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Advanced Quantum Field Theory

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1 Lecture 1: Theory space

The aim of this lecture series is deceptively simple: it is to think about the question, "How do we build theories?". As the whimsical but nevertheless wise Winnie-the-Pooh might say, before one can come to a theory, one might first like to know what a theory is and whether there is any reason to fear such a thing, supposing, as a matter of fact, a theory were ever to jump out from behind a gorse bush and say, "How do you do?". Rest assured, the theories that we will consider in these lectures are all of upright character, but defining what a theory is is not a bad place at all for us to start, so let's start there.

Often, in the spirit of logical positivists like Ernst Mach, we talk of wanting to explain empirical observations. For those of us battling our way through BUSSTEPP, the observed phenomena are likely to be the properties and interactions of elementary particles. But, actually, we might want to do more than explain *observed* phenomena; we might want to predict *new* phenomena for which we (or at least those of us with a flair for experiment or observation) can go off and look. The history of particle physics has seen theories play both these roles.

So, for us, a theory will be a mathematical description of the dynamics of elementary particles that takes some inputs, which we might call model parameters, applies some rules, which we will give some fancy title like *relativistic quantum mechanics* or *quantum field theory*, and makes some prediction. So we have three things to do: (1) set down the mathematical language and rules, (2) decide on the input parameters (i.e., build the model) and (3) derive the relationships between those input parameters and physical observables. We will need to accept that (3) is not independent of (1).

Now, there was a time, not so long ago, when the prevailing attitude was that the only good theories were ones that had a finite number of input parameters and a much larger number of physical observables. This is a win for predictivity, since we can eliminate the input parameters, stick with Mach, and concentrate only on the relationships between physical observables. Theories like this would be highly coveted within the reductionist philosophy that has shaped much of modern physics, and they would be given the less-than-transparent label of being *perturbatively renormalizable*. (Don't worry, we will come back to what this means later.) But times inevitably change — sometimes for the better and sometimes for the worst — and it turns out that there are many physically relevant theories that do not find themselves lucky enough to be part of the perturbatively renormalizable clique. These theories are called *effective*, and they are in good company, alongside General Relativity (still our best description of classical gravitation), chiral perturbation theory (describing the dynamics of hadrons), the Ginzburg–Landau theory of superconductivity (We have much to learn from our condensed-matter colleagues.) and, indeed, the much-celebrated Standard Model of particle physics if we consider it the low-energy limit of some larger theory (that might, for example, offer an explanation of the origins of dark matter or dark energy). The latter is captured in the Standard Model Effective Field Theory that you will

hear about a lot (but not so much in these lectures).

Let's turn then to our little excursion into *advanced quantum field theory*. (In fact, I think all quantum field theory is pretty advanced. And given that we only have four short lectures, we had better rename these lectures *Some Quantum Field Theory*, or better still *Some of the Quantum Field Theory that Pete knows a little about*.[1] Notwithstanding this, it is often the basic foundations that compose the most "advanced" topics.) So what is it that we want our theories to describe? Well, let's restrict ourselves to writing down theories that describe the microscopic origin of the known fundamental forces. We'll leave out gravity — that would need another lecture series — and we'll stick to toy models, too, and just scalar fields, since higher spins are a technical complication to which we do not need to subject ourselves. This will still be more than enough for us to gain some understanding of what we mean by *theory space*.

Now to complete step (1): set down the mathematical language and rules. However, when we said earlier that we would be writing down theories of the dynamics of the elementary particles, we weren't being entirely honest. In fact, we will write down theories of fields — whatever they are — and we will try to interpret the excitations of these fields as elementary particles. It will turn out that defining particles is not the easiest of tasks. We'll worry about this a little more in Exercise 1.

There are a number of ways that we can approach quantum field theory:

- the canonical operator formulation.
- the path integral formulation.
- the operator product expansion.

We will make use of the first two, and leave the third to people who know about conformal field theory.

So, how do we write down a quantum field theory? We would be forgiven for thinking that we just need to write down an action, say

$$S[\Phi] = \int \mathrm{d}^4x \left[\eta^{\mu\nu} Z_{ab} \partial_\mu \Phi_a \partial_\nu \Phi_b - V^{(1)}_a \Phi_a - V^{(2)}_{ab} \Phi_a \Phi_b - V^{(3)}_{abc} \Phi_a \Phi_b \Phi_c - V^{(4)}_{abcd} \Phi_a \Phi_b \Phi_c \Phi_d + \cdots \right] . \tag{1}$$

Here, the square brackets indicate that the action is a *functional* of the field Φ, i.e., it maps functions to a number, $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric, and we consider N real scalar fields Φ_a with $a \in \{1, 2, \ldots, N\}$. We have certainly written down a field theory, but we have not yet written down a *quantum* field theory. The latter is composed of two things:

[1] In fact, a couple of days into preparing these first two lectures, I discovered (much to my annoyance) that I had written almost exactly the same notes several years ago, when I gave a set of lectures on the effective action here in Nottingham as a postdoc, albeit with more of a focus on finite-temperature and non-equilibrium field theory.

1. Some generators of the dynamics. (In Minkowski spacetime, these are the generators of the Poincaré group and, in particular, the Hamiltonian, the generator of time translation, for which the action will serve as a proxy.)

2. Some state space around which the generators of dynamics move us. (This will be something like a Hilbert space, but there are some subtleties related to how we deal with continuous spectra, for example, that we won't go into in these lectures.)

We often forget that there is freedom in how we choose *both* of these ingredients of our quantum field theory. However we choose these ingredients, the challenge is to find useful bases to span this state space.

One choice is the basis of *field eigenstates*, which satisfy the eigenequation

$$\hat{\Phi}(t,\mathbf{x})\,|\Phi(\mathbf{x}),t\rangle = \Phi(\mathbf{x})\,|\Phi(\mathbf{x}),t\rangle \ . \tag{2}$$

The state $|\Phi(\mathbf{x}),t\rangle$ is an instantaneous eigenstate of the Heisenberg-picture field operator $\hat{\Phi}(t,\mathbf{x})$ at time t with "eigenvalue" $\Phi(\mathbf{x})$. These states are orthonormal and complete in a functional sense with

$$\int [\mathrm{d}\Phi(\mathbf{x})]\,|\Phi(\mathbf{x}),t\rangle\,\langle\Phi(\mathbf{x}),t| = \mathbb{I}\ . \tag{3}$$

This basis is useful for constructing the path integral, which we will come to shortly and derive explicitly in the field eigenbasis in Exercise 2. Note that the field eigenstate is not really a function of the spatial coordinate $\mathbf{x}$. We include the argument to remind us that it contains information about the configuration of the system at all points in space.

Another choice is the Fock basis, composed of multiparticle states and generated from the right-action of creation operators on the Fock vacuum state $|0(t)\rangle$. (The definition of this vacuum state is somewhat problematic, as we will see in Exercise 1.) For a single real scalar field, we have

$$\hat{a}^\dagger(\mathbf{p},t)\,|0(t)\rangle = |\mathbf{p}(t)\rangle \qquad \text{with} \qquad \hat{a}(\mathbf{p},t)\,|0(t)\rangle = 0\ . \tag{4}$$

The single-particle momentum states are orthonormal,

$$\langle\mathbf{p}_1(t)|\mathbf{p}_2(t)\rangle = (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)\ , \tag{5}$$

where $E_j(\mathbf{p}) = \sqrt{\mathbf{p}^2+m_j^2}$, and the completeness of the Fock basis takes the unwieldy form

$$\begin{aligned}|0(t)\rangle\,\langle 0(t)| &+ \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2E(\mathbf{p}_1)}\ |\mathbf{p}_1(t)\rangle\,\langle\mathbf{p}_1(t)| \\ &+ \frac{1}{2!}\int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2E(\mathbf{p}_1)}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2E(\mathbf{p}_2}\ |\mathbf{p}_1(t),\mathbf{p}_2(t)\rangle\,\langle\mathbf{p}_1(t),\mathbf{p}_2(t)| + \cdots = \mathbb{I}\ .\end{aligned} \tag{6}$$

Here, we have assumed that the states are orthonormal with respect to the Dirac inner product, i.e., the bra state is just the Hermitian conjugate of the ket state. More generally, we can introduce a metric on the state space, which will play a role similar to a weight function in the orthonormality of the basis. We might be tempted (and, in fact, we should be) to ask why we tend to model build only at the level of the classical action and not also in terms of the way we build the state space.

Either way, we now have our first way of charting our theory space: we can use the parameters of the theory as coordinates, or more precisely the coefficients of the various "operators" appearing in the action. Step (2), building the model, then amounts to deciding which of these operators to include, and how we might be guided to choose them.

However, we are also interested in how the system evolves, and one way to find this out is to kick it and see what happens. The slightly more scientific approach to kicking a quantum field theory is to introduce external sources J_a that perturb the system away from its ground state and measure changes in correlations in the system. A convenient way to generate expressions for these correlation functions is to introduce the particular path integral

$$Z[J] = \mathcal{N} \int \mathcal{D}\Phi \ \exp\left[\frac{i}{\hbar}\left(S[\Phi] - \int \mathrm{d}^4x \ J_a(x)\Phi_a(x)\right)\right] \ , \tag{7}$$

and take functional derivatives with respect to the sources. But let's not go too fast. The functional integral over field configurations Φ is not convergent, because the integrand is oscillatory. We can tame this by taking $V^{(2)}_{ab} \to V^{(2)}_{ab} - \delta_{ab} i\epsilon/2$, where $\epsilon = 0^+$. This is the Feynman prescription, and it will have two effects: first, it will allow us to invert the Klein–Gordon operator

$$\frac{\delta^2 S[\Phi]}{\delta\Phi_{a_1}(x_1)\delta\Phi_{a_2}(x_2)} \tag{8}$$

to obtain the tree-level two-point correlation function; and second, it will give us time ordering of the resulting correlation functions. Moreover, this prescription allows us to construct the Wick contour to rotate to Euclidean space by analytically continuing $t \to -i\tau$. We can then identify the classical Euclidean action with the classical free energy, and it is for this reason that we often refer to $Z[J]$ as the *partition function*.

With this prescription, $Z[J]$ acts as a generating functional of time-ordered correlation functions. More specifically, we can obtain the n-point time-ordered correlation function via

$$\langle \mathrm{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)\rangle_J = \frac{1}{Z[J]}\left(\prod_{j=1}^{n} i\hbar\frac{\delta}{\delta J_{a_j}(x_j)}\right) Z[J] \ , \tag{9}$$

where the operator T ensures that the fields are ordered with the latest time to the left. However, the correlation functions obtained from $Z[J]$ will also contain disconnected pieces. To generate

only connected correlation functions, we can vary the *Schwinger functional*

$$W[J] = i\hbar \ln Z[J] \ , \tag{10}$$

whose Euclidean counterpart is related to the free energy. For example, the connected two-point function is

$$\begin{aligned} \hbar\Delta_{a_1a_2}(x_1,x_2) &= i\hbar\frac{\delta^2 W[J]}{\delta J_{a_1}(x_1)\delta J_{a_2}(x_2)} = \frac{(i\hbar)^2}{Z[J]}\frac{\delta^2 Z[J]}{\delta J_{a_1}(x_1)\delta J_{a_2}(x_2)} - \frac{(i\hbar)^2}{Z^2[J]}\frac{\delta Z[J]}{\delta J_{a_1}(x_1)}\frac{\delta Z[J]}{\delta J_{a_2}(x_2)} \\ &= \langle \mathrm{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)]\rangle_J - \langle\Phi_{a_1}(x_1)\rangle_J \langle\Phi_{a_2}(x_2)\rangle_J \ . \end{aligned} \tag{11}$$

Of course, we can add any sources that we want, and it will prove useful to introduce a source for the two-point correlation function. To this end, we generalise the partition function to

$$Z[J,K] = \int \mathcal{D}\Phi \ \exp\left[\frac{i}{\hbar}\left(S[\Phi] - \int \mathrm{d}^4x \ J_a(x)\Phi_a(x) - \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y \ K^{ab}(x,y)\Phi_a(x)\Phi_b(y)\right)\right] \ , \tag{12}$$

with the Schwinger functional becoming

$$W[J,K] = i\hbar \ln Z[J,K] \ . \tag{13}$$

(We could keep going and add a three-point source and so on. Let's not.) We can now obtain the two-point function from a single variation with respect to the two-point source $K(x,y)$; namely,

$$\langle \mathrm{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)]\rangle_{J,K} = 2\frac{\delta}{\delta K_{a_1a_2}(x_1,x_2)}W[J,K] = \frac{2i\hbar}{Z[J,K]}\frac{\delta}{\delta K_{a_1a_2}(x_1,x_2)}Z[J,K] \ . \tag{14}$$

However we choose to obtain the correlation functions, we now have the second way of charting our theory space: we can use the correlation functions of the theory as coordinates. Moreover, in having written down the generating functionals in the path integral representation, we are now well on our way to completing step (3).

Taking stock, we have two ways to chart our theory space:

1. We can use the parameters of our theory as coordinates and, more precisely, the coefficients of the various operators appearing in the action.

2. We can use the correlation functions of our theory, and we have two useful objects: the partition function and the Schwinger functional for generating them.

In the next lecture, we will introduce the quantum effective action, which will prove to be invaluable for understanding how to move (or rather how quantum mechanics moves us) around in theory space.

2 Lecture 2: The quantum effective action

In the last lecture, we introduced the partition function

$$Z[J,K] = \int \mathcal{D}\Phi \, \exp\left[\frac{i}{\hbar}\left(S[\Phi] - \int \mathrm{d}^4x \; J_a(x)\Phi_a(x) - \frac{1}{2}\int \mathrm{d}^4x \, \mathrm{d}^4y \; K^{ab}(x,y)\Phi_a(x)\Phi_b(y)\right)\right] \; , \tag{15}$$

which we could use to generate n-point correlation functions. However, it would be nice if, instead of having something that was a function of the sources, we had an object that was a function of the correlation functions themselves. It would be even better if this thing were akin to the classical action, whose variation gave us the equations of motion governing those correlation functions. We're in luck: this object exists, or rather a family of such objects exist, and we call them *quantum effective actions* or just *effective actions*.

To construct them, we first need to take a little diversion into differential geometry. Consider a point x on some manifold M. We can define a tangent space at the point x. As an example, suppose the manifold M is a hemisphere. The tangent space at a point x on the hemisphere is just the plane tangential to the surface at that point. Now, we could build the hemisphere by giving the set of points x or from the envelope of the set of all planes tangential to the surface.

We can do the same thing with convex or concave functions: We can build a function $f(x)$ from the coordinates $(x, f(x))$ or we can build it from the set of tangents to the function, specified in terms of their gradients and intercepts. The mapping between these two sets of coordinates is a *Legendre transform*.

Given that the tangents to the Schwinger functional are the correlation functions, this is precisely what we want to do. Thus, we define the quantum effective action

$$\Gamma[\phi,\Delta] = \max_{J,K}\left[-W[J,K] + J_a(x)\phi_a(x) + \frac{1}{2}K_{ab}(x,y)\big(\phi_a(x)\phi_b(y) + \hbar\Delta_{ab}(x,y)\big)\right] \; . \tag{16}$$

In fact, this is the so-called *two-particle-irreducible* or *2PI effective action* — due to Cornwall, Jackiw and Tomboulis [3] — which will prove particularly helpful in understanding the family of effective actions. Note that, having got rather tired of writing so many integrals, we now use a DeWitt-like notation in which repeated arguments are integrated over, e.g.,

$$J_a(x)\phi_a(x) \equiv \sum_a \int \mathrm{d}^4x \; J_a(x)\phi_a(x) \; . \tag{17}$$

The extremisation gives

$$\phi_a(x) = \frac{\delta W[J,K]}{\delta J_a(x)} \ , \tag{18}$$

$$\hbar\Delta_{ab}(x,y) = 2\frac{\delta W[J,K]}{\delta K_{ab}(x,y)} - \phi_a(x)\phi_b(y) \ , \tag{19}$$

and we identify ϕ and Δ with connected one- and two-point functions.

We have, however, committed a slight abuse of notation here to keep things simple. After the extremisation, we have particular sources J and K, which are those that maximise the functional in square brackets in Eq. (16). Thus, the extremisation defines $J = J[\phi,\Delta]$ and $K = K[\phi,\Delta]$ to be functionals of ϕ and Δ, in much the same way as the Legendre transform between Lagrangian and Hamiltonian mechanics makes the momentum p a function of the time derivative of the generalised coordinate q, i.e., $p = p(\dot{q})$.

With this understood, we can drop the extremisation and write the quantum effective action

$$\Gamma[\phi,\Delta] = -W[J,K] + J_a(x)\phi_a(x) + \frac{1}{2}K_{ab}(x,y)\big(\phi_a(x)\phi_b(y) + \hbar\Delta_{ab}(x,y)\big) \ . \tag{20}$$

The gradients of the effective action are

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\phi_a(x)} = J_a(x) + K_{ab}(x,y)\phi_b(y) \ , \tag{21}$$

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta_{ab}(x,y)} = \frac{1}{2}K_{ab}(x,y) \ . \tag{22}$$

Given the presence of the sources on the right-hand side, we infer that these are nothing other than the equations of motion for the one- and two-point functions. They are, however, not the classical equations of motion; they are the quantum-corrected equations of motion, wherein quantum fluctuations have been integrated out to give effective c-number differential equations for the correlation functions. And this brings us to an important facet of the question, "How do we build theories?": *integrating stuff out*.

We are probably all familiar with the maxim, "Sociology is Biology, Biology is Chemistry, Chemistry is Physics, Physics is Maths, and no one knows what Maths is". As crass as this statement is, it is a reflection of what we might call *constructionism*, which is often and incorrectly assumed to be the corollary of reductionism. To be similarly crass: knowing all the physics that there is to know won't help you one bit to explain why certain species of bird are so partial to a mate who can moonwalk. The reason is that the degrees of freedom are all wrong. Trying to understand the mating rituals of the world's fauna in terms of the dynamics of elementary particles is a fool's errand. We might like to think that the rules of social engagement can be derived from fundamental physics, but *complexity* has taught us otherwise.

But we can consider far less extreme hierarchies. For example, we know that the most useful degrees of freedom for describing quantum chromodynamics at high energies are quarks and gluons and, at low energies, they are hadrons. What we learn from this is that the most effective way of describing the physics can depend on the characteristic energy or length scale of the system of interest. We move from shorter length scales to larger ones by averaging over effects on smaller scales. We call this *coarse graining*. In quantum field theory, this process amounts to integrating out quantum fluctuations on particular scales, and we will meet this in more detail next lecture.

To start with though, we are going to integrate out all the quantum fluctuations to get our bearings on the form of the 2PI effective action, but it will already be enough for us to see how the relevant laws of physics can change. This will be the focus of Exercise 4. Let's start with the case $N = 1$ and impose some additional symmetry on the classical action to constrain the number of operators. To this end, we impose $\mathbb{Z}_2$ symmetry, so that the action is symmetric under $\Phi \to -\Phi$. To keep things simpler still, we will do everything in Euclidean space, and we'll stop at operators quartic in the field Φ. Thus, we take the partition function

$$Z[J,K] = \mathcal{N} \int \mathcal{D}\Phi \exp\left[-\frac{1}{\hbar}\left(S[\Phi] - J(x)\Phi(x) - \frac{1}{2}K(x,y)\Phi(x)\Phi(y)\right)\right] , \tag{23}$$

with classical action

$$S[\Phi] = \int \mathrm{d}^4x \left(\frac{1}{2}Z_\Phi\big(\partial\Phi(x)\big)^2 + \frac{1}{2}m^2\Phi^2(x) + \frac{\lambda}{4!}\Phi^4(x)\right) . \tag{24}$$

For the time being, we fix $m^2 > 0$ and $\lambda > 0$, and leave the case $m^2 < 0$ for Exercise 4.

So far, we have allowed for a general wavefunction factor Z_Φ. However, by making the following redefinitions of the fields and sources, we can eliminate Z_Φ from the action:

$$\Phi(x) \longrightarrow Z_\Phi^{-1/2}\Phi(x) , \qquad J(x) \longrightarrow Z_\Phi^{1/2}J(x) , \qquad K(x,y) \longrightarrow Z_\Phi K(x,y) . \tag{25}$$

This is an important observation about our charting of the theory space in terms of parameters (or operators): some parameters (and indeed operators) can be removed by field redefinitions. We call the corresponding operators *inessential* (see [4]), and they are a redundant part of our mathematical description that do not impact physical observables. Making these redefinitions, the classical action becomes

$$S[\Phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\big(\partial\Phi(x)\big)^2 + \frac{1}{2}m^2\Phi^2(x) + \frac{\lambda}{4!}\Phi^4(x)\right) , \tag{26}$$

and thankfully we no longer have to worry about getting our Z's confused.

The functional integral will be dominated by configurations that extremise the exponent. Our aim is to find these configurations and perform a so-called *saddle-point approximation*. We proceed

by writing $\Phi(x) = \varphi(x) + \hbar^{1/2}\delta\Phi(x)$, where the $\hbar^{1/2}$ is for bookkeeping, and expand

$$S[\Phi] = S[\varphi] + \hbar^{1/2}\frac{\delta S[\Phi]}{\delta\Phi(x)}\bigg|_{\Phi=\varphi}\delta\Phi(x) + \frac{\hbar}{2!}\delta\Phi(x)\frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\bigg|_{\Phi=\varphi}\delta\Phi(y) + \frac{\hbar^{3/2}}{3!}S^{(3)}(\varphi;x)\delta\Phi^3(x) + \frac{\hbar^2}{4!}S^{(4)}(\varphi;x)\delta\Phi^4(x)\ , \tag{27}$$

where, e.g.,

$$S^{(3)}(\varphi;x) = \int \mathrm{d}^4y\,\mathrm{d}^4z\,\frac{\delta^n S[\Phi]}{\delta\Phi(x)\delta\Phi(y)\delta\Phi(z)}\bigg|_{\Phi=\varphi} = \lambda\varphi(x)\ . \tag{28}$$

The saddle-point configuration φ is that which satisfies

$$\frac{\delta S[\Phi]}{\delta\Phi(x)}\bigg|_{\Phi=\varphi} = J(x) + K(x,y)\varphi(y)\ , \tag{29}$$

and the partition function can be written

$$Z[J,K] = \mathcal{N}\exp\left[-\frac{1}{\hbar}\left(S[\varphi] - J(x)\varphi(x) - \frac{1}{2}K(x,y)\varphi(x)\varphi(y)\right)\right] \times \int \mathcal{D}\delta\Phi\,\exp\left[-\frac{1}{2}\delta\Phi(x)\mathcal{G}^{-1}(x,y)\delta\Phi(y) - \frac{\hbar^{1/2}}{3!}S^{(3)}(\varphi;x)\delta\Phi^3(x) - \frac{\hbar}{4!}S^{(4)}(\varphi;x)\delta\Phi^4(x)\right]\ , \tag{30}$$

where we have introduced the inverse two-point function

$$\mathcal{G}^{-1}(x,y) = G^{-1}(\varphi;x,y) - K(x,y) = \frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\bigg|_{\Phi=\varphi} - K(x,y)\ . \tag{31}$$

A little pedantry is needed here: We must keep in mind that the sources J and K are still functionals of the one- and two-point functions ϕ and Δ.

Assuming that $S^{(3)}$ and $S^{(4)}$ are controlled by small parameters, we can compute the functional integral over the fluctuations perturbatively. The leading term is

$$\int \mathcal{D}\delta\Phi\,\exp\left[-\frac{1}{2}\delta\Phi(x)\mathcal{G}^{-1}(x,y)\delta\Phi(y)\right] \propto \det{}^{-1/2}\mathcal{G}^{-1}\ , \tag{32}$$

wherein we have omitted constant factors. In Euclidean space, the Schwinger functional is

$$W[J,K] = \hbar\ln Z[J,K] \tag{33}$$

and the quantum effective action up to order $\hbar^2$ can be written as

$$\begin{aligned}\Gamma[\phi,\Delta] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}] + \hbar^2\Gamma_{1\mathrm{PR}}[\varphi,\mathcal{G}] \\ + \big(J(x) + K(x,y)\phi(y)\big)\big(\phi(x)-\varphi(x)\big) \\ - \frac{1}{2}K(x,y)\big[\big(\phi(x)-\varphi(x)\big)\big(\phi(y)-\varphi(y)\big) - \hbar\big(\Delta(x,y)-\mathcal{G}(x,y)\big)\big]\,,\end{aligned} \tag{34}$$

or, more neatly, as

$$\begin{aligned}\Gamma[\phi,\Delta] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}] + \hbar^2\Gamma_{1\mathrm{PR}}[\varphi,\mathcal{G}] \\ + \frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)}\big(\phi(x)-\varphi(x)\big) \\ - \frac{1}{\hbar}\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\big[\big(\phi(x)-\varphi(x)\big)\big(\phi(y)-\varphi(y)\big) - \hbar\big(\Delta(x,y)-\mathcal{G}(x,y)\big)\big]\,,\end{aligned} \tag{35}$$

where

$$\Gamma_0[\varphi] = S[\varphi]\,, \qquad \Gamma_1[\varphi,\mathcal{G}] = \frac{1}{2}\mathrm{tr}\Big[\ln\big(\mathcal{G}^{-1}*G(0)\big) + G^{-1}(\varphi)*\mathcal{G} - 1\Big]\,. \tag{36}$$

The $*$ indicates that the two-point functions are integrated over the volume, and we have introduced a normalisation $G(0)$ to make the logarithm dimensionless. Γ_2 and $\Gamma_{1\mathrm{PR}}$ contain 2PI and one-particle reducible (1PR) diagrams, respectively. We will calculate these explicitly in Exercise 3.

If we assume that the path integral is dominated by a single saddle-point configuration — This is not actually true for general $J(x)$ and $K(x,y)$. — we can eliminate ϕ and Δ for φ and $\mathcal{G}$ (see [5, 6]). To this end, we expand

$$\begin{aligned}\Gamma[\phi,\Delta] = \Gamma[\varphi,\mathcal{G}] + \frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}}\big(\phi(x)-\varphi(x)\big) + \frac{1}{2}\big(\phi(x)-\varphi(x)\big)\frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}}\big(\phi(y)-\varphi(y)\big) \\ + \frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}}(\Delta(x,y)-\mathcal{G}(x,y)) + \mathcal{O}(\hbar^3)\,.\end{aligned} \tag{37}$$

Similarly, we expand

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)} = \frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}} + \frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}}\big(\phi(y)-\varphi(y)\big) + \mathcal{O}(\hbar^2)\,, \tag{38}$$

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)} = \frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}} + \mathcal{O}(\hbar^2)\,. \tag{39}$$

Putting everything together, we are left with (at order $\hbar^2$)

$$\Gamma[\phi,\Delta] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}] + \hbar^2\Gamma_{1\mathrm{PR}}[\varphi,\mathcal{G}] \\ + \frac{1}{2}\big(\phi(x)-\varphi(x)\big)\left[\frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} - \frac{2}{\hbar}\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\right]\big(\phi(y)-\varphi(y)\big)\ . \tag{40}$$

The terms in the second line cancel (see Exercise 3), eliminating the 1PR diagrams, and, after showing an admirable level of patience, we arrive at the final expression

$$\Gamma[\varphi,\mathcal{G}] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}]\ . \tag{41}$$

Notice that we have two ways of obtaining the equation of motion for φ, one obtained by taking the partial functional derivative of $\Gamma[\varphi,\mathcal{G}]$ with respect to φ, and the other obtained from the stationarity condition (29). For these to be consistent, it is clear that we require that

$$\frac{\Gamma[\phi,\Delta]}{\delta\phi(x)}\bigg|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} = \Big[J(x) + K(x,y)\phi(y)\Big]_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} = 0\ , \tag{42}$$

and we can show it to be the case if we recall that J and K are functionals of ϕ and Δ. This approach is particularly useful when the extremum of the quantum effective action and the extremum of the classical action differ by more than some small amount [5]. Thus, φ satisfies the following quantum-corrected equation of motion at order $\hbar$:

$$\frac{\delta\Gamma[\varphi,\mathcal{G}]}{\delta\varphi(x)} = \frac{\delta S[\varphi]}{\delta\varphi(x)} + \frac{\hbar}{2}\frac{\delta G^{-1}(\varphi)}{\delta\varphi(x)}\,\mathcal{G} = -\partial^2\varphi(x) + m^2\varphi(x) + \frac{\lambda}{3!}\varphi^3(x) + \frac{\hbar\lambda}{2}\varphi(x)\mathcal{G}(x,x)\ , \tag{43}$$

wherein we recognise the one-loop tadpole proportional to $\mathcal{G}(x,x)$.

The 2PI effective action is a useful hammer, and it gives us a powerful organising principle for the resummation of loop corrections (see Exercise 3). Moreover, by making various choices of the source K, we can obtain the 1PI effective action ($K \to 0$) or preserve symmetry properties in truncations of the effective action. We can also use it to understand how theories change with energy scale via the exact renormalization group, and this will be the focus of the next lecture.

3 Lecture 3: The geometry of theory space

In the last lecture, we made a sightseeing tour of the 2PI quantum effective action

$$\Gamma[\phi,\Delta] = -W[J,K] + J(x)\phi(x) + \frac{1}{2}K(x,y)\big[\phi(x)\phi(y) + \hbar\Delta(x,y)\big]\ . \tag{44}$$

To kick off this lecture, we are going to write the same effective action in the following more condensed form:

$$\Gamma[\bar{\phi}, \bar{\Delta}; \emptyset] = -W[\bar{J}, \bar{K}] + \bar{J}_a \bar{\phi}^a + \bar{K}_\alpha \bar{\Delta}^\alpha \ , \tag{45}$$

where we have defined $\bar{J} = J$, $\bar{K} = K/2$, $\bar{\phi} = \phi$ and

$$\bar{\Delta}^\alpha = \Delta^{a_1 a_2}(x_1, x_2) + \phi^{a_1}(x_1)\phi^{a_2}(x_2) \ , \tag{46}$$

and we are going to set $\hbar = 1$ from now on. (We worried a lot in previous lectures about working with connected n-point functions and keeping track of factors of $\hbar$, but not any more, and we will see why shortly.) Herein, we also use the full power of the DeWitt notation in which indices contain any number of discrete indices and a spacetime coordinate, and repeated indices are summed and integrated over, as appropriate. For example,

$$\bar{J}_a \bar{\phi}^a \equiv \sum_a \int \mathrm{d}^4 x \ \bar{J}_a(x) \bar{\phi}^a(x) \ , \tag{47}$$

$$\bar{K}_\alpha \bar{\Delta}^\alpha = \sum_{a_1, a_2} \int \mathrm{d}^4 x_1 \, \mathrm{d}^4 x_2 \ \bar{K}_{a_1 a_2}(x_1, x_2) \bar{\Delta}^{a_1, a_2}(x_1, x_2) \ . \tag{48}$$

All of this notation and the discussion that follows are based on [8].

In the previous lecture, we also remarked that there is actually a whole family of effective actions. In fact, even for our choice of two sources J and K, we have four functionals, which are pairwise related by double Legendre transforms. Together with the 2PI effective action above, these four functionals are

$$\Gamma[\emptyset; \bar{J}, \bar{K}] = -W[\bar{J}, \bar{K}] \ , \tag{49}$$

$$\Gamma[\bar{\phi}; \bar{K}] = -W[\bar{J}, \bar{K}] + \bar{J}_a \bar{\phi}^a \ , \tag{50}$$

$$\Gamma[\bar{\Delta}; J] = -W[\bar{J}, \bar{K}] + \bar{K}_\alpha \bar{\Delta}^\alpha \ , \tag{51}$$

$$\Gamma[\bar{\phi}, \bar{\Delta}; \emptyset] = -W[\bar{J}, \bar{K}] + \bar{J}_a \bar{\phi}^a + \bar{K}_{ab} \bar{\Delta}^{ab} \ . \tag{52}$$

The arguments of the functionals are such that any dependence on n-point functions is written to the left of the semicolon and any dependence on the sources is written to the right of the semicolon.

Each of these functionals depends on some number m of the sources and some number $2 - m$ of the n-point functions. Thus, for each functional, we can define a coordinate Q_A whose upper components are the m sources and whose lower components are the $2 - m$ n-point functions. For example, the Schwinger functional depends on the coordinate

$$Q_A = \begin{pmatrix} \bar{J}_a \\ \bar{K}_\alpha \end{pmatrix} \ , \tag{53}$$

on the infinite-dimensional manifold of sources — remember this is all functional — this is our configuration space.

So now we can start making use of all the technology from differential geometry, and our first task is to introduce an affine connection, which is defined by the functional derivatives with respect to the coordinate Q_A. Thus, given some tensor $T^{C_1C_2\cdots}_{B_1B_2\ldots}[Q]$, we can introduce the connection $\bar{D}$ acting as

$$\bar{D}^A T^{C_1C_2\cdots}_{B_1B_2\ldots}[Q] = \frac{\delta}{\delta Q_A} T^{C_1C_2\cdots}_{B_1B_2\ldots}[Q] \ . \tag{54}$$

We can also introduce a metric based on the Hessian of some functional of the coordinates $\bar{F}[Q]$

$$g^{AB} = \bar{D}^A \bar{D}^B \bar{F}[Q] = \frac{\delta^2 \bar{F}[Q]}{\delta Q_A \delta Q_B} \ , \tag{55}$$

where the latter equality holds only in the affine coordinate frame. The geometry that results from this form of metric is called a *Hessian geometry*, and our configuration space is a *Hessian manifold*. Such geometries are common in information theory, so much so that we can refer to information geometry. The Fischer metric is one example.

The affine connection $\bar{D}$ may not be metric compatible, e.g., $\bar{D}^A g^{BC} \neq 0$, in general, and we can introduce a tensor

$$Y^A_{BC} = \frac{1}{2} \bar{D}^A g_{BC} \ , \tag{56}$$

named for Amari and Chentsov (see [9] and references therein), which keeps track of this. We can use the Amari–Chentsov tensor to construct new connections, and two are particularly interesting. The first is the metric-compatible, Levi–Civita connection

$$\nabla = \bar{D} - Y \otimes \mathbb{I} \, , \tag{57}$$

where the tensor product with the identity map $\mathbb{I}$ takes into account that the Amari–Chentsov tensor acts in accordance with $\bar{D}$ on any given tensor [10]. The second is

$$D = \bar{D} - 2Y \otimes \mathbb{I} \, , \tag{58}$$

which is always affine, and the affine coordinates are the functional derivatives of the functional $\bar{\Gamma}[Q]$

$$P^A = \frac{\delta \bar{F}[Q]}{\delta Q_A} \ , \tag{59}$$

such that

$$D_A = \frac{\delta}{\delta P^A} \ . \tag{60}$$

Thus, we see that the coordinates P^A are dual to the coordinates Q_A. Going further, we can define

the inverse of the metric g^{AB} in terms of the Hessian of a functional $F[P]$ via

$$g_{AB} = \frac{\delta F[P]}{\delta P^A \delta P^B} \ . \tag{61}$$

This functional

$$F[P] = -\bar{F}[Q] + Q_A P^A \tag{62}$$

is nothing other than the Legendre transform of $\bar{F}[Q]$. So now we can associate $\bar{F}[Q]$ with the Schwinger functional $W[J,K] = -\Gamma[\emptyset; \bar{J}, \bar{K}]$, $F[P]$ with the 2PI effective action $\Gamma[\bar{\phi}, \bar{\Delta}; \emptyset]$, and

$$P^A = \begin{pmatrix} \bar{\phi}^a \\ \bar{\Delta}^\alpha \end{pmatrix} \ , \tag{63}$$

and we can write

$$g_{AB} = \frac{\delta Q_A}{\delta P^B} \ . \tag{64}$$

With this technology, our n-point correlation functions are now obtained from action of the $\bar{D}$ connection on the Schwinger functional, i.e.,

$$\langle \bar{\phi}^{a_1} \cdot \bar{\phi}^{a_n} \rangle_{\text{conn.}} = \bar{D}^{a_1} \cdots \bar{D}^{a_n} W[\bar{J}, \bar{K}] \ . \tag{65}$$

And now we should be thankful that this connection is not metric compatible, because otherwise we would have no n-point functions higher than the two-point function! Moving away from the affine coordinate frame of barred variables, we can express everything entirely in terms of the the connected two-point function Δ and the field ϕ and write

$$\langle \bar{\phi}^{a_1} \cdots \bar{\phi}^{a_n} \rangle_{\text{conn.}} = \left[\prod_{j=1}^{n-2} g^{a_j B_j} D_{B_j} \right] \Delta^{a_{n-1} a_n} \ , \tag{66}$$

where

$$\begin{aligned} g^{aB} D_B &= \frac{\delta W[J,K]}{\delta \bar{J}_a \delta \bar{J}_b} \frac{\delta}{\delta \bar{\phi}^b} + \frac{\delta^2 W[J,K]}{\delta \bar{J}_a \delta \bar{K}_\beta} \frac{\delta}{\delta \bar{\Delta}^\beta} \\ &= \Delta^{ab} \left[\frac{\delta}{\delta \phi^b} - \frac{\delta^2 \Gamma}{\delta \phi^b \delta \Delta^\gamma} \left(\frac{\delta^2 \Gamma}{\delta \Delta^\gamma \delta \Delta^\delta} \right)^{-1} \frac{\delta}{\delta \Delta^\delta} \right] \ . \end{aligned} \tag{67}$$

This is all very interesting, but what does it mean for the physics. Well, given that the one- and two-point functions $\bar{\phi}$ and $\bar{\Delta}$ are functionals of the sources $\bar{J}$ and $\bar{K}$, changing these sources will move us around in our configuration space or equivalently define trajectories on the associated

Hessian manifold. Suppose then that we let the source K depend on some parameter t, such that

$$\frac{\mathrm{d}\bar{K}_\alpha}{\mathrm{d}t} = \text{constant} \ . \tag{68}$$

This will define a particular trajectory along which the source K increases linearly with t. Given that

$$\bar{K}_\alpha = \frac{\delta\Gamma}{\delta\bar{\Delta}^\alpha} \ , \tag{69}$$

and taking an additional derivative with respect to t

$$\begin{aligned}
\frac{\mathrm{d}^2\bar{K}_\alpha}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\delta\Gamma}{\delta\bar{\Delta}^\alpha} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta^2\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\phi}^b}\frac{\mathrm{d}\bar{\phi}^b}{\mathrm{d}t} + \frac{\delta^2\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\Delta}^\beta}\frac{\mathrm{d}\bar{\Delta}^\beta}{\mathrm{d}t}\right] \\
&= \frac{\delta^2\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\phi}^b}\frac{\mathrm{d}^2\bar{\phi}^b}{\mathrm{d}t^2} + \frac{\delta^2\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\Delta}^\beta}\frac{\mathrm{d}^2\bar{\Delta}^\beta}{\mathrm{d}t^2} + \frac{\delta^3\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\phi}^b\delta\bar{\phi}^c}\frac{\mathrm{d}\bar{\phi}^b}{\mathrm{d}t}\frac{\mathrm{d}\bar{\phi}^c}{\mathrm{d}t} + \frac{\delta^3\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\phi}^b\delta\bar{\Delta}^\gamma}\frac{\mathrm{d}\bar{\phi}^b}{\mathrm{d}t}\frac{\mathrm{d}\bar{\phi}^\gamma}{\mathrm{d}t} \\
&\quad + \frac{\delta^2\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\Delta}^\beta\delta\bar{\phi}^c}\frac{\mathrm{d}\bar{\Delta}^\beta}{\mathrm{d}t}\frac{\mathrm{d}\bar{\phi}^c}{\mathrm{d}t} + \frac{\delta^2\Gamma}{\delta\bar{\Delta}^\alpha\delta\bar{\Delta}^\beta\delta\bar{\Delta}^\gamma}\frac{\mathrm{d}\bar{\Delta}^\beta}{\mathrm{d}t}\frac{\mathrm{d}\bar{\Delta}^\gamma}{\mathrm{d}t} \ .
\end{aligned} \tag{70}$$

Re-expressing this in terms of the geometric objects defined previously, the first term is just

$$g_{\alpha B}\frac{\mathrm{d}^2P^B}{\mathrm{d}t^2} \tag{71}$$

and the second term can be written as

$$D_\alpha g_{CD}\frac{\mathrm{d}P^C}{\mathrm{d}t}\frac{\mathrm{d}P^D}{\mathrm{d}t} = g_{\alpha B}\bar{D}^B g_{CD}\frac{\mathrm{d}P^C}{\mathrm{d}t}\frac{\mathrm{d}P^D}{\mathrm{d}t} = 2g_{\alpha B}Y^B_{CD}\frac{\mathrm{d}P^C}{\mathrm{d}t}\frac{\mathrm{d}P^D}{\mathrm{d}t} \ . \tag{72}$$

Putting everything together, we arrive at the following equation:

$$g_{\alpha B}\left[\frac{\mathrm{d}^2P^B}{\mathrm{d}t^2} + 2Y^B_{CD}\frac{\mathrm{d}P^C}{\mathrm{d}t}\frac{\mathrm{d}P^D}{\mathrm{d}t}\right] = 0 \ . \tag{73}$$

In cases where the bracket itself vanishes, this is nothing other than the geodesic equation of the $\bar{D}$ connection

$$\frac{\mathrm{d}^2P^B}{\mathrm{d}t^2} + 2Y^B_{CD}\frac{\mathrm{d}P^C}{\mathrm{d}t}\frac{\mathrm{d}P^D}{\mathrm{d}t} = 0 \ . \tag{74}$$

These trajectories are just straight lines in the affine coordinate frame of $\bar{D}$, along which the t-derivatives of the sources are, by construction, conserved.

Now, we recall that the source $K(x,y)$ is accompanied by two factors of the field in the path integral. By an appropriate choice K, we can therefore make the field more or less massive. Moreover, since the source K is coordinate dependent, we can use its Fourier transform $\tilde{K}(p)$ to make

certain Fourier modes of the field more or less massive. This allows us to re-weight contributions from particular modes, e.g., freezing modes with momentum p less than some scale k, such that we only integrate out modes with momentum $p > k$ when calculating the partition function. In this way, the Fourier transform of the two-point source can be used to implement a smooth cut-off, and we call it a *regulator*. We can then ask how the effective action changes as a function of the scale k — we call this the *renormalization group* or *RG evolution*.

So returning to Eq. (73), if we associated $t \propto \ln k$ with the so-called RG time, then the geodesics of the $\bar{D}$ connection on the configuration space can be used to specify surfaces of constant RG time or to construct RG trajectories. We can make more sense of this by considering the variation of the effective action with respect to t for fixed ϕ. This gives

$$\partial_t \Gamma[\phi, \Delta] = \frac{\delta \Gamma[\phi, \Delta]}{\delta \Delta^\alpha} \, \partial_t \Delta^\alpha = \frac{1}{2} K_\alpha \partial_t \Delta^\alpha \,, \tag{75}$$

where we have used that Δ is a functional of K. This is the RG flow equation of the 2PI effective action. By acting on it with the $\bar{D}$ connection, we can obtain systems of flow equations for the n-point correlation functions.

Returning then to the geometrical picture, we see that changes in the RG time lead to changes in the two-point correlation function at fixed ϕ. This defines a vector field on the configuration space that connects hypersurfaces of constant RG time. Thus, the renormalization group tells us how we move around in theory space, spanned here by the correlation functions. In the next lecture, we will see how the same renormalization group evolution moves us around in the same theory space when we span it by the parameters (or couplings) of our theory.

4 Lecture 4: The exact renormalization group

In the previous lecture, we wrote down the renormalization group flow equation for the 2PI effective action

$$\partial_t \Gamma[\phi, \Delta] = \frac{\delta \Gamma[\phi, \Delta]}{\delta \Delta^\alpha} \, \partial_t \Delta^\alpha = \frac{1}{2} K_\alpha \partial_t \Delta^\alpha \,, \tag{76}$$

where $t \propto \ln k$ is the RG time. To obtain this, we associated the Fourier transform of the two-point source K_α with a regulator, which allows us to control which of the quantum fluctuations of the fields we integrate out.

We will choose this regulator so that it "freezes" infrared modes below some scale k, such that we only integrate out ultraviolet modes in the path integral. We therefore require that the regulator $R_k(p) = -\tilde{K}(p)$ has the following properties:

- $R_k(p) \to 0$ as $k \to 0$, so that when the scale k is zero, all the modes are integrated out.
- $R_k(p) \to 0$ as $p^2/k^2 \to \infty$, and the UV modes are integrated out as usual in the path integral.

- $R_k(p) \to \infty$ as $p^2/k^2 \to 0$, and the IR modes are *not* integrated out.

In the limit $R_k(p) \to 0$, the scale-dependent 2PI effective action reduces to the 1PI effective action. In the other limit, $R_k(p) \to \infty$, we instead recover the classical action $S[\phi]$, since there are no quantum fluctuations for us to integrate out in the path integral. At some finite k, we integrate out modes with momenta $|p| \gtrsim |k|$. Note that the cut-off k is, in principle, implemented smoothly, and we may still integrate some modes in the vicinity of the scale k. This will depend on the precise form of the regulator, and the choice is not unique.

At this point, we might be worried that there is something unphysical about the effective action at arbitrary values of the scale k, since it will depend on the choice of regulator. We're right to worry. However, we are not interested in the particular form of the effective action, but rather what the evolution in the theory space looks like for a given regulator and, in particular, whether there are any special points in that evolution. We call these *fixed points*, and they don't change under RG evolution. At these points, the correlation length will be zero or infinity, and if it is infinity, we have a critical point (where phase boundaries terminate), we see the emergence of scale invariance, and we have universal behaviour.

The 2PI flow equations are relatively new on the market, and much more common are those derived from the so-called 1PI average effective action, which is a shifted version of the functional $\Gamma[\bar{\phi}; \bar{K}]$ that we introduced in the previous lecture. It has the following form:

$$\Gamma_{\rm av}[\phi, K] = -W[J, K] + J_a\phi^a + \frac{1}{2}K_{ab}\phi^a\phi^b \ , \tag{77}$$

where the maximisation in the Legendre transform has only been performed with respect to J_a. We therefore have the same expression for the one-point function

$$\phi^a = \frac{\delta W[J, K]}{\delta J_a} \ , \tag{78}$$

but we do not have the corresponding expression for the two-point function, as we did in the 2PI case. This means that we do not have the advantage of the implicit resummation of 1PI self-energies implicit to the 2PI approach. Nevertheless, it remains the case that

$$\Delta^{ab} = \frac{\delta W[J, K]}{\delta J_a \delta J_b} = 2\frac{\delta W[J, K]}{\delta K_{ab}} - \phi_a\phi_b \ . \tag{79}$$

At this point, we need to be a bit careful. The 1PI average effective action is *not* the Legendre transform of the Schwinger functional; it is shifted by the term $K_{ab}\phi^a\phi^b/2$. The first two terms in $\Gamma_{\rm av}[\phi, K]$ are, however, the terms present in the 1PI effective action

$$\Gamma_{\rm 1PI}[\phi] = -W[J] + J_a\phi^a \ . \tag{80}$$

Compared to the 2PI case, the convexity of the 1PI effective action follows from the much simpler identity

$$\frac{\delta W[J,K]}{\delta J_a \delta J_b}\frac{\delta^2 \Gamma_{1\mathrm{PI}}[\phi]}{\delta\phi^a\delta\phi^b} = \mathbb{I}\,, \tag{81}$$

where we are being rather blasé about the meaning of the $\mathbb{I}$ on the right-hand side. Even so, we can hopefully convince ourselves that

$$\Delta^{ab} = \left(\frac{\delta^2\Gamma_{1\mathrm{PI}}[\phi]}{\delta\phi^a\delta\phi^b}\right)^{-1}\,. \tag{82}$$

It then follows that the two-point function is obtained from the 1PI average effective action as

$$\Delta^{ab} = \left(\frac{\delta^2\Gamma_{\mathrm{av}}[\phi,K]}{\delta\phi^a\delta\phi^b} - K_{ab}\right)^{-1}\,. \tag{83}$$

In order to obtain the flow equation, we again associate the Fourier transform of K with the regulator, i.e., $\tilde{K}(p) = -R_k(p)$, as we did in the 2PI case. Varying the 1PI average effective action with the respect to the RG time $t \propto \ln k$ for fixed ϕ, we then obtain the flow equation

$$\partial_t\Gamma_{\mathrm{av}}[\phi,K] = \left(-\frac{\delta W[J,K]}{\delta K_{ab}} + \phi^a\phi^b\right)\partial_t K_{ab} = -\frac{1}{2}\Delta^{ab}\partial_t K_{ab}\,, \tag{84}$$

which we notice differs from the 2PI flow equation by a total derivative with respect to the RG time $\partial_t(K_{ab}\Delta^{ab})$. The flow equation (84) is due to Wetterich [13], Morris [14] and Ellwanger [15], and it is more often written in momentum space in the form

$$\partial_k\Gamma_k = \frac{1}{2}\mathrm{STr}\,\partial_k R_k(p)\left[\Gamma_k^{(2)} + R_k\right]^{-1}\,, \tag{85}$$

where $\Gamma_k \equiv \Gamma_{\mathrm{av}}$, STr is the trace in momentum space and over all fields and quantum numbers.

In the previous lecture, we saw that RG evolution moved us around in the configuration space of correlation functions, which we were able to equip with a geometrical structure stolen from information theory. In this lecture, we will see how the flow equation can be used to describe how the couplings in our theory change with energy scale, and therefore allow us to understand how RG evolution moves us around in theory space, when we instead use the couplings as coordinates. To do this, we will solve the flow equation by an Ansatz, using what is commonly called the local potential approximation. We will also work to the lowest order in the derivative expansion, i.e., we consider terms that contain at most two derivatives of the field. We will also neglect the infamous *anomalous dimension*, but there are all manner of excellent reviews of the exact renormalization group that we can dive into to find out more.

One such review, and the one that we will follow relatively closely in what follows, is the review by Berges, Tetradis and Wetterich [16]. Focusing, as these authors did, on the symmetry

breaking case of the ϕ^4 theory that we considered in the exercises to the second lecture, we write the following Ansatz for the Euclidean 1PI average effective action:

$$\Gamma_k = \int \mathrm{d}^d x \, \left[\frac{1}{2} Z_k(\rho) \partial_\mu \phi \partial_\mu \phi + U_k(\rho) \right] \, , \tag{86}$$

where $\rho = \phi^2/2$ and

$$U_k(\rho) = \frac{1}{2} g_k (\rho - \rho_k)^2 + \Lambda_k \, . \tag{87}$$

The parameters g_k, ρ_k and Λ_k are dimensionful in general dimensions d, and it is convenient to scale out their dimensionality by defining

$$\kappa_k = \bar{Z}_k k^{2-d} \rho_k \, , \tag{88}$$

$$\lambda_k = \bar{Z}_k^{-2} k^{d-4} g_k \, , \tag{89}$$

where

$$\bar{Z}_k \equiv Z_k(\rho_k) \, . \tag{90}$$

The scale-dependent potential then takes the form

$$U_k(\rho) = \frac{1}{2} k^d \lambda_k \big(\bar{Z}_k k^{2-d} \rho - \kappa_k \big)^2 + \Lambda_k \tag{91}$$

For constant ρ, the flow equation for the average effective action quickly yields the flow equation for the effective potential

$$\partial_t U_k(\rho) = \frac{1}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \partial_t R_k(q) \Delta_k(\rho, q) \, . \tag{92}$$

Using Eq. (83), the momentum-space two-point function is

$$\Delta_k(\rho, q) = \frac{1}{Z_k(\rho) q^2 + R_k(q) + U_k'(\rho) + 2\rho U_k''(\rho)} \, . \tag{93}$$

For simplicity, we will take $Z_k(\rho) = \bar{Z}_k$ and neglect the scale-dependence of $\bar{Z}_k$. (This is what causes us to neglect the anomalous dimension $\eta = -\partial_t \ln \bar{Z}_k$.) We now just need to pick a regulator. A particularly straightforward choice for our purposes is the Litim regulator [17]:

$$R_k(q) = \bar{Z}_k (k^2 - q^2) \theta(k^2 - q^2) \, , \tag{94}$$

where θ is the unit step function. This regulator has the advantage of making the so-called *threshold functions*, which we will obtain when performing the integral over the momentum q, particular easy to evaluate.

The first of these threshold functions arises in the flow of the effective potential, and we have

$$\frac{1}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\,\partial_t R_k(q)\Delta_k(\rho,q) = \frac{s_d}{2}\int \mathrm{d}q\; q^{d-1}\frac{2\bar{Z}_k k^2\theta(k^2-q^2)}{\bar{Z}_k(\rho)q^2+\bar{Z}_k(k^2-q^2)\theta(k^2-q^2)+U_k'(\rho)+2\rho U_k''(\rho)}$$
$$= \frac{s_d}{d}k^d\left[1+\frac{U_k'(\rho)+2\rho U_k''(\rho)}{\bar{Z}_k k^2}\right]^{-1}\,, \tag{95}$$

where $s_d = \Omega_d/(2\pi)^d$, with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ being the solid angle in d dimensions. Thus,

$$\partial_t U_k(\rho) = \frac{s_d}{d}k^d\left[1+\frac{U_k'(\rho)+2\rho U_k''(\rho)}{\bar{Z}_k k^2}\right]^{-1}\,. \tag{96}$$

Evaluating this at the minimum of the potential $\rho = \rho_k$, we obtain the flow equation for the cosmological constant

$$\partial_t \Lambda_k = \frac{s_d}{d}\frac{k^d}{1+2\kappa_k\lambda_k}\,. \tag{97}$$

To obtain the flow equations for κ_k and λ_k, we take derivatives with respect to ρ, and again evaluate at $\rho = \rho_k$. After a little algebra, we find

$$\partial_t \kappa_k = (2-d)\kappa_k + \frac{6 s_d}{d}\frac{1}{(1+2\kappa_k\lambda_k)^2}\,, \tag{98}$$

$$\partial_t \lambda_k = (d-4)\lambda_k + \frac{18 s_d}{d}\frac{\lambda_k^2}{(1+2\kappa_k\lambda_k)^3}\,. \tag{99}$$

If we were to solve for the RG evolution of this system, we would find that the the potential barrier between the two minima of the potential would lower as we integrated out more and more modes (i.e., lowered the scale k towards the infrared). As we take the scale $k \to 0$, and integrate out all of the quantum fluctuations, we would find a convex potential in agreement with our analysis of the effective potential and the emergence of the Maxwell construction in Exercise 4.

The first thing that we notice is that in $d > 4$ the signs of the leading terms in the flow equations (which result from the canonical scaling dimension of the original dimensionful couplings) are opposite for κ_k and λ_k. For κ_k, the right-hand side of the flow equation, otherwise called the *beta function*, is negative. This means that the corresponding operator (the mass term) increases into the infrared for $d > 2$; we say that the operator is *relevant*. On the other hand, for $d > 4$, the leading term in the beta function for λ_k is positive, and we would say that the operator is *irrelevant*, and it decreases into the infrared. In $d = 4$, the leading term in the beta function for λ_k is actually zero. If it were strictly zero, we would say that the corresponding operator was *marginal*. However, we see that the contribution from the loop corrections (via the threshold functions) is positive and there is the anomalous dimension too that we neglected. In fact, the coupling λ_k is said to be *marginally irrelevant*, and it will evolve comparably slowly in RG time.

We see already from this analysis that we can have theories who share common relevant operators and differ only in their irrelevant operators. Such theories from a *universality class*: they give the same infrared physics. If we think in terms of effective field theories, where we allow higher-dimensional, and therefore a menagerie of irrelevant operators, the same infrared physics can be realised from multiple effective field theories.

Going back to our theory space, and spanning it with the couplings, we also see that if we are close to some fixed point, where the beta function vanishes, we have the possibility to move in irrelevant and relevant directions. If we try to move in an irrelevant direction, the RG flow will drive us back to the fixed point. This is because irrelevant directions do not affect the long-distance physics. On the other hand, we can happily move in a relevant direction, and the RG flow will kindly take us away from the fixed point. This is because relevant operators *do* affect the long-distance physics. (For a much more lucid discussion on these points than I could ever hope to give, I strongly recommend David Tong's lecture notes [18].)

Some concluding remarks

Sadly, this brings us to the end of our summer Smörgåsbord of quantum field theory morsels. And there are many things that we did not get chance to discuss further or, indeed, to meet at all:

- We worried a little about some foundational topics in quantum field theory, but we did not have time to meet the axiomatic field theory or algebraic quantum field theory that were the reaction to the types of concerns raised by Haag and others.

- We didn't get to meet the other kinds of integrating out, in which we trace over some subsystem in order to build an open quantum field theoretic system, and this would have been a lot of fun. We would have introduced the Schwinger–Keldysh closed-time path formalism, which is geared up to calculating equal-time expectation values of local products of operators and readily allows us to treat field-theoretic statistical ensembles. From there, we would have looked at the field-theoretic description of the evolution of reduced density matrices and met the Feynman–Vernon influence functional, where we really would have to worry about the boundary conditions on the path integral. We could have looked at how causality is borne out in the path-integral representation, and how this is naturally framed within the closed-time path formalism, thanks to the presence of both time- and anti-time-ordered prescriptions to ensure convergence of the path integral.

- We only touched the surface of the (exact) renormalization group, which remains one of the most powerful non-perturbative techniques that we have in quantum field theory. And we didn't get to look at the role of non-linear field redefinitions in eliminating redundancies in

our description, allowing us to distinguish between essential and inessential couplings. For this, I point you to the excellent paper by Baldazzi, Zinati and Falls [4].

- We didn't get to worry about quantum field theory in curved spacetime or how all of the tools that we developed allow us to treat non-trivial field configurations, like instantons and solitons, or to worry about the evolution of metastable field configurations.

- We didn't get to play with defining different state spaces to consider much wider classes of quantum field theories, even finding ones with complex actions that nevertheless have real, bounded eigenspectra and a (pseudo-)unitary evolution. Such theories are said to be *pseudo-Hermitian*. Maybe we'll get to discuss these pseudo-Hermitian quantum field theories and what they teach us about "standard" quantum field theory another time.

- And we didn't get to see that the ϕ^4 theory suffers from a Landau pole (divergent coupling at a finite scale) and the problem of triviality. Even the seemingly innocent toy theories can turn out to be less than upright in character in the end. As Winnie-the-Pooh would say, "One never can tell with theories."

And that was just scalar field theories: no fermions, no gauge fields, no gravity.

So thank you for listening, for all the great questions, and for all the great discussions. I look forward to chatting with you all again and seeing all that you achieve. But please, never let anyone try to convince you that quantum field theory is all figured out — it most certainly is not. All the best for your PhDs and the adventures that will follow, wherever they take you.

5 Exercises

5.1 Haag's theorem

In this exercise, following an argument by Rudolf Haag [1], we will uncover an important subtlety of the Fock representation. In doing so, we'll revisit the canonical operator formalism for the real scalar field, come across Bogoliubov transformations, and think about particle creation.

Consider two of our field operators $\hat{\Phi}_1(x)$ and $\hat{\Phi}_2(x)$, with masses m_1 and m_2, respectively, and with corresponding creation and annihilation operators $\hat{a}^\dagger_{1,2}(\mathbf{k}, 0)$ and $\hat{a}_{1,2}(\mathbf{k}, 0)$:

- Choose the fields and their conjugate momenta $\hat{\Pi}_{1,2}(t, \mathbf{x}) = \partial_t \hat{\Phi}_{1,2}(t, \mathbf{x})$ to have the same initial conditions at $t = 0$, i.e., $\hat{\Phi}_1(0, \mathbf{x}) = \hat{\Phi}_2(0, \mathbf{x})$ and $\hat{\Pi}_1(0, \mathbf{x}) = \hat{\Pi}_2(0, \mathbf{x})$.

- Show that the creation and annihilation operators of the two fields are related via

$$\hat{a}_2(\mathbf{k}, 0) = \cosh\theta\, \hat{a}_1(\mathbf{k}, 0) + \sinh\theta\, \hat{a}^\dagger_1(-\mathbf{k}, 0) \,, \tag{100}$$

$$\hat{a}_1(\mathbf{k}, 0) = \cosh\theta\, \hat{a}_2(\mathbf{k}, 0) - \sinh\theta\, \hat{a}^\dagger_2(-\mathbf{k}, 0) \,, \tag{101}$$

and determine θ.

- Consider what this implies for the action of the operator $\hat{a}_2(\mathbf{k}, 0)$ on the vacuum state for the operator $\hat{a}_1(\mathbf{k}, 0)$, i.e., that state for which $\hat{a}_1(\mathbf{k}, 0)\,|0\rangle = 0$.

- Now let $\hat{\Phi}_2$ be the full, interacting (Heisenberg) field and $\hat{\Phi}_1$ be the free, interaction-picture field, such that the masses m_1 and m_2 also coincide at time $t = 0$. The mass m_2 will differ from m_1 for times $t > 0$ due to the interactions — we might call this a fancy name too: *vacuum polarisation*. What do your conclusions above imply for the action of the interaction-picture operators on the interacting vacuum?

- When might this subtlety be relevant?

5.2 The path integral

Often, the path integral is derived for the quantum harmonic oscillator and then extrapolated to quantum fields. In this exercise, we'll derive the path integral for a real scalar field directly, and we'll make sure to keep track of boundary conditions.

- Start by considering discrete time steps. What is the overlap $\langle \Phi_{j+1}(\mathbf{x}), t_{j+1} | \Phi_j(\mathbf{x}), t_j \rangle$ between a field eigenstate at time t_{j+1} and a field eigenstate at time t_j? Rewrite this in terms of the canonical time-evolution operator for the real scalar field, but remember that the subscripts on the field arguments continue to label different integrals over field configurations. We will need to use the eigenequation for the field eigenstates, introduce a basis $|\Pi(\mathbf{x}), t\rangle$ of eigenstates of the conjugate-momentum operator $\hat{\Pi}(t, x)$, and use the corresponding eigenequation, too. The following expression will prove useful:

$$\langle \Phi(\mathbf{x}) | \Pi(\mathbf{x}) \rangle = \exp\left[\frac{i}{\hbar} \int \mathrm{d}^3\mathbf{x}\, \Pi(\mathbf{x}) \Phi(\mathbf{x}) \right] \,. \tag{102}$$

  How can we make the resulting integral over the conjugate-momentum field convergent?

- Now consider the vacuum persistence amplitude for vanishing source

$$Z[0] = \langle 0(+\infty) | 0(-\infty) \rangle \,, \tag{103}$$

  insert N complete sets of field eigenstates at discrete time steps, and make use of your result above for $\langle \Phi_{j+1}(\mathbf{x}), t_{j+1} | \Phi_j(\mathbf{x}), t_j \rangle$.

- Finally, take the limit $N \to \infty$ to obtain the continuum result, and introduce a non-vanishing source to obtain the partition function $Z[J]$.

5.3 The Schwinger–Dyson equation

In the second lecture, we derived the order-$\hbar$ term in the 2PI effective action, which corresponded to the integral over quadratic fluctuations. In this exercise, we will derive the order-$\hbar^2$ corrections to the 2PI effective action, show that the 1PR diagrams cancel, and obtain the Schwinger–Dyson equation for the two-point function:

- Start by expanding the exponential in the partition function as a power series in the vertices $S^{(3)}$ and $S^{(4)}$, being careful with the $\hbar$ power counting.
- Use Wick's theorem to compute the functional integral in terms of the two-point function $\mathcal{G}$.
- Now consider the logarithm of this result and expand again to order $\hbar^2$ in the expression for the effective action.
- By deriving an expression for the one-point function ϕ, show that the 1PR diagrams cancel in the 2PI effective action.
- By varying the final result for the 2PI quantum effective action with respect to the two-point correlation function obtain the Schwinger–Dyson equation for the two-point function. What level of resummation does this amount to?

5.4 Convexity and the Maxwell construction

Since it is defined via a Legendre transform, the effective action must be a convex functional. In this exercise, we will show, in the case of a non-convex classical potential, that this is ensured by the Maxwell construction (which you might have met in thermodynamics):

- Consider the $\lambda\phi^4$ theory with a tachyonic mass term $m^2 < 0$, such that the classical potential has two degenerate minima separated by a potential barrier. Since both minima will contribute equally to the partition function, we must sum over them.
- We want to calculate the effective potential and so are only interested in the zero mode or the infinite-wavelength contribution. This can be projected out by a neat trick, akin to Faddeev and Popov's. We will skip this step, and start from the result

$$Z[J,K] = \sum_{\pm} \exp\left[-\frac{\Omega}{\hbar}\left(\bar{S}(\varphi_\pm) - \bar{J}\varphi_\pm - \frac{1}{2}\bar{K}\varphi_\pm^2 + \frac{\hbar}{2\Omega}\mathrm{tr}\ln\mathcal{G}_\pm^{-1}G(0) + \mathcal{O}(\hbar^2)\right)\right] \ , \qquad (104)$$

  where Ω is a volume factor and $\varphi_\pm = \pm\sqrt{6}m/\sqrt{\lambda}$.
- Using the expression above, determine the one-point function in terms of $\varphi_\pm$ and $Z_\pm$.

- Rearrange this to find the ratio Z_+/Z_- and find an expression for $\bar{J}$. What happens when $\phi < \varphi_-$ or $\phi > \varphi_+$?
- Substitute for $\bar{J}$ in the exponents of the path integrals, working to zeroth order in $\hbar$ to keep things simple. These exponents should now be equal.
- Calculate the effective action and take the limit $\bar{K} \to 0$ to show that the effective potential $U = \Gamma/\Omega$ is convex.

5.5 Obtaining vertex functions

In this exercise, we will prove the convexity of the 2PI effective action, and use the resulting expressions to obtain the operator in Eq. (67). To make our lives easier, we will forget the functional structure and just consider the Schwinger functional and the effective action as *functions* of numbers $\bar{J}$ and $\bar{K}$, and $\bar{\phi}$ and $\bar{\Delta}$, respectively.

- Show that the product of Hessians of the Schwinger "function" with respect to $\bar{J}$ and $\bar{K}$ and the effective action with respect to $\bar{\phi}$ and $\bar{\Delta}$ are inverses of one another. This is encoded in the geometric picture by $g_{AC}g^{CB} = \delta_A^B$. Noting that the Hessian of the Schwinger function is a covariance matrix, argue that the effective action is convex.
- Using the above results, obtain expressions for

$$\frac{\partial W}{\partial \bar{J}\partial \bar{J}} \qquad \text{and} \qquad \frac{\partial^2 W}{\partial \bar{J}\partial \bar{K}} \tag{105}$$

  in terms of the 2PI effective action, $\bar{\phi}$ and $\bar{\Delta}$.
- Using the same results, change coordinates from $(\bar{\phi}, \bar{\Delta})$ to (ϕ, Δ) and write the connected two-point correlation function Δ in terms of variations of the 2PI effective action.
- Hence show that

$$\begin{aligned} g^{aB}D_B &= \frac{\partial W(J,K)}{\partial \bar{J}\partial \bar{J}}\frac{\partial}{\partial \bar{\phi}} + \frac{\partial^2 W(J,K)}{\partial \bar{J}\partial \bar{K}}\frac{\partial}{\partial \bar{\Delta}} \\ &= \Delta\left[\frac{\partial}{\partial \phi} - \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta}\left(\frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta}\right)^{-1}\frac{\partial}{\partial \Delta}\right] . \end{aligned} \tag{106}$$

  This operator is particularly useful for obtaining expressions for the proper n-point vertices from the 2PI effective action by acting on the connected two-point function obtained above.

You might find [6, 11, 12] — all with a Nottingham connection — helpful.

6 Worked solutions

6.1 Haag's theorem

We consider two free scalar field operators $\hat{\Phi}_1$ and $\hat{\Phi}_2$ with masses m_1 and m_2, respectively, which solve the Klein–Gordon equations

$$\left(\Box + m_j^2\right)\hat{\Phi}_j(t,\mathbf{x}) = 0 \;, \qquad j = 1,2 \;. \tag{107}$$

The conjugate-momentum operators are $\hat{\Pi}_j(t,\mathbf{x}) = \partial_t \hat{\Phi}_j(t,\mathbf{x})$.

The field and conjugate-momentum operators can be written in the form

$$\hat{\Phi}_j(t,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_j(\mathbf{p})}\left(\hat{a}_j(\mathbf{p},0)e^{-iE_j(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}} + \mathrm{H.c.}\right) \;, \tag{108}$$

$$\hat{\Pi}_j(t,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2i}\left(\hat{a}_j(\mathbf{p},0)e^{-iE_j(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}} - \mathrm{H.c.}\right) \;, \tag{109}$$

where H.c. indicates the Hermitian-conjugate term. We can therefore express the annihilation and creation operators at the initial time $t = 0$ as

$$\hat{a}_j(\mathbf{k},0) = \int \mathrm{d}^3\mathbf{x}\left(E_j(\mathbf{k})\hat{\Phi}_j(0,\mathbf{x}) + i\hat{\Pi}_j(0,\mathbf{x})\right)e^{-i\mathbf{k}\cdot\mathbf{x}} \;, \tag{110}$$

$$\hat{a}_j^\dagger(\mathbf{k},0) = \int \mathrm{d}^3\mathbf{x}\left(E_j(\mathbf{k})\hat{\Phi}_j(0,\mathbf{x}) - i\hat{\Pi}_j(0,\mathbf{x})\right)e^{+i\mathbf{k}\cdot\mathbf{x}} \;. \tag{111}$$

Now we suppose that the two fields have the same initial conditions, i.e.,

$$\hat{\Phi}_j(0,\mathbf{x}) = \hat{\Phi}(0,\mathbf{x}) \;, \tag{112}$$

$$\hat{\Pi}_j(0,\mathbf{x}) = \hat{\Pi}(0,\mathbf{x}) \;. \tag{113}$$

We can then re-express

$$\hat{\Phi}_2(0,\mathbf{x}) = \hat{\Phi}(0,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_1(\mathbf{p})}\left(\hat{a}_1(\mathbf{p},0)e^{i\mathbf{p}\cdot\mathbf{x}} + \mathrm{H.c.}\right) \;, \tag{114}$$

$$\hat{\Pi}_2(0,\mathbf{x}) = \hat{\Pi}(0,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2i}\left(\hat{a}_1(\mathbf{p},0)e^{i\mathbf{p}\cdot\mathbf{x}} - \mathrm{H.c.}\right) \;, \tag{115}$$

in terms of the annihilation and creation operators of the other field. It then follows that

$$\begin{aligned}\hat{a}_2(0,\mathbf{k}) = \int \mathrm{d}^3\mathbf{x}\Bigg[&E_2(\mathbf{k})\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_1(\mathbf{p})}\left(\hat{a}_1(\mathbf{p},0)e^{i\mathbf{p}\cdot\mathbf{x}} + \mathrm{H.c.}\right)\\ &+ \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2}\left(\hat{a}_1(\mathbf{p},0)e^{i\mathbf{p}\cdot\mathbf{x}} - \mathrm{H.c.}\right)\Bigg]e^{-i\mathbf{k}\cdot\mathbf{x}} \;.\end{aligned} \tag{116}$$

Using

$$\int \mathrm{d}^3\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{k})\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k}) \;, \tag{117}$$

we arrive at the expression

$$\hat{a}_2(\mathbf{k},0) = \frac{E_2(\mathbf{k})+E_1(\mathbf{k})}{2E_1(\mathbf{k})}\,\hat{a}_1(\mathbf{k},0) + \frac{E_2(\mathbf{k})-E_1(\mathbf{k})}{2E_1(\mathbf{k})}\,\hat{a}_1^\dagger(-\mathbf{k},0) \;. \tag{118}$$

Note the relative sign on the momentum of the creation operator.

In the above, we have used a manifestly Lorentz-invariant normalisation of the creation and annihilation operators, meaning that they have inverse mass dimensions. As a result, the commutator $\left[\hat{a}_j(\mathbf{p},t),\hat{a}_\ell^\dagger(\mathbf{k},t)\right] = (2\pi)^3 2\delta_{j\ell}E_\ell(\mathbf{k})\delta^{(3)}(\mathbf{p}-\mathbf{k})$ is not isotropic in the "flavour" space of the two fields, since the right-hand side contains a factor of the energy $E_\ell(\mathbf{k})$ and will not commute with arbitrary rotations amongst the two fields. In order to show that we have a Bogoliubov transformation, we need to choose a normalisation for which the commutator *is* isotropic. This amounts to taking $\hat{a}_j^{(\dagger)}(\mathbf{k},0) \to \sqrt{2E_j(\mathbf{k})}\hat{a}_j^{(\dagger)}(\mathbf{k},0)$.

Our result is then of the form of a Bogoliubov transformation:

$$\hat{a}_2(\mathbf{k},0) = \cosh\theta\,\hat{a}_1(\mathbf{k},0) + \sinh\theta\,\hat{a}_1^\dagger(-\mathbf{k},0) \;, \tag{119}$$

$$\hat{a}_1(\mathbf{k},0) = \cosh\theta\,\hat{a}_2(\mathbf{k},0) - \sinh\theta\,\hat{a}_2^\dagger(-\mathbf{k},0) \;, \tag{120}$$

along with the Hermitian-conjugate expressions, where

$$\theta = \operatorname{arccosh}\frac{E_2(\mathbf{k})+E_1(\mathbf{k})}{2\sqrt{E_1(\mathbf{k})E_2(\mathbf{k})}} \;, \tag{121}$$

and we can readily convince ourselves that it preserves the canonical commutation relations

$$\left[\hat{a}_j(\mathbf{p},t),\hat{a}_\ell^\dagger(\mathbf{k},t)\right] = (2\pi)^3\delta_{j\ell}\delta^{(3)}(\mathbf{p}-\mathbf{k}) \;, \tag{122}$$

with all other commutators vanishing.

We can define an initial Fock vacuum state $|0\rangle$, such that

$$\hat{a}_1(\mathbf{k},0)\,|0\rangle = 0 \;. \tag{123}$$

However, acting on the same state with $\hat{a}_2(\mathbf{k},0)$, we find

$$\hat{a}_2(\mathbf{k},0)\,|0\rangle = \frac{E_2(\mathbf{k})-E_1(\mathbf{k})}{2\sqrt{E_1(\mathbf{k})E_2(\mathbf{k})}}\,\hat{a}_1^\dagger(-\mathbf{k},0)\,|0\rangle \neq 0 \;. \tag{124}$$

Hence, to the second field, the vacuum state looks like an excited state with non-vanishing particle

number

$$\frac{1}{V}\, n_2(\mathbf{k}) = \frac{1}{V}\ \langle 0|\hat{a}_2^\dagger(\mathbf{k},0)\hat{a}_2(\mathbf{k},0)|0\rangle = \left(\frac{E_2(\mathbf{k})-E_1(\mathbf{k})}{2\sqrt{E_1(\mathbf{k})E_2(\mathbf{k})}}\right)^2 = \cosh^2\theta\ , \tag{125}$$

where $V = (2\pi)^3\delta^{(3)}(\mathbf{0})$. It seems we cannot define a state that is simultaneously a vacuum state for both $\hat{\Phi}_1$ and $\hat{\Phi}_2$.

This might seem innocuous enough, but suppose we let $\hat{\Phi}_1$ and $\hat{\Phi}_2$ be the free and interacting (Heisenberg) fields, respectively. We will then have $m_2^2 = m_1^2$ at the initial time, but $m_2^2 \neq m_1^2$ at times $t > 0$, because Φ_1 evolves only with the free part of the Hamiltonian, whereas Φ_2 evolves with the full, interacting Hamiltonian. In this case, we have a problem: there does not exist a vacuum state common to the interaction- and Heisenberg pictures. The two pictures (or representations of the canonical algebra) are said to be *unitarily inequivalent*, because there does not exist a proper unitary mapping between the two pictures. Why do we not worry more about this?

This actually arises whenever we have a time-dependent Hamiltonian, and this includes the case of curved-spacetime quantum field theory, when we consider dynamical geometries (see [2]). It is the route of particle creation in the case of Black Holes (Hawking radiation), the Unruh effect, preheating, and the statement that de Sitter space is unstable.

6.2 The path integral

Let's first consider the overlap between two field eigenstates

$$\langle \Phi_{j+1}(\mathbf{x}), t_{j+1}|\Phi_j(\mathbf{x}), t_j\rangle\ . \tag{126}$$

By introducing the Heisenberg-picture evolution operator, we can write

$$\langle \Phi_{j+1}(\mathbf{x}), t_{j+1}|\Phi_1(\mathbf{x}), t_j\rangle = \langle \Phi_{j+1}(\mathbf{x})|e^{-i\hat{H}\delta t/\hbar}|\Phi_j(\mathbf{x})\rangle\ , \tag{127}$$

where $\delta t = t_{j+1} - t_j$. We recall that the subscripts on the field arguments are to remind us that we are to consider the field configuration $\Phi_j(\mathbf{x})$ independent of $\Phi_{j+1}(\mathbf{x})$. Assuming that the field is canonically normalised, the Hamiltonian will be of the form

$$\hat{H} = \int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\hat{\Pi}^2(\mathbf{x}) + \frac{1}{2}\big(\boldsymbol{\nabla}\hat{\Phi}(\mathbf{x})\big)^2 + V\big(\hat{\Phi}(\mathbf{x})\big)\right]\ . \tag{128}$$

We have not included time arguments on the field and conjugate-momentum operators, because the Hamiltonian is time independent in the Heisenberg picture, and we can therefore consider the operators to be at any time.

Assuming that $1/\delta t$ is larger than any characteristic energy scale in the system, we expand the

time-evolution operator as a power series in δt. Going to first order will be enough, and we take

$$e^{-i\hat{H}\delta t/\hbar} = 1 - \frac{i}{\hbar}\hat{H}\delta t + \mathcal{O}(\delta t^2) \ . \tag{129}$$

We now consider the matrix element of the Hamiltonian

$$\langle \Phi_{j+1}(\mathbf{x})|\hat{H}|\Phi_j(\mathbf{x})\rangle = \int \mathrm{d}^3\mathbf{x}\, \langle \Phi_{j+1}(\mathbf{x})| \left(\frac{1}{2}\hat{\Pi}^2(\mathbf{x}) + \frac{1}{2}\big(\boldsymbol{\nabla}\hat{\Phi}(\mathbf{x})\big)^2 + V\big(\hat{\Phi}(\mathbf{x})\big)\right) |\Phi_j(\mathbf{x})\rangle \ . \tag{130}$$

We insert a complete set of eigenstates of the conjugate-momentum operator and use the eigenvalue equations

$$\hat{\Phi}(t,\mathbf{x})\,|\Phi(\mathbf{x}),t\rangle = \Phi(\mathbf{x})\,|\Phi(\mathbf{x}),t\rangle \ , \tag{131}$$

$$\hat{\Pi}(t,\mathbf{x})\,|\Pi(\mathbf{x}),t\rangle = \Pi(\mathbf{x})\,|\Pi(\mathbf{x}),t\rangle \ , \tag{132}$$

to write

$$\begin{aligned}\langle \Phi_{j+1}(\mathbf{x})|\hat{H}|\phi_j(\mathbf{x})\rangle = \int [\mathrm{d}\Pi_j(\mathbf{x})] \int \mathrm{d}^3\mathbf{x} \left(\frac{1}{2}\Pi_j^2(\mathbf{x}) + \frac{1}{2}\big(\boldsymbol{\nabla}\Phi_j(\mathbf{x})\big)^2 + V\big(\Phi_j(\mathbf{x})\big)\right) \\ \times \langle \Phi_{j+1}(\mathbf{x})|\Pi_j(\mathbf{x})\rangle \, \langle \Pi_j(\mathbf{x})|\Phi_j(\mathbf{x})\rangle \ .\end{aligned} \tag{133}$$

We can now use that

$$\langle \phi(\mathbf{x})|\pi(\mathbf{x})\rangle = \exp\left[\frac{i}{\hbar}\int \mathrm{d}^3\mathbf{x}\,\Pi(\mathbf{x})\Phi(\mathbf{x})\right] \ , \tag{134}$$

which generalises $\langle x|p\rangle = e^{ipx/\hbar}$ from quantum mechanics, to obtain

$$\begin{aligned}\langle \Phi_{j+1}(\mathbf{x})|\hat{H}|\Phi_j(\mathbf{x})\rangle = \int [\mathrm{d}\Pi_j(\mathbf{x})] \int \mathrm{d}^3\mathbf{x} \left(\frac{1}{2}\Pi_j^2(\mathbf{x}) + \frac{1}{2}\big(\boldsymbol{\nabla}\Phi_j(\mathbf{x})\big)^2 + V\big(\Phi_j(\mathbf{x})\big)\right) \\ \times \exp\left[\frac{i}{\hbar}\int \mathrm{d}^3\mathbf{x}\,\Pi_j(\mathbf{x})\big(\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})\big)\right] \ .\end{aligned} \tag{135}$$

Thus, after re-exponentiating the expansion of the time-evolution operator, we find that

$$\begin{aligned}\langle \Phi_{j+1}(\mathbf{x}),t_{j+1}|\Phi_j(\mathbf{x}),t_j\rangle = \int [\mathrm{d}\Pi_j(\mathbf{x})] \exp\Big\{ -\frac{i\delta t}{\hbar}\int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}\Pi_j^2(\mathbf{x}) + \frac{1}{2}\big(\boldsymbol{\nabla}\Phi_j(\mathbf{x})\big)^2 + V\big(\Phi_j(\mathbf{x})\big) \right. \\ \left. - \Pi_j(\mathbf{x})\,\frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta t}\right]\Big\}\end{aligned} \tag{136}$$

In order to perform the integral with respect to Π_j, we make the analytic continuation $\delta t \to -i\delta\tau$

— we'll see what this effects shortly — giving

$$\int [\mathrm{d}\Pi_j(\mathbf{x})] \exp\left\{ -\frac{\delta\tau}{\hbar} \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}\Pi_j^2(\mathbf{x}) + \frac{1}{2}\big(\boldsymbol{\nabla}\Phi_j(\mathbf{x})\big)^2 + V\big(\Phi_j(\mathbf{x})\big) - i\Pi_j(\mathbf{x}) \, \frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta\tau} \right] \right\} . \tag{137}$$

We complete the square by writing

$$\Pi_j^2(\mathbf{x}) - 2i\Pi_j(\mathbf{x}) \, \frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta t} = \left(\Pi_j(\mathbf{x}) - i\frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta\tau} \right)^2 + \left(\frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta\tau} \right)^2 . \tag{138}$$

The remaining Gaussian-like integral over Π_j just gives some normalisation $\mathcal{N}_j$, and we have

$$\begin{aligned} \langle \Phi_{j+1}(\mathbf{x}), -i\tau_{j+1} | \Phi_j(\mathbf{x}), -i\tau_j \rangle = \mathcal{N}_j \exp\left\{ -\frac{\delta\tau}{\hbar} \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2} \left(\frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta\tau} \right)^2 + \frac{1}{2}\big(\boldsymbol{\nabla}\Phi_j(\mathbf{x})\big)^2 \right.\right. \\ \left.\left. + V\big(\Phi_j(\mathbf{x})\big) \right] \right\} . \end{aligned} \tag{139}$$

Now we turn to the vacuum persistence amplitude (in the absence of a source)

$$Z[0] = \langle 0(+\infty) | 0(-\infty) \rangle \ . \tag{140}$$

We proceed by inserting N complete sets of field eigenstates to give

$$Z[0] = \langle 0(+\infty) | \left[\prod_{j=N}^{1} \int [\mathrm{d}\Phi_j(\mathbf{x})] \, |\Phi_j(\mathbf{x}), t_j\rangle \, \langle \Phi_j(\mathbf{x}), t_j| \right] |0(-\infty)\rangle \ . \tag{141}$$

Using our result from above, and after making the same analytic continuation of δt, this is

$$\begin{aligned} Z[0] = \left[\prod_{j=1}^{N} \int [\mathrm{d}\Phi_j(\mathbf{x})] \right] \langle 0(-i\infty) | \Phi_N(\mathbf{x}), -i\tau_N \rangle \, \langle \Phi_1(\mathbf{x}), -i\tau_1 | 0(i\infty) \rangle \\ \times \exp\left\{ -\frac{1}{\hbar} \sum_{j=1}^{N} \delta\tau \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2} \left(\frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta\tau} \right)^2 + \frac{1}{2}\big(\boldsymbol{\nabla}\Phi_j(\mathbf{x})\big)^2 + V\big(\Phi_j(\mathbf{x})\big) \right] \right\} . \end{aligned} \tag{142}$$

All we need to do now — it is a big "All", by the way — is to take the limit $N \to \infty$, such that $\delta\tau \to 0$. We then recognise that

$$\lim_{\delta\tau\to 0} \frac{\Phi_{j+1}(\mathbf{x}) - \Phi_j(\mathbf{x})}{\delta\tau} = \frac{\partial\Phi(\tau, \mathbf{x})}{\partial\tau} \ , \tag{143}$$

and the sum over j becomes an integral over a coordinate τ. In this way, we arrive at

$$Z[0] = \int \mathcal{D}\Phi \exp\left\{ -\frac{1}{\hbar}\int \mathrm{d}^4x \left[\frac{1}{2}\big(\partial\Phi(x)\big)^2 + V\big(\Phi(x)\big)\right]\right\}, \tag{144}$$

where $\big(\partial\Phi\big)^2 = \big(\partial_\tau\Phi\big)^2 + \big(\boldsymbol{\nabla}\Phi\big)^2$ and we have defined the functional integral

$$\int \mathcal{D}\Phi \coloneqq \left[\prod_{j=1}^{N}\int [\mathrm{d}\Phi_j(\mathbf{x})]\right] \langle 0(-i\infty)|\Phi_N(\mathbf{x}), -i\tau_N\rangle \, \langle \Phi_1(\mathbf{x}), -i\tau_1|0(i\infty)\rangle \ , \tag{145}$$

Notice that we have been the usual level of nonchalant about burying boundary conditions in the functional integral. There are cases when we need to be much more careful about this.

The exponent of Eq. (144) contains the Euclidean action

$$S_{\mathrm{E}}[\Phi] = \int \mathrm{d}^4x \left[\frac{1}{2}\big(\partial\Phi(x)\big)^2 + V\big(\Phi(x)\big)\right]. \tag{146}$$

In order to go back to Minkowski spacetime, we Wick rotate by taking $\tau \to it$, giving

$$Z[0] = \int \mathcal{D}\Phi \exp\left\{ \frac{i}{\hbar}\int \mathrm{d}^4x \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi(x)\partial_\nu\Phi(x) - V\big(\Phi(x)\big)\right]\right\}, \tag{147}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. Introducing the source just amounts to a shift in the potential, and we therefore obtain the following result for the partition function:

$$Z[J] = \int \mathcal{D}\Phi \exp\left\{ \frac{i}{\hbar}\int \mathrm{d}^4x \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi(x)\partial_\nu\Phi(x) - V\big(\Phi(x)\big) - J(x)\Phi(x)\right]\right\}. \tag{148}$$

We could have included the source from the outset, but we would have had to worry about the explicit time-dependence that it would introduce, and the derivation above was painful enough.

6.3 The Schwinger–Dyson equation

In order to evaluate the order $\hbar^2$ terms in the effective action, we expand the exponential in the vertices $S^{(3)}$ and $S^{(4)}$:

$$\begin{aligned} Z[J,K] = e^{-\frac{1}{\hbar}\left(S[\varphi] - J(z)\varphi(z) - \frac{1}{2}K(x,y)\varphi(x)\varphi(y)\right)} \int \mathcal{D}\delta\Phi \ e^{-\frac{1}{2}\delta\Phi(x)\mathcal{G}^{-1}(x,y)\delta\Phi(y)} \\ \times \left[1 - \frac{\hbar^{1/2}\lambda}{3!}\varphi(x)\delta\Phi^3(x) - \frac{\hbar\lambda}{4!}\delta\Phi^4(x) + \frac{\hbar\lambda^2}{2!(3!)^2}\varphi(x)\delta\Phi^3(x)\varphi(y)\delta\Phi^3(y) + \mathcal{O}(\hbar^{3/2})\right]. \end{aligned} \tag{149}$$

Now we can apply Wick's theorem, noting that the first term, which is cubic in $\delta\Phi$ will yield a vanishing contribution. Thus, we find

$$\begin{aligned} Z[J,K] &\propto e^{-\frac{1}{\hbar}\left(S[\varphi]-J(z)\varphi(z)-\frac{1}{2}K(x,y)\varphi(x)\varphi(y)\right)}\det{}^{-1/2}\mathcal{G}^{-1} \\ &\times\left[1-\frac{\hbar\lambda}{8}\mathcal{G}^2(x,x)+\frac{\hbar\lambda^2}{12}\varphi(x)\mathcal{G}^3(x,y)\varphi(y)+\frac{\hbar\lambda^2}{8}\mathcal{G}(x,x)\varphi(x)\mathcal{G}(x,y)\varphi(y)\mathcal{G}(y,y)+\mathcal{O}(\hbar^{3/2})\right], \end{aligned} \tag{150}$$

where we have again ignored overall constant factors. Taking the logarithm, and further expanding $\ln(1+x)=1+x+\mathcal{O}(\hbar^2)$, we find

$$\Gamma_2[\varphi,\mathcal{G}]=\frac{\lambda}{8}\mathcal{G}^2(x,x)-\frac{\lambda^2}{12}\varphi(x)\mathcal{G}^3(x,y)\varphi(y)\ , \tag{151}$$

$$\Gamma_{\rm 1PR}[\varphi,\mathcal{G}]=-\frac{\lambda^2}{8}\mathcal{G}(x,x)\varphi(x)\mathcal{G}(x,y)\varphi(y)\mathcal{G}(y,y)\ . \tag{152}$$

We now consider the term

$$\frac{1}{2}\big(\phi(x)-\varphi(x)\big)\left[\frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\Big|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}}-\frac{2}{\hbar}\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\right]\big(\phi(y)-\varphi(y)\big)\ . \tag{153}$$

First, we calculate

$$\begin{aligned} \phi(x)-\varphi(x)&=\frac{\delta W[J,K]}{\delta J(x)}-\varphi(x) \\ &=\frac{1}{Z[J]}\int\mathcal{D}\delta\Phi\ \hbar^{1/2}\delta\Phi(x)\exp\bigg[-\frac{1}{\hbar}\bigg(S[\varphi]-J(z)\varphi(z)-\frac{1}{2}K(x,y)\varphi(x)\varphi(y) \\ &\quad-\frac{\hbar}{2}\delta\Phi(x)\mathcal{G}^{-1}(x,y)\delta\Phi(y)+\frac{\hbar^{3/2}}{3!}\lambda\varphi(x)\delta\Phi^3(x)+\mathcal{O}(\hbar^2)\bigg)\bigg] \\ &=\hbar^{1/2}\int\mathcal{D}\delta\Phi\ \delta\Phi(x)\bigg(1-\frac{\hbar^{1/2}\lambda}{3!}\varphi(y)\delta\Phi^3(y)\bigg)e^{-\frac{1}{2}\delta\Phi(x)\mathcal{G}^{-1}(x,y)\delta\Phi(y)}+\mathcal{O}(\hbar^2) \\ &=-\frac{\hbar\lambda}{2}\mathcal{G}(x,y)\varphi(y)\mathcal{G}(y,y)\ . \end{aligned} \tag{154}$$

Next, we have that

$$\frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\Big|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}}-\frac{2}{\hbar}\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}=\mathcal{G}(x,y)^{-1}+\mathcal{O}(\hbar)\ , \tag{155}$$

and hence

$$\frac{1}{2}\big(\phi(x)-\varphi(x)\big)\left[\left.\frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\right|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}}-\frac{2}{\hbar}\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\right]\big(\phi(y)-\varphi(y)\big)$$
$$=\frac{\hbar^2\lambda^2}{8}\mathcal{G}(x,x)\varphi(x)\mathcal{G}(x,y)\varphi(y)\mathcal{G}(y,y)\;, \tag{156}$$

which cancels $\hbar^2\Gamma_{1\mathrm{PR}}$.

Taking the partial functional derivative of $\Gamma[\varphi,\mathcal{G}]$ with respect to $\mathcal{G}$, we have

$$\frac{\delta\Gamma[\varphi,\mathcal{G}]}{\delta\mathcal{G}(x,y)}=-\frac{\hbar}{2}\mathcal{G}^{-1}(x,y)+\frac{\hbar}{2}G^{-1}(x,y)+\hbar^2\frac{\delta\Gamma_2[\varphi,\mathcal{G}]}{\delta\mathcal{G}(x,y)}\;. \tag{157}$$

If we ask that $\mathcal{G}$ be the extremum of the effective action then this must vanish, and we obtain the Schwinger–Dyson equation

$$\mathcal{G}^{-1}(x,y)=G^{-1}(x,y)+2\hbar\frac{\delta\Gamma_2[\varphi,[G]}{\delta\mathcal{G}(x,y)}\;. \tag{158}$$

The right-most term is the proper 1PI self-energy

$$2\hbar\frac{\delta\Gamma_2[\varphi,\mathcal{G}]}{\delta\mathcal{G}(x,y)}=\frac{\hbar\lambda}{2}\mathcal{G}(x,x)\delta^{(4)}(x-y)-\frac{\hbar\lambda^2}{2}\varphi(x)\mathcal{G}^2(x,y)\varphi(y)\;. \tag{159}$$

Notice that the self-energy is calculated in terms of $\mathcal{G}$, and the 2PI effective action has therefore yielded the Schwinger–Dyson equation for the 1PI-resummed two-point function.

6.4 Convexity and the Maxwell construction

In the lecture, we casually assumed that there was only a single saddle point of the exponent in the path integral. This is certainly not the case if we take $m^2<0$, when we have minima of the classical potential at $\varphi_\pm=\pm\sqrt{6}m/\sqrt{\lambda}$ and a maximum at $\varphi_0=0$, i.e., three saddle points. It is also not the case for $m^2>0$ if $m^2-K<0$. Thus, more generally, we need to sum over all saddle points φ_i when performing the saddle-point approximation. In our case, it turns out that we only need to take into account the local minima.

Since we want to obtain the effective potential, we are actually only interested in the behaviour of the zero mode, i.e., the infinite-wavelength contribution. There is a neat trick to project this mode out, and we proceed by inserting into the path integral a particular "partitioning of unity"

$$1=\int_{-\infty}^{+\infty}\delta\left(\varphi-\Omega^{-1}\int\mathrm{d}^4x\;\Phi(x)\right) \tag{160}$$

to give

$$Z[J,K] = \int \mathcal{D}\Phi \int_{-\infty}^{+\infty} \delta\left(\varphi - \Omega^{-1}\int \mathrm{d}^4x\ \Phi(x)\right) \\ \times \exp\left[-\frac{1}{\hbar}\left(S[\Phi] - J(x)\Phi(x) - \frac{1}{2}K(x,y)\Phi(x)\Phi(y)\right)\right] . \tag{161}$$

The subsequent manipulations are rather involved, and they are described in detail in [6] and the original (and rather lucid) work by Rivers [7]. In the end, we are led to the following expression for the partition function, which now depends only on the zero mode:

$$Z[J,K] = \sum_{\pm} \exp\left[-\frac{\Omega}{\hbar}\left(\bar{S}(\varphi_\pm) - \bar{J}\varphi_\pm - \frac{1}{2}\bar{K}\varphi_\pm^2 + \frac{\hbar}{2\Omega}\mathrm{tr}\ln\mathcal{G}_\pm^{-1}G(0) + \mathcal{O}(\hbar^2)\right)\right] , \tag{162}$$

where

$$\bar{S}(\varphi) = \Omega^{-1}\int \mathrm{d}^4x\ S[\varphi] , \quad \bar{J} = \Omega^{-1}\int \mathrm{d}^4x\ J(x) , \quad \bar{K} = \Omega^{-1}\int \mathrm{d}^4x \int \mathrm{d}^4y\ K(x,y) . \tag{163}$$

This is to say that we now have a zero-dimensional field theory, where the only memory of the original spacetime dependence lies in the volume factor Ω. (The barred quantities here should not be confused with the affine coordinates that we will meet in lecture 3.)

We can now proceed to calculate the one-point function

$$\phi \approx \frac{\varphi_+ Z_+ + \varphi_- Z_-}{Z_+ + Z_-} , \tag{164}$$

and, after some algebra, we can show that

$$\frac{Z_+}{Z_-} = \frac{\phi - \varphi_-}{\varphi_+ - \phi} . \tag{165}$$

Taking the logarithm, we have (at order $\hbar$)

$$-\frac{\hbar}{\Omega}\ln\left[\frac{\phi-\varphi_-}{\varphi_+-\phi}\right] = -\frac{\hbar}{\Omega}\ln Z_+ + \frac{\hbar}{\Omega}\ln Z_- \\ = \bar{S}(\varphi_+) - \bar{S}(\varphi_-) - \bar{J}(\varphi_+ - \varphi_-) - \frac{1}{2}\bar{K}(\varphi_+^2 - \varphi_-^2) + \frac{\hbar}{2\Omega}\mathrm{tr}\ln\mathcal{G}_+^{-1}\mathcal{G}_- , \tag{166}$$

and we can solve for

$$\bar{J} = \frac{\bar{S}(\varphi_+) - \bar{S}(\varphi_-)}{\varphi_+ - \varphi_-} - \frac{1}{2}\bar{K}(\varphi_+ + \varphi_-) + \frac{\hbar}{\varphi_+ - \varphi_-}\left\{\ln\left[\frac{\phi-\varphi_-}{\varphi_+-\phi}\right] + \frac{1}{2\Omega}\mathrm{tr}\ln\mathcal{G}_+^{-1}\mathcal{G}_- +\right\} . \tag{167}$$

We see from the leftmost logarithm in Eq. (167) that there are branch points in $\bar{J}$ when $\phi = \varphi_\pm$,

and this is because outside the range $\varphi_- < \phi < \varphi_+$, we only have one saddle point. Within this range and at zeroth order in $\hbar$, the exponents of $Z_\pm$ are

$$\begin{aligned} -\frac{\hbar}{\Omega}\ln Z_\pm &= \bar{S}(\varphi_\pm) - \frac{\bar{S}(\varphi_+)-\bar{S}(\varphi_-)}{\varphi_+-\varphi_-}\,\varphi_\pm + \frac{1}{2}\bar{K}(\varphi_+ + \varphi_-)\varphi_\pm - \frac{1}{2}\bar{K}\varphi_\pm^2 \\ &= \frac{\varphi_+\bar{S}(\varphi_+)-\varphi_-\bar{S}(\varphi_-)}{\varphi_+-\varphi_-} + \frac{1}{2}\bar{K}\varphi_+\varphi_- \,, \end{aligned} \tag{168}$$

i.e., the same. For constant field configurations, the effective potential can be defined as $U(\phi) = \Gamma[\phi,\Delta]/\Omega$, and we find

$$U(\phi) = \frac{\varphi_+\bar{S}(\varphi_+)-\varphi_-\bar{S}(\varphi_-)}{\varphi_+-\varphi_-} + \frac{1}{2}\bar{K}\varphi_+\varphi_- - \left[\frac{\bar{S}(\varphi_+)-\bar{S}(\varphi_-)}{\varphi_+-\varphi_-} - \frac{1}{2}\bar{K}(\varphi_+ + \varphi_-)\right]\phi - \frac{1}{2}\bar{K}\phi^2 \,. \tag{169}$$

Taking the limit $\bar{K} \to 0$, which actually corresponds to the 1PI effective action — all the better if we are interested in the effective potential — we have

$$U(\phi) = \frac{(\varphi_+-\phi)\bar{S}(\varphi_+) + (\phi-\varphi_-)\bar{S}(\varphi_-)}{\varphi_+-\varphi_-} \,. \tag{170}$$

Thus, for $\varphi_- < \phi < \varphi_+$, the effective potential is a monotonically increasing function of ϕ, connecting the original local minima of the potential. This is the *Maxwell construction*. This implies that the effective potential and also the effective action are convex, despite the non-convexity of the classical potential.

6.5 Obtaining vertex functions

The Hessians of our "Schwinger function" and the 2PI effective action are

$$\mathrm{Hess}\, W = \begin{pmatrix} \frac{\partial^2 W}{\partial\bar{J}\partial\bar{J}} & \frac{\partial^2 W}{\partial\bar{J}\partial\bar{K}} \\ \frac{\partial^2 W}{\partial\bar{K}\partial\bar{J}} & \frac{\partial^2 W}{\partial\bar{K}\partial\bar{K}} \end{pmatrix} \qquad \text{and} \qquad \mathrm{Hess}\,\Gamma = \begin{pmatrix} \frac{\partial^2 \Gamma}{\partial\bar{\phi}\partial\bar{\phi}} & \frac{\partial^2 \Gamma}{\partial\bar{\phi}\partial\bar{\Delta}} \\ \frac{\partial^2 \Gamma}{\partial\bar{\Delta}\partial\bar{\phi}} & \frac{\partial^2 \Gamma}{\partial\bar{\Delta}\partial\bar{\Delta}} \end{pmatrix} \,. \tag{171}$$

Using the expressions that we have encountered in the lectures for the first variations of the Schwinger functional and the 2PI effective action, we can rewrite these as

$$\mathrm{Hess}\, W = \begin{pmatrix} \frac{\partial\bar{\phi}}{\partial\bar{J}} & \frac{\partial\bar{\phi}}{\partial\bar{K}} \\ \frac{\partial\bar{\Delta}}{\partial\bar{J}} & \frac{\partial\bar{\Delta}}{\partial\bar{K}} \end{pmatrix} \qquad \text{and} \qquad \mathrm{Hess}\,\Gamma = \begin{pmatrix} \frac{\partial\bar{J}}{\partial\bar{\phi}} & \frac{\partial\bar{J}}{\partial\bar{\Delta}} \\ \frac{\partial\bar{K}}{\partial\bar{\phi}} & \frac{\partial\bar{K}}{\partial\bar{\Delta}} \end{pmatrix} \,. \tag{172}$$

Recalling that $\bar{\phi}$ and $\bar{\Delta}$ are functions of $\bar{J}$ and $\bar{K}$ then it follows from the chain rule that

$$\mathrm{Hess}\, W \cdot \mathrm{Hess}\,\Gamma = \mathbb{I} \,. \tag{173}$$

This also allows us to prove the convexity of the effective action. The Hessian of the Schwinger functional is just a covariance matrix. It is therefore positive definite. We see from the above that the Hessian of the effective action is its inverse, and it is therefore also positive definite. Thus, the effective action is a convex functional, but note that it is convex with respect to the affine coordinates $\bar{\phi}$ and $\bar{\Delta}$. It need not be convex in terms of ϕ and Δ, the connected two-point function.

For the next part of the question, we need to invert the Hessian of the effective action, to obtain

$$\begin{pmatrix} \frac{\partial^2 W}{\partial \bar{J} \partial J} & \frac{\partial^2 W}{\partial \bar{J} \partial K} \\ \frac{\partial^2 W}{\partial \bar{K} \partial J} & \frac{\partial^2 W}{\partial \bar{K} \partial K} \end{pmatrix} = \frac{1}{\det \operatorname{Hess} \Gamma} \begin{pmatrix} \frac{\partial^2 \Gamma}{\partial \bar{\Delta} \partial \Delta} & -\frac{\partial^2 \Gamma}{\partial \bar{\phi} \partial \Delta} \\ -\frac{\partial^2 \Gamma}{\partial \bar{\Delta} \partial \phi} & \frac{\partial^2 \Gamma}{\partial \bar{\phi} \partial \phi} \end{pmatrix} . \tag{174}$$

We therefore know that

$$\Delta = \frac{\delta W}{\delta \bar{J} \delta \bar{J}} = \frac{1}{\det \operatorname{Hess} \Gamma} \frac{\partial^2 \Gamma}{\partial \bar{\Delta} \partial \bar{\Delta}} . \tag{175}$$

To rewrite this in terms of ϕ and Δ, we need to use the chain rule to change the derivatives, with

$$\frac{\partial}{\partial \bar{\phi}} = \frac{\partial}{\partial \phi} - 2\phi \frac{\partial}{\partial \Delta} \qquad \text{and} \qquad \frac{\partial}{\partial \bar{\Delta}} = \frac{\partial}{\partial \Delta} , \tag{176}$$

such that

$$\frac{\partial^2 \Gamma}{\partial \bar{\Delta} \partial \bar{\Delta}} = \frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} , \tag{177}$$

$$\frac{\partial^2 \Gamma}{\partial \bar{\phi} \partial \bar{\Delta}} = \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} - 2\phi \frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} , \tag{178}$$

$$\frac{\partial^2 \Gamma}{\partial \bar{\phi} \partial \bar{\phi}} = \frac{\partial^2 \Gamma}{\partial \phi \partial \phi} - 2\frac{\partial \Gamma}{\partial \Delta} - 2\phi \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} + 4\phi^2 \frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} . \tag{179}$$

Hence,

$$\begin{aligned} \det \operatorname{Hess} \Gamma &= \left[\frac{\partial^2 \Gamma}{\partial \phi \partial \phi} - 2\frac{\partial \Gamma}{\partial \Delta} - 4\phi \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} + 4\phi^2 \frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} \right] \frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} - \left[\frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} - 2\phi \frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} \right]^2 \\ &= \left[\frac{\partial^2 \Gamma}{\partial \phi \partial \phi} - 2\frac{\partial \Gamma}{\partial \Delta} - \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} \left(\frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} \right)^{-1} \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} \right] \frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} , \end{aligned} \tag{180}$$

and we find the required expression for the two-point function

$$\Delta = \left[\frac{\partial^2 \Gamma}{\partial \phi \partial \phi} - 2\frac{\partial \Gamma}{\partial \Delta} - \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} \left(\frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta} \right)^{-1} \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta} \right]^{-1} . \tag{181}$$

The second term is just the source K, and the third term effects the 2PI resummation, i.e., the resummation of all 1PI self-energy insertions.

For the final part of the question, we write

$$\frac{\partial W}{\partial \bar{J}\partial \bar{J}}\frac{\partial}{\partial \bar{\phi}}+\frac{\partial^2 W}{\partial \bar{J}\partial \bar{K}}\frac{\partial}{\partial \bar{\Delta}}=\frac{\partial W}{\partial \bar{J}\partial \bar{J}}\left[\frac{\partial}{\partial \phi}-2\phi\frac{\partial}{\partial \Delta}\right]+\frac{\partial^2 W}{\partial \bar{J}\partial \bar{K}}\frac{\partial}{\partial \Delta}\,. \tag{182}$$

Using Eq. (175), we can write

$$\det \mathrm{Hess}\,\Gamma=\frac{\partial^2\Gamma}{\partial\bar{\Delta}\partial\bar{\Delta}}\Delta^{-1}\,, \tag{183}$$

such that

$$\frac{\partial^2 W}{\partial \bar{J}\partial \bar{K}}=-\Delta\left(\frac{\partial^2\Gamma}{\partial\bar{\Delta}\partial\bar{\Delta}}\right)^{-1}\frac{\partial^2\Gamma}{\partial\bar{\Delta}\partial\bar{\phi}}=-\Delta\left(\frac{\partial^2\Gamma}{\partial\Delta\partial\Delta}\right)^{-1}\left(\frac{\partial^2\Gamma}{\partial\phi\partial\Delta}-2\phi\frac{\partial^2\Gamma}{\partial\Delta\partial\Delta}\right)\,, \tag{184}$$

where we have changed coordinates at the second equality. Putting everything together, we have

$$\frac{\partial W}{\partial \bar{J}\partial \bar{J}}\frac{\partial}{\partial \bar{\phi}}+\frac{\partial^2 W}{\partial \bar{J}\partial \bar{K}}\frac{\partial}{\partial \bar{\Delta}}=\Delta\left[\frac{\partial}{\partial\phi}-\frac{\partial^2\Gamma}{\partial\phi\partial\Delta}\left(\frac{\partial^2\Gamma}{\partial\Delta\partial\Delta}\right)^{-1}\frac{\partial}{\partial\Delta}\right]\,, \tag{185}$$

as required. We can now use Eq. (66) to obtain the n-point correlation functions directly from functional variations of the effective action.

This was reasonably painful, but this was mostly down to our insistence of working in the non-affine coordinate frame with coordinates ϕ and Δ. The 2PI effective action is an extremely powerful too, but the manipulations needed to make use of it can take some getting used to.

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