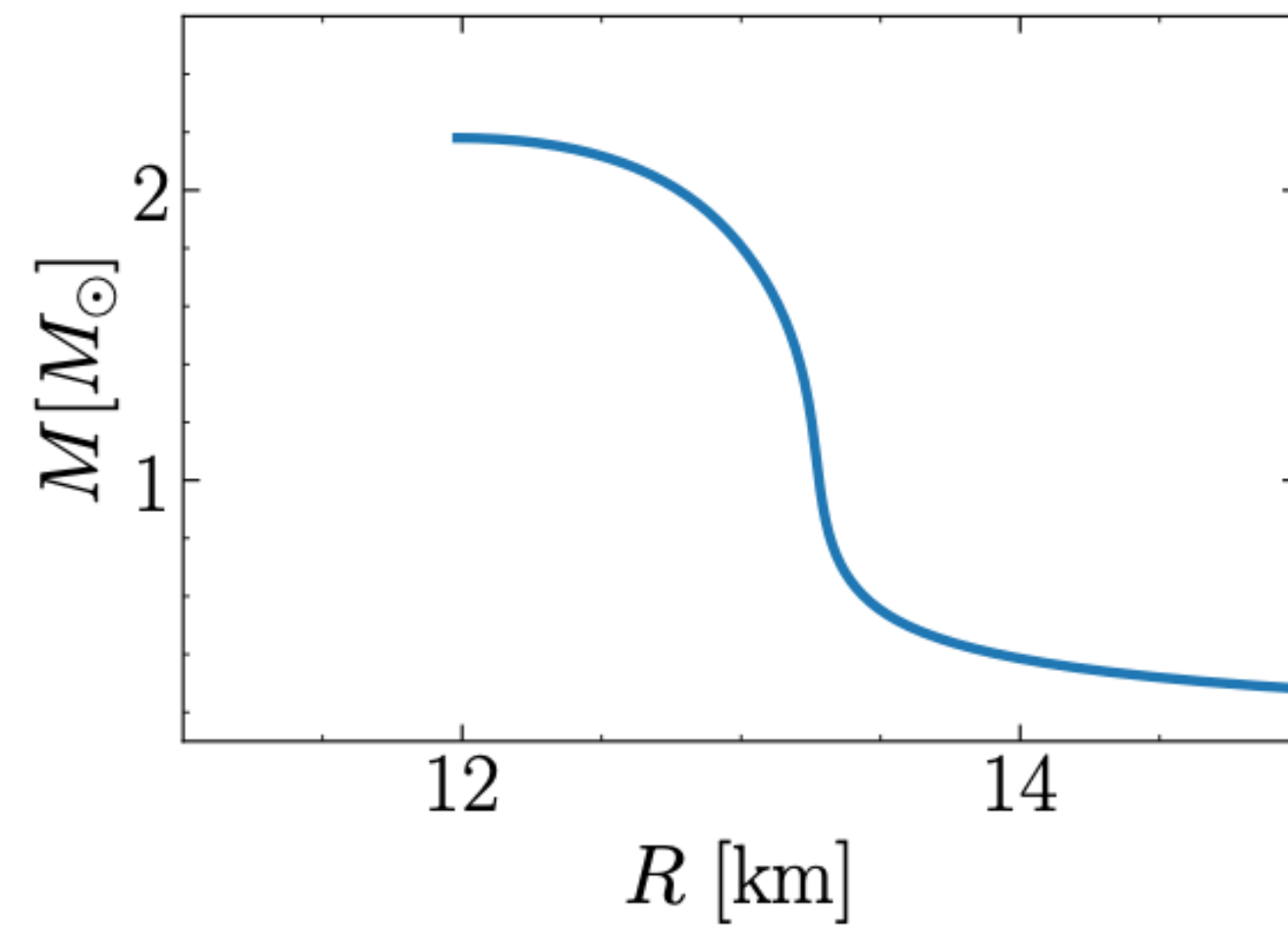
Applications:

- QCD pressure at finite T , μ , and m_s
- Cosmological phase transitions
- Production rates
- Etc.

Summary

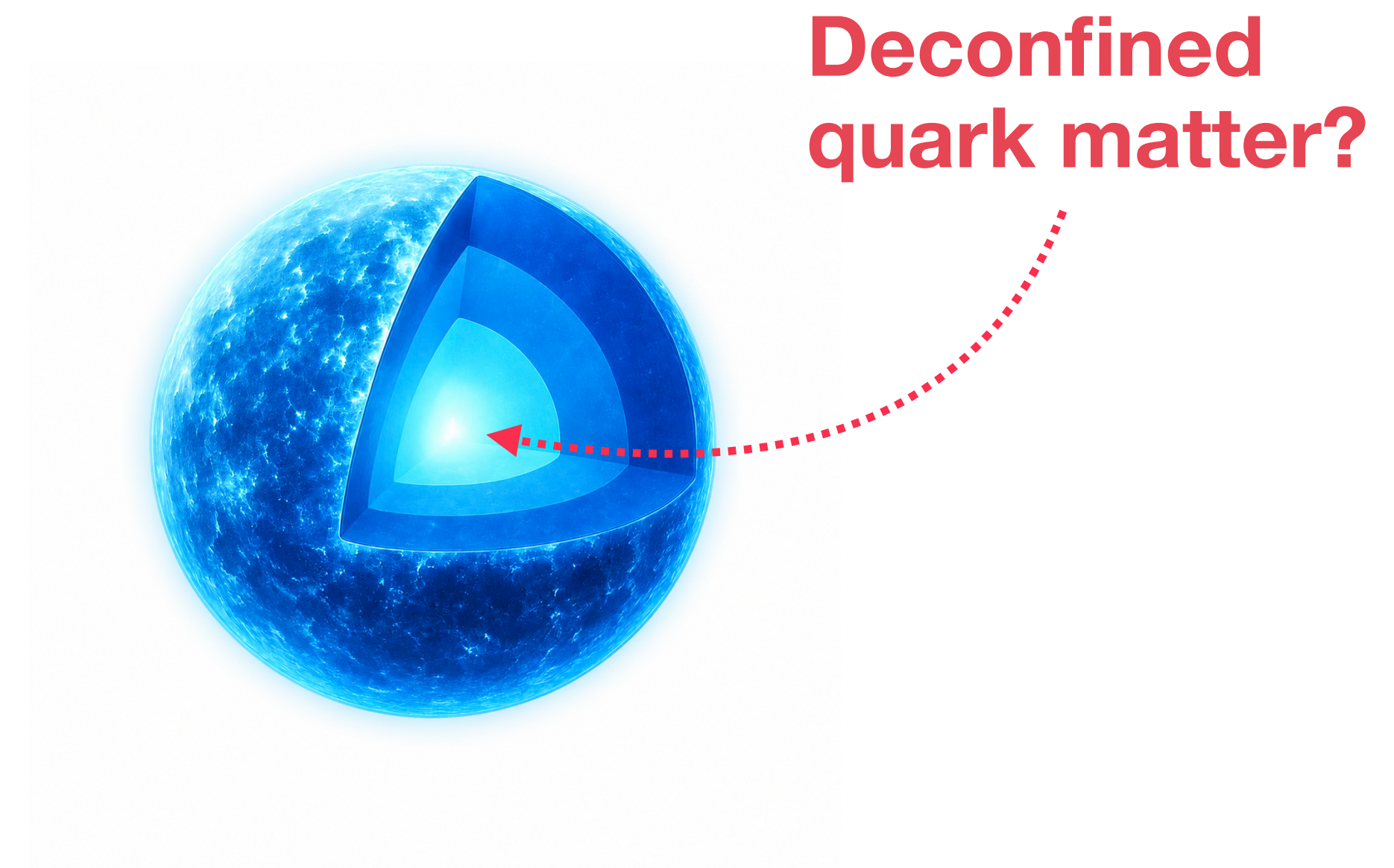
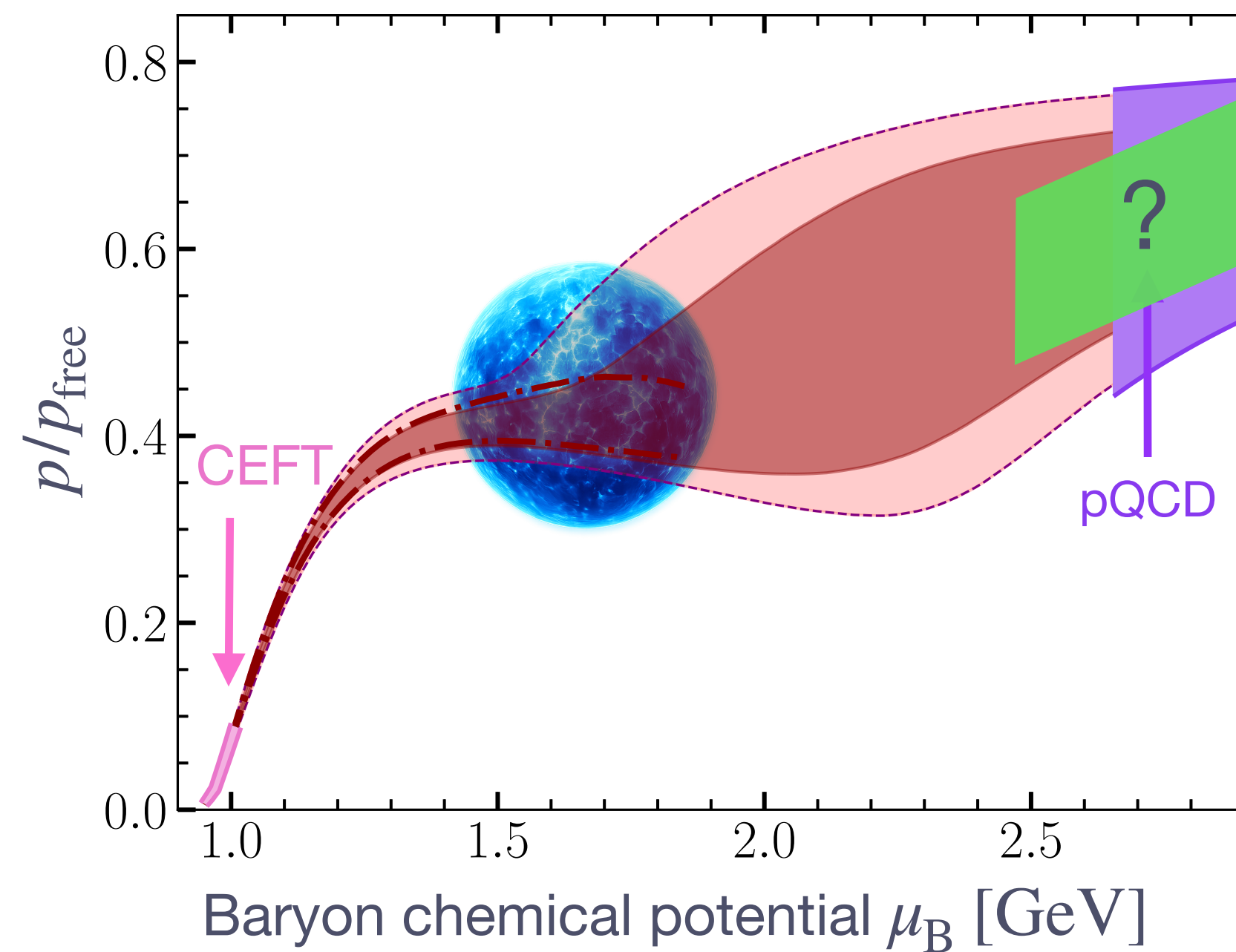
Summary

- EoS of strongly-interacting dense matter connects microscopic QCD, neutron star observables, and the QCD phase diagram



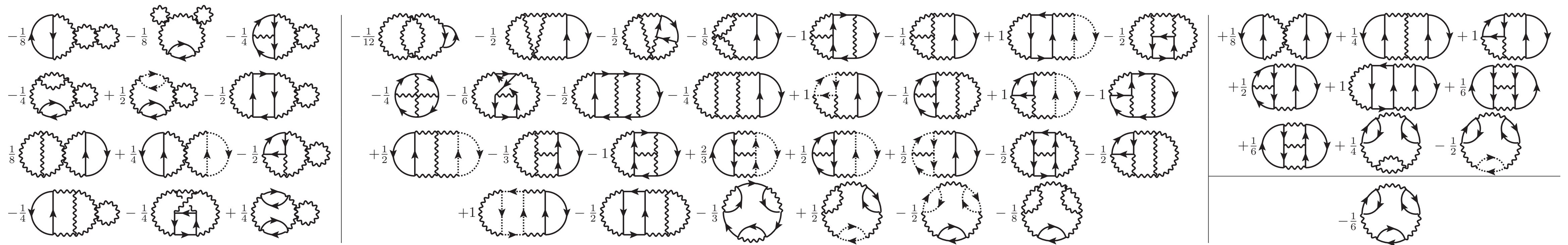
Summary

- pQCD provides an important high-density constraint
⇒ strong motivation to extend the calculation to full **N3LO**



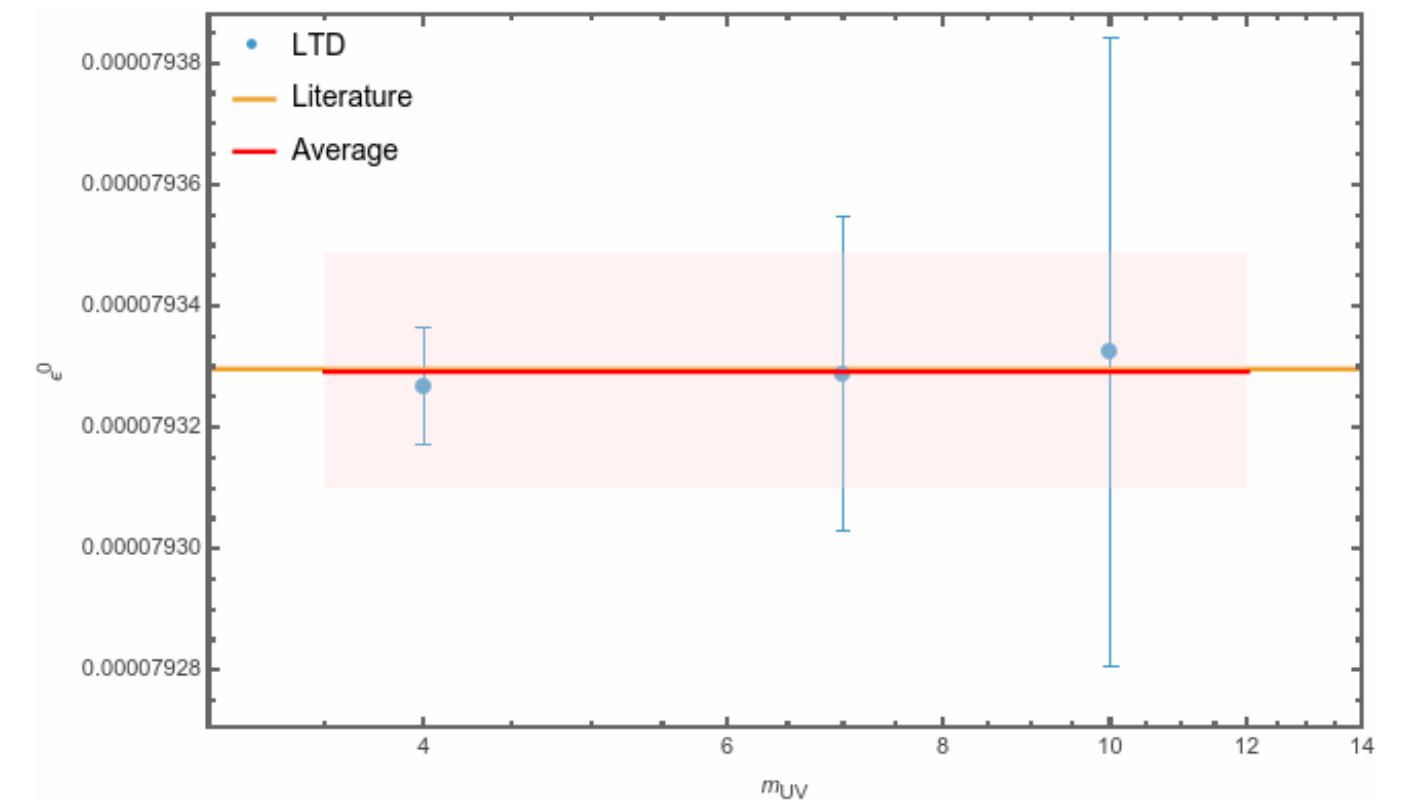
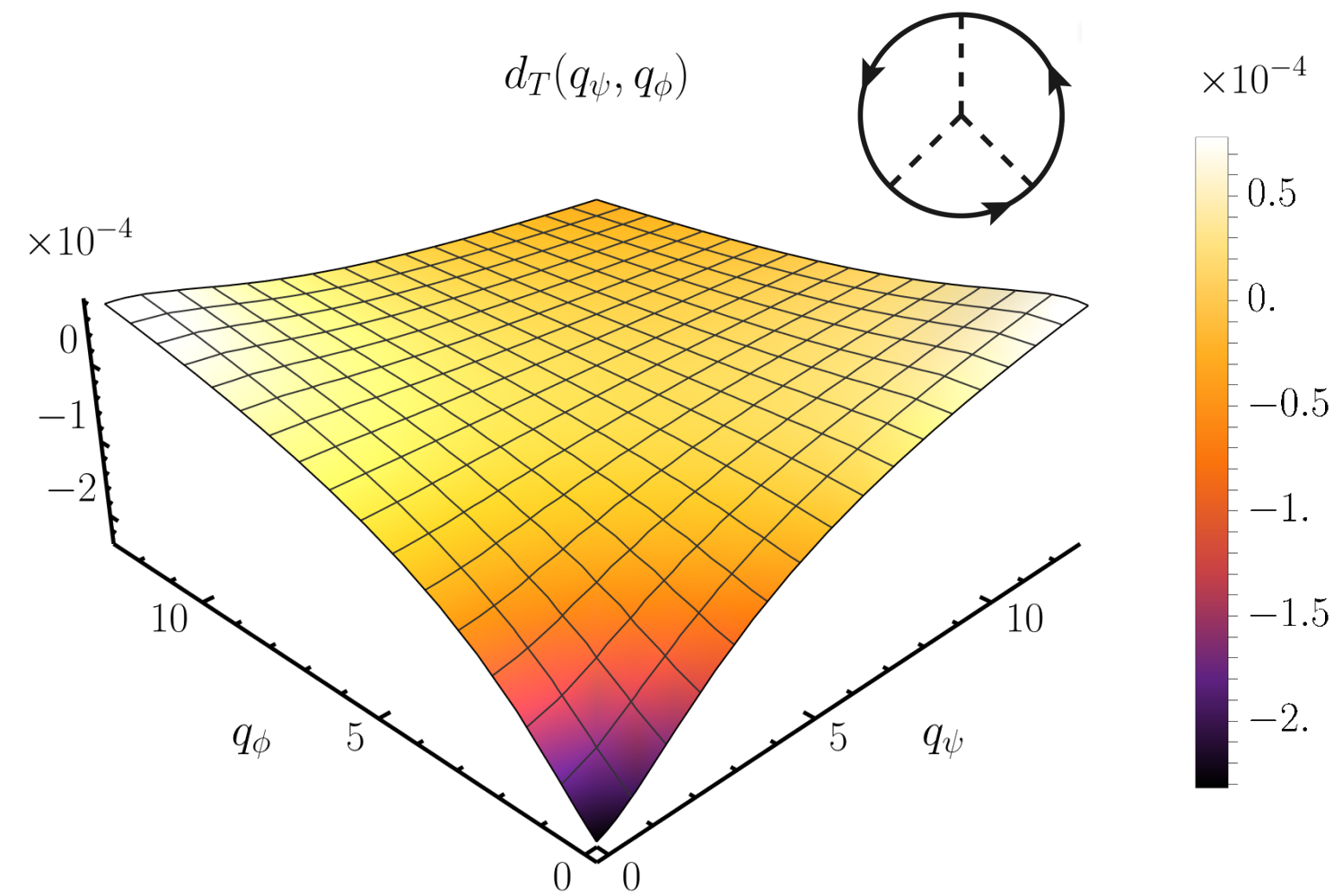
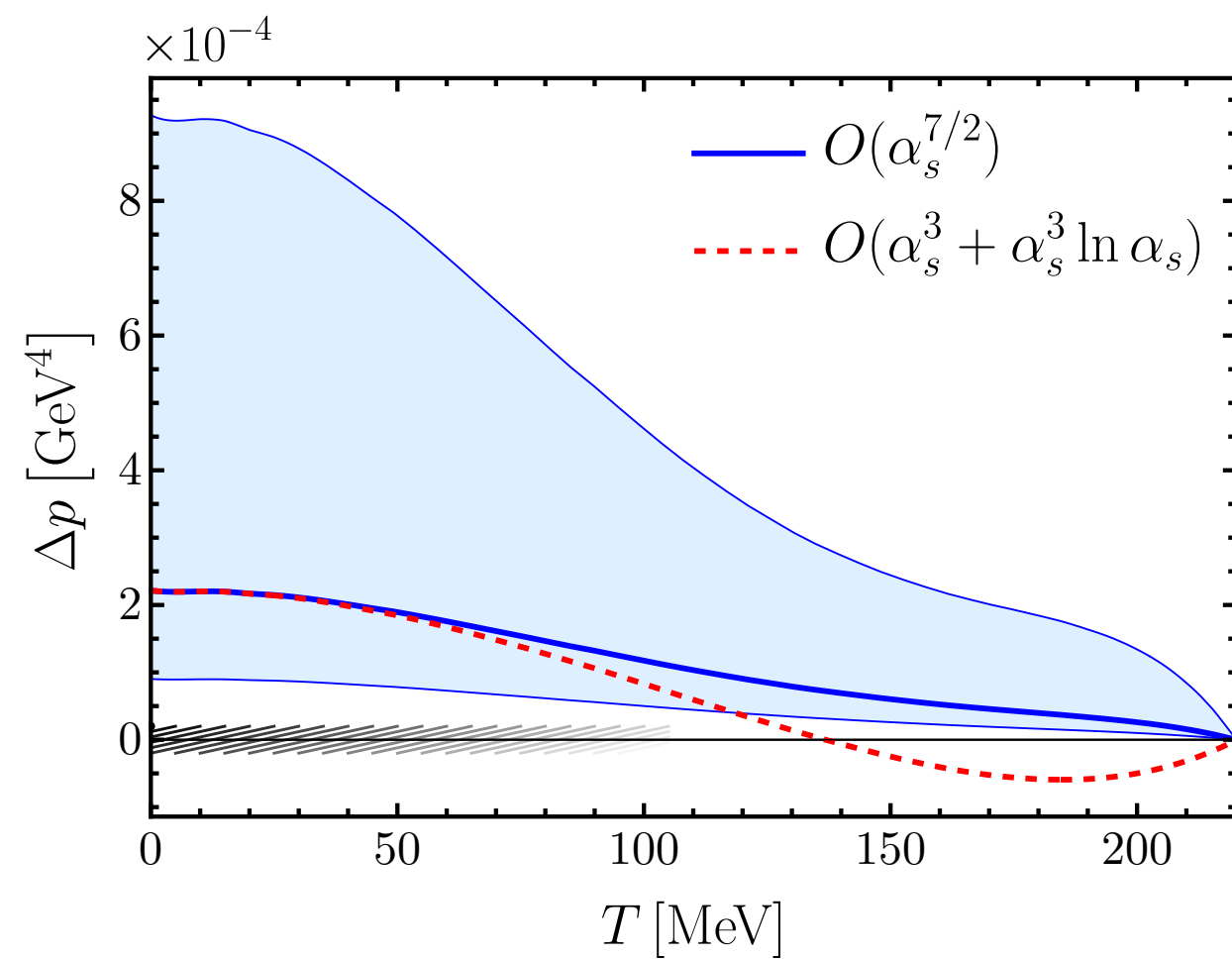
Summary

- After ~10 years of work, the full N3LO EoS of cold quark matter has been completed by evaluating the hard contribution c_0

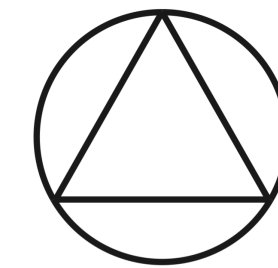


Summary

- New precision era in thermal perturbation theory via dense/thermal LTD: direct numerical integration in momentum space



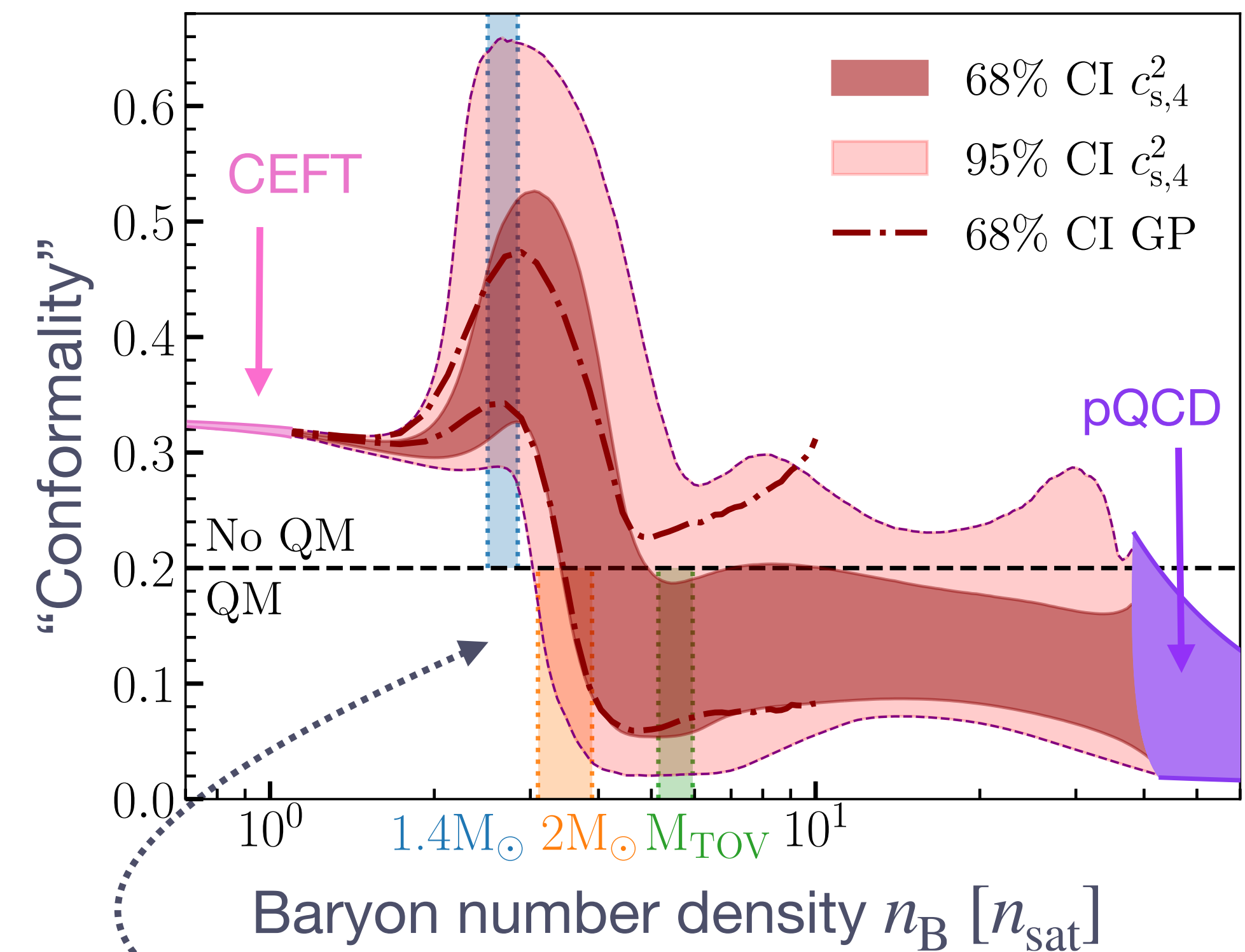
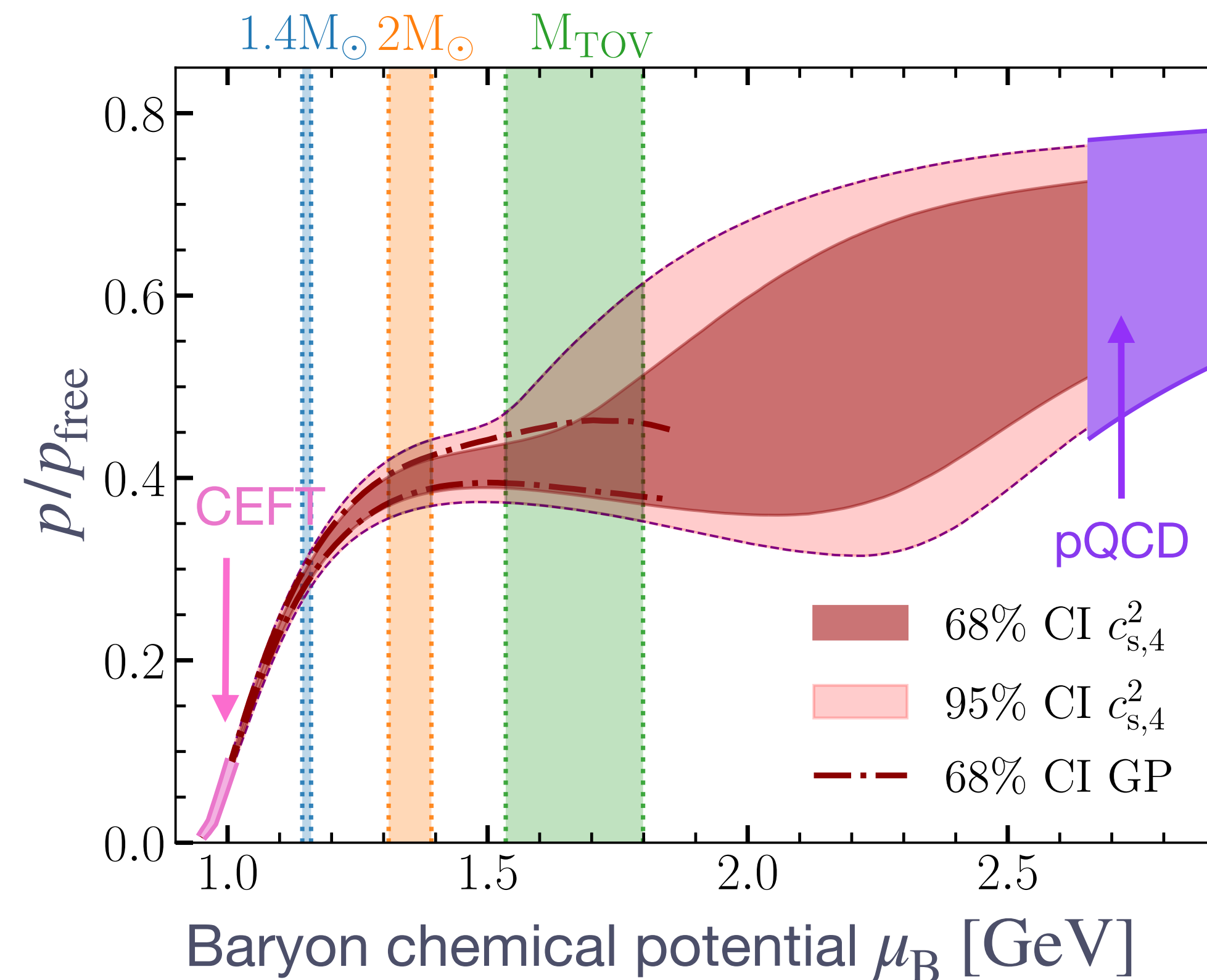
$$p_{\text{PQ}} - p_{\text{QCD}} = \operatorname{Im}[\text{Diagram}] + O(\alpha_s^{7/2})$$



Backup

What have we learned from the EoS inference?

Strong evidence for deconfined quark matter in cores of most massive neutron stars



Annala, Gorda, Hirvonen, Komoltsev, Kurkela, Nättilä, Vuorinen Nat. Comm. 14 (2023)

Conformalization below maximal densities